Witness-Preserving Gluing and Variational Induction in the Collatz Problem

Y.Ueoka
Independent Researcher (corresponding author)

Tsumugi (ChatGPT)
AI Co-researcher

29 November 2026 — Preprint, version 1.0

Abstract

We organize earlier ABN/CPE results on the design and modification of structural scores for the accelerated Collatz map. A common score combines finite compression certificates with analytic descent on occupied-cell-dominant regions. Monotone corrections, bounded labels, and lexicographic rank refinements preserve specified earlier descent witnesses. A three-component ordered score admits an explicit nonnegative integer realization, and coalescence with a lower-ranked state extends the induction criterion beyond forward descent. The aim is to preserve earlier proofs while designing broader induction schemes, not to assert universal descent. Concrete unit-band exit proofs realize a hierarchy of bounded labels. A repaired common score covers all normalized full-C(2) words, and explicit equality-breaking families and a high-count unit-band theorem illustrate the three-component order.

1 Purpose and notation

The ABN/CPE programme distinguishes exact structural dynamics (Paper A) from finite-time descent results (Paper B) [1, 2]. Here the problem is to combine such results using a common well-founded order. The methodological starting point is the Gluing Method introduced by Y. Ueoka in [3], which combines reliable local relations across different regimes. The present paper develops its application to integer dynamics: instead of reconstructing a global model from local relations, we construct a common well-founded score from certified local descent conditions, subject to explicit preservation inequalities. In the present setting, a change of score is admissible for a specified collection of witnesses only if their strict inequalities still hold. A search for a score and a proof of its descent conditions are thus coupled; this is the role of variational induction in the programme. State-dependent weights, discontinuous corrections, and ordered refinements are allowed; no particular parametric family is imposed.

Let $F(n) = (3n+1)/2^{\nu_2(3n+1)}$ on positive odd integers. Write $E(n) = n \text{ XOR } 2n$ for the occupied-cell mask of the signed-power ABN representation (not the ordinary binary digits of n). The low-to-high CPE is $C(r_0)K(d_0)\cdots K(d_{m-1})C(r_m)$ with positive run lengths and even total occupied count. Write $Z = \sum d_i$, $H_C = \sum r_i$, $B = Z + H_C$, $\bar{r}_0 = 2 - (r_0 \bmod 2)$, $H = H_C - r_0 + \bar{r}_0$, and $B_* = Z + H$. Normalizing only the lowest C-run to $\bar{r}_0$ gives ρ, with $\rho^2 = \rho$ and $F\rho = F$ (Paper B, P6-351–353). The base $Z = 0$ has $F(n) = 1$ and $H = 2$.

For a score Q a witness is a pair (n, t) with $t \geq 1$ and $Q(F^t(n)) < Q(n)$. A new score preserves that witness when the strict inequality holds at the same time. Statements about individual scores do not automatically combine into an induction. In particular the endpoint need not remain in the starting region. A nonnegative integer score gives a well-founded order, and a real score with finite sublevel sets on the normalized domain also does so there. For any ρ-invariant score, $G = \rho \circ F$ satisfies $G^t\rho(n) = \rho(F^t(n))$ for $t \geq 0$, so descent proofs on normalized states transfer. Finite sublevel sets are not required on the unnormalized domain for integer-valued scores.

For $m \geq 1$, the exact output word is $V_0 \cdots V_{m-1} 1^{1+p_m}$, where $p_i = r_i \bmod 2$, $\epsilon_i = p_i \,\mathrm{XOR}\, \mathbf{1}_{i=0}$, and

$$V_i = \begin{cases} 1^{1+\epsilon_i} 0^{d_i - 1 - \epsilon_i} 1\, 0^{r_{i+1}-1}, & d_i \geq 2, \\ 1^{2-\epsilon_i} 0^{r_{i+1}-1+\epsilon_i}, & d_i = 1. \end{cases}$$

A zero-length word is empty. Read the concatenated word from low to high; adjacent occupied cells form a single maximal C-run, even across block boundaries. For a K-run of length $d \geq 3$, the first zero interval in its block has length $d - 1 - \epsilon \geq d - 2 \geq 1$ and is bounded by occupied cells. Thus a capped long band stays identifiable through T updates when its initial length is at least $2T + 1$: before update $t < T$ its length is at least $2(T - t) + 1$. Increasing such an initial band changes only the length of this descendant interval at each step, preserving the C-run list and all other band lengths. Induction therefore preserves the initial difference of vacancy counts through time T. This explains the K-band compression used below. We use the established update and finite compression theorems of Papers A and B, with the explicit source numbers below. These two earlier papers supply the structural update and compression lemmas; the finite replay in Appendix A tests the present finite certificates independently of the CPE update implementation.

2 A common score and variations preserving its proofs

Set $Q_0 = Z + 2(H - 28)_+$. It is a nonnegative integer score, zero precisely on the base; its normalized sublevel sets are finite since Z and H are bounded there. Its expression as a state-dependent weighted combination of Z and $Z + 2H$ uses $w_2 = (H - 28)_+/H$, $w_1 = 1 - w_2$.

Theorem 2.1 (P6-569; finite computation)**.** *If $Z > 0$ and $H \leq 10$, there exists $1 \leq t \leq 54$ with $Q_0(F^t(n)) < Q_0(n)$. The selected witnesses have both endpoint normalized occupied counts H at most 28.*

Proof. Use the adaptive finite partition for vacancy descent of Paper B (P6-565/566). On a normalized ordered composition of H with $r_0 \leq 2$ and at least one K-run, cap K-lengths at $2T + 1$ at stage T. A representative with a Z witness by T certifies its whole cell; otherwise subdivide each tail coordinate $2T + 1$ into the two singleton lengths $2T + 1, 2T + 2$ and the new tail $\geq 2T + 3$. The compression theorem preserves Z increments and the entire C-run length list through T. The original R104 program checks the normalized H at the first Z witness in each certified cell. The replay in R25 reproduces its saved results exactly:

Initial H	Checks	Certified cells	Last T	Max. endpoint H
2	3	3	1	4
4	77	64	5	10
6	1554	1114	40	16
8	21518	18152	54	22
10	335288	296262	51	28

No unresolved cells remain. The endpoint bounds apply to every state in the respective cells by preservation of C-run lists. Both endpoints have $Q_0 = Z$. Normalization transfers the witnesses to the original states. This result is computer-assisted; Appendix A provides a standalone replay, including the endpoint bounds. □

Theorem 2.2 (P6-570/571; analytic common-score descent)**.** *For $Z > 0$, let $I_i = \mathbf{1}_{d_i=1}$ and put*

$$U = 1 + p_m + \sum_{i<m} \{2 + \epsilon_i(1 - 2I_i)\}, \qquad \delta = 1 + p_m - \bar{r}_0.$$

If $H > \max(28, 54 + \delta, U + \delta)$, then $Q_0(F(n)) < Q_0(n)$. In particular this holds for $H > \max(55, 3m + 3)$.

