Partial Progress Toward the Asymptotic 2n Upper Bound for Erdős Problem #708

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Abstract. Let A be a set of n distinct integers at least 2 with maximum $M = \max A$. For any interval I of M consecutive integers, denote by $\tau(A, I)$ the minimum cardinality of a subset $B \subseteq I$ such that $\prod_{a\in A} a \mid \prod_{b\in B} b$, and set $g(n) := \sup_{A,I} \tau(A, I)$ under the standard "at most" convention (Erdős Problem #708). While the general asymptotic conjecture $g(n) \le (2 + o(1))n$ remains open, we prove four uniform partial upper bounds by purely analytic and combinatorial methods without numerical computation or external datasets: (i) $M \le 2.0001n \implies \tau(A, I) \le 2n$ for large n; (ii) $M \ge n^2 \implies \tau(A, I) \le 2n + 3$ (sharpening to $\tau(A, I) \le 2n$ when $M \ge (n + 3)^2$, and yielding $\tau(A, I) \le 2n + o(n)$ as $M/n^2 \to 1$ from below), pushing Wang's $M \ge 16n^2$ threshold down to n^2; (iii) for all $M \ge 2n$ without an asymptotic threshold, $\tau(A, I) \le 2n$ whenever the high-demand prime part $H(a)$ of each $a \in A$ satisfies $H(a) \le \lfloor M/(2n)\rfloor$; and (iv) $\tau(A, I) \le \lceil kn/(k - 1)\rceil$ whenever each input is a product of k primes in $[X, CX]$ for large X, with a matching construction showing that $k/(k - 1)$ is optimal.

Overview and Main Results

Let $n \ge 1$, let $A \subseteq \{2, \dots, M\}$ be a set of n distinct integers with $M = \max A$, and let $I \subset \mathbb{Z}$ be any interval of M consecutive integers. Define $\tau(A, I)$ to be the smallest cardinality $|B|$ of a subset $B \subseteq I$ satisfying $\prod_{a\in A} a \mid \prod_{b\in B} b$, and write $g(n) := \sup_{A,I} \tau(A, I)$ under the standard "at most" cardinality convention of Erdős Problem #708. As of this note, the full conjecture $g(n) \le (2+o(1))n$ remains open.

We establish four partial upper bounds (summarized in Table 1) together with a matching lower-bound family for Section 4. In Theorems 1 and 2, "sufficiently large n" depends only on an absolute constant, uniformly across all target sets A and all starting positions of I. Notice that whenever I contains 0 we may take $B = \{0\}$, and if $I \subset \mathbb{Z}_{<0}$ we may replace I by $-I \subset \mathbb{Z}_{>0}$; hence it suffices throughout to work with positive integer intervals $I \subset \mathbb{Z}_{\ge 1}$.

Table 1: Summary of partial upper bounds proved in this paper

Regime / Hypothesis	Conclusion	Location
$n \ge n_0$ and $M \le 2.0001n$	$\tau(A, I) \le 2n$	Theorem 1 (Sec. 1)
$n \ge n_0$, $A \subseteq \{2, \dots, M\}$, $\lvert A\rvert = n$, $M \ge n^2$	$\tau(A, I) \le 2n + 3$	Theorem 2 (Sec. 2)
$n \ge n_0$, $A \subseteq \{2, \dots, M\}$, $\lvert A\rvert = n$, $M \ge (n + 3)^2$	$\tau(A, I) \le 2n$	Theorem 2 (Sec. 2)
$M/n^2 \to 1$ (including $M < n^2$)	$\tau(A, I) \le 2n + o(n)$	Corollary 2.1 (Sec. 2)
$M \ge 2n$ and $\forall a \in A$, $H(a) \le \lfloor M/(2n)\rfloor$	$\tau(A, I) \le 2n$ (all n)	Theorem 3 (Sec. 3)
$k \ge 2, C \ge 1$ fixed, $X \ge X_0(k, C)$, $a \in \mathbb{P}^{(k)}_{[X,CX]}$	$\tau(A, I) \le \lceil kn/(k - 1)\rceil$ (sharp)	Theorem 4 (Sec. 4)

Relationship to prior work. Erdős and Surányi [2] (Section 9) constructed inputs showing $g(n) \ge (2 - o(1))n$. Recently, Wang [3] obtained the unrestricted linear bound $g(n) \le 8.82828125\, n + O(n/\log n)$ (building on Haoyu Chen's written H883 bound) and proved in Appendix A that $\tau(A, I) \le 2n$ holds for all $M \ge 16n^2$. In the long-interval regime (Section 2), we adopt Wang's decomposition-and-matching framework while tightening the cofactor count ($c \ge u$) and three-type union bound to lower the quadratic threshold from $16n^2$ down to n^2 (with an additive $+3$ slack that vanishes at $(n + 3)^2$). In the short-interval regime (Section 1), we introduce an L^2 prime-union estimate coupled with Hahn–Banach separation and linear-programming vertex rounding. All results here are purely theoretical and have not yet undergone external peer review; no claim of priority is made over concurrent preprints.

1 The Near-Linear Regime $M \le 2.0001n$

Theorem 1 (Near-Linear Upper Bound)

Theorem 1. *There exists an absolute constant $n_0 \in \mathbb{Z}_{\ge 1}$ such that for all $n \ge n_0$, every n-element set $A \subseteq \{2, \dots, M\}$ with $M \le 2.0001n$, and every interval I of M consecutive integers,*

$$\tau(A, I) \le 2n.$$

Throughout Section 1, every $o(1)$ term depends solely on M and is completely uniform with respect to the prime subset P, weight vector z, target set A, and interval starting point. We use only the prime number theorem ($\pi(x) = o(x)$) and Mertens' second theorem ($\sum_{p \le x} 1/p = \log\log x + B_1 + o(1)$). Notice first that if $M \le 2n$, then $B = I$ directly gives $|B| = M \le 2n$ because $\prod_{a \in A} a \mid M! \mid \prod_{b \in I} b$. Thus we assume throughout the proof that $2n < M$, and as noted above that $I \subset \mathbb{Z}_{\ge 1}$.

Lemma 1.1 (Prime-Multiple Union Density). *Let $P \subseteq \mathbb{P} \cap [2, M]$ be any set of primes with $S := \sum_{p \in P} 1/p \ge 0.9$, and let*

$$U := \frac{1}{M} \#\{j \in \{1, \dots, M\} : \exists p \in P,\ p \mid j\}$$

be the proportion of integers in $\{1, \dots, M\}$ divisible by at least one prime in P. Then for all sufficiently large M (independent of P),

$$U \ge 0.5061.$$

Proof. Pick $j \in \{1, \dots, M\}$ uniformly at random and set $X := \sum_{p \in P} 1_{p \mid j}$, so $U = \mathbb{P}(X > 0)$. For each $p \in P$, the count $\lfloor M/p \rfloor$ lies in $(M/p - 1, M/p]$, so putting $\varepsilon_M := \pi(M)/M = o(1)$ gives

$$\mathbb{E}[X] = \sum_{p \in P} \frac{\lfloor M/p \rfloor}{M} \ge S - \frac{|P|}{M} \ge S - \varepsilon_M, \qquad \mathbb{E}[X^2] = \sum_{p \in P} \frac{\lfloor M/p \rfloor}{M} + \sum_{\substack{p,q \in P \\ p \ne q}} \frac{\lfloor M/(pq) \rfloor}{M} \le S + S^2.$$

If $S \ge 1.05$, then for M large enough that $\varepsilon_M < 0.002$, the Paley–Zygmund (Cauchy–Schwarz) inequality gives uniformly across all $S \ge 1.05$

$$U \ge \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]} \ge \frac{(S - \varepsilon_M)^2}{S(1+S)} \ge \frac{S^2 - 2S\varepsilon_M}{S(1+S)} = \frac{S}{1+S} - \frac{2\varepsilon_M}{1+S} \ge \frac{1.05}{2.05} - 2\varepsilon_M > 0.5121 - 2\varepsilon_M > 0.5061.$$

