EVERY CUBIC PLANAR GRAPH CONTAINS C_4, C_8, C_{16}, OR C_{32}: ARBITRARY CONNECTIVITY, PLANAR CLAW-FREE C_{64}, AND GENERAL PLANAR C_{16} THEOREMS

SETH REHWALDT

Abstract. The Erdős–Gyárfás conjecture (1995) asserts that every finite graph with minimum degree at least 3 contains a simple cycle whose length is a power of two. For 3-connected cubic planar graphs, Heckman and Krakovski (2013) proved the existence of a power-of-two cycle of cycle length at most 128; for planar induced claw-free graphs of minimum degree at least 3, Daniel and Shauger (2001) proved the existence of a power-of-two cycle.

We establish four main theorems for finite nonempty simple planar graphs of *arbitrary connectivity* (permitting disconnected graphs, bridges, cut-vertices, and 2-cuts):

(1) **All Cubic Planar Graphs (Sharp C_{32}):** Every finite nonempty simple cubic planar graph contains a simple cycle in $\{C_4, C_8, C_{16}, C_{32}\}$. This drops 3-connectivity from Heckman–Krakovski and improves the cutoff fourfold from 128 to the sharp threshold 32, matching Exoo's $m \geq 5$ lower bound (G_{420}) and witnessed by an elementary 168-vertex 6-face-free 3-connected cubic planar graph $G_{168} = C_{24} * [T_1 + P + T_2]$ with no C_4, C_8, C_{16}.

(2) **All Planar Induced Claw-Free Graphs (Explicit C_{64}):** Every finite nonempty simple planar induced claw-free (induced $K_{1,3}$-free) graph with $\delta(G) \geq 3$ contains a simple cycle in $\{C_4, C_8, C_{16}, C_{32}, C_{64}\}$, via planar bowtie-splitting and a universal cubic-to-claw-free transfer theorem applied to (1).

(3) **General Planar $\delta \geq 3$ C_{16} Trilogy (Arbitrary Δ):** Every finite nonempty simple planar graph with $\delta(G) \geq 3$ that is C_3-free, C_5-free, or C_6-free contains a simple cycle in $\{C_4, C_8, C_{16}\}$. Each part is sharp, witnessed by the truncated icosahedron C_{60}, the truncated dodecahedron TD_{60}, and their 119-vertex $\Delta = 6$ planar 1-sums.

(4) **Alternating-C_6 Planar Degree-Density Bound:** Every finite nonempty simple planar graph with $\delta(G) \geq 3$, independent high-degree set $V_{\geq 4}$, and no C_4, C_8 satisfies $16h + 4\varepsilon \leq 5(n-2) + 4\tau_6$, where $h = |V_{\geq 4}|$, $\varepsilon = \sum_{v \in V_{\geq 4}} (d(v) - 4)$, and τ_6 is the 6-cycle edge-transversal number of the mixed bipartite subgraph $G[V_{\geq 4}, V_3]$. In particular, when C_6-free, $|V_3| \geq \lceil (11n + 10)/16 \rceil > 2n/3$ for all $n \geq 7$, improving the 2/3 density benchmark by 1/48.

Our verification artifact separates four transparent layers: (i) core discrete algebraic, discharging, Bellman–Ford/DP soundness, and density lemmas proved in the pure Lean 4 kernel (`v4.36.0`, zero `sorry`, zero custom axioms, zero `native_decide`); (ii) proof-carrying Lean 4 certificates for packed dual potentials, DP/BWD upper bounds, 6,772 strict-schema explicit simple dyadic-cycle patch entries (counted with multiplicity across certificate groups), and cellular-rotation sharpness witnesses; (iii) standalone zero-float standard-library Python replay (`cert/verify_planar_eg.py`) of all underlying finite searches and uncollapsed DPs, paired with a runnable zero-float parity and weighted state-collapse verifier (`cert/verify_parity_and_dp.py`) checking all 12 weighted DP simulations and literal Lean–Python–JSON parity; and (iv) full topological and graph-theoretic proofs in this manuscript.

Contents

1. Introduction and Main Results 2
1.1. Statements of Main Theorems 3
1.2. Honest Verification and Formalization Summary 4
2. Preliminaries, Planar Embeddings, and Connectivity-Cap Reductions 5
2.1. Spherical Embeddings, Rotation Systems, and Euler's Formula 5

Date: November 25, 2026.

2.2. Connectivity-Cap Reductions 6
3. Main Theorem 1: All Cubic Planar Graphs Contain C_4, C_8, C_{16}, or C_{32} 9
3.1. Boundary-Weld Quotient Soundness 10
3.2. Clean Clusters, 51 Mixed Chains, and Complete Charge-Conserving Discharging Rules 12
3.3. Graph-to-Token Mapping, Transition/Flag Derivations, Conservative Enumeration, and Closure 17
4. Main Theorem 2: All Planar Induced Claw-Free Graphs ($\delta \geq 3$) Contain $C_4..C_{64}$ 24
5. Main Theorem 3: General Planar ($\delta \geq 3$) C_{16} Trilogy 25
5.1. Part (a): All C_3-Free General Planar Graphs ($\delta \geq 3$) 27
5.2. Part (b): All C_5-Free General Planar Graphs ($\delta \geq 3$, Zero R_{10}) 28
5.3. Part (c): All C_6-Free General Planar Graphs ($\delta \geq 3$) 33
5.4. Sharpness Witnesses: C_{60}, TD_{60}, and 119-Vertex $\Delta = 6$ 1-Sums 36
6. Main Theorem 4: Alternating-C_6 Planar Degree-Density Bound 37
7. Formal Verification, Reproducibility, and Honest Scope 38
Declaration of Generative AI and AI-Assisted Technologies in the Manuscript Preparation Process 39
References 40

1. Introduction and Main Results

Formulated by Erdős and Gyárfás in 1995 and published by Erdős in 1997 [4], the Erdős–Gyárfás conjecture asserts that every finite graph with minimum degree at least 3 contains a simple cycle of length 2^k for some integer $k \geq 2$. While the conjecture is now known to hold for graphs of sufficiently large average degree—following the order-dependent average-degree bound $\bar{d}(G) \geq e^{C \log^* n}$ of Sudakov and Verstraëte [11] and culminating in the constant average-degree theorem of Liu and Montgomery [8]—the original sparse regime $\delta(G) \geq 3$ remains open even for general planar graphs.

Three lines of prior work directly anchor our results:

- **Cubic planar graphs and sharpness:** Heckman and Krakovski [7] proved that every 3-connected cubic planar graph contains a simple cycle of length 2^m with $2 \leq m \leq 7$ (cycle length at most 128), asking whether the upper bound $m \leq 7$ could be reduced. Exoo [5] responded by constructing a 420-vertex 3-connected cubic planar graph G_{420} (derived from the truncated icosahedron C_{60}, Section 2 of [5]) with no C_4, C_8, or C_{16}, establishing $m \geq 5$ (cutoff at least 32). Earlier, Markström [9] found a 24-vertex 3-connected cubic planar graph avoiding C_4 and C_8; Garcia [6] recently certified minimum order 24 for avoiding $\{C_4, C_8\}$ across all $\delta \geq 3$ graphs, matching Markström's 24-vertex cubic planar example and the cubic threshold $f(3) = 24$.
- **Claw-free and planar claw-free graphs:** Shauger [10] proved the conjecture for $K_{1,m}$-free graphs with $\delta(G) \geq m + 1$ (including claw-free graphs with $\delta \geq 4$), and Daniel and Shauger [2] proved that every planar induced claw-free graph with $\delta(G) \geq 3$ contains a power-of-two cycle (see also [7]).
- **Degree-density in minimal counterexamples:** Using Markström's [9] observation that any minimal counterexample to the Erdős–Gyárfás conjecture has an independent high-degree set $V_{\geq 4}$ together with new structural results, Carr [1] proved $|V_3| \geq \frac{4}{7}|V|$; Ducoffe and Dumitru [3] sharpened this to $|V_3| > \frac{2}{3}|V|$ for minimal counterexamples generally.

Throughout this paper, all graphs are *finite, nonempty ($V(G) \neq \emptyset$), and simple* (no loops or parallel edges), unless explicitly noted otherwise. We write $\delta(G)$ and $\Delta(G)$ for the minimum and maximum vertex degrees of G, $V_3(G) = \{v \in V(G) : d_G(v) = 3\}$, and $V_{\geq 4}(G) = \{v \in V(G) : d_G(v) \geq 4\}$.

A *dyadic cycle* is a simple cycle C_{2^k} of length 2^k with $k \geq 2$. A graph is *claw-free* if it contains no *induced* subgraph isomorphic to $K_{1,3}$. Theorems 1.1–1.4 below (numbered Theorems 1.1–1.4 in this section) correspond respectively to Main Theorems 1–4 in the accompanying verification artifact.