Proof. Counting the occupied cells and width in the output word gives $H_C(F(n)) = U$, $B(F(n)) - B(n) = -r_0 + 1 + p_m$. Thus $2 \le H' \le U$ and $B'_* - B_* \le \delta$. With $g(h) = 2(h-28)_+ - h$, we have $Q_0 = B_* + g(H)$. Convexity gives $g(H') \le \max(-2, U - 56)$, and $g(H) = H - 56$ by hypothesis. Hence $\Delta Q_0 \le \max(54 + \delta, U + \delta) - H < 0$. Also $U \le 3m + 2$ and $\delta \le 1$, proving the stated simpler criterion. □

Theorem 2.3 (P6-572; proof-preserving variation)**.** *For any nonnegative nondecreasing real function u on $[2, \infty)$ with $u(h) = 0$ for $h \le 28$, the score $Q_0 + u(H)$ preserves the witnesses of Theorems 2.1 and 2.2 at the same times. It retains the base and finite normalized sublevel sets.*

Proof. For the low-count certificates the correction vanishes at both endpoints. For the analytic certificates $H > U + \delta$ with $\delta \ge -1$ and integral H, U implies $H \ge U \ge H'$. Hence $u(H') \le u(H)$. Nonnegativity preserves the sublevel bound and the base. The proof does not assert preservation for arbitrary variations. □

Theorem 2.4 (P6-928; localized modification)**.** *Define $Q = Z + p(H)$, where for even $h \ge 2$*

$$p(h) = \begin{cases} 0, & h \le 28, \\ 25(h-29)/24, & 30 \le h \le 52, \\ 2(h-28), & h \ge 54. \end{cases}$$

Then $24Q$ is a nonnegative integer score with base $Z = 0$ and finite normalized sublevel sets. Every Q_0 witness from an initial state with $H \notin [30, 52]$ is preserved, including Theorems 2.1 and 2.2.

Proof. On the allowed even counts, p is nonnegative, nondecreasing, unbounded, and $p(h) \le 2(h-28)_+$. Equality holds outside the modified band. Thus $Q(w) \le Q_0(w) < Q_0(n) = Q(n)$ for any such old witness with endpoint w. The condition in Theorem 2.2 implies $H > 54 + \delta \ge 53$, hence $H \ge 54$. The remaining assertions follow from the definition and normalization. □

Local changes can lose witnesses whose initial counts lie inside the modified band; no universal monotonic improvement is asserted.

2.1 Repair of a common score: the full-C(2) domain

The earlier R220 candidate was $Q_{220} = Z + [25(H-29)_+ + 23(H-52)_+]/24$. It is not an order-preserving replacement of Q_0 on every earlier domain. For example the normalized word with $r = (1, 10, 12, 9, 6, 13, 9)$ and $d = (14, 4, 14, 10, 5, 4)$ has $H = 60$, $H' = 16$, $\Delta Q_0 = -19$, but $24\Delta Q_{220} = 121$. The localized repair Q in Theorem 2.4 restores this witness. This is a loss of one selected one-step witness, not a counterexample to finite-time descent.

Lemma 2.5 (P6-573; full-C(2) balances)**.** *For a normalized word with all $r_i = 2$ and $m \ge 1$, $H = 2m + 2$. If $d_0 \ge 2$, then $\Delta Z = -1$ and $H' \le H$. If $d_0 = 1$, put $a = \#\{i \ge 1 : d_i \ge 3\}$. Then*

$$Z' - Z = 1, \quad H' = H - 2, \qquad Z'' - Z = -2a, \quad H'' = H + 2a.$$

Primes denote successive iterates of F, and counts are normalized as above.

Proof. The flags are $\epsilon_0 = 1$, $\epsilon_i = 0$ for $i \ge 1$. Counting output zeros gives $\Delta Z = 2\mathbf{1}_{d_0=1} - 1$, and counting occupied cells gives $H' \le H - 2\mathbf{1}_{d_0=1}$. For $d_0 = 1$ the first output word starts with $C(1)K(2)$; each subsequent band contributes $C(2)K(1)$ when $d_i = 1$, or $C(1)K(d_i - 1)C(1)K(1)$ otherwise. The last run is $C(1)$. Hence $H' = 2m$, with no bottom normalization needed. Let $b = \#\{i \ge 1 : d_i \ge 2\}$. The new band count is $m + b$. In the next zero balance the parity correction $\sum_j \epsilon'_j(2\mathbf{1}_{d'_j=1} - 1)$ is $2(b - a)$: the initial K(2) contributes zero, the blocks from $d_i = 1$ contribute zero, those from $d_i = 2$ contribute two, and those from $d_i \ge 3$ contribute zero. Thus

$Z'' - Z' = 2m - 1 - 2(m + b) + 2(b - a) = -1 - 2a$. The bottom run of the second output is also one, since the first residual zero interval of the initial K(2) has length one. The normalized width changes by -1 and $+1$ in the two updates (counting output length); consequently $H'' = H + 2a$. □

Theorem 2.6 (P6-926/929; common-score descent on full-C(2))**.** *Every nonbase word whose normalized C-runs all have length two has a Q descent witness by time 95, with no bounds on band lengths or their number. Together with Theorem 2.4, the same Q retains the guarantees of Theorems 2.1 and 2.2.*

Proof. If $d_0 \geq 2$, Lemma 2.5 and monotonicity of p give one-step descent. If $d_0 = 1$ and $m \geq 14$, then $H \geq 30$ and $p(H) - p(H-2) \geq 25/24$, so $\Delta Q \leq -1/24$. If $d_0 = 1$, $m \leq 13$ and $a > 0$, then $H \leq 28$, $H'' = H + 2a \leq 52$, and

$$Q'' - Q = -2a + \tfrac{25}{24}(H + 2a - 29)_+ \leq -2a + \tfrac{25}{24}(2a - 1) < 0.$$

Here $1 \leq a \leq 12$. The remaining states have $1 \leq m \leq 13$, $d_0 = 1$ and $d_i \in \{1, 2\}$ for $i \geq 1$. There are exactly $\sum_{m=1}^{13} 2^{m-1} = 8191$ normalized words. Exact integer iteration through time 95 verifies a strict Q_{220} witness for each, with the following maximal first times by m (R27 replay):

m	1	2	3	4	5	6	7
Time	5	40	12	14	17	30	28

m	8	9	10	11	12	13
Time	33	42	40	55	55	95

At these initial states $Q = Q_{220} = Z \leq 25$. A strict Q_{220} witness cannot end at $H_t \geq 54$, where the penalty alone is at least $671/24 > 25$. For all even $H_t \leq 52$ the two penalties coincide, so the same witnesses transfer to Q. The R27 exact replay also checks them directly; the largest endpoint H is 38. Normalization transfers all claims. The only finite computational branch of this theorem is the explicitly bounded 8191-word set; no orbit invariance of the full-C(2) domain is assumed. The other two guarantees follow from Theorem 2.4. □