Now assume $0.9 \le S < 1.05$. Partition P into small primes $L := \{p \in P : p \le M^{1/3}\}$, large primes $H := \{p \in P : p > \sqrt{M}\}$, and the middle range $K := P \setminus (L \cup H) \subseteq (M^{1/3}, \sqrt{M}]$. Write $S_L := \sum_{p \in L} 1/p$, $S_H := \sum_{p \in H} 1/p$, $S_K := \sum_{p \in K} 1/p$, and let $e_k(L) := \sum_{T \subseteq L, |T| = k} \prod_{p \in T} 1/p$ be the k-th elementary symmetric polynomial in $\{1/p : p \in L\}$. By Mertens' second theorem,

$$S_K \le \sum_{M^{1/3} < p \le \sqrt{M}} \frac{1}{p} = \log \frac{\log \sqrt{M}}{\log(M^{1/3})} + o(1) = \log \frac{3}{2} + o(1) < 0.41$$

for sufficiently large M (as $\log(3/2) \approx 0.40547$), whence

$$S_L + S_H = S - S_K \ge 0.9 - 0.41 = 0.49. \tag{1}$$

Define $R(m) := 1_{m>0} - m + \binom{m}{2}$ for $m \in \mathbb{Z}_{\geq 0}$. Then $R(0) = 0$ and $R(m) = \binom{m-1}{2}$ for $m \geq 1$, so R is nonnegative and nondecreasing on $\mathbb{Z}_{\geq 0}$. Moreover, for every $m \geq 1$, Pascal's identity gives

$$R(m) - \left(\binom{m}{3} - \binom{m}{4}\right) = \binom{m-1}{2} - \binom{m}{3} + \binom{m}{4} = \binom{m}{4} - \binom{m-1}{3} = \binom{m-1}{4} \geq 0,$$

and the same inequality holds at $m = 0$. Letting $X_L := \sum_{p \in L} 1_{p|j} \leq X$, we obtain

$$R(X) \geq R(X_L) \geq \binom{X_L}{3} - \binom{X_L}{4}.$$

Taking expectations of $1_{X>0} = X - \binom{X}{2} + R(X)$ over $j \in \{1, \ldots, M\}$ gives three bounds:

(1) $\mathbb{E}[X] \geq S - \pi(M)/M = S - o(1)$;

(2) $\mathbb{E}\binom{X}{2} = \sum_{\{p,q\} \subseteq P} \frac{\lfloor M/(pq) \rfloor}{M} \leq \sum_{\{p,q\} \subseteq P} \frac{1}{pq} - \sum_{\{p,q\} \subseteq H} \frac{1}{pq}$ because $pq > M$ for every pair $\{p, q\} \subseteq H$, so $\lfloor M/(pq) \rfloor = 0$; expanding both symmetric sums gives

$$\mathbb{E}\binom{X}{2} \leq \frac{1}{2}\left(S^2 - \sum_{p \in P} \frac{1}{p^2} - S_H^2 + \sum_{p \in H} \frac{1}{p^2}\right) = \frac{S^2}{2} - \frac{1}{2}\left(S_H^2 + \sum_{p \in P \setminus H} \frac{1}{p^2}\right);$$

(3) for triples $\{p_1, p_2, p_3\} \subseteq L$, each denominator satisfies $\lfloor M/(p_1p_2p_3) \rfloor / M \geq (p_1p_2p_3)^{-1} - 1/M$ while there are at most $\binom{|L|}{3} \leq \pi(M^{1/3})^3 = o(M)$ triples (since $\pi(M^{1/3}) = O(M^{1/3}/\log M)$), and every quadruple satisfies $\lfloor M/(p_1p_2p_3p_4) \rfloor / M \leq (p_1p_2p_3p_4)^{-1}$, yielding $\mathbb{E}[R(X)] \geq e_3(L) - e_4(L) - o(1)$.

Combining (1)–(3) produces

$$U \geq S - \frac{S^2}{2} + \frac{1}{2}\left(S_H^2 + \sum_{p \in P \setminus H} \frac{1}{p^2}\right) + e_3(L) - e_4(L) - o(1). \tag{2}$$

Now note two algebraic inequalities on L. First, expanding $S_L e_3(L)$ counts each 4-element subset of L four times (plus nonnegative diagonal terms), so $4e_4(L) \leq S_L e_3(L)$; as $S_L \leq S < 1.05$, we get $e_3(L) - e_4(L) \geq (1 - S_L/4)e_3(L) \geq 0.7375\, e_3(L)$. Second, expanding $S_L^3 = (\sum_{p \in L} 1/p)^3$ yields

$$S_L^3 = 6e_3(L) + 3 \sum_{\substack{p,q \in L \\ p \neq q}} \frac{1}{p^2 q} + \sum_{p \in L} \frac{1}{p^3} \leq 6e_3(L) + 3S_L \sum_{p \in L} \frac{1}{p^2}.$$

Multiplying by 0.12 and noting that $6(0.12) = 0.72 \leq 0.7375$ and $3(0.12)S_L < 3(0.12)(1.05) = 0.378 < 0.5$ gives

$$0.12\, S_L^3 \leq e_3(L) - e_4(L) + \frac{1}{2}\sum_{p \in L} \frac{1}{p^2} \leq e_3(L) - e_4(L) + \frac{1}{2}\sum_{p \in P \setminus H} \frac{1}{p^2}.$$

Moreover, for all $x \geq 0$, the tangent inequality at $x = 0.4$ reads $x^3 - 0.48x + 0.128 = (x - 0.4)^2(x + 0.8) \geq 0$. Because $x \mapsto 0.48x - 0.128$ is nondecreasing on $\mathbb{R}$ and (1) implies $S_L \geq 0.49 - S_H$ regardless of whether $S_H \leq 0.49$ or $S_H > 0.49$, we have $S_L^3 \geq 0.48\, S_L - 0.128 \geq 0.48(0.49 - S_H) - 0.128 = 0.1072 - 0.48\, S_H$ unconditionally. Thus

$$\frac{1}{2}S_H^2 + 0.12\, S_L^3 \geq \frac{1}{2}S_H^2 - 0.0576\, S_H + 0.012864 = \frac{1}{2}(S_H - 0.0576)^2 + 0.01120512 > 0.0112.$$

Finally, $S \mapsto S - S^2/2 = \frac{1}{2} - \frac{1}{2}(1 - S)^2$ attains its minimum on $[0.9, 1.05]$ at $S = 0.9$, namely 0.495. Substituting into (2) gives $U \geq 0.495 + 0.011205 - o(1) > 0.5061$ for all sufficiently large M. □

Lemma 1.2 (Weight Comparison Between Initial and Shifted Intervals). *Let M be sufficiently large, let $2n < M$, let $I \subset \mathbb{Z}_{\geq 1}$ be an interval of length M, and set $\delta := 0.000055$ and $k := M - \lfloor \delta M \rfloor$. For every prime weight vector $z = (z_p)_{p \leq M} \in \mathbb{R}_{\geq 0}^{\pi(M)}$, define the truncated prime-power weight of $j \in \mathbb{Z}_{\geq 1}$ by*

$$W(j) := \sum_{p \leq M} z_p \sum_{\substack{\ell \geq 1 \\ p^\ell \leq M}} 1_{p^\ell | j}.$$

Then the sum of the k largest values of W on I is at least the sum of the n largest values of W on $\{1, \ldots, M\}$.

Proof. Since I is an interval of M consecutive integers, every prime power p^ℓ has at least $\lfloor M/p^\ell \rfloor$ multiples in I (and exactly $\lfloor M/p^\ell \rfloor$ multiples in $\{1, \ldots, M\}$), so $\sum_{j \in I} W(j) \geq \sum_{j=1}^{M} W(j)$.

