# The Mathematical Structure of Accessibility Geometry: State Space, Equivalence, Invariants, and Cosmographic Correspondence

## Definitive Mathematical Edition

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## Abstract

We develop the representation-theoretic structure of accessibility geometry independently of any particular dynamical completion. A homogeneous density is written as $\rho = Q/S_{D,X}$ with $S_{D,X} \propto R_X^D$, yielding the dilution law $n = D\gamma_X - \eta$, where $\gamma_X = \mathrm{d}\ln R_X/\mathrm{d}\ln a$ and $\eta = \mathrm{d}\ln Q/\mathrm{d}\ln a$. On a connected interval, equal dilution histories fix the shape of the density history, and equality at one reference epoch fixes its normalization. The compensating transformations of $\eta$ and $Q$ therefore preserve the full density history when that normalization is fixed. We distinguish standard cosmographic variables $(H, q, j, w_{\mathrm{eff}})$, which describe expansion kinematics, from structural variables $(R_X, \gamma_X, D, \eta, Q)$, which resolve a density history into measure growth and charge evolution. We define the accessibility coordinate space and its constant-dilution foliation; establish dilution and density equivalence with finite transformation laws for $\eta$ and $Q$; identify the invariant content and the background identifiability limit; classify positive, regular, bounded, and asymptotically complete charge histories; derive the generalized law for a running index $D(a)$; and extend the construction to multiple sectors. The exact apparent-horizon dictionary $R_A = c/H$, $\gamma_A = 1+q$, and $\gamma_A' = 2\gamma_A^2 - 3\gamma_A + 1 - j$ connects the structural and kinematic layers. Numerical particle-, apparent-, and event-horizon histories provide concrete trajectories through the state space. The result completes the mathematical and representational layer of the COSMOS framework while making explicit that predictive content begins only when an independently motivated law restricts the horizon choice, scaling index, charge evolution, or perturbations.

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## I. INTRODUCTION

The COSMOS accessibility framework begins from a deliberately simple representation of a homogeneous density,

$$\rho(a) = \frac{Q(a)}{S_{D,X}(a)}, \qquad S_{D,X}(a) = S_{D,X0}\left[\frac{R_X(a)}{R_{X0}}\right]^D. \tag{1}$$

Here $R_X$ is a selected proper cosmological horizon or physically selected length scale, $D$ is an effective scaling index, and $Q$ is an accessibility charge. The logarithmic slopes

$$\gamma_X(a) \equiv \frac{\mathrm{d}\ln R_X}{\mathrm{d}\ln a}, \qquad \eta(a) \equiv \frac{\mathrm{d}\ln Q}{\mathrm{d}\ln a} \tag{2}$$

give the central decomposition

$$\boxed{n(a) = D\gamma_X(a) - \eta(a)} \tag{3}$$

and therefore, for an effective separately conserved fluid,

$$\boxed{w(a) = \frac{D\gamma_X(a)}{3} - 1 - \frac{\eta(a)}{3}}. \tag{4}$$

Equations (3) and (4) are mathematically exact once the representation (1) is adopted. They do not, by themselves, specify a microscopic theory, a dynamical equation for $Q$, a preferred horizon, or a perturbative completion. This distinction defines the task of the present paper.

The question considered here is not immediately "what is $Q$?" Instead we ask:

> What mathematical properties, equivalences, admissibility conditions, and cosmographic correspondences follow from the accessibility representation before any microscopic interpretation is imposed?

This question is necessary because the framework contains both information and freedom. The observable background history fixes $n(a)$, while the decomposition into $D$, $X$, and $\eta$ is generally non-unique. A useful mathematical theory must therefore identify:

1. the state space on which an accessibility representation lives;
2. the transformations that preserve the background history;
3. the quantities invariant under those transformations;
4. the minimum regularity and asymptotic requirements on $Q$ and $\eta$;

5. the relation between standard cosmographic variables and accessibility variables;

6. the exact boundary between kinematic classification and genuine dynamics.

**Scope.** This paper establishes the mathematical architecture of accessibility geometry. It does not claim that positivity, monotonicity, entropy growth, causal bounds, fixed points, or attractors follow universally from COSMOS. Such statements are derived only when their additional assumptions are stated explicitly.

The analysis proceeds in four layers. First, we construct an exact dictionary between the standard cosmographic hierarchy $(H, q, j, \ldots)$ and the apparent-horizon quantities $(R_A, \gamma_A, \gamma_A', \ldots)$. Second, we define the representation space and its background-equivalence classes. Third, we classify admissible charge histories and generalized scaling sectors. Fourth, we use the continuous three-horizon timeline developed in the companion numerical analysis as a set of concrete trajectories through the abstract state space.

On a connected interval, equal dilution histories fix the shape of the density history, and equality at one reference epoch fixes its normalization. The compensating transformations of $\eta$ and $Q$ therefore preserve the full density history when that normalization is fixed. This equivalence is the basis for the identifiability analysis in Sec. XII.

## II. KINEMATIC FOUNDATION

### A. Dilution and effective equation of state

For a positive homogeneous density, define the instantaneous dilution exponent

$$n(a) \equiv -\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a}. \tag{5}$$

For an effective separately conserved perfect fluid,

$$w(a) = \frac{n(a)}{3} - 1. \tag{6}$$

The standard constant-fluid ladder is summarized in Table I.

The component-level boundary for the sign of its contribution to acceleration is

$$n = 2 \quad \Longleftrightarrow \quad w = -\frac{1}{3} \quad \Longleftrightarrow \quad \rho + 3p = 0 \tag{7}$$

for a positive-density component. In a multicomponent universe, the total acceleration is determined by the sum over all sectors.

TABLE I. Kinematic classification of constant dilution histories.

| $n$ | $w = n/3 - 1$ | Effective background behavior |
|---|---|---|
| 0 | $-1$ | Vacuum-like |
| 1 | $-2/3$ | Domain-wall-like |
| 2 | $-1/3$ | Bridge-, string-, or curvature-like |
| 3 | 0 | Pressureless-matter-like |
| 4 | $1/3$ | Radiation-like |

### B. The accessibility decomposition

Taking the logarithmic derivative of Eq. (1) gives

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a} = \eta - D\gamma_X, \tag{8}$$

which yields Eq. (3). The decomposition contains three logically distinct ingredients:

$$D: \text{ the scaling response of the measure to the selected length,} \tag{9}$$

$$\gamma_X: \text{ the growth rate of that length relative to cosmic expansion,} \tag{10}$$

$$\eta: \text{ the logarithmic evolution of the charge.} \tag{11}$$

The observed quantity $n$ is the combination of these ingredients. This immediately suggests a geometric state space and an associated equivalence relation.

## III. THE STANDARD-COSMOLOGY–COSMOS DICTIONARY

Standard cosmography describes expansion through the Hubble, deceleration, jerk, and higher parameters,

$$H = \frac{\dot{a}}{a}, \qquad q = -\frac{\ddot{a}}{aH^2}, \qquad j = \frac{\dddot{a}}{aH^3}. \tag{12}$$

Accessibility geometry does not replace these quantities. It reorganizes the same background information in terms of selected radii and their logarithmic growth.

### A. Hubble rate and apparent-horizon radius

For a spatially flat FRW background,

$$\boxed{R_A = \frac{c}{H}}. \tag{13}$$

Thus $H$ and $R_A$ contain the same one-function background information. Their interpretations differ: $H$ is a local expansion rate, while $R_A$ is an instantaneous proper dynamical-horizon scale.

### B. Deceleration and horizon growth

Differentiating Eq. (13),

$$\gamma_A = -\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}. \tag{14}$$

Using

$$\dot{H} = -H^2(1+q), \tag{15}$$

we obtain

$$\boxed{\gamma_A = 1+q}. \tag{16}$$

Under standard flat Friedmann evolution,

$$q = \frac{1}{2}(1+3w_{\mathrm{eff}}), \tag{17}$$

so

$$\boxed{\gamma_A = \frac{3}{2}(1+w_{\mathrm{eff}})}. \tag{18}$$

This dictionary turns familiar cosmological epochs into apparent-horizon growth states:

| epoch | radiation | matter | bridge | de Sitter |
|---|---|---|---|---|
| $q$ | 1 | 1/2 | 0 | $-1$ |
| $\gamma_A$ | 2 | 3/2 | 1 | 0 |

(19)

### C. Jerk and the evolution of horizon growth

The jerk satisfies

$$j = q + 2q^2 - \frac{\mathrm{d}q}{\mathrm{d}\ln a}. \tag{20}$$

Because $\gamma_A = 1+q$,

$$\boxed{\gamma_A' = 2\gamma_A^2 - 3\gamma_A + 1 - j}, \qquad ' \equiv \frac{\mathrm{d}}{\mathrm{d}\ln a}. \tag{21}$$

The jerk therefore measures the departure of apparent-horizon growth from a constant-$q$ trajectory. Higher cosmographic parameters map similarly to higher derivatives of $\gamma_A$.

### D. Combined apparent-horizon relation

Substitution of Eq. (16) into the master equation gives

$$\boxed{n = D(1+q) - \eta}. \tag{22}$$

The effective equation of state becomes

$$\boxed{w = \frac{D(1+q) - \eta}{3} - 1}. \tag{23}$$

For $D = 2$ and $\eta = 0$,

$$w = \frac{2q - 1}{3} = w_{\text{eff}}, \tag{24}$$

which is the exact apparent-horizon tracking identity.