Theorem 2.7 (P6-930; intermediate-band certificates)**.** *Let $28 \leq H \leq 52$. A one-step Q descent holds if either $28 \leq H' \leq 52$ and $\Delta B_* \leq -1$, or $H' \leq 26$ and $\Delta B_* \leq H' - 29$. Independently of the initial restriction $28 \leq H \leq 52$, on a word with all C-runs even, one-step Q descent holds when $m \geq 14$, or when $m = 13$ and $d_0 \geq 2$.*

Proof. For $g(h) = p(h) - h$ one has $-695/24 \leq g(h) \leq -28$ on the even counts $28 \leq h \leq 52$, and $g(h) = -h$ for $h \leq 26$. Since $Q = B_* + g(H)$, either hypothesis gives $\Delta Q \leq -1/24$. For the all-even domain normalize r_0 to 2. The flags $\epsilon_0 = 1, \epsilon_{i>0} = 0$ in the output word give $H'_C = 2m + 2 - 2\mathbf{1}_{d_0=1}$ and raw bottom lengths $1, 3, 2$ for $d_0 = 1, 2, \geq 3$ respectively. The pairs $(H', \Delta B_*)$ are therefore $(2m, -1), (2m, -3), (2m+2, -1)$. Also $H \geq 2m+2$. For $H \geq 54$, $\delta = -1$ and $U \leq H$ in Theorem 2.2, so Theorem 2.4 applies. For $H \leq 52$, $m \geq 14$ gives both counts in $[28, 52]$. When $m = 13$ and $d_0 \geq 3$, $H' = 28$; when $d_0 = 2$, $H' = 26$ and $\Delta B_* = -3 = H' - 29$. This exhausts the stated cases. □

Corollary 2.8 (P6-931; compression of common-score witnesses)**.** *At fixed $T \geq 1$, replacing each d_i by $\min(d_i, 2T + 1)$ preserves every Q increment relative to the initial state up to time T. The same holds for any score $Z + f(H)$ with f defined on the even occupied counts.*

Proof. Paper B's finite band compression theorem (P6-565) preserves the complete C-run list and gives a constant difference $\sum_i (d_i - 2T - 1)_+$ in Z at all times $0 \leq t \leq T$. The normalized H values therefore agree at corresponding times. Subtract the initial scores. For fixed H, the normalized ordered C-run compositions and capped K lengths form a finite set for each fixed T; existence of a common successful T for general H is not asserted. □

3 Bounded labels and ordered refinement

Theorem 3.1 (P6-640/641; bounded labels)**.** *Let $k \geq 1$ be integral and let $\ell(n) \in \{0, \ldots, k\}$, with $\ell = 0$ on $Z = 0$. Then $R = (k+1)Z + \ell$ is a nonnegative integer score, with the same base. Every Z witness is preserved, and every equality-Z witness with strict label decrease is added.*

Proof. A strict integer Z decrease is at least one, so $\Delta R \leq -(k+1) + k = -1$. At equal Z, $\Delta R = \Delta \ell$. The base condition follows from the definition. □

3.1 Concrete exit certificates and successive refinement

We now construct the R127–130 label used in Theorem 3.6. The inputs in this subsection, including all six named regions, require $m \geq 1$ and $d_i = 1$ for every i. Thus $Z = m$ on their union. Write $p_i = r_i \bmod 2$ and

$$P = \sum_{i=1}^{m} (\lfloor r_i/2 \rfloor - 1)_+, \qquad q = F(n), \quad w = F(q).$$

Lemma 3.2 (P6-643/652; two-step balance for unit bands)**.** *For the above unit-band input,*

$$Z(w) - Z(n) = \begin{cases} 2(P + p_m - \mathbf{1}_{r_1=1}), & p_0 = 0, \\ 2(P + p_m - 1 + \mathbf{1}_{r_1=2}), & p_0 = 1,\ r_1 \geq 2, \\ 2(P + p_m) - 1 - 2\mathbf{1}_{r_2=1}, & p_0 = 1,\ r_1 = 1,\ m \geq 2, \\ -1, & p_0 = 1,\ r_1 = 1,\ m = 1. \end{cases}$$

Proof. The first output consists of blocks $1^{2-\epsilon_i}0^{r_{i+1}-1+\epsilon_i}$ followed by 1^{1+p_m}. A zero-free block has length two and does not alter the parity of the occupied run it joins. Except for the first surviving band, the next flag is therefore ϵ_i for the band arising from block i. For q with a band, let h be its first C-length, and let g_j, e_j denote its band lengths and flags. Counting width and occupied cells in the second update gives

$$Z(w) - m = \sum_{i=1}^{m} r_i + 1 + p_m - h - \sum_{j} [2 + e_j(1 - 2\mathbf{1}_{g_j=1})].$$

For an interior block with flag p_i, its contribution to r_{i+1} minus the corresponding bracket (zero bracket if the block is zero-free) equals

$$r_{i+1} - 2 - p_i + 2\mathbf{1}_{r_{i+1}=1} = 2(\lfloor r_{i+1}/2 \rfloor - 1)_+ + p_{i+1} - p_i.$$

These identities follow by considering $r_{i+1} = 1, 2$ and $r_{i+1} \geq 3$. The parity terms telescope. If $p_0 = 0$, the first band has length r_1, $h = 1$, and its next flag is zero, hence its bracket is 2. Substitution yields the first formula. If $p_0 = 1$ and $r_1 \geq 2$, the first band has length $r_1 - 1$, $h = 2$, and flag one, with bracket $3 - 2\mathbf{1}_{r_1=2}$. If $p_0 = 1, r_1 = 1, m \geq 2$, the first two blocks join to give $h = 3$, first band length r_2, flag zero and bracket 2; the remaining sum begins at $i = 2$. This gives the third formula. In the final case q has no bands and $w = 1$. All computations concern the unnormalized output, which is legitimate because Z and the forward orbit are unchanged by normalization. □

Define $\mathcal{A}$ by the additional requirement $r_i \in \{1, 2\}$ for $i \geq 1$; r_0 is unrestricted.

Theorem 3.3 (P6-635/639/640; exit without vacancy increase)**.** *Every $n \in \mathcal{A}$ has some $t \in \{1, 2\}$ with $F^t(n) \notin \mathcal{A}$ and $Z(F^t(n)) \leq Z(n)$. Consequently $2Z + \mathbf{1}_{\mathcal{A}}$ preserves every strict Z witness and descends on $\mathcal{A}$ within two steps.*