Sort the values $W(1), \ldots, W(M)$ in nondecreasing order and let $t \geq 0$ be the r-th value, where $r := \lceil 0.494M \rceil$. Put $P := \{p \leq M : z_p > t\}$. Every $j \in \{1, \ldots, M\}$ divisible by some $p \in P$ has $W(j) \geq z_p > t$; yet at least $r \geq 0.494M$ integers in $\{1, \ldots, M\}$ satisfy $W(j) \leq t$. Hence P has multiple-union density $U \leq 1 - 0.494 = 0.506 < 0.5061$. By Lemma 1.1, for large M we must have $\sum_{p \in P} 1/p < 0.9$. In any shifted interval I, the number of integers divisible by at least one prime in P is at most $\sum_{p \in P} \lceil M/p \rceil \leq M \sum_{p \in P} 1/p + |P| < 0.9M + \pi(M)$, so at least $M - (0.9M + \pi(M)) \geq 0.099M$ elements of I avoid every prime in P.

If $t = 0$, every element $j \in I$ avoiding P satisfies $z_p = 0$ whenever $p \mid j$ and $p \leq M$, so $W(j) = 0$; deleting $\lfloor \delta M \rfloor < 0.099M$ such elements incurs zero weight loss.

Assume now $t > 0$. Define normalized truncated coefficients $b_p := z_p/t \in [0, 1]$ if $z_p \leq t$ and $b_p := 0$ if $p \in P$, and set

$$F(j) := \sum_{p \leq M} b_p \sum_{\substack{\ell \geq 1 \\ p^\ell \leq M}} 1_{p^\ell | j}, \qquad A_0 := \sum_{p \leq M} b_p \sum_{\substack{\ell \geq 1 \\ p^\ell \leq M}} p^{-\ell}.$$

Let $\pi^*(M) := \sum_{p \leq M} \lfloor \log M / \log p \rfloor = O(M/\log M) = o(M)$ count all prime powers at most M, and note $A_0 \leq \sum_{p \leq M} \frac{1}{p-1} = \log\log M + O(1)$. On $\{1, \ldots, M\}$, since $0 \leq b_p \leq 1$, we have $A_0 - \pi^*(M)/M \leq \mathbb{E}[F] \leq A_0$. For the second moment $\mathbb{E}[F^2]$:

- cross terms with distinct primes $p \neq q$ and powers p^{ℓ_1}, q^{ℓ_2} satisfy $\lfloor M/(p^{\ell_1} q^{\ell_2}) \rfloor / M \leq p^{-\ell_1} q^{-\ell_2}$;
- same-prime terms with $p^{\ell_1}, p^{\ell_2} \leq M$ contribute $\lfloor M/p^{\max(\ell_1, \ell_2)} \rfloor / M \leq p^{-\max(\ell_1, \ell_2)}$. For any prime $p \geq 2$ and integer $k_p \geq 1$, the exact grouping $\sum_{1 \leq \ell_1, \ell_2 \leq k_p} p^{-\max(\ell_1, \ell_2)} = \sum_{\ell=1}^{k_p} (2\ell - 1) p^{-\ell}$ satisfies

$$\frac{\sum_{\ell=1}^{k_p} (2\ell - 1) p^{-\ell}}{\sum_{\ell=1}^{k_p} p^{-\ell}} \leq \lim_{K \to \infty} \frac{\sum_{\ell=1}^{K} (2\ell - 1) 2^{-\ell}}{\sum_{\ell=1}^{K} 2^{-\ell}} = \frac{3}{1} = 3,$$

  because $p \geq 2$ and the partial-sum ratio increases with k_p and decreases with p. Using $b_p^2 \leq b_p$, same-prime terms contribute at most $3A_0$.

Therefore $\mathbb{E}[F^2] \leq A_0^2 + 3A_0$, whence

$$\mathbb{E}\big[(F - A_0)^2\big] = \mathbb{E}[F^2] - 2A_0 \mathbb{E}[F] + A_0^2 \leq (3 + o(1))A_0$$

(since $2A_0 \cdot \pi^*(M)/M = o(A_0)$). If $A_0 > 1$, then since at least $r \geq 0.494M$ elements $j \in \{1, \ldots, M\}$ satisfy $F(j) \leq W(j)/t \leq 1$, Chebyshev's inequality gives

$$0.494(A_0 - 1)^2 \leq \mathbb{E}\big[(F - A_0)^2\big] \leq (3 + o(1))A_0,$$

forcing $A_0 \leq 7.97 + o(1) \leq 8.1$ for large M.

Now evaluate the expectation $\mathbb{E}_I[F]$ on the shifted interval I: every prime power $p^\ell \leq M$ divides at most $M/p^\ell + 1$ elements of I, so $\mathbb{E}_I[F] \leq A_0 + \pi^*(M)/M \leq 8.2$ for large M. By Markov's inequality,

at most $(8.2/100)M = 0.082M$ elements of I satisfy $F(j) > 100$. Intersecting with the $\geq 0.099M$ elements avoiding P leaves at least $(0.099 - 0.082)M = 0.017M$ elements $j \in I$ satisfying both $F(j) \leq 100$ and $W(j) = tF(j) \leq 100t$.

In both cases ($t = 0$ and $t > 0$), deleting $\lfloor \delta M \rfloor < 0.017M$ such elements from I removes total weight at most $100t \cdot \delta M = 0.0055Mt$, leaving $k = M - \lfloor \delta M \rfloor$ elements in I of total weight at least $\sum_{j=1}^{M} W(j) - 0.0055Mt$. On $\{1, \ldots, M\}$, all $M - r$ elements above the r-th order statistic have weight at least t; since $n < M/2$ and $r < 0.494M + 1$, the $M - n$ lightest elements of $\{1, \ldots, M\}$ include at least $(M-n)-r > (0.5-0.494)M-1 = 0.006M-1$ elements of weight at least t, so their sum is at least $(0.006M - 1)t \geq 0.0055Mt$ (for large M). Subtracting from $\sum_{j=1}^{M} W(j)$ shows that the sum of the k largest weights on I is at least the sum of the n largest weights on $\{1, \ldots, M\}$. □

Proof of Theorem 1. Assume n (hence $M > 2n$) is sufficiently large. Note that for every $a \in A$, $a \leq M$, so the p-adic valuation of a equals the truncated sum $v_p(a) = \sum_{\ell \geq 1,\, p^\ell \leq M} 1_{p^\ell | a}$; whereas for $j \in I$, $\sum_{\ell \geq 1,\, p^\ell \leq M} 1_{p^\ell | j} \leq v_p(j)$. For each $j \in I$, define the vector $v_j := \left(\sum_{\ell \geq 1,\, p^\ell \leq M} 1_{p^\ell | j}\right)_{p \leq M} \in \mathbb{Z}_{\geq 0}^{\pi(M)}$, and put $D := \left(\sum_{a \in A} v_p(a)\right)_{p \leq M}$. Consider the bounded polytope

$$\mathcal{P} := \left\{ \sum_{j \in I} x_j v_j : \; x \in [0,1]^I, \; \sum_{j \in I} x_j \leq k \right\} \subset \mathbb{R}^{\pi(M)},$$

where $k = M - \lfloor \delta M \rfloor$. Since A has size n, the top-n weight sum on $\{1, \ldots, M\}$ is at least $\sum_{a \in A} W(a) = z \cdot D$ for every $z \in \mathbb{R}_{\geq 0}^{\pi(M)}$. Choosing x to put mass 1 on the k elements of I with largest weight $z \cdot v_j = W(j)$ and mass 0 elsewhere, Lemma 1.2 gives

$$z \cdot D \leq \max_{y \in \mathcal{P}} (z \cdot y) \qquad \forall\, z \in \mathbb{R}_{\geq 0}^{\pi(M)}.$$

Consider the closed convex downward closure $\mathcal{K} := \mathcal{P} - \mathbb{R}_{\geq 0}^{\pi(M)} \subset \mathbb{R}^{\pi(M)}$. Notice that any linear functional $z \cdot (\cdot)$ bounded above on $\mathcal{K}$ must have $z \in \mathbb{R}_{\geq 0}^{\pi(M)}$ (otherwise $z \cdot (-te_p) \to +\infty$ as $t \to +\infty$), for which $\sup_{y \in \mathcal{K}} (z \cdot y) = \max_{y \in \mathcal{P}} (z \cdot y) \geq z \cdot D$. By the Hahn–Banach separation theorem in $\mathbb{R}^{\pi(M)}$, $D \in \mathcal{K}$; that is, there exists a fractional cover $x \in [0,1]^I$ with $\sum_{j \in I} x_j \leq k$ and $\sum_{j \in I} x_j v_j \geq D$ coordinatewise.