1.1. **Statements of Main Theorems.**

Theorem 1.1 (Main Theorem 1 — All Cubic Planar Graphs, Sharp C_{32})**.** *Every finite nonempty simple cubic planar graph G (of arbitrary connectivity) contains a simple cycle of length $4, 8, 16,$ or 32.*

Moreover, the threshold 32 is sharp: there exists an explicit 168-vertex 3-connected simple cubic planar graph $G_{168} = C_{24} * [T_1 + P + T_2]$ containing a simple C_{32} and no simple $C_4, C_8,$ or C_{16}.

Theorem 1.1 strengthens [7] simultaneously in two orthogonal directions: it removes the 3-connectivity hypothesis completely (covering disconnected graphs, bridged graphs, and 2-edge-cut graphs via a frozen-edge cap discharging method) and sharpens the dyadic cutoff fourfold from 128 to the exact optimum 32, closing the $m \in \{5, 6, 7\}$ gap left between [7] and Exoo's construction G_{420} [5]. Alongside Exoo's 420-vertex C_{60}-derived construction G_{420} [5], our smaller elementary 168-vertex, 6-face-free $G_{168} = C_{24} * [T_1 + P + T_2]$ supplies a sharpness witness fully kernel-verified in Lean 4.

Theorem 1.2 (Main Theorem 2 — All Planar Induced Claw-Free Graphs, Explicit C_{64})**.** *Every finite nonempty simple planar induced claw-free (induced $K_{1,3}$-free) graph G with $\delta(G) \geq 3$ (of arbitrary connectivity and arbitrary maximum degree Δ) contains a simple cycle of length $4, 8, 16, 32,$ or 64.*

Theorem 1.2 gives an independent proof of the planar claw-free theorem of Daniel and Shauger [2] with the explicit universal bound 64. It follows directly from Theorem 1.1 combined with our *Planar Bowtie-Splitting Lemma* and *Universal Cubic-to-Claw-Free Transfer Theorem* (Section 4).

Theorem 1.3 (Main Theorem 3 — General Planar $\delta \geq 3$ C_{16} Trilogy)**.** *Let G be a finite nonempty simple planar graph with $\delta(G) \geq 3$ (of arbitrary connectivity and arbitrary maximum degree Δ). If G is*

(a) C_3-free (triangle-free),
(b) C_5-free (pentagon-free), or
(c) C_6-free (hexagon-free),

then G contains a simple cycle of length $4, 8,$ or 16.

Moreover, every part is sharp at C_{16} for both cubic and non-cubic planar graphs: the truncated icosahedron C_{60} (60 vertices, cubic, no C_3, C_4, C_8) and its planar 1-sum $C_{60} \cdot C_{60}$ (119 vertices, $\delta = 3$, $\Delta = 6$) witness sharpness for (a); the truncated dodecahedron TD_{60} (60 vertices, cubic, no $C_4, C_5, C_6, C_7, C_8, C_9$) and its planar 1-sum $TD_{60} \cdot TD_{60}$ (119 vertices, $\delta = 3$, $\Delta = 6$) witness sharpness for both (b) and (c).

As an immediate corollary of Theorem 1.3, any finite simple planar graph with $\delta(G) \geq 3$ avoiding $\{C_4, C_8, C_{16}\}$ must simultaneously contain triangles (C_3), pentagons (C_5), and hexagons (C_6).

Theorem 1.4 (Main Theorem 4 — Alternating-C_6 Planar Degree-Density Bound)**.** *Let G be a finite nonempty simple planar graph on $n = |V(G)|$ vertices with $\delta(G) \geq 3$ such that $V_{\geq 4}(G)$ is an independent set and G contains no C_4 and no C_8. Set $h := |V_{\geq 4}(G)|$, $\varepsilon := \sum_{v \in V_{\geq 4}(G)} (d_G(v) - 4)$, and let τ_6 denote the minimum cardinality of an edge set $T \subseteq E(Q)$ meeting every 6-cycle in the mixed bipartite subgraph $Q := G[V_{\geq 4}(G), V_3(G)]$. Then $n \geq 7$ and*

$$16h + 4\varepsilon \;\leq\; 5(n-2) + 4\tau_6. \tag{1}$$

In particular:

(i) If G is C_6-free (so $\tau_6 = 0$), then

$$|V_3(G)| \;\geq\; \left\lceil \frac{11n+10}{16} \right\rceil \;>\; \frac{2}{3}n \tag{2}$$

for every $n \geq 7$, strictly surpassing the $2/3$ coefficient by $11/16 - 2/3 = 1/48$.

(ii) Even when 6-cycles are permitted, any such graph with $|V_3(G)| \leq 2n/3$ must satisfy $\tau_6 \geq \lceil (n+30)/12 + \varepsilon \rceil$ and therefore contains at least $\lceil \tau_6/6 \rceil$ pairwise edge-disjoint alternating 6-cycles in Q.

1.2. **Honest Verification and Formalization Summary.** Our proofs combine traditional structural graph theory and topological embedding arguments with extensive finite discharging casework (boundary-weld quotients, noncrossing outer-rim partitions, cyclic dynamic programming, and exact rational Bellman–Ford dual potentials). To guarantee mathematical reliability without overstating the scope of machine formalization, we explicitly partition every claim across four transparent verification layers:

(i) **Proved in the Lean 4 kernel (`lean/PlanarEG/`, pure Lean `v4.36.0`, zero Mathlib):** Core discrete simple-graph predicates, Euler discharging conservation on combinatorial `PlaneMap` structures, local cap degree/avoiding-lift lemmas, universal vertex-absorption inequalities, exact budget identities, exhaustive F_{10} 5-triangle score maximization (252×123 bitmask pairs, max $= 47/12 < 4$), Bellman–Ford/DP/BWD soundness theorems, bowtie-forest exclusion arithmetic and inductive forest bounds, and alternating-C_6 density algebra. All 106 audited declarations in `AxiomCheck.lean` depend solely on Lean's standard axioms {`propext`, `Classical.choice`, `Quot.sound`}, with zero `sorry`, zero custom axioms, and zero `native_decide`.

(ii) **Lean 4 proof-carrying finite certificates:** Kernel-checked dual potentials for all clean/frozen Bellman–Ford and DP state-graphs; kernel-checked strict face-schema validity (`WeldSchemaValid`) and explicit simple dyadic cycles on all 6,772 eliminated weld and outer-rim entries (counted with multiplicity across certificate groups: $114 + 5 + 1{,}327 + 520 = 1{,}966$ for Theorem 1.1, where the 5 cubic R_{10} seals are part of the 520 seals and reused in the $d = 17$ check; and $1{,}260 + 3{,}501 + 45 = 4{,}806$ for Theorem 1.3); and kernel-checked exact dart bijections, single cyclic vertex rotations, 3-connectivity (G_{168}), forbidden-cycle exclusions, and explicit target cycles for G_{168}, C_{60}, and TD_{60}.

(iii) **Standalone zero-float Python replay (`cert/verify_planar_eg.py`) and parity/simulation verifier (`cert/verify_parity_and_dp.py`):** Pure-stdlib Python 3 scripts (both AST-guarded against floating-point literals and true division `/`). First, `cert/verify_planar_eg.py` replays all underlying finite searches, boundary-weld and rim-quotient enumerations, 51-chain BFS, conservative $d \in \{14, 17, 18\}$ enumerations, exact uncollapsed cyclic DPs, Bellman–Ford potentials, and 119-vertex 1-sum witnesses in ∼21 s and ∼43 MiB RSS. Second, `cert/verify_parity_and_dp.py` directly verifies literal parity across Lean 4 data modules, `cert/witnesses/*.json`, and live Python replay across all 6,772 weld/rim entries, confirms the Lemma 3.1 boundary-regularity invariant across all 844/844 production calls, verifies all 12 weighted forward-simulation checks linking the uncollapsed C_5-free (7 degrees: $d \in \{9..14, 17\}$) and C_6-free (5 degrees: $d \in \{9, 11..14\}$) DPs to the packed Lean automata, and replays all supplemental manuscript subaudits in ∼43 s and ∼70 MiB RSS (Section 7).

(iv) **Proved in this manuscript:** Topological spherical embeddings, Jordan curve arguments, Euler's formula with bridges/cut-vertices/components, Minimal 2-Edge-Bond Cap and Universal 2-Vertex-Cut Cap reductions, edge-minimality $V_{\geq 4}$-independence, charge-conserving discharging rules by full configuration, graph-to-token maps and transition/flag derivations, completeness of hereditary boundary-weld and noncrossing outer-rim enumerations, conservative enumeration start/symmetry/shift/splice soundness, bowtie-forest/planar-skeleton/cycle-lift

constructions, mixed bipartite Q/transversal/forest-Euler proofs for Theorem 1.4, and planar 1-sum gluing for the 119-vertex sharpness witnesses.

The complete Lean 4 library, Python verifiers, JSON witnesses, and SHA-256 manifest are publicly archived at https://github.com/lollipoll/erdos-gyarfas (directory `paper1_planar/`); a complete theorem-by-theorem four-column traceability matrix appears in Section 7 and in the repository `README.md`.