Proof. On $\mathcal{A}$, the first output gives $Z(q) = m + 1 - p_0 - p_m$ and $m(q) = m - v$, where v counts zero-free blocks. If $q \in \mathcal{A}$, equality $Z(q) = m(q)$ implies $v = p_0 + p_m - 1$. When $p_m = 0$ this forces $p_0 = 1, v = 0$. But with $p_0 = 1$ and no zero-free block, the first run above the bottom must have length two, and a run of length two cannot be followed by length one. Hence all higher runs are even, contradicting even total occupied count. Thus $p_m = 1$. If the last block were zero-free it would merge with the final 11, giving a top run of length at least four (there is a band since $q \in \mathcal{A}$). Therefore q has top run length two, so the same necessity just proved ensures $w \notin \mathcal{A}$. If $q \notin \mathcal{A}$ and $p_0 + p_m \geq 1$, use q. If $q \in \mathcal{A}$ and $p_0 = 1$, then $Z(q) = m - 1$ and $Z(w) \leq m$. If $q \in \mathcal{A}$ and $p_0 = 0$, its first occupied run has length one, its top run length two, and its vacancy count is m; hence $Z(w) = m$. In the remaining case $p_0 = p_m = 0$, Lemma 3.2 gives $Z(w) - m = -2\mathbf{1}_{r_1=1}$. If $m \geq 2$, the first and last output blocks have positive zero lengths, so the last band of q is not the first. Its length is $1 + \epsilon_{m-1}$, its lower occupied parity is ϵ_{m-1}, and its upper run has length one. In the second update it disappears and joins the final two occupied cells, giving either no band or a top run of length at least four. Thus $w \notin \mathcal{A}$. If $m = 1$, normalization gives the input 9, with orbit $9 \to 7 \to 11$ and $11 \notin \mathcal{A}$. The score claim follows from the exit and Theorem 3.1. □

For the next regions exclude $\mathcal{A}$ and impose the additional conditions in the table; j is the indicated uppermost C-length.

Region	Conditions	Allowed j
$\mathcal{B}_j$	$p_0 = 0,\ r_m = j,\ P \leq \mathbf{1}_{r_1=1}$	$2, 4$
$\mathcal{C}_j$	$p_0 = 0,\ r_m = j,\ r_1 = 1,\ r_i \leq 3\ (i \geq 1)$	$1, 3$

Each has $m \geq 2$: a single band is incompatible with the conditions and exclusion of $\mathcal{A}$. The five regions including $\mathcal{A}$ are disjoint. Define

$$\ell_{128} = 3\mathbf{1}_{\mathcal{A}} + 2\mathbf{1}_{\mathcal{B}_2} + \mathbf{1}_{\mathcal{B}_4},$$
$$\ell_{129} = 5\mathbf{1}_{\mathcal{A}} + 4\mathbf{1}_{\mathcal{B}_2} + 3\mathbf{1}_{\mathcal{B}_4} + 2\mathbf{1}_{\mathcal{C}_3} + \mathbf{1}_{\mathcal{C}_1}.$$

Theorem 3.4 (P6-644/645/647–649; successive label refinement)**.** *The score $4Z + \ell_{128}$ preserves all witnesses for $2Z + \mathbf{1}_{\mathcal{A}}$ and descends within two steps on all three of its positive-label regions. The score $6Z + \ell_{129}$ preserves all witnesses for $4Z + \ell_{128}$, with descent within two steps on the former three regions and $\mathcal{C}_1$, and within four steps on $\mathcal{C}_3$.*

Proof. On $\mathcal{B}_j$ the balance in Lemma 3.2 is nonpositive. At equality $Z(w) = m > 0$, the last band of q is not its first: the first and last output blocks both have positive zeros. The final band length is $j - 1 + \epsilon_{m-1}$, its lower parity ϵ_{m-1} and the top run length one. For $j = 2$ the last band vanishes in the second update, leaving a top run of length at least four. For $j = 4$ the last block is $1^{1+\epsilon_{m-1}}0^21$, followed by 11, giving a top run of length three. Hence at equal vacancy count the first case has label at most one and the second label zero for ℓ_{128}. For $\mathcal{C}_j$ the balance is zero and the first block has a positive zero length. If $j = 3$, the last band of q has length $2 + \epsilon_{m-1}$ and top run length two. The next update ends in a positive newly produced band followed by a top run of length one. If $j = 1$ and $\epsilon_{m-1} = 1$, the last band of q has length one, lower parity one and top run length two, so the second update creates a band of length two. If $j = 1$ and $\epsilon_{m-1} = 0$, the last block has no zeros and joins the final 11; the previous block has positive zeros since $r_{m-1} = 2$. The top run of q has length four, so the next update creates a band of length at least three. In all cases the top run of w has length one; for $\mathcal{C}_1$ a band of length at least two ensures label zero. For $\mathcal{C}_3$, the endpoint w has label at most one unless it belongs to $\mathcal{A}$. For any $u \in \mathcal{A}$ with top run length one, if its bottom run is odd, $Z(F(u)) = Z(u) - 1$. If the bottom run is even, $Z(F(u)) = Z(u)$ and the next bottom run is one. If $F(u) \notin \mathcal{A}$, it is outside all five regions since the other four require an even bottom run. Otherwise the first output band must have length one, forcing $r_1 = 1$, and successively each higher input run has length one. Their number m is even and at least two. The output is $(10)^m11$; the next output has a band of length two at its

upper end, with $Z(F^2(u)) = Z(u)$ by Lemma 3.2. This gives exit within the additional two steps. Finally, both score refinements strictly preserve the order of distinct old labels: only the old zero label is split. Strict Z witnesses remain strict by Theorem 3.1; all old equal-Z witnesses remain strict as well. The stated time bounds follow. □

Define $\mathcal{D}$ by requiring odd r_0, $r_1 \geq 2$, $\max_{i\geq 1} r_i \geq 3$, $P + p_m - 1 + \mathbf{1}_{r_1=2} = 0$, and either $r_1 \in \{2, 5\}$ or $r_m \in \{1, 4\}$. The region is disjoint from the previous five. All six regions are taken to be empty outside the nonbase unit-band domain. The final label is

$$\ell = \ell_{130} = \begin{cases} \ell_{129} + 1, & \ell_{129} > 0, \\ \mathbf{1}_{\mathcal{D}}, & \ell_{129} = 0. \end{cases} \tag{1}$$

Theorem 3.5 (P6-653; final unit-band label)**.** *The score $7Z + \ell_{130}$ preserves every witness for $6Z + \ell_{129}$ and descends on $\mathcal{D}$ within two steps. Its previous five positive-label regions retain their time bounds. The label vanishes on the base, is invariant under ρ, and has $\ell > 0$ only when $m = Z$.*

Proof. Lemma 3.2 gives $Z(w) = Z(n)$. We prove that w has a band of length at least two. Here q starts with $C(2)K(r_1 - 1)$. If $r_1 = 2$, parity implies $m \geq 2$ and the next occupied run of q has length $R \geq 2$, so the first new band of w has length R. If $r_1 = 5$, the first band of q has length four and leaves a band of length two. If $r_m = 4$, parity implies $m \geq 2$; the last band of q is distinct from its first, has length $3 + p_{m-1}$ and lower parity p_{m-1}, leaving a band of length two. If $r_m = 1$, $m \geq 2$ since $r_1 \geq 2$. When $p_{m-1} = 1$, the final band of q has length one, lower parity one and upper length two, producing a band of length two. When $p_{m-1} = 0$, the last block joins the final 11, giving a top run of length four and a new band of length at least three in w. Thus w lies outside all six regions. Refinement again splits only the old zero label, preserving all old strict witnesses. The label properties follow from its definition: only the parity of the bottom length occurs, all positive regions have unit bands, and none contains the base. □

For example, the original R129 certificate $89 \to 67 \to 101 \to 19 \to 29$ has equal initial and final $Z = 2$, with $101 \in \mathcal{A}$. Transient increases of a label do not invalidate the selected finite-time strict witness. These results establish a concrete hierarchy of six regions, not a proof of descent on every unit-band input. The construction of ℓ_{130} is exactly the label required for the subsequent band-count and three-score refinements. In the R26 check, all normalized unit-band words with $1 \leq m \leq 7$ and higher lengths in $\{1, \ldots, 5\}$, together with 5,000 seeded longer words (including unnormalized inputs), gave 102,655 balance checks without discrepancy. An additional 262,144 pairs of orbit states were checked for all four successive strict-score implications. These finite checks corroborate, rather than replace, the proofs.