Choose an extreme point (vertex) x^* of the nonempty compact polytope $\{x \in [0,1]^I : \sum_j x_j \leq k, \; \sum_j x_j v_j \geq D\}$. Since there are $\pi(M)$ covering inequalities and 1 budget inequality, at most $\pi(M) + 1$ coordinates of x^* can lie strictly inside $(0,1)$ (otherwise the subspace of fractional coordinates has dimension at least $\pi(M) + 2$, admitting a nonzero perturbation preserving all active constraints). Rounding every positive coordinate up gives $B := \{j \in I : x_j^* > 0\}$, which covers all target prime powers ($\prod_{a \in A} a \mid \prod_{b \in B} b$) and satisfies

$$|B| \leq \sum_{j \in I} x_j^* + \pi(M) + 1 \leq M - \lfloor \delta M \rfloor + \pi(M) + 1 \leq (1 - 0.000051)M$$

for sufficiently large M (as $\delta = 0.000055$ and $\pi(M) = o(M)$). Finally, $M \leq 2.0001n$ implies $2n \geq \frac{2}{2.0001} M > 0.99995\, M = (1 - 0.00005)M > (1 - 0.000051)M \geq |B|$, completing the proof. □

Remark. Because $\delta = 0.000055$ leaves a margin of only $4 \times 10^{-6} M$ to absorb $\pi(M) + 1 \sim M/\log M$, the implicit threshold n_0 in Theorem 1 is astronomically large ($\log M \gtrsim 2.5 \times 10^5$), making this result purely asymptotic rather than computational.

2 Quadratic Length: $M \geq n^2$ Yields $\tau(A, I) \leq 2n + 3$

In this section we prove a slightly more general formulation in which $A \subseteq \{2, \ldots, M\}$ is any n-element set without requiring $M \in A$ (so Corollary 2.1 applies directly to subsets $A' \subseteq A$).

Theorem 2 and Corollary 2.1 (Quadratic Threshold at $M = n^2$)

Theorem 2. *There exists an absolute constant $n_0 \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq n_0$, every n-element subset $A \subseteq \{2, \ldots, M\}$, and every interval I of length M,*

$$M \geq n^2 \implies \tau(A, I) \leq 2n + 3,$$

and likewise

$$M \geq (n+3)^2 \implies \tau(A, I) \leq 2n.$$

Corollary 2.1. *Along any sequence of inputs (A, I) with $|A| = n \to \infty$ and $M/n^2 \to 1$ (including from below, $M < n^2$), we have $\tau(A, I) \leq 2n + o(n)$.*

We adapt the divisor-decomposition and weighted Hall matching strategy from Wang [3, Appendix A] (which achieved $\tau(A, I) \leq 2n$ for $M \geq 16n^2$). As before, assume $I \subset \mathbb{Z}_{\geq 1}$.

2.1 Three-Way Decomposition

Put $Y := M/n \geq n$ and $D := M^{2/3}$. Partition A into three types:

- **Type 1 (large-prime inputs).** If $a \in A$ has a prime factor $p > Y$, write $a = pc$. Then $c \leq M/p < M/Y = n$, and $p^2 > Y^2 \geq M$, so $p^2 \nmid a$. Set p aside for Step 2, leaving the single cofactor $c < n$.
- **Type 2 (balanced / ordinary inputs).** Among the remaining elements of A, call a *ordinary* if $a = uv$ for some integers $1 \leq u, v \leq Y$ (choose one such pair). In particular, any a with a prime factor $p \in (n, Y]$ is ordinary via $u = p$, $v = a/p \leq M/n = Y$.
- **Type 3 (composite-core inputs).** Every remaining $a \in A$ has all prime factors at most n and possesses no factorization into two factors at most Y. List the prime factors of a with multiplicity and repeatedly merge any two parts whose product is at most D, until no two remaining parts multiply to at most D. At most two parts can remain: if three parts remained, each pairwise product would exceed D, making the product of the three parts exceed $D^{3/2} = M \geq a$, impossible. Because a is not ordinary, one of the remaining parts must exceed Y (and hence every divisor at most n already has product at most D). Let d be the smallest divisor of a strictly exceeding n, and write $a = dc$. Then $d \leq D = M^{2/3}$ and $d > Y$ (if $n < d \leq Y$, then $c = a/d \leq M/d < M/n = Y$, contradicting that a is not ordinary), whence $c \leq M/d < M/Y = n$.

For each Type 3 divisor $d > Y$, d is composite (all prime factors of a are at most $n \leq Y < d$), and by minimality every proper divisor of d is at most n. Let p be the smallest prime factor of d, and let $u := d/p \leq n \leq Y$ be its largest proper divisor. The crucial lower bound on its Type 3 cofactors c is

$$c \geq u.$$

Indeed, $a = u(pc)$ with $u \leq Y$. If $pc \leq Y$, then (u, pc) would give an ordinary factorization of a; hence $pc > Y \geq n$. Because $pc \mid a$ and $d = up$ is the smallest divisor of a exceeding n, we conclude $d = up \leq pc$, i.e., $c \geq u$. Letting $q_d := \lfloor M/d \rfloor$ and writing $m_d \geq 0$ for the number of Type 3 elements $a \in A$ assigned to divisor d, distinctness of those elements implies that their cofactors c are distinct integers in $[u, q_d]$, yielding

$$m_d \leq q_d - u + 1. \tag{3}$$

In particular, whenever a Type 3 divisor d is active ($m_d \geq 1$), we necessarily have $q_d \geq u$.

2.2 Selecting Distinct Multiples for Type 1 and Type 3 Divisors

For each Type 3 divisor d, let $C_d := \{b \in I : d \mid b\}$, $N_d := |C_d|$, and $N_{de} := |C_d \cap C_e|$ for $d \neq e$. Because distinct minimal divisors $d \neq e$ satisfy $Y < d, e \leq M^{2/3}$ while all proper divisors of d are at most $n \leq Y$, neither d nor e can divide the other; hence $\gcd(d, e)$ is a proper divisor of d, so $\gcd(d, e) \leq u = d/p$. Moreover, $N_d \geq q_d$ and $N_{de} \leq \lceil M \gcd(d, e)/(de) \rceil < M \gcd(d, e)/(de) + 1$. Using $d, e \leq M^{2/3}$ and $\gcd(d, e) \leq u$, we have $N_d \geq M/d - 1 \geq (M/d)(1 - M^{-1/3})$ (likewise for N_e) and $de/(Mu) = pe/M \leq pM^{2/3}/M = p/M^{1/3}$, which yields uniformly for every active pair $d \neq e$:

$$\frac{N_{de}}{N_d N_e} \leq \big(1 + O(M^{-1/3})\big) \left(\frac{u}{M} + \frac{de}{M^2}\right) \leq \big(1 + O(M^{-1/3})\big) \frac{u}{M} \left(1 + \frac{p}{M^{1/3}}\right). \tag{4}$$