2. Preliminaries, Planar Embeddings, and Connectivity-Cap Reductions

2.1. **Spherical Embeddings, Rotation Systems, and Euler's Formula.** Let $G = (V, E)$ be a finite nonempty simple graph ($|V| \geq 1$). A *planar embedding* (or *plane graph*) is a drawing of G on the 2-sphere S^2 in which vertices map to distinct points and edges map to internally disjoint simple Jordan arcs connecting their endpoints. Every connected plane graph with at least one edge induces a *combinatorial rotation system*: letting $D(G) = \{(u, v), (v, u) : uv \in E(G)\}$ be the set of $2|E|$ *darts*, the embedding specifies at each vertex $v \in V$ a single cyclic permutation σ_v of the outgoing darts $\{(v, u) : uv \in E\}$ (clockwise cyclic order on S^2), together with the fixed-point-free dart-reversal involution $\alpha(u, v) = (v, u)$. The orbits of the face-traversal permutation $\phi := \sigma \circ \alpha$ are the *facial boundary walks* of the embedding, bounding the connected components (*faces*) of $S^2 \setminus G$ (with a bridge traversed once in each direction by the same facial walk; for a one-vertex component K_1, the unique face has facial walk length 0).

For a plane graph G with $c \geq 1$ connected components, $|V|$ vertices, $|E|$ edges, and $|F|$ faces, Euler's formula reads

$$|V| - |E| + |F| \;=\; 1 + c. \tag{3}$$

Indeed, if $c = 1$ this is the standard connected spherical Euler formula $|V| - |E| + |F| = 2$; if $c \geq 2$, adding $c - 1$ bridge edges iteratively between distinct components across a common face connects the embedding while increasing $|E|$ by $c - 1$, reducing c to 1, and leaving $|V|$ and $|F|$ unchanged, whence $|V| - (|E| + c - 1) + |F| = 2$, yielding (3). Every edge contributes two darts to the facial walks so that $\sum_{f \in F} d(f) = 2|E|$, where $d(f)$ is the length of the facial walk of f. Throughout the paper, for a collection of faces $\mathcal{P} = (F_1, \ldots, F_m)$ of a plane graph, we refer to the subgraph formed by all edges on $\partial F_1, \ldots, \partial F_m$ as the *edge union* of those facial boundaries (noting that the parent graph may contain additional edges between the same vertices).

Lemma 2.1 (Facial Boundaries and Whitney Intersection Properties)**.** *Let G be a finite simple plane graph embedded on S^2.*

(i) If G is 2-connected with $|V(G)| \geq 3$, then every facial boundary walk of G is a simple cycle of length at least 3. Conversely, a bridge appears twice in the same facial walk, and a cut-vertex repeats on at least one facial walk.

(ii) If G is 3-connected, then any two distinct faces f_1, f_2 share either $\emptyset$, a single vertex, or a single edge (together with its two endpoints).

(iii) If G is 3-connected and cubic, then any two distinct faces share either $\emptyset$ or a single edge; equivalently, the planar dual G^* is a simple triangulation.

(iv) For any connected plane graph G on S^2 with $|V| - |E| + |F| = 2$:

$$\sum_{v \in V} \big(6 - 2d(v)\big) + \sum_{f \in F} \big(6 - d(f)\big) \;=\; 12, \tag{4}$$

$$\text{and if } G \text{ is cubic,} \qquad \sum_{f \in F} \big(6 - d(f)\big) \;=\; 12. \tag{5}$$

Proof. (i) Suppose a facial boundary walk of a face f visits a vertex v twice (in distinct steps of the ϕ-orbit). These two visits occupy two distinct angular wedges of f in the cyclic rotation σ_v around v; even when two distinct wedges are adjacent in σ_v (at a bridge-incident vertex), the two cyclic

intervals between their interiors on σ_v each contain at least one edge (when $d(v) = 1$ the single incident edge is a bridge; when $d(v) \geq 2$ each interval contains at least one incident edge). Joining the interiors of these two wedges by a simple closed Jordan curve Γ inside $f \cup \{v\}$ meeting G only at v separates the two intervening edge-arcs of v into opposite sides of Γ, so v is a cut-vertex (or incident to a bridge if $d(v) = 1$). Conversely, if v is a cut-vertex of G, the outgoing edges at v belong to at least two distinct components of $G - v$, so there exist two consecutive edges in σ_v belonging to distinct components of $G - v$; the facial walk entering the wedge between them cannot cross between components without revisiting v, so v repeats on that facial walk. Likewise a bridge e has both of its opposite darts in the same ϕ-orbit. When G is simple and 2-connected with $|V| \geq 3$, it has neither bridges nor cut-vertices, so every facial walk is a simple cycle of length ≥ 3.

(ii) Suppose two distinct faces f_1, f_2 of a 3-connected plane graph G share two distinct vertices x, y. Draw a simple arc inside f_1 from x to y and a simple arc inside f_2 from y to x; their union is a simple closed Jordan curve $\Gamma \subset S^2$ intersecting G only at $\{x, y\}$. If both open regions of $S^2 \setminus \Gamma$ contain vertices of G, then $\{x, y\}$ is a 2-vertex-cut of G, contradicting 3-connectivity. Hence one open side U of Γ contains no vertices of G. Every edge of G drawn inside U must have both endpoints on $\partial U \cap V(G) = \{x, y\}$, so U can contain at most the single edge xy (by simplicity of G). If U contained no edge of G (whether $xy \notin E(G)$ or xy were drawn outside U), then $U \subset S^2 \setminus G$ would be empty of G and the interior arcs of f_1 and f_2 could be joined inside U, forcing $f_1 = f_2$, contradiction. Thus $xy \in E(G)$ lies directly inside U, $U \setminus xy$ is empty of G, and xy lies on both ∂f_1 and ∂f_2. Applying this to every pair of common vertices rules out a third common vertex $z \in \partial f_1 \cap \partial f_2$: otherwise xy, yz, zx would all lie on both ∂f_1 and ∂f_2, forcing both faces to be bounded by the same triangle xyz and connected $G = K_3$, contradicting 3-connectivity ($|V| \geq 4$). Hence $f_1 \cap f_2$ is $\emptyset$, a single vertex, or a single edge with its two endpoints.

(iii) If G is cubic and two faces f_1, f_2 share a vertex v, then among the three incident edges at v, each face uses two of them, so by pigeonhole f_1 and f_2 share at least one edge incident to v, and by (ii) they share exactly that edge.

(iv) Using $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} d(f)$ and $|V| - |E| + |F| = 2$:

$$\sum_{v \in V} (6 - 2d(v)) + \sum_{f \in F} (6 - d(f)) = 6|V| - 4|E| + 6|F| - 2|E| = 6(|V| - |E| + |F|) = 12.$$

When G is cubic, $6 - 2d(v) = 0$ for every $v \in V$, yielding (5). □

2.2. **Connectivity-Cap Reductions.** A central obstacle in dropping 3-connectivity for discharging proofs is that a planar graph with bridges, cut-vertices, or 2-cuts does not enjoy Lemma 2.1(i)–(iii). Rather than attempting to exclude dyadic cycles passing through an artificial cap edge, our reduction adds a single artificial edge ab to an innermost cut-shore, *freezes* the two faces f_1, f_2 incident to ab, and permits cycles through ab completely freely (Figure 1). Every cycle avoiding ab lifts directly to the original graph G, so all clean faces and clean edge-unions obey every forbidden-cycle exclusion of G; the charge bounds on the frozen faces are proved separately in Sections 3 and 5.

Lemma 2.2 (Minimal 2-Edge-Bond Cap Lemma — Cubic Planar Graphs)**.** *Let G be a finite nonempty simple cubic planar graph avoiding a set $\mathcal{L}$ of cycle lengths. Then there exists a finite simple 3-connected cubic plane graph F^+ together with a (possibly empty) set $\mathcal{F}_{\mathrm{frz}}(F^+)$ of at most two* frozen *faces such that:*

- *(i) Either $\mathcal{F}_{\mathrm{frz}}(F^+) = \emptyset$ and F^+ is a connected component of G (in which case "avoids the cap edge" holds vacuously for all subgraphs of F^+); or $\mathcal{F}_{\mathrm{frz}}(F^+) = \{f_1, f_2\}$ consists of the exact two faces incident to a distinguished artificial* cap edge *$ab \in E(F^+)$ such that $F^+ - ab$ is an induced subgraph of G.*
- *(ii) Every simple cycle of F^+ that avoids the cap edge ab is a simple cycle of G, and therefore has length outside $\mathcal{L}$.*

(iii) Every face $f \notin \mathcal{F}_{\mathrm{frz}}(F^+)$ (called a clean *face) is bounded by a simple cycle of $F^+ - ab \subseteq G$, and every subgraph formed by an edge union of clean facial boundaries avoids ab and hence contains no cycle of length in $\mathcal{L}$.*