Theorem 3.6 (P6-659; insertion of band count)**.** *Suppose $\ell \in \{0, \ldots, 6\}$ vanishes on the base and $\ell > 0$ implies $m = Z$. Then*

$$L = 7\left[\frac{Z(Z+1)}{2} + m\right] + \ell$$

is equivalent to the lexicographic order on (Z, m, ℓ). It preserves all strict witnesses for $7Z + \ell$ and all lexicographic strict witnesses for (Z, m), with base $L = 0$.

Proof. For fixed $Z = z$ we have $0 \leq m \leq z$; the triangular intervals for consecutive z are disjoint and ordered, and the factor 7 separates distinct values of the first two entries. If an old $7Z + \ell$ witness has equal Z, then $\ell(n) > \ell(w)$ and $m(n) = Z(n) \geq m(w)$. Either the band count decreases or it is equal and the label decreases. Other comparisons follow directly from lexicographic order. The R130 label satisfies the support condition because all its positive regions have only K(1) bands. The explicit definition is given in equation (1). □

4 Three-score integration

Theorem 4.1 (P6-1041; integer realization of a common order)**.** *Assume the label in Theorem 3.6 is invariant under ρ. Let $C = 24Q$, $W = H - 2 + \sum_{j=0}^{5} Z(F^j(n))$, and L be as above. The triple (C, L, W) in lexicographic order has an exact nonnegative integer realization*

$$A(c) = 7c(c+3)/2 + 7, \quad D(c) = 7c + 205, \qquad \Phi = CA(C)D(C) + LD(C) + W.$$

Its base is $Z = 0$. Every strict C witness is preserved unconditionally, every strict L witness with C unchanged is preserved, and every strict W witness with C, L unchanged is preserved.

Proof. The output word gives $B'_* \le B_* + 1$ also at the base. Since $p(H) \ge H - 29$, $B_* \le C/24 + 29$. Thus $Z(F^j(n)) \le C/24 + 27 + j$ and $H - 2 \le C/24 + 27$. It follows that $0 \le W \le 7C/24 + 204 < D(C)$. Using $m \le Z \le C$ gives $0 \le L < A(C)$. The functions A, D are positive, nondecreasing and integer valued on nonnegative integers. At fixed C, $0 \le LD + W < AD$; at $c' < c$ the upper endpoint bound $(c'+1)A(c')D(c') \le cA(c)D(c)$ separates the ranks. This proves the claimed order, and hence the precise witness-preservation statements. On the base all three components vanish. Conversely $C > 0$ off the base. The normalized sublevel sets are finite by the first component. □

This is not the unconditional union of the witness sets of the three different scores. For example a W decrease accompanied by a C increase is not a descent in this order.

Corollary 4.2 (P6-1042; band-count decrease at equality of vacancy count)**.** *If $Z(F(n)) = Z(n)$, $H(F(n)) \le H(n)$ and $m(F(n)) < m(n)$, then $\Phi(F(n)) < \Phi(n)$.*

Proof. Monotonicity of p gives $C' \le C$. At equality the second component satisfies $L' - L \le -7 + 6 < 0$. □

Theorem 4.3 (P6-1050; coalescence induction)**.** *Suppose for each positive odd n with $Z(n) > 0$ there are a positive odd z and integers $T, J \ge 0$ such that $F^T(n) = F^J(z)$ and $\Phi(z) < \Phi(n)$. Then every positive odd integer converges to 1.*

Proof. Apply strong induction on the nonnegative integer $\Phi(n)$. At rank zero, $F(n) = 1$. Otherwise the lower-ranked z converges by induction, and the common forward state shows n converges as well. A forward descent is the special case $J = 0, z = F^T(n)$. A coalescence witness need not be a forward descent witness. □

Write $\Psi = (C, L, W)$ for the common order. The all-state hypothesis of Theorem 4.3 has not been established. In particular the three-score order retains only the conditional witness sets described in Theorem 4.1.

4.1 Equality-breaking certificates for the common order

Theorem 4.4 (P6-1043/1044; two infinite families)**.** *Write $\Psi = (C, L, W)$ for the order of Theorem 4.1.*

1. *For $J \ge 1$ and arbitrary $D_j \ge 3$, let $r = (2, (2,1,2,1)^J)$ and $d = ((1, D_j, 1, 1))_{j=1}^{J}$, where repeated tuples denote concatenation. At time one, $\Delta C = 0$ and $\Delta L = -7J$, so Ψ strictly decreases. For $J \ge 9$, the first score Q is unchanged at time one and increases by $2J - 2$ relative to time zero at time two.*

2. *Let $r = (2,3,4,2,1)$ and $d = (2,2,1,D)$, $D \ge 13$. At time one, $\Delta C = \Delta L = 0$ and $\Delta W = -10$, so Ψ strictly decreases.*

Proof. In (1), the output word on each four-band block is $1001\,0^{D_j-1}110011$, with a single additional 11 at the overall upper end. Thus $r' = (1, 1, 2, (3, 1, 2)^{J-1}, 4)$ and $d' = ((2, D_j - 1, 2))_{j=1}^{J}$. Both endpoints have $H = 6J + 2$, $Z = \sum_j (D_j + 3)$ and $\ell = 0$ since there are bands of length at least two. The band count changes from $4J$ to $3J$. For the second update all $d'_i \geq 2$, the sum of flags is $2J - 1$, and the top and bottom lengths of the first output are 4 and 1. The zero balance from the output word gives $Z'' - Z' = 2 - 2J$. The width change of this step is zero with no normalization loss, because its new bottom run is one. Hence $H'' = H + 2J - 2$. For $J \geq 9$ both counts lie in the linear high branch, yielding the stated score difference. This does not preclude a later Q witness.