We now show that for sufficiently large n and $M \geq n^2$, every active Type 3 divisor d (i.e., with $m_d \geq 1$, so $q_d \geq u$) satisfies the weighted row-sum bound

$$\frac{m_d}{N_d} + \sum_{e \neq d} m_e \frac{N_{de}}{N_d N_e} < 1. \tag{5}$$

Suppose for contradiction that (5) fails at some active d along a sequence of inputs with $n \to \infty$ and $M \geq n^2$. Since $N_d \geq q_d$ and $x \mapsto 1 - m_d/x$ is nondecreasing for $x \geq q_d \geq m_d$ (note $q_d \geq u \geq 1$ and $m_d \leq q_d - u + 1 \leq q_d$), (3) gives

$$1 - \frac{m_d}{N_d} \geq 1 - \frac{m_d}{q_d} \geq \frac{u - 1}{q_d} > 0,$$

as $u \geq \sqrt{d} > \sqrt{n} \to \infty$ (since $p \leq u$ is the smallest prime factor of $d = up > n$). Write $E_d := \sum_{e \neq d} m_e N_{de}/(N_d N_e)$ for the off-diagonal sum. Combining $q_d \leq M/d$, $u/(u-1) = 1 + o(1)$ uniformly in d, $\sum_{e \neq d} m_e \leq n - m_d$, and $p/d = 1/u$, (4) yields

$$\frac{E_d}{1 - m_d/N_d} \leq \frac{q_d}{u - 1} E_d \leq \big(1 + o(1)\big) \frac{n - m_d}{d} \left(1 + \frac{p}{M^{1/3}}\right) \leq \big(1 + o(1)\big) \left(\frac{n - m_d}{d} + \frac{n}{uM^{1/3}}\right).$$

Hence failure of (5) implies

$$1 \leq \big(1 + o(1)\big) \left(\frac{n - m_d}{d} + \frac{n}{uM^{1/3}}\right). \tag{6}$$

Since $d > Y = M/n \geq n$ and $n/(uM^{1/3}) \leq n/(n^{1/2} n^{2/3}) = O(n^{-1/6})$, (6) forces $d/n \to 1$, $m_d/n \to 0$, and $M/n^2 \to 1$. In this regime, $q_d = \lfloor M/d \rfloor \sim n$, so $m_d/N_d \leq m_d/q_d \to 0$. At the same time, $u = d/p \leq d/2$, $d \sim n$, and $M \sim n^2$ give $p \leq \sqrt{d} = O(n^{1/2})$ and $M^{1/3} \asymp n^{2/3}$, so $p/M^{1/3} = O(n^{-1/6}) = o(1)$. Therefore (4) yields directly

$$E_d \leq \big(1 + o(1)\big) \frac{nu}{M} \left(1 + \frac{p}{M^{1/3}}\right) \leq \big(1 + o(1)\big) \frac{n(d/2)}{n^2} = \frac{1}{2} + o(1),$$

so $m_d/N_d + E_d \leq \frac{1}{2} + o(1) < 1$ for large n, contradicting failure of (5). This proves (5) (even at $M = n^2$).

For any nonempty subset T of active Type 3 divisors, define $f(b) := \sum_{d \in T} \frac{m_d}{N_d} 1_{C_d}(b)$ for $b \in I$. Then $\sum_{b \in I} f(b) = \sum_{d \in T} m_d$, while expanding the square and applying (5) yields

$$\sum_{b \in I} f(b)^2 = \sum_{d \in T} m_d \left(\frac{m_d}{N_d} + \sum_{\substack{e \in T \\ e \neq d}} m_e \frac{N_{de}}{N_d N_e}\right) < \sum_{d \in T} m_d$$

whenever $\sum_{d\in T} m_d > 0$. By Cauchy–Schwarz,

$$\Big(\sum_{d\in T} m_d\Big)^2 \le \Big|\bigcup_{d\in T} C_d\Big| \sum_{b\in I} f(b)^2 < \Big|\bigcup_{d\in T} C_d\Big| \sum_{d\in T} m_d,$$

whence $\big|\bigcup_{d\in T} C_d\big| \ge \sum_{d\in T} m_d$. Hall's marriage theorem therefore provides pairwise distinct multiples in I, one for each Type 3 occurrence.

Next, for each Type 1 prime $p > Y$, if p divides r_p elements of A, then since those elements are distinct multiples of p in $[1, M]$ we have $\lfloor M/p \rfloor \ge r_p$, so I also contains at least r_p multiples of p. We choose r_p distinct multiples of p for each such p (allowing collisions across different Type 1 primes p and with the selected Type 3 multiples; every prime factor of a Type 3 divisor d is at most $n \le Y < p$, so Type 1 and Type 3 divisors are pairwise coprime and thus can share interval positions without reducing valuations). If A has m Type 2 (ordinary) elements, then the total number R of distinct interval positions chosen so far satisfies

$$R \le n - m.$$

2.3 Assigning Positions to Remaining Factors (and Tightening to $2n + 3$)

It remains to cover two factors at most Y for each of the m Type 2 inputs, and one cofactor $c < n \le Y$ for each of the $n - m$ Type 1 and Type 3 inputs ($n + m$ remaining factors in total). Any remaining factor lying in $(Y/2, Y]$ is called a *long factor*. Whenever a cofactor $c \in (Y/2, Y]$ appears s_c times as a Type 1 or Type 3 cofactor, all s_c original elements $a \in A$ equal cd_i with pairwise distinct multipliers $d_i > Y$; hence all s_c multipliers lie in $(\lfloor Y \rfloor, \lfloor M/c \rfloor]$, giving

$$s_c \le \lfloor M/c \rfloor - \lfloor Y \rfloor. \tag{7}$$

Lemma 2.2 (Three-Element Multiple Union). *Let $L \ge 1$ and $L/2 < a < b < c \le L$. In any interval I of M consecutive integers, the set $M_a \cup M_b \cup M_c$ of elements divisible by at least one of a, b, c satisfies*

$$|M_a \cup M_b \cup M_c| \ge \left(2 + \frac{1}{9}\right)\frac{M}{L} - 6. \tag{8}$$

Proof. Write $M_x := \{j \in I : x \mid j\}$. For each pair $x < y$ in $\{a, b, c\}$, write the reduced fraction $x/y = i/j$ ($\gcd(i, j) = 1$), so $\operatorname{lcm}(x, y) = iy$. Since $x/y > 1/2$ and $x < y$, the only reduced fraction with $i < 3$ is $i/j = 2/3$. Notice that at most one of the pairs (a, b), (b, c), (a, c) can equal $2/3$: if $a/b = b/c = 2/3$, then $a/c = 4/9 < 1/2$, contradicting $a > L/2 \ge c/2$.

By Bonferroni's inequality and $\lfloor M/d \rfloor \le |M_d| \le M/d + 1$,

$$|M_a \cup M_b \cup M_c| \ge \frac{M}{a} + \frac{M}{b} + \frac{M}{c} - \frac{M}{\operatorname{lcm}(a, b)} - \frac{M}{\operatorname{lcm}(b, c)} - \frac{M}{\operatorname{lcm}(a, c)} - 6.$$

If no pair has ratio $2/3$, then $\operatorname{lcm}(a, b) \ge 3b$, $\operatorname{lcm}(a, c) \ge 3c$, and $\operatorname{lcm}(b, c) \ge 3c$, giving a main term at least $M\big(\frac{1}{a} + \frac{2}{3b} + \frac{1}{3c}\big)$ (note that if $a \le 0.9L$, then $\frac{1}{a} + \frac{2}{3b} + \frac{1}{3c} \ge \frac{10}{9L} + \frac{1}{L} = \frac{19}{9L} = (2 + \frac{1}{9})\frac{1}{L}$; if $a > 0.9L$, then every ratio $x/y > 0.9$, forcing $i \ge 10$ for all three pairs and hence $\ge (3 - \frac{3}{10})\frac{M}{L} = 2.7\frac{M}{L} > (2 + \frac{1}{9})\frac{M}{L}$).