Proof. Since G is nonempty, pick any connected component G_0 of G, embedded in S^2. We first show that if G_0 is 3-edge-connected, then because G_0 is cubic it is also 3-vertex-connected (note $|V(G_0)| \geq 4$ since G_0 is simple and cubic). Indeed, a 1-vertex-cut x would leave at least two components of $G_0 - x$ sharing the 3 edges at x, so some component attaches to x by at most 1 edge (a bridge, contradiction). Suppose instead that $\{x, y\}$ is a 2-vertex-cut of G_0. By 3-edge-connectivity, every connected component of $G_0 - \{x, y\}$ attaches to $\{x, y\}$ by at least 3 edges. Since $d_{G_0}(x) + d_{G_0}(y) = 6$, there are exactly two components A, B of $G_0 - \{x, y\}$, x and y are nonadjacent, and each of A, B receives exactly 3 edges from $\{x, y\}$. Neither x nor y can send all 3 of its edges to one component (that would disconnect G_0 or leave the other vertex as a 1-cut), so x and y have opposite attachment splits $(2, 1)$ and $(1, 2)$ across (A, B): without loss of generality, x has 2 neighbors in A and 1 in B, while y has 1 neighbor in A and 2 in B. Then the bipartition $(A \cup \{x\},\ B \cup \{y\})$ has crossing edge set of size $1 + 1 = 2$, contradicting 3-edge-connectivity. Thus if G_0 is 3-edge-connected, setting $F^+ := G_0$ and $\mathcal{F}_{\mathrm{frz}}(F^+) := \emptyset$ satisfies (i)–(iii).

Assume now that G_0 is not 3-edge-connected. Define a 2-*edge-bond* of G_0 to be a partition $V(G_0) = V(F) \sqcup V(R)$ such that both $F := G_0[V(F)]$ and $R := G_0[V(R)]$ are connected, $|V(F)| \geq 2$, $|V(R)| \geq 1$, and $|E_{G_0}(F, R)| = 2$. We first show that G_0 admits at least one 2-edge-bond:

- If G_0 has a bridge, consider the bridge-block tree of G_0 and let B be a leaf bridge-block (a maximal 2-edge-connected subgraph incident to a single bridge vw with $v \in V(B)$). Then $\deg_B(v) = 2$ and $\deg_B(u) = 3$ for all $u \in V(B) \setminus \{v\}$. By handshaking, $3|V(B)| - 1 = 2|E(B)|$ is even, so $|V(B)|$ is odd; since B is simple and has vertices of degree 3, $|V(B)| \geq 5$. Let a, b be the two neighbors of v in B. Because B is 2-edge-connected and $\deg_B(v) = 2$, removing v cannot disconnect B (any component of $B - v$ would attach to v by only one edge, which would be a bridge of B). Hence $F := B - v$ is connected of order $|V(B)| - 1 \geq 4 \geq 2$, $R := G_0 - V(F)$ is connected (containing v, the bridge vw, and $G_0 - B$), and $E_{G_0}(F, R) = \{va, vb\}$ has size 2. Thus (F, R) is a 2-edge-bond.
- If G_0 is bridgeless, then it has a 2-edge-cut $\{e_1, e_2\}$; removing $\{e_1, e_2\}$ splits G_0 into exactly two connected components F, R, each of order ≥ 2 since every vertex has degree 3. Thus (F, R) is a 2-edge-bond.

Now choose a 2-edge-bond (F, R) of G_0 *minimizing* $|V(F)|$, and let $E_{G_0}(F, R) = \{e_1, e_2\}$.

Claim 1. *The endpoints $a, b \in V(F)$ of e_1, e_2 are distinct.* Suppose instead $a = b$. Then $\deg_F(a) = 1$, so the unique edge $ax \in E(F)$ is a bridge of G_0 separating $F - a$ from $R \cup \{a\}$. Any leaf bridge-block B' of G_0 contained in $F - a$ (with bridge vertex $v' \in V(B')$) yields a 2-edge-bond $(B' - v', G_0 - (B' - v'))$ with $|V(B' - v')| < |V(F)|$, contradicting minimality. Hence $a \neq b$, $\deg_F(a) = \deg_F(b) = 2$, and $\deg_F(u) = 3$ for all $u \in V(F) \setminus \{a, b\}$.

Claim 2. $ab \notin E(F)$. Suppose for contradiction that $ab \in E(F)$, and set $F' := F - \{a, b\}$. Since $\deg_F(a) = \deg_F(b) = 2$, each of a, b sends exactly one edge into F', so $|E_{G_0}(F', R \cup \{a, b\})| = 2$ and $\sum_{u \in V(F')} \deg_{F'}(u) = 3|V(F')| - 2$. By parity $|V(F')|$ is even; $|V(F')| = 0$ is impossible since $\deg_F(a) = 2 > 1$, and $|V(F')| = 2$ is impossible since a simple 2-vertex graph has degree sum $\leq 2 < 4$. Thus $|V(F')| \geq 4$. Moreover $R \cup \{a, b\}$ is connected via e_1, e_2. If F' is connected, $(F', R \cup \{a, b\})$ is a 2-edge-bond with $|V(F')| = |V(F)| - 2 < |V(F)|$, contradiction. If F' is disconnected, each component F_1 of F' has $|E_{G_0}(F_1, G_0 - F_1)| = 1$ (a bridge of G_0), so a leaf bridge-block B' inside F_1 gives a 2-edge-bond $(B' - v', G_0 - (B' - v'))$ strictly smaller than F, contradiction.

Claim 3. *$F^+ := F + ab$ is simple, cubic, planar, and 3-connected.* Simplicity and cubicity follow immediately from Claims 1–2. Since R is connected and e_1, e_2 attach a, b to R, there

is a simple a–b path P_{out} whose internal vertices lie in R. Replacing R in the spherical embedding of G_0 by the arc of P_{out} yields a planar embedding of $F^+ = F + ab$ on S^2.

It remains to prove F^+ is 3-edge-connected (hence 3-vertex-connected by the opening paragraph). First, F^+ is bridgeless: ab lies in a cycle since F is connected, and any bridge $e^* \neq ab$ of F^+ separates a component $K \subsetneq F$ avoiding both cap endpoints a, b with $|E_{G_0}(K, G_0 - K)| = 1$, giving a leaf bridge-block $B' \subseteq K$ and a strictly smaller 2-edge-bond, contradiction. Second, suppose F^+ has a 2-edge-cut $\{g_1, g_2\}$ splitting F^+ into connected components K_1, K_2 (each of order ≥ 2).

- If $ab \notin \{g_1, g_2\}$, assume without loss of generality $a, b \in V(K_2)$. Then $K_1 \subsetneq F$ is connected, $G_0 - K_1 = (K_2 - ab) \cup R \cup \{e_1, e_2\}$ is connected (replacing ab by P_{out} preserves connectedness of K_2 with R), and $|E_{G_0}(K_1, G_0 - K_1)| = 2$, contradicting minimality of $|V(F)|$.
- If $ab \in \{g_1, g_2\}$, say $g_1 = ab$ with $a \in V(K_1)$ and $b \in V(K_2)$, then $K_1 \subsetneq F$ is connected ($|V(K_1)| \geq 2$), $G_0 - K_1 = K_2 \cup R \cup \{e_2\}$ is connected, and $E_{G_0}(K_1, G_0 - K_1) = \{e_1, g_2\}$ has size 2, again contradicting minimality of $|V(F)|$.

Let $\mathcal{F}_{\text{frz}}(F^+) = \{f_1, f_2\}$ be the two distinct faces incident to ab in the spherical embedding of 3-connected F^+. Every cycle avoiding ab is a cycle of $F = F^+ - ab \subseteq G$, proving (ii), and every clean face $f \notin \{f_1, f_2\}$ avoids ab, proving (iii). □

Lemma 2.3 (Edge-Minimality and Universal 2-Vertex-Cut Cap — General Planar $\delta \geq 3$)**.** *Let $\mathcal{L}$ be any set of forbidden cycle lengths, and let G be an edge-minimal counterexample among finite nonempty simple planar graphs with $\delta(G) \geq 3$ avoiding $\mathcal{L}$ (minimizing $|V(G)| + |E(G)|$). Then:*

(i) ***Independence of*** *$V_{\geq 4}(G)$: No two vertices in $V_{\geq 4}(G)$ are adjacent in G.*

(ii) ***Universal 3-Connected Cap:*** *There exists a finite simple 3-connected plane graph H^+ with $\delta(H^+) \geq 3$ and a set $\mathcal{F}_{\text{frz}}(H^+)$ of either 0 or 2 frozen faces (sharing a single cap edge $ab \in E(H^+)$) such that:*

- *(a) $H := H^+ - ab$ (or $H := H^+$ when $\mathcal{F}_{\text{frz}} = \emptyset$) is a subgraph of G, so every cycle of H^+ avoiding ab avoids $\mathcal{L}$;*
- *(b) $d_{H^+}(u) \leq d_G(u)$ for every $u \in V(H^+)$, whence $V_{\geq 4}(H^+) \subseteq V_{\geq 4}(G)$ is an independent set in H (no clean edge, and hence no edge of any clean face, joins two vertices of $V_{\geq 4}(H^+)$);*
- *(c) Every clean face $f \notin \mathcal{F}_{\text{frz}}(H^+)$ avoids ab and is bounded by a simple cycle of $H \subseteq G$.*