In (2) the representative at $D = 13$ is $n = 536851921$. Direct application of the displayed output rule six times gives

t	0	1	2	3	4	5	6
Z	18	18	17	15	14	13	10
H	12	10	12	10	12	10	12

The first output has $r' = (3, 3, 2, 1, 3)$ and $d' = (2, 3, 1, 12)$. The first two components agree at the endpoints ($C = 432$, $L = 1225$ at this representative), because $p = 0$, $m = 4$ and $\ell = 0$. The sliding-window difference is $W' - W = Z_6 - Z_0 + H_1 - H_0 = -10$. At $T = 6$ the band cap is 13. Extending D preserves the C-run lists and band counts through time six and adds $D - 13$ to every Z_t; the two endpoints still have bands of length at least two. Thus C, L agree across the step and the same window difference holds for every extension. The equality-breaking one-step witness is new to the common order relative to the first score at that time, not a claim that the first score never descends. □

Theorem 4.5 (P6-1063; high-H unit-band domain)**.** *For every nonbase normalized word with all $d_i = 1$ and $H \geq 56$, there exists $t \in \{1, 2\}$ with $\Psi(F^t(n)) <_{\text{lex}} \Psi(n)$, equivalently $\Phi(F^t(n)) < \Phi(n)$. There is no bound on the number or length of C-runs. The same conclusion holds before bottom normalization.*

Proof. Let $o = \sum_{i=0}^{m} p_i$, $e = \sum_{i=1}^{m} (r_i - (2 - p_i))/2$, and $\delta = p_0 + p_m - 1$. Thus $H = 2(m+1) - o + 2e$. If $H_t \leq H - 2$ and $B_{*,t} - B_* \leq 1$, then $Q_t < Q$: indeed $Q = B_* + g(H)$, $g(H) = H - 56$ for $H \geq 54$, and $g(h) \leq -2$ for even $h \leq 52$. Any visit to the base also suffices. The first output consists of the unit-band blocks from Section 1. Let v count zero-free blocks, exactly those with $\epsilon_i = 0, r_{i+1} = 1$, and let γ be the even number removed in bottom normalization of the first output. Counting the output width and occupied cells, and using $\sum_{i<m} \epsilon_i = o - p_m + 1 - 2p_0$, gives

$$m' = m - v, \quad H' - H = 2\delta - 2e - \gamma, \quad B'_* - B_* = \delta - \gamma.$$

All subsequent arguments can stop at the base.

Case 1: $p_0 = p_m = 0$. Then $\delta = -1$ and the two bounds for strict Q descent hold at once.

Case 2: $p_0 + p_m = 1$. If $e > 0$ or $\gamma > 0$ the bounds again hold. Otherwise $H' = H$ and $Z' = Z$. If $v > 0$, Corollary 4.2 gives strict descent of the common order. If $v = 0$, all higher runs have length 1 or 2, with transitions from 2 to 1 forbidden. When $p_0 = 1$, an initial higher run of length 1 is also forbidden, so all higher runs have length two, contradicting even total occupied count. When $p_0 = 0$, the upper run is odd and all higher runs must be one. The output has $r' = (1^m, 2)$, $d' = (1^m)$ (repetition notation), and even $m \geq 2$. Its second output starts with at least three occupied cells since the first block is zero-free 11 and the next begins with an occupied cell. Its normalization removes at least two cells. Applying the same counting formula with $e' = 0, \delta' = 0$ yields $H'' \leq H - 2$ and $B''_* \leq B_*$.

Case 3: $p_0 = p_m = 1$. Then $\delta = 1$. The required bounds hold at time one if $e \geq 2$, $\gamma \geq 4$, or both $e \geq 1$ and $\gamma \geq 2$. If $r_1 = 1$, $m \geq 2$ follows from $H \geq 56$ and bottom normalization. The output begins with 11 followed by a one-cell occupied run and a nonempty zero interval (the next

flag is one), so $\gamma = 2$. If $e = 0$, this gives $H' = H$ and $B'_* - B_* = -1$, also a strict Q descent. The remaining possibilities have $r_1 \geq 2$ and $e \leq 1$.

Now the output has bottom length two, and top length even (zero-free blocks have length two and preserve the parity of the run they join, including the terminal 11). The first width increment is one; the next normalized width increment is at most minus one by the general output width formula. Thus $B''_* - B_* \leq 0$. Put $a = \#\{i \geq 1 : r_i \text{ odd},\ r_i \geq 3\} \leq e$. Zero-free blocks cannot be consecutive because the next flag after such a block is one. The sum of flags in the first output is $o - 1$: its even bottom contributes one, its even top is excluded, and the remaining odd runs correspond to the $o - 2$ odd internal flags. In that output let $s' = \sum_j \epsilon'_j \mathbf{1}_{d'_j=1}$. Its contributions are $\mathbf{1}_{r_1=2}$ from the first block and one for each internal block with $\epsilon_i = 1, r_{i+1} = 1$. For each higher input run of length one (necessarily internal to this count since $r_1 \geq 2$), its preceding block contributes either one to v if its flag is zero or one to s' if its flag is one. An odd higher run of length at least three contributes to neither of these counts. The number of length-one higher runs is $o - 1 - a$, since the bottom run is odd and excluded. The separate first-block contribution to s' is $\mathbf{1}_{r_1=2}$. Adding gives $v + s' = o - 1 - a + \mathbf{1}_{r_1=2}$. Counting the raw occupied cells after the second update gives

$$H''_C = 1 + 2(m - v) + (o - 1) - 2s' = H + 2(a - e - \mathbf{1}_{r_1=2}).$$

If $r_1 = 2$ or $a < e$, we obtain $H'' \leq H - 2$. Otherwise $r_1 \geq 3$, $e \leq 1$ and $a = e$ force $r_1 = 3, e = a = 1$ (length four would consume one even excess). Then $r'_0 = 2, d'_0 = 2$, so the first occupied run of the second output has length at least three, by the output formula with first flag one. Normalization removes at least two occupied cells, giving $H'' \leq H - 2$ also here. Together with the two-step width bound this proves strict Q descent. The cases exhaust the possible parities, and the normalization identities transfer the result. □

For instance the R270 family $r = (1, 2, 1^{2J-1}), d = (1^{2J})$, $J \geq 27$, has $H = 2J + 2$, $Q' - Q = 3$ and $Q'' - Q = -2$. Thus the two-step certificate is genuinely useful when the first score increases initially. This theorem closes the high-H all-unit-band domain in the common order, not the entire high-H domain.

5 Coalescence witnesses under upper connections

The preceding induction criterion permits additional certificates without changing the common order. We now give the analytic lifting and contraction results of R265. Write $h(x) = H_C(x)$ for the raw occupied count, $b(x) = B(x)$ for the width, and $p_{24} = 24p$. The letters a, v, q here denote positive odd integers, not scores.

Theorem 5.1 (P6-1051; lifting a lower coalescence certificate)**.** *Suppose $n = 2^k q + a$, $k > b(a)$, and $F^T(a) = F^J(v)$ for nonnegative integers T, J. Here k and K are integer shift positions. Write $S_a = \sum_{i=0}^{T-1} \nu_2(3F^i(a) + 1)$ and $S_v = \sum_{i=0}^{J-1} \nu_2(3F^i(v) + 1)$ for the respective cumulative division exponents; an empty sum at time zero is zero. Assume $S_a < k$, $q' = 3^{T-J} q$ is a positive odd integer, and $K = k - S_a + S_v > b(v)$. Then $z = 2^K q' + v$ satisfies $F^T(n) = F^J(z)$, and*

$$\begin{aligned} C(z) - C(n) = 24[&S_v - S_a + h(a) - h(v) + Z(q') - Z(q)] \\ &+ p_{24}(H(v) + h(q')) - p_{24}(H(a) + h(q)). \end{aligned}$$

In particular a negative right-hand side certifies a lower-ranked coalescing state, independently of any further increase of k (with K increased by the same amount).