If one pair (v, w) has ratio $v/w = 2/3$, then replacing $\operatorname{lcm}(v, w) = 2w$ by $3w$ deducts an extra $M/(2w) - M/(3w) = M/(6w)$ from the bound above; however, v appears with coefficient at least $2/3$ in the bound above (specifically 1 if $v = a$, or $2/3$ if $v = b$), so relative to replacing denominators by L, replacing $1/v$ by $1/w = \frac{2}{3}/v$ gained an extra $\ge \frac{2}{3}M\big(\frac{1}{v} - \frac{1}{w}\big) = \frac{2}{3}M \cdot \frac{1}{2w} = \frac{M}{3w} > \frac{M}{6w}$, keeping the bound strictly valid (indeed yielding $(2 + 1/6)M/L - 6$). □

We now check Hall's matching condition for assigning each remaining long factor in $(Y/2, Y]$ to a *fresh* interval position in I avoiding the $R \le n - m$ positions already used above. To test any subset of long-factor values $S \subseteq (Y/2, Y]$, it suffices to check all occurrences of the values in S simultaneously (taking fewer occurrences relaxes demand while keeping candidates unchanged). Note that $M/Y = n$:

(a) $|S| \ge 3$**.** Lemma 2.2 (with $L = Y$) guarantees at least $(2 + 1/9)n - 6$ candidate positions in I. Excluding at most $R \le n - m$ prior positions leaves at least $(1+1/9)n+m-6$ fresh candidates, whereas total long-factor demand is at most $n + m$, which is covered for all $n \ge 54$.

(b) $|S| \in \{1, 2\}$**: Type 2 contribution.** Note that any Type 2 input $a \in A$ has at most two remaining factors in S. Moreover, if both factors of $a = uv$ lie in S, the unordered pair $\{u, v\}$ can take at most three values (when $S = \{x, y\}$, only $\{x, x\}, \{x, y\}, \{y, y\}$). Because elements of A are distinct, at most three Type 2 elements contribute 2 factors to S, so the total number of Type 2 factors in S is at most $m + 3$.

(c) $|S| \in \{1, 2\}$ **with zero Type 1/Type 3 cofactors in** S**.** If Type 2 elements supply at most m factors in S, a single factor $x \in S$ already has at least $\lfloor M/x \rfloor \ge \lfloor Y \rfloor \ge n$ candidates in I, leaving at least $n - R \ge m$ fresh positions. If Type 2 elements supply $m + 1, m + 2$, or $m + 3$ factors in S, at least one Type 2 element $a = uv \le M$ has both $u, v \in S$; its smaller factor satisfies $\min(u, v) \le \sqrt{M}$, giving at least $\lfloor M/\sqrt{M} \rfloor = \lfloor \sqrt{M} \rfloor$ candidates in I, hence at least $\lfloor \sqrt{M} \rfloor - R$ fresh positions against demand at most $m + 3$.

(d) $|S| \in \{1, 2\}$ **with one or two Type 1/Type 3 cofactors in** S**.** Let $s(S)$ be the total Type 1/Type 3 cofactor demand in S. If S contains one such cofactor c, (7) gives $|M_c| \ge \lfloor M/c \rfloor \ge \lfloor Y \rfloor + s_c$, so fresh candidates number at least $\lfloor Y \rfloor + s(S) - R \ge \lfloor Y \rfloor + s(S) - n + m$ against demand at most $s(S) + m + 3$.

If $S = \{c_1, c_2\}$ $(c_1 < c_2)$ contains two Type 1/Type 3 cofactors, put $L_0 := \operatorname{lcm}(c_1, c_2)$. Since $c_1, c_2 \in (Y/2, Y]$ are distinct, $L_0 \ge 2c_2 > Y = M/n$, so $M < nL_0$. If I contained t multiples of L_0, their span would satisfy $(t - 1)L_0 \le M - 1 < M < nL_0$, forcing $t - 1 < n$ and thus $|M_{c_1} \cap M_{c_2}| = t \le n$. Applying (7) to both cofactors gives

$$|M_{c_1} \cup M_{c_2}| = |M_{c_1}| + |M_{c_2}| - |M_{c_1} \cap M_{c_2}| \ge \big(\lfloor Y \rfloor + s_{c_1}\big) + \big(\lfloor Y \rfloor + s_{c_2}\big) - n = 2\lfloor Y \rfloor + s(S) - n.$$

Subtracting $R \le n - m$ leaves at least $2\lfloor Y \rfloor - 2n + s(S) + m$ fresh candidates against demand at most $s(S) + m + 3$.

Observe that across *all* subcases (a)–(d):

- When $M \ge (n + 3)^2$, we have $\lfloor \sqrt{M} \rfloor \ge n + 3$ and $\lfloor Y \rfloor \ge \lfloor (n + 3)^2/n \rfloor \ge n + 6$ (hence $2\lfloor Y \rfloor - 2n \ge 12 > 3$). Thus every subcase has zero Hall deficit, all long factors can be matched to pairwise distinct fresh positions in I, and the remaining short factors $(\le Y/2)$ each have at least $\lfloor M/(Y/2) \rfloor = 2n$ multiples in I and can therefore be greedily placed on fresh positions one by one, giving total size at most $2n$.
- When only $M \ge n^2$ holds, we have $\lfloor Y \rfloor \ge n$ and $\lfloor \sqrt{M} \rfloor \ge n$, so the Hall deficit across subcases (c) and (d) is at most 3 (explicitly: $(m + 3) - (\lfloor \sqrt{M} \rfloor - (n - m)) \le 3$, $(s(S) + m + 3) - (\lfloor Y \rfloor + s(S) - n + m) \le 3$, and $(s(S) + m + 3) - (2\lfloor Y \rfloor - 2n + s(S) + m) \le 3$). Equivalently, adjoining three dummy vertices adjacent to every long factor satisfies Hall's condition, so all but $h \le 3$ long factors receive distinct fresh positions in I. Greedily placing all short factors $(\le Y/2$, each having $\ge 2n$ multiples in $I)$ on fresh positions yields a partial cover $B_0 \subseteq I$ of size $|B_0| \le 2n - h$ covering every factor except $h \le 3$ omitted long factors $v_1, \ldots, v_h \in (Y/2, Y]$.

Completion of Theorem 2 ($M \ge n^2$ deficit repair) and Corollary 2.1. Put $H := \lfloor Y/3 \rfloor$. For $n \ge 30$, $Y \ge n \ge 30$ gives $H \ge Y/3 - 1 \ge 0.3Y$, so $H^{3/2} \ge (0.3)^{3/2} Y^{3/2} > 0.16\sqrt{30}\, Y > Y$ (as $Y \ge 30$

and $0.16\sqrt{30} \approx 0.876$, and for $n \geq 40$, $(0.3)^{3/2}\sqrt{40} \approx 1.039 > 1$; simply take $n_0 \geq 40$). For each omitted value $v_i \leq Y$:

- if v_i has a prime factor $p > H$, then $v_i/p \leq Y/(H+1) < 3$, so $v_i/p \in \{1, 2\}$ and $v_i = pc_i$ with $c_i \leq 2 \leq H$;
- if all prime factors of v_i are at most H, repeatedly merging pairs of prime factors whose product is at most H yields at most two factors (three factors pairwise multiplying to $> H$ would multiply to $> H^{3/2} > Y \geq v_i$), both at most H.

Thus every v_i factors as $p_i c_i$ where $c_i \leq H$ and $p_i \leq H$ or p_i is prime ($p_i > H$). Since each factor $\leq H$ has at least $\lfloor M/H \rfloor \geq 3n > 2n+3$ multiples in I, we first append one fresh multiple of c_i (and also p_i if $p_i \leq H$) to B_0 for each $i = 1, \ldots, h$. Finally, for each remaining prime $p_i > H$, because $\prod_{a\in A} a \mid M! \mid \prod_{b\in I} b$, if the current cover B still lacks the required p_i-adic exponent, at least one multiple of p_i in $I \setminus B$ remains unselected; appending one such multiple increases the p_i-adic valuation by at least 1. Performing this at most once per remaining p_i adds at most $2h$ total positions to B_0, yielding a complete cover $B \subseteq I$ of size $|B| \leq (2n-h) + 2h = 2n + h \leq 2n + 3$.