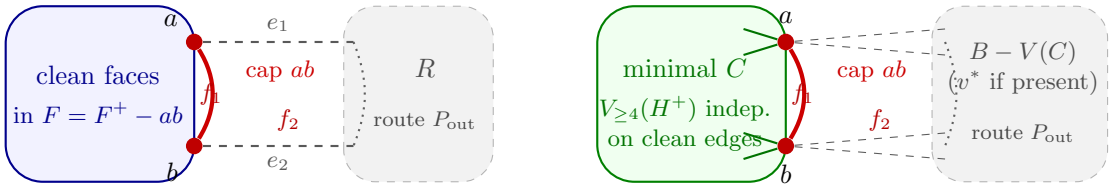
Proof. (i) Suppose $uv \in E(G)$ with $d_G(u) \geq 4$ and $d_G(v) \geq 4$. Deleting uv produces a finite nonempty simple planar graph $G - uv$ with $\delta(G - uv) \geq 3$. Since $G - uv$ is a subgraph of G, no new cycles are created, so $G - uv$ still avoids $\mathcal{L}$, contradicting minimality of $|V(G)| + |E(G)|$. (In particular, bridging two copies of C_{60} creates two adjacent degree-4 vertices whose joining bridge is immediately deleted by edge-minimality.)

(ii) Pick a connected component G_0 of G, and let B be a leaf block in the block-cut tree of G_0 (possibly the sole block of G_0, in which case B has no cut-vertex of G_0). A leaf block cannot be a single bridge K_2 (its non-cut endpoint would have degree $1 < 3$ in G), nor can it be K_3 (its at least two non-cut vertices would have degree $2 < 3$ in G). Thus B is 2-connected with $|V(B)| \geq 4$, contains at most one cut-vertex v^* of G_0, and satisfies $d_B(x) = d_G(x) \geq 3$ for every $x \in V(B) \setminus \{v^*\}$.

If B is already 3-connected, then $\delta(B) \geq 3$ automatically and setting $H^+ := B$, $\mathcal{F}_{\text{frz}}(H^+) := \emptyset$ satisfies (a)–(c) directly. Otherwise B contains a 2-vertex-cut. Because B is 2-connected, removing any 2-vertex-cut $\{x, y\}$ yields at least two connected components, at most one of which contains v^* (if v^* exists). Choose a 2-vertex-cut $\{a, b\}$ of B and a connected component C of $B - \{a, b\}$ avoiding v^* (i.e., $v^* \notin V(C)$ if v^* exists) *minimizing* $|V(C)|$.

Observe that $|V(C)| \geq 2$: if $V(C) = \{w\}$, then $N_B(w) \subseteq \{a, b\}$ implies $d_B(w) \leq 2$, contradicting $w \neq v^* \implies d_B(w) \geq 3$. Set $H := B[V(C) \cup \{a, b\}] - ab$ (removing the edge ab if it was present in B) and $H^+ := H + ab$.

(a) Minimal 2-Edge-Bond Cap (Cubic) **(b) Universal 2-Vertex-Cut Cap ($\delta \geq 3$)**



Discharging (Sec. 3): $\text{ch}_{\text{fin}}(f_1) + \text{ch}_{\text{fin}}(f_2) \leq 32/3 < 12$
(strict `CAP`: $\leq 10 < 12$)

$d_{H^+}(u) \leq d_G(u)$, $\delta(H^+) \geq 3$
Discharging (Sec. 5): $\text{ch}_{\text{fin}}(f_1) + \text{ch}_{\text{fin}}(f_2) \leq 8$ or $10 < 12$

FIGURE 1. Connectivity-cap reductions: (a) Minimal 2-Edge-Bond Cap (Lemma 2.2) for cubic planar graphs, and (b) Universal 2-Vertex-Cut Cap (Lemma 2.3) for general planar graphs with $\delta \geq 3$ (where v^* exists only if the leaf block B has a cut-vertex of G_0). The terminals a, b are joined by the artificial cap edge ab routed along the external path P_{out}. Cycles using ab are *not* excluded; only the two incident faces f_1, f_2 are frozen while every clean face avoids ab. The displayed frozen-charge bounds are proved in Sections 3 and 5.

- **Simplicity and planarity:** By construction $ab \notin E(H)$, so $H^+ = H + ab$ is simple. Since B is 2-connected, every component of $B - \{a, b\}$ attaches to both a and b, so $B - V(C)$ is connected (even if ab is removed) and contains a simple a–b path P_{out} internally disjoint from H. Contracting P_{out} to the single edge ab realizes H^+ as a minor of the planar graph B (drawing ab along P_{out} yields a spherical embedding of H^+).
- **3-connectivity and $\delta(H^+) \geq 3$:** Clearly $|V(H^+)| = |V(C)| + 2 \geq 4$. Moreover H^+ is 2-connected: deleting a or b leaves connected C together with the surviving terminal (which attaches to C), and deleting $z \in V(C)$ preserves connectivity since any path in $B - z$ can replace outside excursions through $B - V(C)$ by the edge ab. Suppose for contradiction that H^+ had a 2-vertex-cut $\{x, y\}$. Because $ab \in E(H^+)$, at most one component of $H^+ - \{x, y\}$ intersects $\{a, b\} \setminus \{x, y\}$, so some component C' of $H^+ - \{x, y\}$ satisfies $V(C') \subseteq V(C)$. Because C' has no neighbors in B outside $V(C) \cup \{a, b\}$, no edge of $B - \{x, y\}$ leaves $V(C')$; meanwhile, $B - \{a, b\}$ has at least one other component $C_2 \neq C$, whose vertices lie in $V(B) \setminus (V(C) \cup \{a, b\}) \subseteq V(B) \setminus (V(C') \cup \{x, y\})$. Thus $B - \{x, y\}$ is genuinely disconnected, so $\{x, y\}$ is a 2-vertex-cut of B and C' is a component of $B - \{x, y\}$ avoiding v^*. Furthermore $V(C') \subsetneq V(C)$ (if $V(C') = V(C)$, then $\{x, y\} = \{a, b\}$, impossible since $H^+ - \{a, b\} = C$ is connected). This contradicts minimality of $|V(C)|$. Hence H^+ is 3-connected, which implies $\delta(H^+) \geq 3$.
- **Degree monotonicity and $V_{\geq 4}$-independence on clean edges:** Every $u \in V(C)$ satisfies $d_{H^+}(u) = d_B(u) = d_G(u)$. For $u \in \{a, b\}$, u has at least one neighbor in another component C_2 of $B - \{a, b\}$ (by 2-connectivity of B), so replacing all edges from u to $B - (V(C) \cup \{a, b\})$ (and any original edge ab) by the single cap edge ab yields $d_{H^+}(u) \leq d_B(u) \leq d_G(u)$. Therefore $V_{\geq 4}(H^+) \subseteq V_{\geq 4}(G)$, so by (i) no clean edge $e \in E(H) = E(H^+ - ab) \subseteq E(G)$ joins two vertices of $V_{\geq 4}(H^+)$.

Taking $\mathcal{F}_{\text{frz}}(H^+) = \{f_1, f_2\}$ to be the two faces incident to ab completes the proof. □

3. Main Theorem 1: All Cubic Planar Graphs Contain C_4, C_8, C_{16}, or C_{32}

Suppose for contradiction that G is a finite nonempty simple cubic planar graph containing no simple cycle in $\{C_4, C_8, C_{16}, C_{32}\}$. Apply Lemma 2.2 to obtain a 3-connected cubic plane graph F^+ with at most two frozen faces $\mathcal{F}_{\text{frz}}(F^+) \subseteq \{f_1, f_2\}$ sharing the cap edge ab, such that every cycle avoiding ab avoids $\{C_4, C_8, C_{16}, C_{32}\}$. By Lemma 2.1(iii), any two distinct faces of F^+ share

at most one edge. Assign every face f initial Euler charge

$$\mathrm{ch}_0(f) := 6 - d(f), \qquad \text{so that } \sum_f \mathrm{ch}_0(f) = 12 \quad \text{by (5)}.$$

Clean faces have degrees in $\{3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, \dots, 31, 33, \dots\}$ (no clean 4-, 8-, 16-, or 32-faces). Only clean 3-faces (+3) and clean 5-faces (+1) carry positive initial charge.