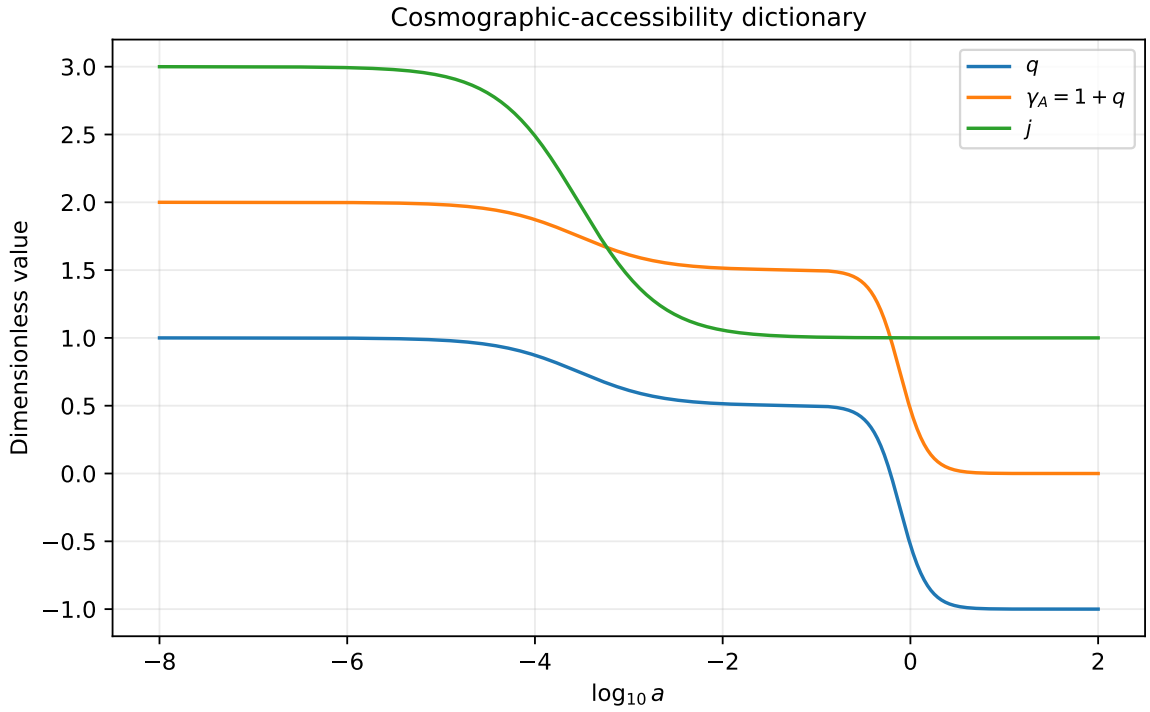


FIG. 1. Cosmographic correspondence between standard expansion variables and apparent-horizon accessibility variables. The dictionary is exact for flat FRW geometry; the relation to $w_{\text{eff}}$ additionally assumes the standard Friedmann equations.

### E. Structural versus kinematic coordinates

The correspondence should not be interpreted as a competition between two descriptions. The variables

$$(H, q, j, w_{\text{eff}}, \ldots) \tag{25}$$

are *kinematic coordinates*: they specify the expansion rate and its successive derivatives. The variables

$$(R_X, \gamma_X, D, \eta, Q) \tag{26}$$

are *structural coordinates*: they resolve a density history into a selected accessibility scale, the growth of its measure, and the evolution of the quantity distributed over that measure.

For the apparent horizon, $R_A$ and $\gamma_A$ are fixed by $H$ and $q$. The additional quantities $D$, $\eta$, and $Q$ are not fixed by cosmography alone. They form the structural fiber over a given kinematic background. This motivates a fibered interpretation of representation space: the cosmographic history supplies the base trajectory, while the set of accessibility representations compatible with that history forms a family of fibers. We use this language as a mathematical organization, not as a claim that a differentiable fiber bundle with a physical connection has already been constructed.

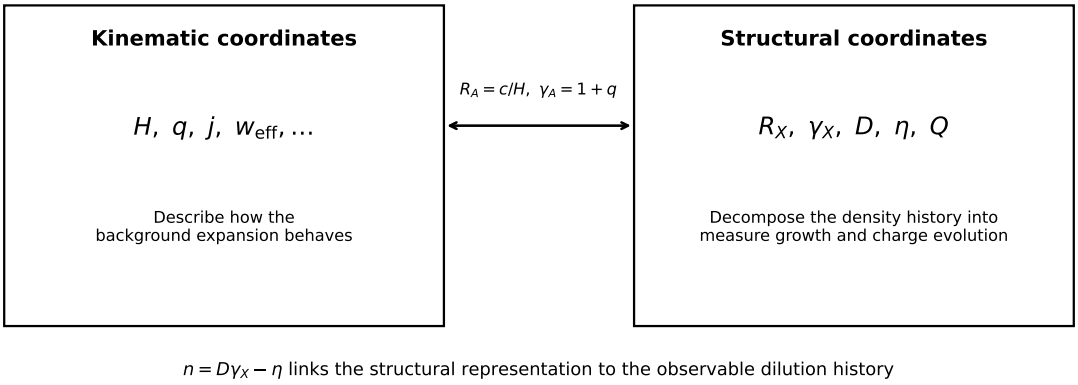


FIG. 2. Two complementary coordinate layers. Standard cosmography describes how the background expansion behaves. Accessibility geometry decomposes a density history into measure growth and charge evolution. For the apparent horizon, the exact relations $R_A = c/H$ and $\gamma_A = 1 + q$ connect the layers, while $D$, $\eta$, and $Q$ remain additional structural coordinates.

**Coordinate-layer interpretation.** Cosmography provides kinematic coordinates on the background history. Accessibility geometry provides structural coordinates over that history. The structural layer adds explanatory decomposition, but it adds predictive content only when part of that decomposition is restricted independently.

## IV. ACCESSIBILITY REPRESENTATION SPACE

### A. Definition

For a fixed epoch and selected horizon, define the local representation coordinates

$$\mathcal{A} = (D, \gamma_X, \eta, Q). \tag{27}$$

Since $Q$ fixes the normalization while $D\gamma_X - \eta$ fixes the logarithmic slope, it is often useful to separate the slope space

$$\mathcal{A}_{\rm slope} = (D, \gamma_X, \eta) \tag{28}$$

from the positive normalization coordinate $Q$.

The map from representation space to observable dilution is

$$\pi : \ (D, \gamma_X, \eta, Q) \mapsto (n, Q/S_{D,X}), \qquad n = D\gamma_X - \eta. \tag{29}$$

At background-slope level, the fibers of $\pi$ contain all representations that yield the same $n$.

### B. Constant-$n$ surfaces

For fixed $n$, the slope-space surface is

$$\eta = D\gamma_X - n. \tag{30}$$

Thus vacuum-, bridge-, matter-, and radiation-like histories form surfaces rather than isolated points. At fixed $\eta$, these become curves in the $(D, \gamma_X)$ plane. Figure 3 displays representative slices.

### C. Trajectories

A cosmological model traces a curve

$$\Gamma : \ \ln a \mapsto [D(a), \gamma_X(a), \eta(a), Q(a)]. \tag{31}$$

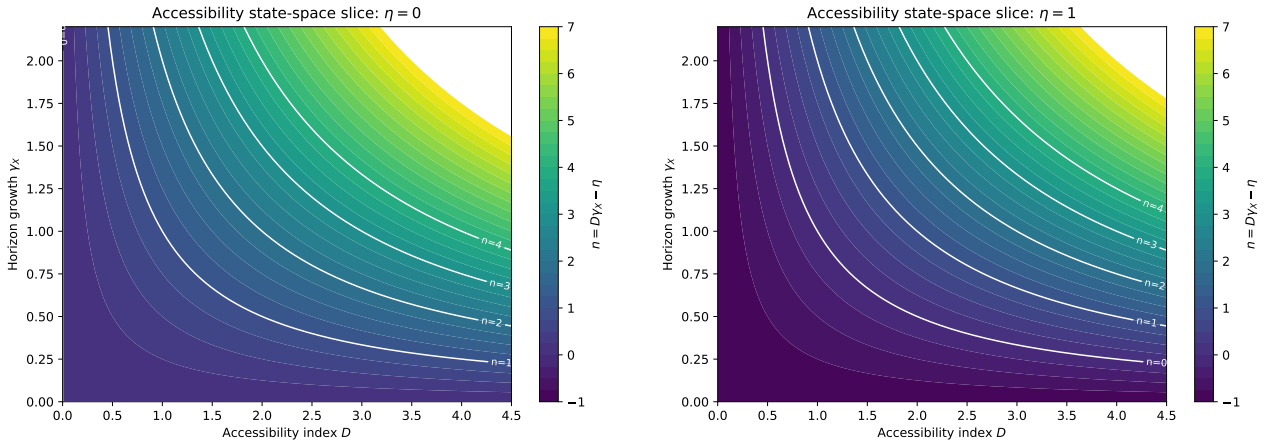


FIG. 3. Slices of the accessibility slope space. White contours show the standard constant-$n$ classes. Left: constant charge, $\eta = 0$. Right: growing charge with $\eta = 1$, which shifts every effective dilution class by one unit relative to the geometric contribution.

When $D$ and $\eta$ are fixed, cosmic evolution moves only through $\gamma_X(a)$. When $Q$ evolves, the trajectory acquires motion in the $\eta$ direction. If $D$ runs, the trajectory also moves along the scaling-index direction.

The important distinction is that a trajectory is not automatically a dynamical-system orbit. It becomes one only when differential equations determine its evolution from initial data.

### D. Foliation by effective-fluid class

The level sets

$$\mathcal{L}_n = \{(D, \gamma_X, \eta) : D\gamma_X - \eta = n\} \tag{32}$$

foliate the regular part of slope space into constant-dilution leaves. At fixed $D$, each leaf is a straight line in the $(\gamma_X, \eta)$ plane,

$$\eta = D\gamma_X - n, \tag{33}$$

with slope $D$. At fixed $\gamma_X$, the leaves are straight lines in the $(D, \eta)$ plane. The familiar vacuum, bridge, matter, and radiation classes are therefore not privileged points; they are codimension-one leaves selected by $n = 0, 2, 3, 4$.

A history crosses from one effective-fluid class to another when its trajectory crosses these leaves. Such a crossing is kinematic. It does not by itself imply a phase transition, instability, or change of microscopic degrees of freedom. Those interpretations require additional dynamics.