For Corollary 2.1, suppose $|A| = n \to \infty$ and $M/n^2 \to 1$ with $M < n^2$. Set $k := \lfloor \sqrt{M} \rfloor = n - o(n)$ and pick any k-element subset $A' \subseteq A$. Because $A' \subseteq \{2, \ldots, M\}$ satisfies $M \geq k^2$ (note Theorem 2 did not require $M \in A'$), Theorem 2 covers A' in I using at most $2k+3$ positions for large n. Put $t := n - k = o(n)$ and $H' := \lfloor M/(3n) \rfloor \asymp n$. Each of the t remaining elements $a \in A \setminus A'$ has at most two prime factors (with multiplicity) exceeding H' (since $H' > M^{1/3}$ for large n so $(H'+1)^3 > M$), and repeatedly merging prime factors $\leq H'$ leaves at most four parts (five parts pairwise multiplying to $> H'$ would exceed $(H')^{5/2} > M$ for large n), each at most H'. First append one fresh multiple of each small part (each has at least $\lfloor M/H' \rfloor \geq 3n$ multiples in I), and then top up each deficient prime $p > H'$ one by one using $\prod A \mid \prod I$. This adds at most $6t$ positions, valid since $2k + 3 + 6t < 3n$ for large n. Hence $\tau(A, I) \leq 2k + 3 + 6t = 2n + o(n)$. □

3 An Explicit Sufficient Criterion Using High-Demand Primes

For an arbitrary input $A \subseteq \{2, \ldots, M\}$ of size n with $M \geq 2n$, define for each prime p its total target demand $D_p := \sum_{a\in A} v_p(a)$, define the set of *high-demand primes*

$$\mathcal{H} := \{p \in \mathbb{P} : \ D_p > \lfloor M^{1/3} \rfloor\},$$

and extract the high-demand factor of each $a \in A$ via $H(a) := \prod_{p\in\mathcal{H}} p^{v_p(a)}$. Notice that any prime $p > M^{2/3}$ has at most $\lfloor M/p \rfloor < M^{1/3}$ multiples in $\{2, \ldots, M\}$ and $p^2 > M$, so $v_p(a) \leq 1$ and $D_p < M^{1/3}$; thus no prime above $M^{2/3}$ belongs to $\mathcal{H}$.

Theorem 3 (Uniform Intermediate $2n$ Bound)

Theorem 3. *Let $M \geq 2n$ and $A \subseteq \{2, \ldots, M\}$ with $|A| = n$. If $H(a) \leq \lfloor M/(2n) \rfloor$ for every $a \in A$, then for every interval I of M consecutive integers,*

$$\tau(A, I) \leq 2n,$$

uniformly for all $n \geq 1$ without any asymptotic threshold.

Proof. As before, assume $I \subset \mathbb{Z}_{\geq 1}$. Factor each $a \in A$ into at most two parts $(d_{a,1}, d_{a,2})$: if a has a prime factor $p > M^{2/3}$ (which satisfies $v_p(a) = 1$ and $p \notin \mathcal{H}$), set $d_{a,1} := a/p \leq M^{1/3} \leq M^{2/3}$ and set p aside for Step 2; otherwise all prime factors of a are at most $M^{2/3}$, so repeatedly merging

pairs of prime factors with product at most $M^{2/3}$ splits $a = d_{a,1}d_{a,2}$ into two factors $\le M^{2/3}$ (three remaining parts would multiply to $> (M^{2/3})^{3/2} = M \ge a$).

Step 1: Covering all parts $d \le M^{2/3}$ while maintaining a global invariant. Let $\mathcal{L} := (d_1, \dots, d_s)$ ($s \le 2n$) list all parts $d \le M^{2/3}$ from the above factorizations. Maintain a growing set $B \subseteq I$ and a set $\mathcal{S} \subseteq \mathbb{P} \setminus \mathcal{H}$ of primes whose *entire* target demand D_p is already satisfied by $\prod_{b\in B} b$. Initialize $B = \emptyset$ and $\mathcal{S} = \emptyset$. For $r = 1, \dots, s$, let $d = d_r \mid a$ be the current part:

- If $I \setminus B$ contains a multiple b of d, add b to B (paying d in full while adding 1 element).
- Otherwise, every multiple of d in I already lies in B, so $\prod_{b\in B} b$ already contains at least $\lfloor M/d \rfloor \ge \lfloor M^{1/3} \rfloor$ multiples of d. Consequently, for every prime $p \mid d$ with $p \notin \mathcal{H}$, $v_p(\prod_{b\in B} b) \ge \lfloor M^{1/3} \rfloor \ge D_p$, so we may add all such primes $p \notin \mathcal{H}$ dividing d into $\mathcal{S}$. Write $d_{\mathcal{H}} := \prod_{p\in\mathcal{H}} p^{v_p(d)}$. Since $d \mid a$, we have $d_{\mathcal{H}} \mid H(a)$, so $d_{\mathcal{H}} \le H(a) \le \lfloor M/(2n) \rfloor$. Thus $d_{\mathcal{H}}$ has at least $2n$ multiples in I, while $|B| \le r - 1 < s \le 2n$; pick an unused multiple $b \in I \setminus B$ of $d_{\mathcal{H}}$ and add it to B. This pays the full $\mathcal{H}$-part of d, while all non-$\mathcal{H}$ primes dividing d already belong to $\mathcal{S}$ and hence have their entire demand D_p satisfied.

Notice that once $p \in \mathcal{S}$, enlarging B preserves $v_p(\prod_{b\in B} b) \ge D_p$. At termination, $\prod_{b\in B} b$ covers all factors $\le M^{2/3}$ using $|B| \le s$ positions.

Step 2: Adjusting deficient large primes $p > M^{2/3}$. For each of the t primes $p > M^{2/3}$ set aside initially (so $s + t \le 2n$), if $v_p(\prod_{b\in B} b) < D_p$, the identity $\prod_{a\in A} a \mid M! \mid \prod_{b\in I} b$ guarantees an unselected multiple $b \in I \setminus B$ of p; adding it increments the p-adic valuation by at least 1. Repeating once per deficient slot adds at most t elements, giving $|B| \le s + t \le 2n$. □

Remark. In general inputs, Theorem 3 does not apply directly: for instance, if $p \asymp \sqrt{n}$ is prime and $A = \{p, 2p, \dots, np\}$, then $M = np \asymp n^{3/2}$, $D_p \ge n > \lfloor M^{1/3} \rfloor$, and $H(a) \ge p > M/(2n)$ for all $a \in A$. That specific structured family is trivially covered in n positions by taking all n multiples of p in I (whose quotients are n consecutive integers with product divisible by $n!$), whereas the genuine obstruction for arbitrary intermediate inputs lies in the simultaneous interaction and contention across multiple such high-demand primes.

4 Sharp Constant $\lceil kn/(k-1) \rceil$ for Same-Scale k-Prime Products

We next determine the exact leading worst-case constant for the structured family of products of k primes at a common scale $[X, CX]$, which naturally populates intermediate regimes such as $M \asymp n^{3/2}$ ($k = 3$).

Theorem 4 (Optimal Bound for Same-Scale k-Prime Products)

Theorem 4. *Fix an integer $k \ge 2$ and a real constant $C \ge 1$. For all sufficiently large $X \ge X_0(k, C)$, if $A \subseteq \{2, \dots, M\}$ is any n-element set in which every element $a \in A$ is a product of exactly k primes (counted with multiplicity) lying in $[X, CX]$, then for every interval I of length $M = \max A$,*

$$\tau(A, I) \le \lceil \frac{kn}{k-1} \rceil.$$

Furthermore, for every fixed $k \ge 2$, the leading coefficient $k/(k-1)$ is sharp: there exist sequences of such sets A ($n \to \infty$) and intervals I with $\tau(A, I) \ge \big(k/(k-1) - o(1)\big)n$.

Proof. Assume $I \subset \mathbb{Z}_{\ge 1}$ and note $M \ge X^k$.