Proof. For $0 \leq i \leq T$ write S_i for the cumulative division exponent of a through time i. Since $S_i \leq S_a < k$, induction gives $F^i(n) = 2^{k-S_i} 3^i q + F^i(a)$. Indeed, the valuation of $3F^i(a) + 1$ is $S_{i+1} - S_i < k - S_i$, so adding the upper term of valuation at least $k - S_i$ does not change that

valuation. Likewise $K - S_v = k - S_a > 0$ yields $F^J(z) = 2^{K-S_v}3^J q' + F^J(v)$. Both expressions agree by the hypotheses. For the structural difference, the two masks $E(a)$ and $2^k E(q)$ are separated by a positive gap, hence $Z(n) = k - h(a) + Z(q)$ and $H(n) = H(a) + h(q)$. The analogous identities for z give the displayed expression on subtraction, using $K - k = S_v - S_a$. Increasing k and K leaves every term of this difference unchanged and preserves the isolation assumptions. If the difference is negative, the first coordinate alone gives $\Psi(z) <_{\text{lex}} \Psi(n)$, with no restriction on the other coordinates. □

Theorem 5.2 (P6-1052; three-step contraction)**.** *Let the normalized input word be*

$$C(2)K(4)C(r)K(d)\,U,$$

where $r \geq 4$ is even, $d \geq 1$, and U is the CPE of an arbitrary positive odd q. The replacement word is

$$C(2)K(1)C(r-2)K(d)\,U.$$

Its odd integer z satisfies

$$F^3(n) = F^3(z), \qquad Z(z) = Z(n) - 3, \qquad H(z) = H(n) - 2,$$
$$C(z) - C(n) \leq -72.$$

The corresponding unnormalized inputs with this normalized prefix have the same coalescence certificate and rank inequality.

Proof. The signed-power expansion gives $n = 2^k q + a$, $a = 1 + 64(2^r - 1)/3$, $k = r + 6 + d$. Set $v = (2^{r+1} - 5)/3 = 1 + 8(2^{r-2} - 1)/3$. The lower trajectories, with division exponents above the arrows, are

$$a \xrightarrow{2} 16 \cdot 2^r - 15 \xrightarrow{2} 12 \cdot 2^r - 11 \xrightarrow{5} 9 \cdot 2^{r-3} - 1,$$
$$v \xrightarrow{2} 2^{r-1} - 1 \xrightarrow{1} 3 \cdot 2^{r-2} - 1 \xrightarrow{1} 9 \cdot 2^{r-3} - 1.$$

All displayed quotients are positive and odd for even $r \geq 4$; each arrow is thus the stated accelerated step. The width of a is $r + 6$ and that of v is $r + 1$. Consequently $S_a = 9 < k$, $S_v = 4$, $K = k - 5 = r + 1 + d > b(v)$ and $q' = q$. Theorem 5.1 gives the equality of iterates. The word of v is $C(2)K(1)C(r-2)$, so the upper gap d is unchanged in $z = 2^K q + v$ and the two structural changes are exactly -3 and -2. Since p_{24} is nondecreasing, $C(z) - C(n) = -72 + p_{24}(H(n) - 2) - p_{24}(H(n)) \leq -72$. Before normalization the initial rank is unchanged and the first forward iterate is the same; thus the three-step certificate also transfers to those inputs. □

This construction is an induction certificate and not a claim that the forward iterate $F^3(n)$ has smaller rank than n. R265 additionally records a finite-certificate completion of the all-even two-band domain (P6-1053); that result is outside the present paper's scope, pending separate replay of its full dependent finite cover and certificate list.

6 Relation to earlier methods

The original Gluing Method [3] supplies the local-to-global research strategy. In the present discrete setting the local inputs are proved inequalities, rather than approximate descriptions, and every claimed preservation of a witness is subject to an exact order comparison. This distinction is essential: a weighted combination of locally valid scores does not by itself preserve descent across domains. The constructions in this paper give sufficient conditions, not a general theorem of successful gluing. Classical Collatz stopping-time theory uses decreases of the integer itself [5, 6]. Well-founded ranking functions, including lexicographic rankings, are standard tools in

termination analysis; linear ranking synthesis and lexicographic synthesis with reachability information have been studied by Podelski and Rybalchenko [7] and Bradley, Manna and Sipma [8]. The contribution here is not the invention of strong induction, lexicographic order, or ranking functions in general. It is the explicit construction and revision of structural scores using ABN/CPE, together with verified finite-time witnesses and the exact arithmetic of their preservation or transfer. In particular, a witness need not decrease at every intervening step, and the selected coalescence witness may be on a different initial trajectory. The present results neither provide an unconditional union of arbitrary witness sets nor establish the existence of a witness for every state. The comparison is contextual, not an exhaustive historical-priority investigation.

7 Status and provenance

This preprint draws on Phase 6 R104/105 (common scores and their variations), R127/132 (label and count refinement), R221 (local modification), R262 (three-score order), and R265 (coalescence induction). The ABN/CPE framework and variational-induction research direction originate with Y. Ueoka; the general Gluing Method was presented in his earlier preprint [3]. Historical derivations, proofs and checks of the listed Phase 6 results were recorded under ChatGPT; the current compilation and R104 replay are by Tsumugi (ChatGPT). Earlier records are not retroactively renamed. The unit-band balance and label proofs follow Phase 6 R126–130 (P6-635, 639–641, 643–645, 647–649, 652–653); the original records credit Y.Ueoka with the state-dependent step-function gluing and parity-transition research directions, and ChatGPT with the specific exit proofs and label construction. Ordered tuples, integer rankings, and strong induction are standard mathematical tools; the contribution here is their use with explicit ABN/CPE structural certificates. An initial comparison with standard ranking-function methods is given above; comprehensive priority investigation remains open.

The full-C(2) balances and repaired common score follow R106 and R220/221 (P6-573, 926, 929–931); the two equality-breaking families and high-H unit-band theorem follow R262 (P6-1043/1044) and R270 (P6-1063). Y.Ueoka proposed the ABN/CPE framework, residual-domain score reconsideration and the variational-induction/gluing direction; the concrete constructions and original proofs were recorded under ChatGPT. R27 compilation, exact finite replay and sample checks are by Tsumugi (ChatGPT), without retroactive renaming. R27 reproduced the 8191-word finite branch and maximum endpoint $H = 38$, checked 5000 full-C(2) sample words and 1500 equality-breaking examples, and replayed the original R270 code (35731 checks, identical archived output). These samples do not replace the analytic proofs. R28 adds the analytic R265 lifting and contraction theorems (P6-1051/1052), following the original attribution of the ABN/CPE and coalescence direction to Y.Ueoka and the concrete structural formulas and proofs to ChatGPT. The present exposition and standalone replay are by Tsumugi (ChatGPT). The appendix reproduces both complete finite branches and checks 2500 contraction samples; the latter are not a substitute for the analytic proof. A separate ChatGPT referee report on the R28 draft found no major defect and recommended minor revision. Its extracted Appendix A code agrees structurally with the present code, and its reported finite branch counts agree with R28. The archived review PDF hash differs from the referee's recorded hash, so exact byte identity of the reviewed document is not asserted. R29 clarifies normalized counts, word reading, compression, cumulative exponents and the final unit-band count, following the referee's recommendations. The revised manuscript itself has not undergone a further independent review. No all-state descent or convergence claim is made.