### E. Intersections, branch crossings, and closed paths

Two projected trajectories may intersect at the same $(D, \gamma_X, \eta)$ while differing in $Q$, the horizon label $X$, or higher derivatives. An intersection in slope space therefore need not mean that the complete representations are identical. The following hierarchy is useful:

1. a *slope-space crossing* matches $(D, \gamma_X, \eta)$ at one epoch;
2. a *dilution crossing* matches $n$ at one epoch;
3. an *interval equivalence* matches $n(a)$ over a finite interval;
4. a *normalized density equivalence* matches the full density history after one normalization is fixed.

Closed paths are mathematically possible if $D(a)$, $\gamma_X(a)$, and $\eta(a)$ are nonmonotonic. A closed projected loop does not imply a cyclic universe: the scale factor may remain monotonic, $Q$ may not return to its original value, and the trajectory may close only after projection. Likewise, the word *bifurcation* should be reserved for a dynamical system whose solution structure changes when a control parameter crosses a critical value. Before an evolution law is supplied, COSMOS admits branch crossings and multiple representations, but not demonstrated dynamical bifurcations.

## V. ACCESSIBILITY AND DENSITY EQUIVALENCE CLASSES

### A. Definition

Two accessibility representations

$$\mathcal{R} = (D, X, \eta, Q), \qquad \widetilde{\mathcal{R}} = (\widetilde{D}, \widetilde{X}, \widetilde{\eta}, \widetilde{Q}) \tag{34}$$

are *dilution equivalent* over an interval if

$$D\gamma_X - \eta = \widetilde{D}\gamma_{\widetilde{X}} - \widetilde{\eta} \tag{35}$$

throughout that interval. We write

$$\mathcal{R} \sim_n \widetilde{\mathcal{R}}. \tag{36}$$

The relation $\sim_n$ is reflexive, symmetric, and transitive, and therefore defines equivalence classes. The term *accessibility equivalence* may be used when the context makes clear that the equivalence concerns the COSMOS dilution representation, not equality of the horizons themselves.

Dilution equivalence preserves the logarithmic density slope but permits a constant normalization difference. A stronger relation is *density equivalence*,

$$\mathcal{R} \sim_\rho \widetilde{\mathcal{R}} \quad \Longleftrightarrow \quad \frac{Q(a)}{S_{D,X}(a)} = \frac{\widetilde{Q}(a)}{S_{\widetilde{D},\widetilde{X}}(a)} \tag{37}$$

throughout the interval. If two representations are dilution equivalent and agree at one reference epoch, they are density equivalent everywhere on that connected interval.

### B. Transformation law

Given an initial representation and a new choice $(\widetilde{D}, \widetilde{X})$, background equivalence requires

$$\boxed{\widetilde{\eta} = \eta + \widetilde{D}\gamma_{\widetilde{X}} - D\gamma_X}. \tag{38}$$

This is the compensating charge-slope transformation that preserves $n$.

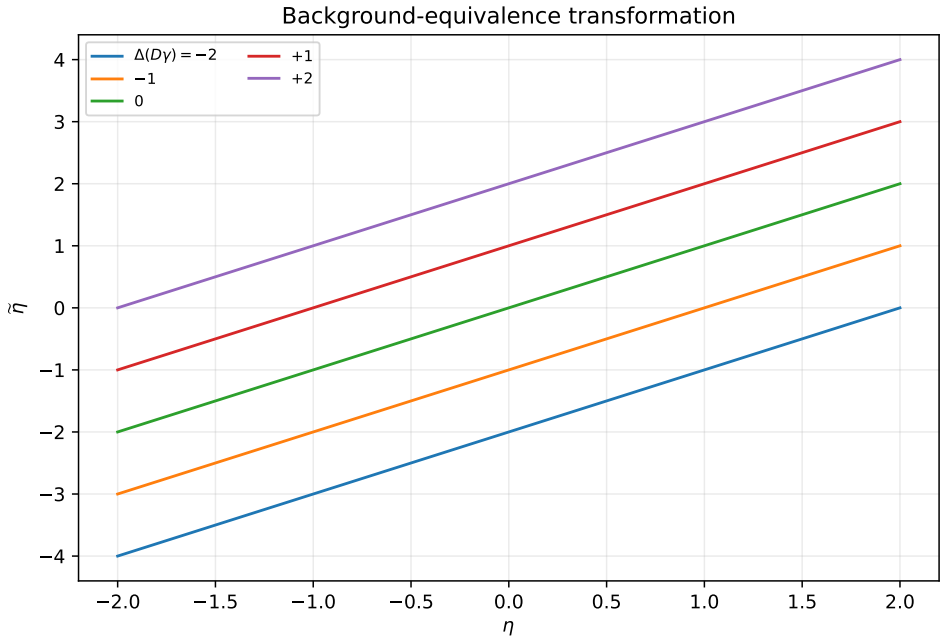


FIG. 4. The background-equivalence map. A change in the geometric contribution $D\gamma_X$ must be compensated by the same change in $\eta$ to preserve the observable dilution exponent.

### C. Invariant and non-invariant quantities

At the level of the background logarithmic slope,

$$\boxed{n = D\gamma_X - \eta} \tag{39}$$

is invariant under Eq. (38). Consequently $w = n/3 - 1$ is also invariant for a separately conserved effective fluid.

The following quantities are not generally invariant:

$$D, \quad X, \quad \gamma_X, \quad \eta, \quad Q, \quad R_X, \quad S_{D,X}. \tag{40}$$

Density equivalence requires the finite transformation

$$\boxed{\widetilde{Q}(a) = Q(a)\,\frac{S_{\widetilde{D},\widetilde{X}}(a)}{S_{D,X}(a)}} \tag{41}$$

which preserves the ratio $Q/S$ pointwise. Taking a logarithmic derivative of Eq. (41) reproduces Eq. (38). Thus the slope transformation is the differential form of the finite charge transformation.

The invariant content depends on which equivalence is imposed:

| Equivalence | Invariant | May change |
|---|---|---|
| Dilution equivalence $\sim_n$ | $n(a)$ and $w(a)$ | density normalization, $Q$, $S$, $D$, $X$, $\eta$ |
| Density equivalence $\sim_\rho$ | $\rho(a)$, $n(a)$, $w(a)$ | $Q$, $S$, $D$, $X$, $\eta$ |

In particular, $Q/S$ is not a second invariant independent of $\rho$; it is the density itself. The mathematical value of Eq. (41) is that it identifies exactly how numerator and measure must co-transform to leave the physical density unchanged.

### D. Representation degeneracy is not yet gauge symmetry

The equivalence relation resembles a redundancy: several triples $(D, X, \eta)$ describe the same $n$. However, it should not yet be called a gauge symmetry. A gauge symmetry normally acts on a dynamical theory with an action, constraint structure, or field equations. Present COSMOS establishes a representational degeneracy, not a demonstrated gauge group. If a future action or complete set of field equations is invariant under the finite transformation (41) and its slope form (38), then the redundancy could acquire the status of a genuine gauge symmetry.

**Terminology boundary.** We use "background-equivalence class" and "representation degeneracy." The term "gauge" is reserved for a future dynamical completion in which the transformations are shown to be symmetries of the full theory, including perturbations and observables.

## VI. ADMISSIBLE CHARGE HISTORIES

### A. Reconstruction from the charge slope

The definition of $\eta$ integrates to

$$\boxed{Q(a) = Q(a_0) \exp\left[\int_{a_0}^{a} \eta(\tilde{a})\, \mathrm{d}\ln \tilde{a}\right]}. \tag{42}$$

This formula is the starting point for all admissibility statements.

### B. Positivity

If

$$Q(a_0) > 0 \tag{43}$$

and the integral in Eq. (42) is finite on every compact interval, then

$$Q(a) > 0 \tag{44}$$

throughout that interval. Positivity alone therefore places only a weak restriction on $\eta$: local integrability is sufficient.

### C. Local regularity

The following hierarchy is useful:

1. If $\eta \in L^1_{\mathrm{loc}}(\mathrm{d}\ln a)$, then $Q$ is positive and absolutely continuous.
2. If $\eta$ is continuous, then $Q$ is continuously differentiable.
3. If $\eta \in C^k$, then $Q \in C^{k+1}$ for $Q > 0$.
4. A jump in $\eta$ produces a continuous $Q$ with a kink in its logarithmic derivative.

These are mathematical statements, not physical prohibitions. Whether a kink is acceptable depends on the intended microscopic interpretation.

### D. Boundedness and asymptotic completeness

Let $x = \ln a$. Then

$$\ln \frac{Q(x)}{Q(x_0)} = \int_{x_0}^{x} \eta(u)\,\mathrm{d}u. \tag{45}$$

As $a \to \infty$, the charge approaches a finite positive limit if and only if the improper integral

$$\int_{x_0}^{\infty} \eta(u)\,\mathrm{d}u \tag{46}$$

converges to a finite real number. If the integral tends to $+\infty$, then $Q \to \infty$; if it tends to $-\infty$, then $Q \to 0$.

Similarly, a finite nonzero early-time limit requires convergence of

$$\int_{-\infty}^{x_0} \eta(u)\,\mathrm{d}u. \tag{47}$$

Thus asymptotically constant charge requires more than $\eta \to 0$: it requires an integrable approach to zero.

### E. Constant-slope sectors

If

$$\eta = \eta_* \tag{48}$$

is constant, then

$$Q(a) = Q_0 a^{\eta_*}. \tag{49}$$

This is a constant-slope state, not automatically a fixed point. The associated dilution is

$$n = D\gamma_X - \eta_*. \tag{50}$$

For constant $\gamma_X$ as well, the effective equation of state is constant.