Part 1: Upper bound via cofactor LP rounding. Let $P \subseteq \mathbb{P} \cap [X, CX]$ be the set of primes dividing elements of A, and put $r := |P| \le \pi(CX) = O_C(X/\log X) = o(X)$ (and recall $D_p := \sum_{a\in A} v_p(a)$). For each $a = p_1 \cdots p_k \in A$ (listing its k prime factors in $[X, CX]$ with multiplicity), form the k cofactors $a/p_1, \ldots, a/p_k$, assigning fractional weight $1/(k-1)$ to each. For every $p \in P$, the total p-adic contribution across these k weighted cofactors of a equals

$$\frac{1}{k-1}\sum_{i=1}^{k} v_p(a/p_i) = \frac{1}{k-1}\Big(kv_p(a) - \sum_{i=1}^{k} 1_{p_i=p}\Big) = \frac{kv_p(a) - v_p(a)}{k-1} = v_p(a).$$

Aggregating identical cofactor values d into x^0_d gives a fractional vector $x^0 = (x^0_d)_{d\in\mathcal{D}}$ satisfying

$$\sum_{d\in\mathcal{D}} x^0_d\, v_p(d) = D_p \quad (\forall\, p \in P), \qquad 0 \le x^0_d \le \frac{kr}{k-1} \quad (\forall\, d \in \mathcal{D}),$$

where the upper bound holds because for any fixed $(k-1)$-prime product d, every element $a \in A$ yielding d has the form $a = dp$ for some $p \in P$ (at most r such $a \in A$) and each a contributes d at most k times with weight $1/(k-1)$. Notice that summing the r linear equations over all $p \in P$ forces $(k-1)\sum_d x_d = \sum_p D_p = kn$ on any solution x of $\sum_d x_d v_p(d) = D_p$.

Let x^* be a vertex of the nonempty compact polytope $\big\{x \in \mathbb{R}^{\mathcal{D}} : 0 \le x_d \le \lceil x^0_d\rceil,\ \sum_d x_d v_p(d) = D_p\ (\forall\, p \in P)\big\}$. Because there are only r equality constraints, at most r coordinates of x^* can satisfy $0 < x^*_d < \lceil x^0_d\rceil$ (otherwise the nullspace of the r equations on those strict coordinates allows a nonzero bidirectional perturbation). Round every coordinate down to $\lfloor x^*_d\rfloor$. For each $p \in P$, the remaining deficit $D'_p := D_p - \sum_d \lfloor x^*_d\rfloor v_p(d) \ge 0$ is a nonnegative integer, and summing over $p \in P$ gives total remaining prime-factor multiplicity

$$T := \sum_{p\in P} D'_p = kn - (k-1)\sum_{d\in\mathcal{D}} \lfloor x^*_d\rfloor.$$

Because $(k-1)\sum_d x^*_d = kn$, we have $T = (k-1)\sum_d (x^*_d - \lfloor x^*_d\rfloor) < (k-1)r$ whenever $T > 0$ (as at most r fractional parts $\{x^*_d\}$ are nonzero and each is strictly below 1). Now partition the T remaining prime-factor tokens (with multiplicity) arbitrarily into $\lceil T/(k-1)\rceil$ multisets of size $k-1$ (with the final multiset possibly of size $< k-1$ and padded with arbitrary primes from P or left shorter; note padding never reduces valuations). Let $y_d \in \mathbb{Z}_{\ge 0}$ denote the total count of each $(k-1)$-or-fewer prime product d from both $\lfloor x^*_d\rfloor$ and the extra $\lceil T/(k-1)\rceil$ groups. Then:

- $\sum_d y_d v_p(d) \ge D_p$ for every $p \in P$;
- the total number of slots required is

$$\sum_{d\in\mathcal{D}} \lfloor x^*_d\rfloor + \Big\lceil \frac{kn - (k-1)\sum_{d\in\mathcal{D}} \lfloor x^*_d\rfloor}{k-1}\Big\rceil = \Big\lceil \frac{kn}{k-1}\Big\rceil;$$

- each single value d has multiplicity $y_d \le \lceil x^0_d\rceil + \lceil T/(k-1)\rceil \le \lceil \frac{kr}{k-1}\rceil + r$.

Part 2: Disjoint P-clean multiple pools. Every group product d satisfies $d \le (CX)^{k-1}$, whereas $M \ge X^k$. Therefore the multiples of d in I, after dividing by d, form a set of at least $L := \lfloor M/d\rfloor \ge \lfloor X/C^{k-1}\rfloor$ consecutive integers. Deleting from these L quotients every integer divisible by at least one prime $p \in P$ leaves at least

$$L - \sum_{p\in P}\left(\frac{L}{p} + 1\right) \ge L\left(1 - \frac{r}{X}\right) - r$$

surviving quotients q (using $p \geq X$). Since C is fixed and $r \leq \pi(CX) = O_C(X/\log X) = o(X)$ as $X \to \infty$, we have $L(1 - r/X) - r \geq \lceil \frac{kr}{k-1} \rceil + r \geq y_d$ for all sufficiently large $X \geq X_0(k, C)$. For any surviving quotient q, the interval element $b = dq \in I$ satisfies $\gcd(q, \prod_{p \in P} p) = 1$, hence

$$\prod_{p \in P} p^{v_p(b)} = d.$$

Thus the candidate pools $\{dq \in I\}$ for distinct values d are *pairwise disjoint*! Selecting y_d distinct elements from the pool of d for each active d produces a subset $B \subseteq I$ of size $\sum_d y_d = \lceil \frac{kn}{k-1} \rceil$ satisfying $\prod_{a \in A} a \mid \prod_{b \in B} b$.

Part 3: Matching lower bound. Fix $k \geq 2$ and choose $1 < c < 2^{1/k}$. Let $P := \mathbb{P} \cap [X, cX]$ and $r := |P| \to \infty$. Take A to be a collection of n distinct squarefree k-prime products from P with $n/r \to \infty$ (for instance, all $\binom{r}{k}$ distinct products, or for $k = 3$ take $n = \lfloor X^2 \rfloor$ distinct products so $M \asymp n^{3/2}$ and $r = o(n)$). Note $M < (cX)^k < 2X^k$. Center the interval I of length M around $T_0 := \prod_{p \in P} p^k$, so every $b \in I \setminus \{T_0\}$ satisfies $0 < |b - T_0| \leq M/2 < X^k$. Consequently, no $b \in I \setminus \{T_0\}$ can be divisible by k primes from P (counted with multiplicity): if $q = \prod_{i=1}^{k} p_i \mid b$ for $p_i \in P$, then $q \mid T_0$ as well, forcing $q \mid |b - T_0| < X^k \leq q$, contradiction. Thus every $b \in I \setminus \{T_0\}$ contributes at most $k - 1$ prime-factor tokens from P (counted with multiplicity), while T_0 itself contributes kr tokens. Since the target product $\prod_{a \in A} a$ requires kn tokens from P, any valid cover $B \subseteq I$ must have size at least

$$|B| \geq \frac{kn - kr}{k - 1} = \left(\frac{k}{k-1} - \frac{kr}{(k-1)n} \right) n = \left(\frac{k}{k-1} - o(1) \right) n. \qquad \square$$

5 Remaining Obstacles Toward $g(n) \leq (2 + o(1))n$

Establishing the full asymptotic bound $g(n) \leq (2 + o(1))n$ requires bounding all n-element sets $A \subseteq \{2, \ldots, M\}$ and all intervals I uniformly by $2n + o(n)$. The results proved here leave open the intermediate mixed-scale regimes, notably arbitrary inputs with $M \asymp n^{3/2}$ as well as fixed linear aspect ratios $M/n > 2.0001$. By Erdős and Surányi's 1959 lower bound [2] (Section 9), if the conjecture holds, the leading constant 2 is best possible; settling those intermediate mixed regimes will likely require sharper control of colliding prime-power neighborhoods across heterogeneous factor scales.

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