A Standalone verification of the finite branches

The following complete Python program (Python 3.10 or newer, standard library only; do not use `-O`) replays the adaptive partition in Theorem 2.1 and the finite branch in Theorem 2.6, using ex-

act integer iteration and signed-cell reconstruction. The analytic lifting from these representative cells to their infinite domains is the compression theorem of Paper B. Its applicability and the partition refinement are part of the mathematical proof, not conclusions of numerical sampling. The additional 2500 seeded tests check Theorem 5.2 only as finite corroboration. The program was executed in R28 and reproduced all five low-count table rows and all thirteen full-C(2) time bounds; the maximum endpoint H for the latter was 38. All assertions passed. The raw source and its R28 execution output are also embedded as PDF attachments when supported by the viewer. Extract `verify_public_r28.py` and run `python verify_public_r28.py` with assertions enabled. The R29 local verification environment uses Python 3.13.5; the attached finite-branch log is the retained R28 run, not a new full replay.

```
# Complete finite branches of Paper C and finite coalescence checks.
# Python 3.10+, standard library only. Do not run with -O.
# Infinite-domain lifting is proved in the text, not by these checks.
from itertools import product
from random import Random

def F(n):
    x=3*n+1
    return x//(x&-x)

def Z(n):
    e=n^(n<<1)
    return e.bit_length()-e.bit_count()

def H(n):
    e=n^(n<<1)
    r=(e^(e+1)).bit_length()-1
    return e.bit_count()-r+2-r%2

def num(r,d):
    # Alternating signed occupied cells.
    n,j,s=0,0,-1
    for i,a in enumerate(r):
        for _ in range(a):
            n+=s*(1<<j); j+=1; s=-s
        if i<len(d): j+=d[i]
    assert n>0 and n%2 and s==-1
    return n

def compositions(h):
    for cut in range(1<<(h-1)):
        r,a=[],1
        for i in range(h-1):
            if (cut>>i)&1: r.append(a); a=1
            else: a+=1
        yield tuple(r+[a])

def p24(h):
    if h<=28: return 0
    if h<=52: return 25*(h-29)
    return 48*(h-28)

def C(n): return 24*Z(n)+p24(H(n))

def C220(n):
    h=H(n)
    return 24*Z(n)+25*max(h-29,0)+23*max(h-52,0)
```

```
def low_cover(h):
    active=[(r,d) for r in compositions(h)
            if len(r)>1 and r[0]<=2
            for d in product(range(1,4),repeat=len(r)-1)]
    checked=certified=endmax=0
    for T in range(1,55):
        cap=2*T+1; fail=[]; checked+=len(active)
        for r,d in active:
            n=num(r,d); x=n; z=Z(n)
            for t in range(1,T+1):
                x=F(x)
                if Z(x)<z:
                    assert H(x)<=28 and C(x)<C(n)
                    certified+=1; endmax=max(endmax,H(x))
                    break
            else: fail.append((r,d))
        if not fail: return checked,certified,T,endmax
        active=[(r,new) for r,d in fail
                for new in product(*[(v,) if v<cap else
                    (cap,cap+1,cap+2) for v in d])]
    raise AssertionError('unresolved low-count cells')

def full_c2():
    times=[]; count=endmax=0
    for m in range(1,14):
        mt=0
        for d in product((1,2),repeat=m-1):
            n=num([2]*(m+1),(1,)+d); x=n
            for t in range(1,96):
                x=F(x)
                if C220(x)<C220(n):
                    assert C(x)<C(n)
                    count+=1; mt=max(mt,t)
                    endmax=max(endmax,H(x)); break
            else: raise AssertionError((m,d))
        times.append(mt)
    assert count==8191 and endmax==38
    assert times==[5,40,12,14,17,30,28,33,42,40,55,55,95]
    print('full-C(2):',count,times,endmax,flush=True)

def iterate(n,t):
    s=0
    for _ in range(t):
        x=3*n+1; a=x&-x
        s+=a.bit_length()-1; n=x//a
    return n,s

def coalescence():
    rng=Random(1052)
    for _ in range(2500):
        r=2*rng.randrange(2,101); d=rng.randrange(1,101)
        q=2*rng.getrandbits(rng.randrange(1,100))+1
        m=1+64*((1<<r)-1)//3; v=((1<<(r+1))-5)//3
        k=r+6+d; K=k-5
        n=(q<<k)+m; w=(q<<K)+v
        assert iterate(m,3)==(9*(1<<(r-3))-1,9)
        assert iterate(v,3)==(9*(1<<(r-3))-1,4)
        assert iterate(n,3)[0]==iterate(w,3)[0]
```

```
        assert Z(w)-Z(n)==-3 and H(w)-H(n)==-2
        assert C(w)-C(n)==-72+p24(H(n)-2)-p24(H(n))<=-72
    print('three-step coalescence: 2500 sample checks PASS',flush=True)

if __name__=='__main__':
    expected=[(3,3,1,4),(77,64,5,10),(1554,1114,40,16),
              (21518,18152,54,22),(335288,296262,51,28)]
    for h,want in zip((2,4,6,8,10),expected):
        got=low_cover(h)
        assert got==want,(h,got,want)
        print('low H:',h,got,flush=True)
    full_c2(); coalescence(); print('PASS',flush=True)
```

B Source concordance

The following table identifies the Phase 6 research records for the numbered results in this paper. Proposition numbers are the original archive identifiers, not a claim that every result in the archive has been independently checked. The methodological source for gluing is the published preprint [3]; the source of the individual Collatz certificates is the Phase 6 archive [4]. Historical concrete derivations and checks are credited as recorded under ChatGPT, not retroactively attributed to the current pen name. The present compilation and replay are credited to Tsumugi (ChatGPT). Y. Ueoka is the human corresponding author.

Result	Original propositions	Phase 6 records
2.1	P6-569	R104
2.2	P6-570–571	R104–105
2.3	P6-572	R105
2.4	P6-928	R221
2.5	P6-573	R106
2.6	P6-926, 929	R220–221
2.7	P6-930	R221
2.8	P6-931	R221
3.1	P6-640–641	R127
3.2	P6-643, 652	R128, R130
3.3	P6-635, 639–640	R126–127
3.4	P6-644–645, 647–649	R128–129
3.5	P6-653	R130
3.6	P6-659	R132
4.1	P6-1041	R262
4.2	P6-1042	R262
4.3	P6-1050	R265
4.4	P6-1043–1044	R262
4.5	P6-1063	R270
5.1	P6-1051	R265
5.2	P6-1052	R265

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