### F. Monotonicity and convexity

Because

$$\frac{\mathrm{d}Q}{\mathrm{d}\ln a} = \eta Q, \tag{51}$$

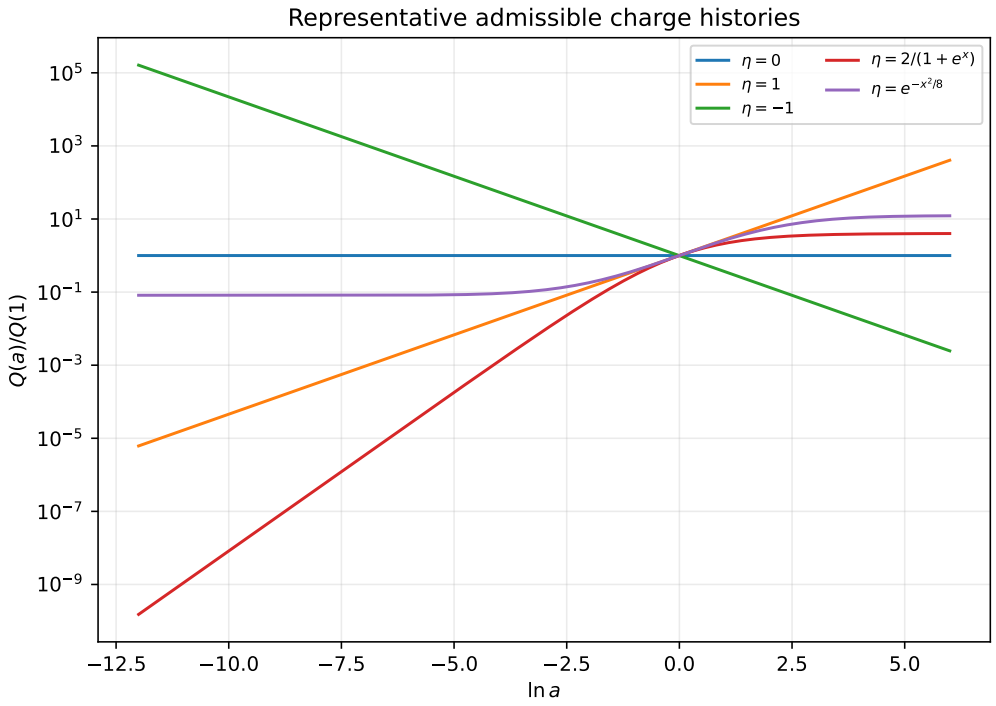


FIG. 5. Representative positive charge histories reconstructed from prescribed $\eta(\ln a)$. Positivity is automatically preserved by the exponential reconstruction, illustrating why positivity alone does not sharply restrict the charge slope.

for $Q > 0$:

$$\eta > 0 \iff Q \text{ increases with } a, \tag{52}$$

$$\eta = 0 \iff Q \text{ is locally stationary}, \tag{53}$$

$$\eta < 0 \iff Q \text{ decreases with } a. \tag{54}$$

The second logarithmic derivative is

$$\frac{\mathrm{d}^2 \ln Q}{\mathrm{d}(\ln a)^2} = \eta'. \tag{55}$$

The ordinary second derivative satisfies

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}(\ln a)^2} = Q(\eta^2 + \eta'). \tag{56}$$

Thus convexity of $Q$ in $\ln a$ requires $\eta' + \eta^2 \geq 0$.

## VII. THE ACCESSIBILITY INDEX $D$

### A. Constant and fractional values

Mathematically, $D$ need not be an integer. Any real value is admissible in the scaling law

$$S_{D,X} \propto R_X^D \tag{57}$$

provided the reference normalization carries the appropriate dimensions. Integer values admit familiar labels such as length-, area-, volume-, or radiation-like scaling, but fractional $D$ may describe effective anomalous scaling.

No present COSMOS principle selects a unique set of allowed $D$. The area value $D = 2$ is physically motivated for gravitational horizons by entropy-area scaling, but that motivation is not a derivation of universality.

### B. Running $D(a)$

If $D$ varies, the measure must be defined with a dimensionless radius ratio,

$$S(a) = S_0 \exp\left[D(a) \ln \frac{R_X(a)}{R_{X0}}\right]. \tag{58}$$

Differentiating gives

$$\frac{\mathrm{d} \ln S}{\mathrm{d} \ln a} = D\gamma_X + D' \ln \frac{R_X}{R_{X0}}, \tag{59}$$

where $D' = \mathrm{d}D/\mathrm{d} \ln a$. The generalized dilution law is therefore

$$\boxed{n = D\gamma_X + D' \ln \frac{R_X}{R_{X0}} - \eta}. \tag{60}$$

The extra term depends on the reference scale because a running exponent changes the normalization convention. This is not a defect; it signals that a variable-dimensional scaling law requires a fully specified reference prescription.

### C. Interpretive caution

Allowing arbitrary $D(a)$ and arbitrary $\eta(a)$ makes the representation extremely flexible. A running $D$ should therefore be introduced only when one of the following is supplied:

1. a microscopic scaling law;

2. a renormalization- or coarse-graining interpretation;

3. a geometric flow;

4. a phenomenological restriction with independently testable consequences.

Otherwise the additional freedom weakens identifiability.

## VIII. CONDITIONAL PHYSICAL RESTRICTIONS

The mathematical framework itself permits many charge histories. Additional restrictions may be useful, but they must be labeled by their assumptions.

### A. Conserved-charge condition

A strictly conserved accessibility charge corresponds to

$$\eta = 0. \tag{61}$$

Then

$$n = D\gamma_X. \tag{62}$$

This is the cleanest geometric sector. It is not universally required.

### B. Constant-total-charge visibility condition

If the accessible charge is decomposed as

$$Q_{\mathrm{acc}} = F(a)Q_{\mathrm{tot}}(a), \tag{63}$$

then

$$\eta_{\mathrm{acc}} = \frac{\mathrm{d}\ln F}{\mathrm{d}\ln a} + \frac{\mathrm{d}\ln Q_{\mathrm{tot}}}{\mathrm{d}\ln a}. \tag{64}$$

Under the diagnostic assumption $Q_{\mathrm{tot}} = \text{constant}$,

$$\eta_{\mathrm{acc}} = \frac{\mathrm{d}\ln F}{\mathrm{d}\ln a}. \tag{65}$$

Monotonic visibility, $F' \geq 0$, would then imply $\eta \geq 0$, but only under this particular interpretation.

### C. Entropy-motivated condition

If $Q$ is postulated to be proportional to an entropy-like state function satisfying a generalized second law, one may impose

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \geq 0. \tag{66}$$

For an expanding universe and $Q > 0$, this gives

$$\eta \geq 0. \tag{67}$$

Present COSMOS does not identify $Q$ universally with entropy. Therefore $\eta \geq 0$ is a conditional prior, not an inherent theorem.

### D. Non-growing-density condition

If one chooses to require

$$n \geq 0, \tag{68}$$

then

$$\eta \leq D\gamma_X. \tag{69}$$

This excludes densities that increase with expansion and therefore excludes effective phantom behavior. Such behavior is not mathematically forbidden by the accessibility representation itself, so Eq. (69) is again a model prior.

### E. Regular-future condition

A future-complete model may require finite $\rho$, $Q$, and $S$ as $a \to \infty$, or may allow one factor to diverge provided their ratio remains finite. The accessibility representation makes these cancellation possibilities explicit:

$$\ln \rho = \ln Q - \ln S. \tag{70}$$

Regular density requires the difference of the two logarithms to remain finite, not necessarily each term separately.

## IX. KINEMATIC STATES VERSUS DYNAMICAL FIXED POINTS

### A. Why constant slope is not an attractor

A constant value

$$\eta = \eta_* \tag{71}$$

identifies a power-law charge history. It does not establish that nearby histories evolve toward $\eta_*$. To discuss stability, one must supply an evolution equation such as

$$\eta' = F(\eta, \gamma_X, D, \Omega_i, \ldots). \tag{72}$$

Only then does a fixed point satisfy

$$F(\eta_*, \gamma_*, D_*, \ldots) = 0. \tag{73}$$

Linearization yields

$$\delta\eta' = \left.\frac{\partial F}{\partial \eta}\right|_* \delta\eta + \cdots . \tag{74}$$

The sign of the eigenvalues determines stability.

### B. Minimal dynamical completion

A future dynamical version of accessibility geometry requires at least:

1. an evolution law for $Q$ or $\eta$;

2. an evolution law or selection principle for $D$ if it is not constant;

3. a background equation determining $H$ and therefore the horizon growth;

4. perturbation equations if structure formation is to be predicted.

**Boundary theorem.** The kinematic relation $n = D\gamma_X - \eta$ classifies histories but does not define a flow on representation space. Fixed points, attractors, stability eigenvalues, and basins of attraction are undefined until an evolution law is added.

## X. THREE-HORIZON TRAJECTORIES AS CONCRETE EXAMPLES

The companion numerical paper computed $R_p$, $R_A$, and $R_e$ from a single flat radiation–matter–$\Lambda$ background [18]. A complementary analysis treats an inverse-square particle-horizon component as a dynamical source in flat general relativity with conserved dust and derives a sharp upper bound on the duration of acceleration under a finite coefficient ceiling [19]. The exact horizon-growth relations are

$$\gamma_p = 1 + \frac{c}{H R_p}, \tag{75}$$

$$\gamma_A = -\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}, \tag{76}$$

$$\gamma_e = 1 - \frac{c}{H R_e}. \tag{77}$$

### A. Three distinct paths

For fixed $D = 2$ and $\eta = 0$, each horizon traces a one-dimensional path through the $\gamma$ coordinate. Figure 6 displays those paths over the post-inflationary reference history.

The state-space figure makes the horizon split explicit. The apparent-horizon path moves from the radiation leaf through matter and the bridge toward the vacuum leaf. The particle-horizon path tends toward the bridge in the late future, whereas the event-horizon path tends toward the vacuum leaf.

The apparent-horizon path follows

$$\gamma_A = \frac{3}{2}(1 + w_{\mathrm{eff}}) \tag{78}$$

exactly. The particle horizon remains above unity for finite $R_p$, while the event horizon remains below unity when finite. Thus the apparent horizon alone crosses the bridge surface exactly at the acceleration boundary.

### B. Growth equivalence in ideal eras

For a decelerating power law $a \propto t^p$ with $0 < p < 1$,

$$\gamma_p = \gamma_A = \frac{1}{p}. \tag{79}$$

The particle and apparent horizons are therefore background-growth equivalent in pure radiation and matter eras. Their radii and causal meanings nevertheless differ.