3.1. **Boundary-Weld Quotient Soundness.** To certify local structural exclusions on clean patches inside F^+ without risking hidden vertex or edge coincidences, we enumerate planar boundary-weld quotients via `enum_planar_welds` in `cert/verify_planar_eg.py`. For any oriented face-list $\mathcal{P} = (F_1, \dots, F_m)$ (where every face is a simple cycle of length ≥ 3, every shared undirected edge appears with opposite dart orientations in its two incident faces, and every vertex v has corner multiplicity $c(v) \in \{1, 2, 3\}$), let $p(v)$ be the number of internal edges (used by two faces of $\mathcal{P}$) incident to v, and let $e(v)$ be the number of exposed boundary edges (used by one face of $\mathcal{P}$) incident to v. Because each facial corner at v contributes two incident darts,

$$2p(v) + e(v) \;=\; 2c(v), \qquad \text{hence } e(v) \;=\; 2\big(c(v) - p(v)\big) \;\in\; 2\mathbb{Z}_{\geq 0}. \tag{6}$$

Call $\mathcal{P}$ *boundary-regular* if (a) its face-edge contact graph is connected, and (b) every vertex v satisfies $c(v) - p(v) \in \{0, 1\}$ (equivalently $e(v) \in \{0, 2\}$) and the one-way implication $e(v) = 0 \implies c(v) = 3$. (Note that an intermediate state may have $c(v) = 3$ with $e(v) = 2$ prior to boundary zipping; the stronger biconditional $e(v) = 0 \iff c(v) = 3$ holds for every fully realized quotient $\Phi(\mathcal{P}_0)$ in cubic F^+.) Direct instrumentation in `cert/verify_parity_and_dp.py` verifies that all 844/844 production seeds passed to `enum_planar_welds` in `cert/verify_planar_eg.py` satisfy boundary-regularity (722/844 also satisfy the stronger biconditional).

Lemma 3.1 (Soundness and Completeness of Planar Boundary-Weld Enumeration)**.** *Let $\mathcal{P}_0 = (F_1, \dots, F_m)$ be a boundary-regular oriented face-list representing a candidate local configuration of m distinct clean faces of F^+ glued along prescribed shared edges, and let $\Phi : V(\mathcal{P}_0) \to V(F^+)$ be the map induced by an actual embedding of those m faces in 3-connected cubic F^+. Then:*

- *(i) Every vertex identification $\Phi(u) = \Phi(v)$ $(u \neq v)$ forces the identification of two oppositely oriented exposed boundary darts at that vertex. Moreover, every partial boundary-weld quotient $\mathcal{Q}$ subordinate to Φ (obtained from $\mathcal{P}_0$ by welding a subset of boundary-edge pairs identified under Φ) preserves boundary-regularity through every merge wave of `weld_boundary`: specifically, every vertex satisfies $e(v) \in \{0, 2\}$ and $e(v) = 0 \implies c(v) = 3$, and whenever $c(v) = 3$ with $e(v) = 2$, cubicity uniquely forces the two exposed boundary darts at v to be opposite sides of the same edge of F^+.*
- *(ii) The realized image $\Phi(\mathcal{P}_0)$ is connected, satisfies $e(v) = 0 \iff c(v) = 3$ at every vertex, and its exposed boundary edges form a vertex-disjoint union of $h \geq 0$ simple boundary circles bounding h open disks in $S^2 \backslash \Phi(\mathcal{P}_0)$, whence $\Phi(\mathcal{P}_0)$ satisfies the spherical Euler equation $V - E + F = 2 - h$ as well as $|F_i \cap F_j| \in \{0, 2\}$ matching a unique shared edge (Lemma 2.1(iii)).*
- *(iii) Separating* hereditary pruning *(invalid `patch_data`: non-simple face, corner multiplicity > 3, repeated face-pair contact, non-opposite dart orientation, or `weld_boundary` failure) from* emit-only validity filters *(the Euler test $V - E + F = 2 - h$ and exact face-intersection check) guarantees that $\Phi(\mathcal{P}_0)$ is isomorphic to one of the quotients emitted by `enum_planar_welds`$(\mathcal{P}_0)$.*
- *(iv) Conversely, if every emitted quotient W satisfies strict face-schema validity (`WeldSchemaValid`: dense labels $0, \dots, n_W - 1$, all faces simple of length ≥ 3) and contains a simple cycle C of length $L \in \{4, 8, 16, 32\}$ in the edge union of its face boundaries, then the image of C in F^+ is a genuine simple cycle of length L avoiding ab in the edge union of the m clean facial boundaries in $F^+ - ab \subseteq G$.*

Proof. (i) Because F^+ is 3-connected and cubic, Lemma 2.1(iii) implies that two distinct faces of F^+ share a vertex z if and only if they share an edge incident to z. Hence if Φ identifies two distinct vertices of a quotient $\mathcal{Q}$, their incident faces share an edge in F^+ that is not yet welded in $\mathcal{Q}$, so two oppositely oriented exposed boundary edges of $\mathcal{Q}$ map to the same edge of F^+.

We prove by induction on iterations of `weld_boundary` that for any boundary-regular quotient $\mathcal{Q}$ subordinate to Φ and any pair of exposed boundary darts (i,u,v) and (j,w,x) mapping to opposite darts of the same edge of F^+, every merge wave preserves $e(z) \in \{0,2\}$ and $e(z) = 0 \implies c(z) = 3$. In wave 1, identifying (u,v) with (x,w) merges $u \sim x$ and $v \sim w$. Because $\Phi(u) = \Phi(x)$ and $\Phi(v) = \Phi(w)$ are the two distinct endpoints of the edge $\Phi(uv) = \Phi(xw)$ in simple F^+, the two endpoint merge classes $\{u,x\}$ and $\{v,w\}$ have distinct Φ-images and are disjoint. Since u,x each have an exposed dart before the merge, $c(u) - p(u) = c(x) - p(x) = 1$; after merging, the newly paired edge $(u,v) \sim (x,w)$ increases the internal edge count by at least 1, so the merged vertex $z = \{u,x\}$ satisfies

$$c(z) - p(z) \;\le\; \big(c(u) - p(u)\big) + \big(c(x) - p(x)\big) - 1 \;=\; 1 + 1 - 1 \;=\; 1.$$

By (6), $e(z) = 2(c(z) - p(z)) \in \{0,2\}$. Whenever a vertex z has $c(z) = 3$ and $e(z) = 2$, its three facial corners already account for all three faces around $\Phi(z)$ in cubic F^+, so its two exposed boundary edges (z,t_1) and (z,t_2) must be the two opposite darts of the third edge of F^+ at $\Phi(z)$, forcing $\Phi(t_1) = \Phi(t_2)$ (so the zipping pair (t_1,t_2) in `pending` is subordinate to Φ). In any subsequent zipping wave, let K be a connected component of vertices merged together by the pairs in `pending`. Every vertex $y \in K$ is a tip of an exposed dart, so $c(y) - p(y) = 1$, and no $c = 3$ vertex z can belong to K (merging it with another vertex would give $c \ge 4$, impossible under Φ). A spanning tree of K has $|K| - 1$ edges, each emitted by a distinct $c = 3$ vertex $z_r \notin K$ (since each $c = 3$ vertex emits at most one pair (t_1,t_2)), yielding $|K| - 1$ distinct newly paired internal edges (z_r, K) incident to the merged vertex K. Hence

$$c(K) - p(K) \;\le\; \sum_{y \in K} \big(c(y) - p(y)\big) - (|K| - 1) \;=\; |K| - (|K| - 1) \;=\; 1,$$

so $e(K) = 2(c(K) - p(K)) \in \{0,2\}$. Finally, if $e(K) = 0$ then $c(K) = p(K)$, meaning the $c(K)$ facial corners at K form a closed cyclic fan of distinct simple faces around $\Phi(K)$ in cubic F^+; in a simple cubic plane graph, every vertex has degree 3 and lies in exactly 3 faces, forcing $c(K) = 3$. Thus `weld_boundary` never rejects a partial quotient subordinate to Φ.

(ii) Because $\mathcal{P}_0$ is face-edge-connected, the union $U := \bigcup_{i=1}^{m} \overline{\Phi(F_i)} \subset S^2$ is connected. In the fully realized quotient $\Phi(\mathcal{P}_0)$, all three incident edges at any $c(v) = 3$ vertex are already identified under Φ, so $c(v) = 3 \implies e(v) = 0$ as well, giving $e(v) = 0 \iff c(v) = 3$. At every vertex z of $\Phi(\mathcal{P}_0)$, the $c(z) \in \{1,2,3\}$ incident facial wedges of U form a contiguous subset in the cyclic rotation σ_z around the cubic vertex z (every nonempty subset of a 3-element cycle is contiguous); hence U is locally a topological 2-manifold with boundary, and its exposed boundary edges ($e(v) = 2$) form a vertex-disjoint union of $h \ge 0$ simple closed boundary circles, each separating an empty region of $S^2 \setminus U$ from the connected face union U. Because U is connected on S^2, every connected component of $S^2 \setminus U$ is simply connected (an open topological disk) bounded by one of these h simple boundary circles. Capping each of the h boundary circles with a disk face yields a cellular decomposition of S^2 with V vertices, E edges, and $F + h$ faces, so Euler's formula (3) gives $V - E + (F + h) = 2$, i.e., $V - E + F = 2 - h$. Exact face intersection $|F_i \cap F_j| \in \{0,2\}$ holds by Lemma 2.1(iii).