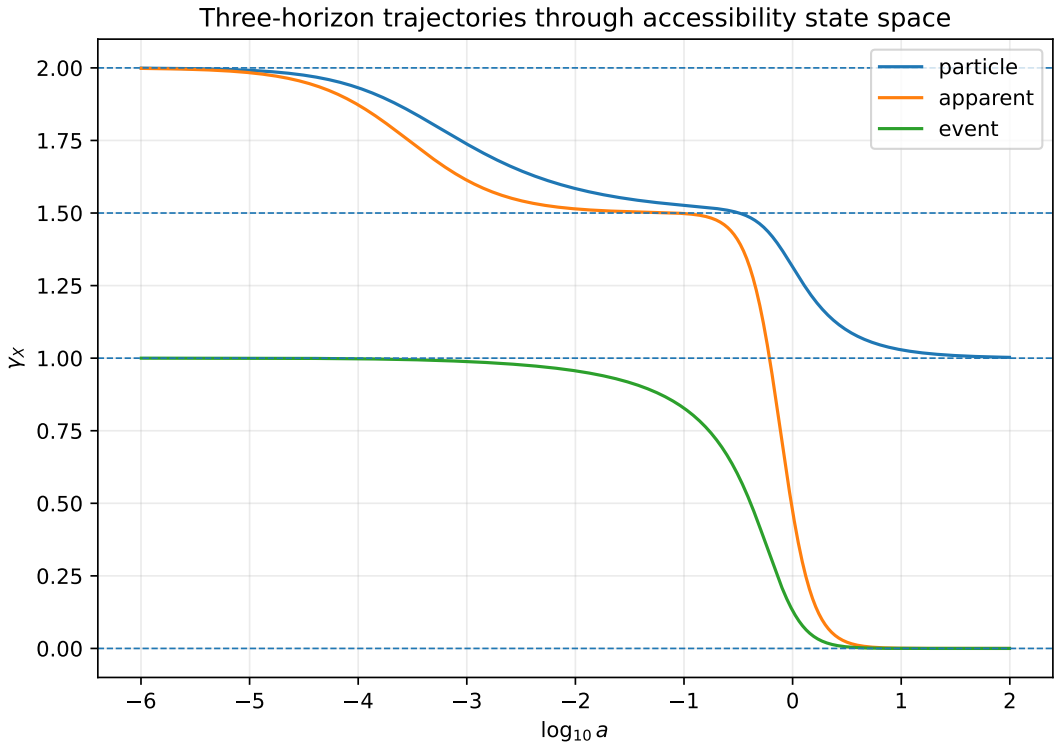


FIG. 6. Particle-, apparent-, and event-horizon growth trajectories over a common flat radiation–matter–$\Lambda$ background. Horizontal reference lines correspond to radiation-like ($\gamma = 2$), matter-like ($3/2$), bridge-like ($1$), and vacuum-like ($0$) states for $D = 2$, $\eta = 0$.

In the mixed late-time universe, all three growth rates separate. At the reference present epoch,

$$\gamma_p \simeq 1.314, \qquad \gamma_A \simeq 0.473, \qquad \gamma_e \simeq 0.130. \tag{80}$$

For the clean area branch these correspond to

$$w_p \simeq -0.124, \qquad w_A \simeq -0.685, \qquad w_e \simeq -0.913. \tag{81}$$

### C. Required charge slopes for a target history

For a chosen horizon and $D$, the charge slope required to reproduce a target $n_{\rm tar}$ is

$$\boxed{\eta_{\rm req} = D\gamma_X - n_{\rm tar}}. \tag{82}$$

Thus a fixed observable history corresponds to different charge histories in different horizon representations. This is the operational form of background equivalence.

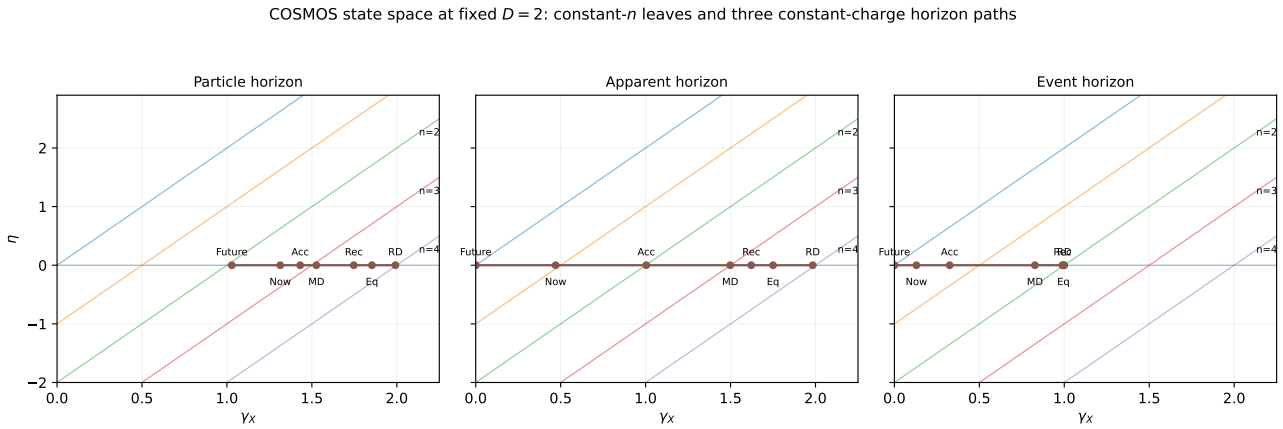


FIG. 7. The defining state-space view for fixed $D = 2$. Each panel shows the constant-$n$ leaves $\eta = 2\gamma_X - n$ together with the constant-charge path $\eta = 0$, sampled at radiation domination (RD), equality (Eq), recombination (Rec), matter domination (MD), acceleration onset (Acc), the present epoch, and the far future. The paths are generated by the reference background and are not dynamical-system orbits until an evolution law for $\eta$ or $D$ is supplied.

For a target equation of state $w_{\rm tar}$,

$$\eta_{\rm req} = D\gamma_X - 3(1 + w_{\rm tar}). \tag{83}$$

## XI. A UNIFIED STATE VECTOR

The two descriptive languages may be combined into

$$\boxed{\mathcal{S} = (H, q, j, \ldots; R_X, \gamma_X, D, \eta, Q)}. \tag{84}$$

The semicolon separates the standard background kinematics from the accessibility decomposition.

For the apparent horizon, $(R_A, \gamma_A, \gamma_A', \ldots)$ is determined by $(H, q, j, \ldots)$. The genuinely additional information lies not in $R_A$ alone but in the choice of $D$, the charge normalization, and the charge evolution.

This observation sharpens the status of COSMOS:

1. $R_A$ and $\gamma_A$ are alternative coordinates on standard flat-FRW kinematics.

2. $D$ and $\eta$ provide a decomposition of the observed dilution.

3. The decomposition becomes physically informative only when $D$ or $Q$ is independently restricted.

4. Other horizons introduce nonlocal information about past or future accessibility and therefore are not algebraic replacements for local cosmography.

## XII. IDENTIFIABILITY AND INFORMATION CONTENT

### A. Background theorem

Suppose a positive sector density $\rho(a)$ and the background geometry are specified on the interval of interest. Then $n(a)$ is fixed, and $\gamma_X(a)$ can be computed for each horizon whose defining history is supplied. The particle horizon requires a past history or an initial radius, and the event horizon requires a future continuation for which the horizon integral converges. For any chosen $D$, Eq. (3) reconstructs

$$\eta(a) = D\gamma_X(a) - n(a). \tag{85}$$

Therefore background data alone do not identify $D$, $X$, and $\eta$ separately.

For example, suppose the density history fixes $n(a)$ and the apparent-horizon growth $\gamma_A(a)$ is known. The two representations

$$D = 2: \qquad \eta_2(a) = 2\gamma_A(a) - n(a), \tag{86}$$

$$D = 3: \qquad \eta_3(a) = 3\gamma_A(a) - n(a) \tag{87}$$

produce exactly the same dilution history. Their charge slopes differ by $\gamma_A(a)$, and their charges differ according to Eq. (41). No background measurement of $\rho(a)$ can select between them without an independent rule fixing $D$ or $Q$.

**Background identifiability theorem.** Given a positive sector density history on a specified background, its logarithmic slope fixes the invariant combination $n = D\gamma_X - \eta$. The horizon choice, scaling index, and charge slope remain underdetermined without additional restrictions. Inferring the sector density itself from $H(a)$ requires separate information about the sector decomposition.

### B. How identifiability may be restored

The degeneracy can be reduced by:

1. deriving $D$ from a microscopic or thermodynamic principle;
2. specifying a conservation or evolution law for $Q$;
3. relating multiple sectors through shared charges or shared measures;
4. introducing perturbation observables sensitive to the horizon choice;
5. imposing independently testable asymptotic or regularity conditions.

A restriction is scientifically valuable only if it is motivated independently of the background history being reconstructed.

## XIII. INVARIANTS BEYOND THE DILUTION EXPONENT

At present, $n$ is the primary invariant of background-slope equivalence. Additional candidates can be constructed by differentiating:

$$n' = D'\gamma_X + D\gamma_X' - \eta' \tag{88}$$

for constant-reference conventions. If $D$ is fixed,

$$n' = D\gamma_X' - \eta'. \tag{89}$$

Under an equivalence transformation that preserves the complete function $n(a)$, all derivatives $n^{(k)}$ are also invariant.

This yields an invariant hierarchy

$$\mathcal{I}_0 = n, \quad \mathcal{I}_1 = n', \quad \mathcal{I}_2 = n'', \quad \ldots \tag{90}$$

which is equivalent to the derivative hierarchy of the density history. These are not new observables, but they provide a precise way to compare representations order by order. The functions $n, n', n'', \ldots$ are fixed by $n(a)$ on the interval wherever the derivatives exist. Integrating $n(a)$ over the interval and fixing one density value recovers the density history. Derivatives at a single epoch determine a local Taylor expansion when the history is analytic; smoothness alone does not ensure such a reconstruction.

At density-equivalence level, $\rho = Q/S$ itself is invariant, while $Q$ and $S$ separately are representation dependent. At dilution-equivalence level, only the shape of the density history is invariant and its normalization may differ. A genuinely new invariant would need to remain unchanged while carrying information not already contained in $\rho(a)$. Such an invariant likely requires perturbations, cross-sector relations, conserved integrals, or a dynamical completion.

## XIV. MULTI-SECTOR ACCESSIBILITY GEOMETRY

The single-component representation extends directly to a collection of homogeneous sectors,

$$\rho_i = \frac{Q_i}{S_{i,X_i}}, \qquad S_{i,X_i} = S_{i,X_i 0}\left(\frac{R_{X_i}}{R_{X_i 0}}\right)^{D_i}, \tag{91}$$

with

$$n_i = D_i \gamma_{X_i} - \eta_i. \tag{92}$$

The total density and pressure are

$$\rho_{\rm tot} = \sum_i \rho_i, \qquad p_{\rm tot} = \sum_i p_i, \tag{93}$$

so the effective total equation of state is

$$w_{\rm eff} = \sum_i \Omega_i w_i, \qquad \Omega_i = \frac{\rho_i}{\rho_{\rm tot}}. \tag{94}$$

The total dilution exponent is not a simple unweighted sum. Differentiation gives

$$\boxed{n_{\rm tot} = \sum_i \Omega_i n_i} = \sum_i \Omega_i (D_i \gamma_{X_i} - \eta_i). \tag{95}$$

Thus the total background history is the density-weighted projection of the sector trajectories.

### A. Cross-sector degeneracy

Even if $n_{\rm tot}(a)$ is known, the decomposition into sector histories is not unique. Freedom exists at two levels:

1. each sector admits its own accessibility-equivalence class;

2. the density fractions $\Omega_i(a)$ can redistribute the contribution of the sectors to the total history.

Consequently a multicomponent model requires more than one background observable to identify its internal structure. Separate density measurements, interaction laws, perturbations, or shared-charge relations are needed.

### B. Shared measures and shared charges

Additional structure can reduce this freedom. If several sectors use a common horizon $X$, then their geometric slopes share $\gamma_X$ and

$$n_i - n_j = (D_i - D_j)\gamma_X - (\eta_i - \eta_j). \tag{96}$$

If they also share the same index, the relative dilution becomes

$$n_i - n_j = -(\eta_i - \eta_j), \tag{97}$$

so relative background evolution directly measures the difference of charge slopes.

Alternatively, suppose sectors share a total charge through fractions $F_i$,

$$Q_i = F_i Q_{\rm tot}, \qquad \sum_i F_i = 1. \tag{98}$$

Then

$$\eta_i = \eta_{\rm tot} + \frac{\mathrm{d}\ln F_i}{\mathrm{d}\ln a}. \tag{99}$$

Cross-sector relations can therefore turn otherwise arbitrary charge functions into a constrained redistribution problem. Whether such a shared charge is physical must be established independently.

### C. Sector-space invariants

Under independent density-equivalence transformations in every sector, each $\rho_i$ is invariant, and therefore so are

$$\rho_{\rm tot}, \qquad \Omega_i, \qquad n_i, \qquad n_{\rm tot}, \qquad w_{\rm eff}. \tag{100}$$

Under dilution equivalence alone, the individual normalizations may change and the density fractions need not remain invariant. This distinction is especially important in multicomponent cosmology: preserving every sector slope does not preserve the total expansion unless the relative normalizations are also fixed.

**Multisector lesson.** Single-sector dilution equivalence is not sufficient to preserve a multicomponent cosmology. At fixed curvature and gravitational coupling in general relativity, the same total density history gives the same $H(a)$ on the same expansion branch. Preserving each sector density also preserves the sector fractions. A common rescaling of all sector densities can keep their relative fractions while changing $H(a)$.

## XV. LOCAL GEOMETRY OF STATE-SPACE MOTION

The master function

$$f(D, \gamma, \eta) = D\gamma - \eta \tag{101}$$

defines the dilution leaves. Along a differentiable trajectory $\Gamma(x)$ with $x = \ln a$,

$$\frac{\mathrm{d}n}{\mathrm{d}x} = \gamma D' + D\gamma' - \eta'. \tag{102}$$

This quantity is the component of state-space motion normal to the constant-$n$ foliation, up to normalization of the normal vector. Motion satisfying

$$\gamma D' + D\gamma' - \eta' = 0 \tag{103}$$

is tangent to a constant-dilution leaf and changes the representation without changing the effective fluid class.

### A. Crossing speed

When $n$ crosses a reference value $n_\star$, the local crossing speed is $n'(x_\star)$. A transverse crossing has

$$n'(x_\star) \neq 0, \tag{104}$$

whereas a tangential contact has $n'(x_\star) = 0$. For the acceleration boundary $n_\star = 2$, this distinction separates a genuine change from decelerating to accelerating effective behavior from a trajectory that merely touches the bridge leaf and returns.

At fixed $D$,

$$n' = D\gamma' - \eta'. \tag{105}$$

A transition can therefore be driven by changing horizon growth, changing charge slope, or an exact cancellation between them. Standard cosmography supplies $\gamma'_A$ through the jerk, while COSMOS assigns the remainder to $\eta'$.

### B. Curvature of projected trajectories

For a fixed-$D$ trajectory in the $(\gamma, \eta)$ plane, define

$$\mathbf{v} = (\gamma', \eta'), \qquad \mathbf{a} = (\gamma'', \eta''). \tag{106}$$

The Euclidean curvature of the plotted curve is

$$\kappa = \frac{|\gamma'\eta'' - \eta'\gamma''|}{(\gamma'^2 + \eta'^2)^{3/2}}. \tag{107}$$

This is a visualization diagnostic, not an invariant physical curvature: it depends on the chosen coordinates and metric. Nevertheless, it can locate rapid turns in a prescribed trajectory and distinguish smooth tracking from sharp charge transitions.

### C. No kinematic criterion for stability

The tangent and normal decomposition tells us how a specified history moves through state space, but not whether nearby histories converge toward it. Stability requires a vector field on state space, not merely one curve. If a dynamical completion supplies

$$\Gamma' = \mathbf{F}(\Gamma), \tag{108}$$

then the Jacobian $\partial\mathbf{F}/\partial\Gamma$ determines local stability. Until then, crossing speeds and trajectory curvature are descriptive properties only.

## XVI. CLASSIFICATION OF HORIZON-GENERATED PATHS

For the clean sector $D = 2$, $\eta = 0$, the effective class is determined entirely by $\gamma_X$:

$$n_X = 2\gamma_X. \tag{109}$$

The following intervals provide a useful path classification:

| Growth interval | Dilution interval | Effective behavior |
|---|---|---|
| $\gamma_X < 0$ | $n_X < 0$ | Phantom-like density growth |
| $0 \le \gamma_X < 1$ | $0 \le n_X < 2$ | Accelerating, non-phantom range |
| $\gamma_X = 1$ | $n_X = 2$ | Bridge |
| $1 < \gamma_X < 3/2$ | $2 < n_X < 3$ | Intermediate decelerating range |
| $\gamma_X = 3/2$ | $n_X = 3$ | Matter-like |
| $3/2 < \gamma_X < 2$ | $3 < n_X < 4$ | Matter–radiation intermediate range |
| $\gamma_X = 2$ | $n_X = 4$ | Radiation-like |
| $\gamma_X > 2$ | $n_X > 4$ | Faster-than-radiation dilution |

On the reference timeline, the apparent horizon moves monotonically from $\gamma_A \simeq 2$ toward zero. The particle horizon moves from approximately two toward one. The event horizon begins near one in the post-inflationary radiation era and tends toward zero in the de Sitter future. These statements refer to the chosen reference cosmology and constant-charge branch; charge evolution displaces every path vertically in the $(\gamma, \eta)$ representation.

### A. Charge displacement of a horizon path

For a target dilution $n_\star(a)$ and fixed $D$, the required path is

$$\eta_X(a) = D\gamma_X(a) - n_\star(a). \tag{110}$$

Thus changing the target history translates the horizon path relative to the constant-charge axis. If $n_\star$ is constant, the path lies exactly on the corresponding constant-$n$ leaf. If the target evolves, it crosses the foliation according to $n'_\star(a)$.

## XVII. SELF-CONSISTENCY QUESTIONS

### A. Tracking sectors

For $X = A$, $D = 2$, and $\eta = 0$,

$$w_A = w_{\text{eff}}. \tag{111}$$

If this component contributes a fraction $\Omega_A$ to the total cosmic fluid, self-consistency requires

$$w_{\text{eff}} = \Omega_A w_A + \sum_{i \neq A} \Omega_i w_i. \tag{112}$$

Using $w_A = w_{\text{eff}}$,

$$(1 - \Omega_A) w_{\text{eff}} = \sum_{i \neq A} \Omega_i w_i. \tag{113}$$

This identity shows that the tracking relation alone does not determine $\Omega_A$ or produce acceleration. It is a compatibility condition: for any chosen $\Omega_A < 1$, the remaining sectors must combine to reproduce the same $w_{\text{eff}}$. In the limit $\Omega_A \to 1$, Eq. (113) becomes the identity $0 = 0$ and supplies no additional dynamics. Thus exact tracking neither fixes the abundance of the component nor explains why it comes to dominate.

### B. Why charge evolution matters

To obtain behavior different from exact tracking, one requires

$$\eta = 3(w_{\text{eff}} - w) \tag{114}$$

for the apparent-horizon area branch. A realistic independent dark-energy sector therefore needs either charge evolution or a different $D$. The mathematical paper does not choose that evolution law; it identifies the required function.

## XVIII. WHAT CAN BE PROVED BEFORE MICROPHYSICS?

The following results are available without a microscopic theory:

1. reconstruction of $Q$ from a chosen $\eta$;
2. regularity and asymptotic criteria based on integrability;
3. classification of constant-$n$, constant-$\eta$, and constant-$D$ sectors;
4. background-equivalence relations;
5. exact cosmographic correspondence for the apparent horizon;
6. conditional inequalities once additional physical priors are stated;
7. no-go statements that follow from mutually inconsistent priors.