(iii) Every hereditary pruning condition is preserved under subordination to Φ: any partial quotient subordinate to Φ inherits face simplicity, corner multiplicity ≤ 3, at most one shared edge per face pair, opposite dart orientations, and (by (i)) `weld_boundary` closure. Starting from $\mathcal{P}_0$ and repeatedly picking any pair of distinct exposed boundary edges identified under Φ yields a finite chain of valid `weld_boundary` transitions terminating at $\Phi(\mathcal{P}_0)$ (since when no two exposed boundary edges map to the same edge, Lemma 2.1(iii) implies Φ is injective on vertices). Applying

the non-hereditary Euler and exact-contact filters strictly at emission time guarantees that $\Phi(\mathcal{P}_0)$ is reached and emitted.

(iv) By (iii), `enum_planar_welds`$(\mathcal{P}_0)$ emits a representative quotient W_0 isomorphic to the realized patch $\Phi(\mathcal{P}_0)$, whose face-boundary edge union is isomorphic to the edge union of the m clean facial boundaries in $F^+ - ab \subseteq G$. Applying the universal hypothesis over all emitted quotients to W_0 supplies a strict-schema simple cycle C of length $L \in \{4, 8, 16, 32\}$ in the edge union of the face boundaries of W_0, which lifts through the isomorphism $W_0 \cong \Phi(\mathcal{P}_0)$ to a genuine simple cycle of length L in $F^+ - ab \subseteq G$ (additional edges of F^+ between vertices of $\Phi(\mathcal{P}_0)$ only add chords and never affect simplicity of C). □

3.2. **Clean Clusters, 51 Mixed Chains, and Complete Charge-Conserving Discharging Rules.** In F^+, no two clean 3-faces share an edge ($3 + 3 - 2 = 4$), and no two clean 5-faces share an edge ($5 + 5 - 2 = 8$). Moreover, no clean 3-face T_2 can share edges with two distinct clean 5-faces P_1, P_2: those two edges meet at a cubic vertex of T_2, so P_1 and P_2 share that cubic vertex and hence by Lemma 2.1(iii) share an edge, forming a simple C_8 ($5 + 5 - 2 = 8$). Likewise, a clean 5-face cannot touch three clean 3-faces (by pigeonhole on C_5, two of the 3-faces would share a vertex and hence an edge, creating C_4). Consequently, the maximal edge-connected clusters of clean $\{3, 5\}$-faces have size at most 3, are pairwise vertex-disjoint (by Lemma 2.1(iii), if two clusters shared a vertex they would share an edge), and fall into five types:

- Isolated clean 3-faces T ($\mathrm{ch}_0 = +3$) and isolated clean 5-faces P ($\mathrm{ch}_0 = +1$);
- Pairs TP ($\mathrm{ch}_0 = +4$): a clean 3-face T and clean 5-face P sharing an edge uv. Its four outer faces are the two *side faces* S_1, S_2 meeting opposite ends u, v of the shared edge uv (each sharing one edge with T and one edge with P, and meeting each other along an edge at the third vertex of T), and the two *bottom faces* B_1, B_2 on the remaining two edges of P (with B_1 adjacent to S_2 and B_2 adjacent to S_1, meeting each other at the vertex of P opposite uv);
- Triples TPT ($\mathrm{ch}_0 = +7$): a clean 5-face flanked by two non-adjacent clean 3-faces, with one *apex face* A (width 3 on TPT, sharing an edge with both triangles and the pentagon edge between them) and two *flank faces* B, C (each width 2 on TPT, sharing one edge with a triangle and one with the pentagon, and sharing an edge with each other at the fifth pentagon vertex);
- Super-gadgets $S = TPT_1 + F_9 + TPT_2$ ($\mathrm{ch}_0 = 7 + 7 - 3 = +11$): formed whenever a clean 9-face F_9 serves as the A-face of two distinct TPT clusters, with three outer ≥ 17/frozen receiver faces X (width 5, bridging both TPTs across one edge of F_9) and Y, Z (each width 3, meeting the other two free edges of F_9 at a single common cubic vertex w_8 incident only to F_9, Y, Z).

Clean 6-faces carry initial charge 0 and act as relays. Because adjacent $(3, 3)$ (C_4), adjacent $(5, 5)$ (C_8), and clean 6-faces on either side face S_1, S_2 of TP or any of the 7 boundary edges of TPT (C_8) are forbidden, every clean 6-face H_6 touches at most one clean 3-face ($6 + 2 = 8$) and belongs to at most one positive-cluster charge source: *Corannulene pent-hex* 6_c (adjacent to a $(6, 6, 6, 6, 6)$ isolated P), *tri-hex* (6_s sharing a triangle T directly along a receiver boundary, or 6_o transmitting charge from a triangle T across an outer edge of H_6), or *TP-bottom hex* TP_6 (on $B_1 = 6$ of a $TP(S_2 = 9, B_2 = 9)$ cluster with $S_1 \geq 33$ or frozen and no triad triangle on B_1). In particular, if H_6 forms a clean $(3, 6, 5)$-triad (touching both a clean 3-face and a clean 5-face), structural welding shows that the 5-face (P or TP) always sends 0 charge to H_6, so H_6 receives charge exclusively from its unique clean 3-face.

Deterministic Charge-Conserving Discharging Rules (Three Stages). Fix an arbitrary canonical total ordering $\prec$ on $F(F^+)$ to break ties deterministically. Every rule below applies uniformly whether 0, 1, or 2 frozen faces appear in the cluster neighborhood, and every rule preserves total charge identically.

Stage 1A — Isolated clean 3-face T ($\mathrm{ch}_0 = +3$, neighbors X, Y, Z):

(T1) *Two frozen neighbors (f_1, f_2, Z; token T_{frz}):* This occurs only when T contains an endpoint a or b of ab (at most two such triangles). Send $3/2$ to f_1, $3/2$ to f_2, and 0 to Z. (Total $3/2 + 3/2 = 3$.)

(T2) *Three clean 6-faces $(6,6,6)$ (Mini-Corannulene):* Send 1 to each of the three 6-faces (routed in Stage 3 across its three valleys and flanking opposite faces as below). (Total $1+1+1 = 3$.)

(T3) *Two clean 6-faces $(6,6,X)$ (token T_2 on X):* Clean X has $d(X) \in \{9, 10, 11, 14\} \cup [17, \infty)$ (since $3 + 6 + d - 6 \in \{4, 8, 16\}$ or $6 + d - 2 = 16$ forbids $7, 12, 13, 15$), or $X \in \mathcal{F}_{\mathrm{frz}}$. Send $(1, 1, 1)$ to $(6, 6, X)$ if X is clean 9; send $(0, 0, 3)$ if X is clean 11; send $(1/2, 1/2, 2)$ if X is clean $10, 14, \geq 17$ or frozen. (Total 3.)

(T4) *One clean 6-face $(6, X, Y)$ (token T_1 on X, Y):* Clean X, Y lie in $\{9, 10, 11, 14\} \cup [17, \infty)$ or $\mathcal{F}_{\mathrm{frz}}$; neither $(9, 9)$ (C_{16}) nor $(9, 10)$ ($3 + 9 + 10 - 6 = 16$) occurs. Define $w_1(d) = 3/4$ ($d = 9$), 1 ($d = 10$), $3/2$ ($d \in \{11, 14\} \cup [17, \infty)$ or frozen). If at least one of X, Y is clean 11 (say $X = 11$, choosing the $\prec$-first if both are 11), send 0 to 6, $w_1(Y)$ to Y, and $3 - w_1(Y) \in \{9/4, 2, 3/2\} \leq 9/4$ to X. Otherwise send $w_1(X)$ to X, $w_1(Y)$ to Y, and $3 - w_1(X) - w_1(Y) \in \{0, 1/2, 3/4, 1\}$ to 6 (nonnegative since $w_1(X) + w_1(Y) \leq 3$, and ≤ 1 since $(9, 9)$ and $(9, 10)$ are forbidden). (Total 3.)

(T5) *Zero clean 6-faces (X, Y, Z) (tokens $T, T_{10}, T_{L9}, T_{R9}$):* Each neighbor is frozen or clean in $\{9, 10, 11, 12, 13, 14\} \cup [17, \infty)$ (no $7, 15$). At most one is clean 9 ($9 + 9 - 2 = 16$), and if one is clean 9 neither other is clean 10 ($3 + 9 + 10 - 6 = 16$). If one neighbor is clean 9, send $3/4$ to it and $9/8$ to each of the other two ($3/4 + 9/8 + 9/8 = 3$); otherwise send 1 to each of X, Y, Z ($1 + 1 + 1 = 3$; note $1 \leq 9/8$, and when adjacent to a 5-triangle clean 10-face R_{10} the other two neighbors record T_{10} of weight 1).