The following cannot be established without additional dynamics:

1. attractors or dynamical stability;
2. a unique preferred $D$ or horizon;
3. a microscopic identity for $Q$;
4. perturbation propagation and clustering;
5. observational predictions beyond the background reconstruction;
6. a gauge symmetry of the complete theory.

## XIX. PROGRAM FOR A DYNAMICAL COMPLETION

The mathematical structure suggests a disciplined sequence.

### A. Step 1: restricted kinematic models

Define named restricted sectors, for example:

$$\text{A2C}: \quad X = A,\ D = 2,\ \eta = 0, \tag{115}$$

$$\text{A2F}: \quad X = A,\ D = 2,\ Q = FQ_{\text{tot}},\ Q_{\text{tot}} = \text{const}, \tag{116}$$

$$\text{A2D}: \quad X = A,\ D = 2,\ \eta' = F(\eta, \gamma_A, \ldots). \tag{117}$$

Each added restriction decreases representational freedom.

### B. Step 2: derive an evolution law

Candidate sources include:

1. a conservation equation;
2. a variational principle;
3. horizon thermodynamics;
4. a coarse-graining or renormalization flow;
5. a microscopic field realization.

### C. Step 3: perturbations and observations

Only after a restricted background model is defined should one derive or specify perturbative quantities such as $\mu(a, k)$, gravitational slip, and sound speed. The natural apparent-horizon scale implies that ultra-large-scale observables are the first direct empirical target unless a second physical scale emerges.

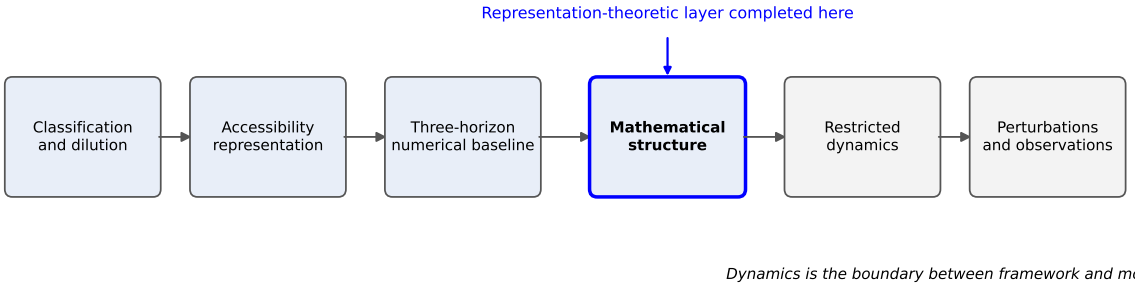


FIG. 8. Logical architecture of the COSMOS program. The present paper completes the mathematical and representation-theoretic layer. A restricted evolution law is the first step that converts an admissible path into a dynamical model.

### D. Priority ordering

The recommended sequence is summarized below. The phase labels are programmatic rather than institutional; the scientific boundary is the introduction of dynamics.

**Priority ordering.**

1. **Immediate: restricted-model catalogue.** Fix $X$, $D$, and the status of total versus accessible charge.
2. **First dynamics: evolution law for $\eta$, $Q$, or $F$.** Initial data must determine a unique background trajectory.
3. **Consistency: fixed points, stability, and asymptotics.** Analyze the derived flow rather than prescribed functions.
4. **Covariant layer: action or closed field equations.** Derive background and perturbation equations consistently.
5. **Empirical layer: numerical implementation and likelihoods.** Produce observables that can distinguish the restricted model.

## XX. RELATION TO NEIGHBORING COSMOLOGICAL FORMALISMS

Accessibility geometry overlaps conceptually with several established approaches but is not identical to any of them. The comparison below is intended to delimit scope rather than claim superiority.

**Cosmography:**

Uses $H, q, j, \ldots$ to describe expansion kinematics without selecting a structural decomposition. The apparent-horizon dictionary maps these variables exactly to $R_A, \gamma_A, \gamma'_A, \ldots$ [6, 7].

**Effective fluids:**

Use $\rho_i, p_i, w_i(a)$ to classify homogeneous sectors by stress-energy behavior. Accessibility geometry further decomposes the dilution rate into measure growth and charge evolution.

**Holographic dark energy:**

Uses a horizon or infrared cutoff $L$, typically with $\rho_{\rm de} \propto L^{-2}$. It shares horizon and area scaling, but generally selects one cutoff and density prescription; accessibility geometry instead treats $D$, $X$, and $Q$ as a representation space [8, 9].

**Entropic cosmology:**

Uses horizon entropy, temperature, and thermodynamic balance. It motivates area-like measures and apparent-horizon thermodynamics but does not by itself derive the general charge decomposition used here [10, 13].

**Dynamical systems:**

Uses autonomous variables and evolution equations to study fixed points, stability, and attractors. Accessibility paths are not dynamical-system orbits until a vector field is supplied [11].

**Accessibility geometry:**

Uses $R_X, \gamma_X, D, \eta, Q$ to provide a structural decomposition of a density history, formal equivalence classes, admissibility criteria, and a precise boundary between representation and dynamics.

The closest formal resemblance occurs with holographic dark-energy constructions, where inverse powers of a cosmological length define an energy density. The distinction is that accessibility geometry does not initially identify the denominator with one unique infrared cutoff or assume a fixed numerator. Instead it separates the geometric exponent, horizon choice, and charge history, then studies which combinations are equivalent at background level. This broader freedom is also the source of the identifiability problem formalized above.

Likewise, cosmography and accessibility geometry should be viewed as complementary coordinate layers. Cosmography is deliberately kinematic; accessibility geometry is deliberately structural. Neither layer supplies microphysics by itself. The structural layer becomes predictive only when a dynamical or constitutive law selects a restricted subset of representations.

## XXI. DISCUSSION

The mathematical structure developed here clarifies both the value and the limitations of accessibility geometry.

First, the framework is not simply a new name for $H$, $q$, or $w$. The apparent-horizon variables are exactly related to cosmography, but the decomposition into $D$ and $\eta$ adds a new representational layer. This layer can reveal whether an observed dilution history is attributed to geometric growth, charge evolution, or a mixture of both.

Second, that additional layer is underdetermined. Without independent restrictions, any smooth positive density history can be represented by many accessibility triples. The correct response is not to deny this freedom but to formalize it through equivalence classes and identify the invariant content.

Third, the mathematical restrictions available without microphysics are real but limited. Positivity is weak; boundedness and asymptotic convergence are stronger. Monotonicity, entropy growth, and causal bounds require additional interpretive assumptions. Stability requires dynamics.

Fourth, the three-horizon timeline demonstrates that accessibility channels are not interchangeable. Particle and apparent horizons share growth rates only in ideal decelerating power-law eras. The event horizon depends on the future history. In the mixed late universe all three channels separate, producing distinct paths through representation space.

Finally, the framework now has a precise research boundary. The foundational stage estab-

lished the representation, horizon extension, and continuous three-horizon baseline. The present work completes the late-foundational mathematical layer: state space, equivalence, invariants, admissibility, and correspondence. The next scientifically decisive stage begins when an evolution law selects a subset of the mathematically admissible histories. In this operational sense, dynamics marks the boundary between framework and model.

## XXII. CONCLUSION

We have formulated the mathematical structure of accessibility geometry around the central relation

$$n = D\gamma_X - \eta. \tag{118}$$

The principal results are:

1. the exact apparent-horizon cosmographic dictionary

$$R_A = \frac{c}{H}, \qquad \gamma_A = 1 + q, \qquad \gamma_A' = 2\gamma_A^2 - 3\gamma_A + 1 - j; \tag{119}$$

2. the definition of dilution- and density-equivalence classes, with the transformations

$$\widetilde{\eta} = \eta + \widetilde{D}\gamma_{\widetilde{X}} - D\gamma_X, \qquad \widetilde{Q} = Q\frac{S_{\widetilde{D},\widetilde{X}}}{S_{D,X}}; \tag{120}$$

3. the identification of $n(a)$ and its existing derivatives as invariants of dilution equivalence. A single reference density then fixes the full density history on the connected interval;

4. precise admissibility criteria for positive, regular, bounded, and asymptotically complete charge histories;

5. the generalized running-index law

$$n = D\gamma_X + D' \ln(R_X/R_{X0}) - \eta; \tag{121}$$

6. a strict separation between kinematic states and dynamical fixed points;

7. the integration of the continuous three-horizon timeline as concrete trajectories through the representation space.

**Central conclusion.** Accessibility geometry has a well-defined mathematical structure before microphysics is chosen. That structure organizes the freedom of the representation, but it does not remove it. Predictive physical content begins when an independently motivated law restricts $D$, $Q$, the horizon choice, or the perturbations.

### Appendix A: Proof that background equivalence is an equivalence relation

Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be representations with dilution functions $n_1, n_2, n_3$. Define $\mathcal{R}_i \sim_n \mathcal{R}_j$ if $n_i(a) = n_j(a)$ on an interval.

Reflexivity follows because $n_i = n_i$. Symmetry follows because $n_i = n_j$ implies $n_j = n_i$. Transitivity follows because $n_i = n_j$ and $n_j = n_k$ imply $n_i = n_k$. Hence $\sim_n$ partitions the representation set into equivalence classes.

### Appendix B: Dimensional status of the charge

If $[\rho] = EL^{-3}$ and $[S_{D,X}] = L^D$, then

$$[Q] = EL^{D-3}. \tag{B1}$$

Representative cases are shown in Table II.

TABLE II. Dimensions of the accessibility charge for representative $D$.

| $D$ | $[Q]$ |
|---|---|
| 0 | $EL^{-3}$ |
| 1 | $EL^{-2}$ |
| 2 | $EL^{-1}$ |
| 3 | $E$ |
| 4 | $EL$ |

Thus $Q$ is ordinary energy only for $D = 3$. Any microscopic interpretation must respect this sector dependence.

### Appendix C: Energy-depth relation

For a homogeneous volume $V$, define

$$\chi_{D,X} = \frac{V}{S_{D,X}}. \tag{C1}$$

Then

$$E = \rho V = Q\chi_{D,X}. \tag{C2}$$

The dimensions are $[\chi_{D,X}] = L^{3-D}$. The term "depth" is literal only for $D = 2$.

### Appendix D: Asymptotic classification of charge slopes

Let $x = \ln a$. The future behavior of $Q$ is classified by

$$I_+(x) = \int_{x_0}^{x} \eta(u)\,\mathrm{d}u. \tag{D1}$$

Then:

$$I_+ \to I_\infty \in \mathbb{R} \Rightarrow Q \to Q_0 e^{I_\infty}, \tag{D2}$$

$$I_+ \to +\infty \Rightarrow Q \to \infty, \tag{D3}$$

$$I_+ \to -\infty \Rightarrow Q \to 0, \tag{D4}$$

$$I_+ \text{ oscillatory without limit} \Rightarrow Q \text{ has no asymptotic limit.} \tag{D5}$$

A similar classification holds as $x \to -\infty$.

### Appendix E: Representative charge laws

#### 1. Power law

For $Q = Q_0 a^s$,

$$\eta = s. \tag{E1}$$

#### 2. Saturating law

For

$$Q = Q_\infty \frac{a^s}{a^s + a_t^s}, \tag{E2}$$

we obtain

$$\eta = \frac{s}{1 + (a/a_t)^s}. \tag{E3}$$

The charge grows as $a^s$ at early times and approaches a constant at late times.

### 3. Localized transition

For a Gaussian logarithmic slope

$$\eta(x) = A \exp\left[-\frac{(x - x_t)^2}{2\sigma^2}\right], \tag{E4}$$

$Q$ undergoes a finite multiplicative transition because $\eta$ is integrable over $x$.

## Appendix F: Generalized derivative hierarchy

For constant $D$,

$$n = D\gamma - \eta, \tag{F1}$$

$$n' = D\gamma' - \eta', \tag{F2}$$

$$n'' = D\gamma'' - \eta'', \tag{F3}$$

$$\vdots \tag{F4}$$

For variable $D$ and the convention in Eq. (58), derivatives include additional terms from $D'$ and the logarithmic radius ratio.

## Appendix G: Three-horizon definitions and growth rates

For flat FRW:

$$R_p(a) = a \int_{a_i}^{a} \frac{c\,\mathrm{d}a'}{a'^2 H(a')}, \tag{G1}$$

$$R_A(a) = \frac{c}{H(a)}, \tag{G2}$$

$$R_e(a) = a \int_{a}^{\infty} \frac{c\,\mathrm{d}a'}{a'^2 H(a')}. \tag{G3}$$

Differentiation gives

$$\gamma_p = 1 + \frac{c}{H R_p}, \tag{G4}$$

$$\gamma_A = -\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}, \tag{G5}$$

$$\gamma_e = 1 - \frac{c}{H R_e}. \tag{G6}$$

These identities show immediately that finite particle and event horizons lie on opposite sides of the bridge-growth value $\gamma = 1$.

### Appendix H: Geometry of the constant-dilution foliation

For fixed horizon label $X$, define the smooth map

$$f(D, \gamma, \eta) = D\gamma - \eta. \tag{H1}$$

Its gradient is

$$\nabla f = (\gamma, D, -1), \tag{H2}$$

which never vanishes because the third component is $-1$. Therefore every level set $f^{-1}(n)$ is a regular two-dimensional surface in the three-dimensional slope space. The constant-$n$ classes form a smooth foliation of the full slope space. This statement is purely kinematic and does not require a metric on the representation space.

A tangent displacement $(\delta D, \delta\gamma, \delta\eta)$ remains on the same leaf when

$$\gamma\,\delta D + D\,\delta\gamma - \delta\eta = 0. \tag{H3}$$

This is the infinitesimal equivalence condition. Integrating it along a path gives the finite compensation law when the horizon growth and index are changed continuously.

One may introduce the Euclidean metric

$$\mathrm{d}s^2 = \mathrm{d}D^2 + \mathrm{d}\gamma^2 + \mathrm{d}\eta^2 \tag{H4}$$

for visualization, but no physical significance should be assigned to distances or angles without a dynamical principle selecting a metric. The foliation is invariantly defined by $f$; the chosen metric is not.

### Appendix I: Dilution equivalence and normalized density equivalence

Suppose two positive densities satisfy

$$-\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a} = -\frac{\mathrm{d}\ln\widetilde{\rho}}{\mathrm{d}\ln a} \tag{I1}$$

throughout a connected interval. Then

$$\frac{\mathrm{d}}{\mathrm{d}\ln a}\ln\frac{\widetilde{\rho}}{\rho} = 0, \tag{I2}$$

so

$$\widetilde{\rho} = C\rho \tag{I3}$$

for a positive constant $C$. Dilution equivalence therefore fixes the density history up to normalization. If the densities agree at one epoch, $C = 1$ and the representations are density equivalent.

The same result follows directly from the finite transformation of the charge. Given two measures $S$ and $\widetilde{S}$, define

$$\widetilde{Q} = CQ\frac{\widetilde{S}}{S}. \tag{I4}$$

Then $\widetilde{Q}/\widetilde{S} = CQ/S$. The case $C = 1$ gives exact density equivalence, while arbitrary $C > 0$ gives dilution equivalence with a normalization offset.

### Appendix J: Worked identifiability examples

The principal examples have been promoted to Sec. XII of the main text. This appendix records the compact transformation formulas:

$$\eta_{D_2} - \eta_{D_1} = (D_2 - D_1)\gamma_X, \tag{J1}$$

$$\eta_X - \eta_Y = D(\gamma_X - \gamma_Y). \tag{J2}$$

These relations quantify the change in charge slope required to preserve a target dilution history when either the scaling index or accessibility channel is changed.

### Appendix K: Conditional admissible regions in fixed-$D$ slices

At fixed positive $D$, several commonly discussed priors define half-planes in the $(\gamma, \eta)$ slice:

$$\text{non-growing density:} \quad \eta \leq D\gamma, \tag{K1}$$

$$\text{accelerating effective behavior:} \quad D\gamma - \eta < 2, \tag{K2}$$

$$\text{non-phantom effective behavior:} \quad D\gamma - \eta \geq 0, \tag{K3}$$

$$\text{entropy-like nondecreasing charge:} \quad \eta \geq 0. \tag{K4}$$

Their intersections produce conditional admissible wedges. None is inherent to the accessibility representation. For example, combining nondecreasing charge with non-phantom behavior gives

$$0 \leq \eta \leq D\gamma. \tag{K5}$$

Combining acceleration with non-phantom behavior gives

$$D\gamma - 2 < \eta \leq D\gamma. \tag{K6}$$

These regions are useful for model triage once the relevant priors are justified, but they should not be described as universal COSMOS bounds.

### Appendix L: Projection effects and apparent loops

Let the complete history be

$$\Gamma : x \mapsto (D(x), \gamma_X(x), \eta(x), Q(x), X), \qquad x = \ln a. \tag{L1}$$

A projection onto slope space discards $Q$ and the horizon label. The projected curve may close even when the complete curve does not. If

$$(D, \gamma, \eta)(x_2) = (D, \gamma, \eta)(x_1) \tag{L2}$$

but

$$Q(x_2) \neq Q(x_1), \tag{L3}$$

then the full representation has not returned to its original state. Likewise, two trajectories associated with different horizons can cross in the projected space without being physically equivalent. This is why topological statements must specify the space and projection under consideration.

### Appendix M: Reference-scale dependence for a running index

For constant $D$, changing the reference radius from $R_{X0}$ to $\widehat{R}_{X0} = \lambda R_{X0}$ rescales the constant normalization of the measure and can be absorbed into $S_{D,X0}$ or $Q_0$. The logarithmic slope $D\gamma_X$ is unchanged.

For running $D(a)$, write

$$\ln S = \ln S_0 + D(a) \ln \frac{R_X}{R_{X0}}. \tag{M1}$$

Under $R_{X0} \to \lambda R_{X0}$,

$$\ln S \to \ln S - D(a) \ln \lambda. \tag{M2}$$

Its logarithmic derivative changes by

$$\frac{\mathrm{d} \ln S}{\mathrm{d} \ln a} \to \frac{\mathrm{d} \ln S}{\mathrm{d} \ln a} - D' \ln \lambda. \tag{M3}$$

Therefore the decomposition

$$n = D\gamma_X + D' \ln \frac{R_X}{R_{X0}} - \eta \tag{M4}$$

is reference-prescription dependent unless the charge transformation is specified simultaneously. Density invariance requires

$$Q \to Q\lambda^{-D(a)}, \tag{M5}$$

which shifts

$$\eta \to \eta - D' \ln \lambda \tag{M6}$$

and restores the same $n$. Running $D$ is therefore mathematically consistent, but only as part of a complete measure-and-charge convention.

**Running-index caution.** A varying accessibility index is not an additional scalar function that can be inserted without convention. Its meaning depends on the reference measure, and a change of reference induces a compensating change in the charge slope.

### Appendix N: Symbol glossary

$a$**:** Cosmological scale factor.

$H$**:** Hubble expansion rate.

$q$**:** Deceleration parameter.

$j$: Jerk parameter.

$R_X$: Selected proper horizon or physical length scale.

$X$: Horizon label; commonly $p$, $A$, or $e$.

$D$: Effective accessibility scaling index.

$S_{D,X}$: Accessibility measure scaling as $R_X^D$.

$Q$: Accessibility charge.

$\gamma_X$: Logarithmic growth rate $\mathrm{d}\ln R_X/\mathrm{d}\ln a$.

$\eta$: Charge slope $\mathrm{d}\ln Q/\mathrm{d}\ln a$.

$n$: Density dilution exponent $-\mathrm{d}\ln\rho/\mathrm{d}\ln a$.

$w$: Effective component equation of state.

$w_{\mathrm{eff}}$: Total effective equation of state of the background.

$F$: Accessible fraction or visibility weight when $Q_{\mathrm{acc}} = FQ_{\mathrm{tot}}$.

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