Stage 1B — Isolated clean 5-face P ($\mathrm{ch}_0 = +1$):

(P1) *Corannulene $(6, 6, 6, 6, 6)$:* Send $1/5$ to each of its 5 adjacent clean 6-faces (6_c). (Total $5 \cdot 1/5 = 1$.)

(P2) *Non-Corannulene P:* If P touches at least one frozen face or clean face X with $d(X) \in \{10\} \cup [12, \infty)$, send $+1$ to the $\prec$-first such face X and 0 to all other neighbors. Otherwise all 5 neighbors of P are clean in $\{6, 7, 9, 11\}$: because $(6, 6, 6, 6, 7)$ (0/9 welds), $(6, 6, 6, 6, 9)$ (0/18 welds), $(6, 6, 6, 6, 11)$ (0/31 welds), and every clean $\{6, 7, 9, 11\}$-ring containing ≤ 1 neighbor in $\{9, 11\}$ (with a 7-face or triad 6-face) die at $\{C_4, C_8, C_{16}, C_{32}\}$ (by cubic boundary welding, Lemma 3.1), P must touch at least two clean $\{9, 11\}$-faces; send $1/2$ to each of the $\prec$-first two clean $\{9, 11\}$-neighbors and 0 to all others. (Total 1; every clean 6-face and clean 7-face receives 0, and frozen neighbors always take the first branch directly without invoking clean cycle exclusions.)

Stage 1C — Cluster TP ($\mathrm{ch}_0 = +4$, side faces S_1, S_2, bottom faces B_1, B_2): Each side face S_i shares an edge with T and an edge with P, so S_i is frozen or clean ≥ 9 ($3, 4, 5, 6, 7, 8$ are excluded by maximality, C_4, and C_8 since $3+6-2 = 7$ and $3+5+6-6 = 8$ forbid $S_i = 6$, and $3+7-2 = 8$ forbids $S_i = 7$), and at most one of S_1, S_2 is clean 9 ($9 + 9 - 2 = 16$). Similarly, at most one of B_1, B_2 is clean 9.

(TP1) *Neither S_1 nor S_2 is clean 9:* Send $(2, 2, 0, 0)$ to (S_1, S_2, B_1, B_2) (regardless of B_1, B_2, including when either is $6, 7$, or frozen). (Total $2 + 2 = 4$.)

(TP2) *One side face is clean 9 (say $S_2 = 9$, so S_1 is frozen or clean ≥ 11 by $3 + 9 + 10 - 6 = 16$):* Send $3/4$ to $S_2 = 9$ and distribute the remaining $13/4$ as follows:

- If neither B_1 nor B_2 is clean 6 or 7: send 2 to S_1; if one bottom is clean 9 and the other is ≥ 10/frozen, send $(1/4, 1)$ to them; if both bottoms are ≥ 10/frozen, send 1 to the $\prec$-first and $1/4$ to the other. (Total $3/4 + 2 + 1 + 1/4 = 4$.)
- If one bottom is clean 7: the other bottom B' is frozen or clean ≥ 14 (since B' shares a vertex with $B = 7$ and $P = 5$, clean $6, 7, 9, 10, 11, 12, 13$ yield $5 + 7 + d - 6 \in \{16\}$ or $C_4/C_8/C_{16}/C_{32}$ welds, while a frozen B' satisfies this branch directly); send 2 to

S_1, 0 to the 7-face, and 5/4 to B' (token TP_{b54}, or fused length-3 token HH of weight 5/2 when two TP_{b54} clusters share the same separating 7-face edge along B'). (Total $3/4 + 2 + 5/4 = 4$.)

- If one bottom is clean 6: by the 16 $S_2 = 9, B_2 = 6$ seal welds (Proposition 3.2(iii)), only B_1 (adjacent to $S_2 = 9$) can be clean 6, and B_2 is either clean 9 or frozen/clean ≥ 14 (clean $6, 7, 10, 11, 12, 13$ yield C_{16}/C_{32} perimeters with $B_1 = 6, P = 5, S_2 = 9$, while frozen B_2 takes the first subbranch directly). If B_2 is frozen or clean ≥ 14, send 2 to S_1, 0 to $B_1 = 6$, and 5/4 to B_2 (TP_{b54}; total 4). If $B_2 = 9$, then $TP(B_1 = 6, S_2 = 9, B_2 = 9)$ has unique surviving weld shift spectrum $\texttt{sh_TP699} = (0..3, 5..14)$ on S_1 (out of 21 welds), eliminating clean $S_1 \in \{18..20, 22, 24..27, 29..31\}$ (and $11, 13, 14, 15$ by C_{16}), leaving $S_1 \in \{12, 17, 21, 23, 28\} \cup [33, \infty) \cup \mathcal{F}_{\text{frz}}$: if $S_1 \in \{12, 17, 21, 23, 28\}$, send 3 to S_1 ($TP_s = 3$ at $d = 12$; width-4 $TP_{s3} = 3$ at $d \in \{17, 21, 23, 28\}$) and 0 to B_1; if $S_1 \geq 33$ or frozen and $B_1 = 6$ touches a triad triangle on its $(8, 9)$ edge, send 3 to S_1 (width-4 $TP_{s3,t89}$) and 0 to B_1; otherwise send 7/3 to S_1 (width-2 TP_{s73}) and 2/3 to $B_1 = 6$ (relayed via TP_6 in Stage 3). (Total $3/4 + 1/4 + 7/3 + 2/3 = 4$.)

Stage 1D — Cluster TPT ($\text{ch}_0 = +7$) and Super-Gadget S ($\text{ch}_0 = +11$): Every outer face A, B, C of TPT shares edges with a clean triangle and clean pentagon, so each is frozen or clean ≥ 9 ($3, 4, 5, 6, 7, 8$ excluded by maximality, C_4, C_8), and at most one is clean 9 ($9 + 9 - 2 = 16$). Whenever a flank or apex face is frozen, it automatically satisfies every $\geq 10/\mathcal{F}_{\text{frz}}$, $\geq 11/\mathcal{F}_{\text{frz}}$, or $\geq 17/\mathcal{F}_{\text{frz}}$ condition below without requiring any clean cycle witness on that face:

(TPT1) *Super-gadget $S = TPT_1 + F_9 + TPT_2$ (+11):* Its three outer faces X, Y, Z are clean ≥ 17 or frozen (clean 9..15 on X, Y, Z produce C_{16} or C_{32} with the 7-face cluster S whose shift spectra are $\texttt{sh_X}, \texttt{sh_Y}$). Send 9/2 to X (S_X, width 5) and 13/4 to each of Y, Z (S_{YZ}, width 3). (Total $9/2 + 13/4 + 13/4 = 11$.)

(TPT2) *Standard TPT (+7, not in S):*

- $A = 9$ ($B, C \geq 17$ or frozen, since $TPT + F_9$ has shift spectrum $\texttt{sh_A9}$ killing clean 10..15): send 3/2 to A and 11/4 each to B, C (TPT_{A9}, width 3). (Total 7.)
- $A = 10$ ($B, C \geq 17$ or frozen, since $TPT + F_{10}$ has shift spectrum $\texttt{sh_A10} = (0..9)$ killing clean 9..15): send 2 to A and 5/2 each to B, C (TPT_{A10}, width 3). (Total 7.)
- $B = 9$ w.l.o.g. ($A \geq 17$/frozen and $C \in \{11\} \cup [17, \infty) \cup \mathcal{F}_{\text{frz}}$, since $TPT + F_{B9}$ has shift spectra $\texttt{sh_B9_A}, \texttt{sh_B9_C}$ killing all other clean degrees ≤ 15): send 3/2 to B, 23/8 to A (TPT_{B9A}, width 4), and 21/8 to C (TPT_{B9C}, width 3; or $TPT_{BC}(11) = 21/8$ at $d = 11$). (Total $3/2 + 23/8 + 21/8 = 7$.)
- $B = 10$ w.l.o.g. ($A \in \{12\} \cup [17, \infty) \cup \mathcal{F}_{\text{frz}}$ by TPT shift (0..3) plus $B = 10$ detours, $C \geq 10$/frozen): send 12/5 to A, 23/10 to B, and 23/10 to C (note $23/10 \leq w(TPT_{BC})$ for all $d \geq 10$ and frozen). (Total $12/5 + 23/10 + 23/10 = 7$.)
- All of A, B, C clean ≥ 11 or frozen: send 7/3 to each of A, B, C (note $7/3 \leq w(TPT_A), w(TPT_{BC})$ for all $d \geq 11$ and frozen). (Total 7.)

Stage 2 — Clean 10-Face 5-Triangle Relay R_{10}: If a clean 10-face F_{10} ($\text{ch}_0 = -4$) is adjacent to 5 clean 3-faces, those 5 triangles are non-adjacent (C_4) and therefore alternate around ∂F_{10}. For each of the 5 non-triangle gap neighbors Q of F_{10}: clean $d(Q) \in \{3, 4, 8, 16, 32\}$ are excluded by C_4 and forbidden lengths; clean $d(Q) \in \{5, 6, 7, 9, 10\}$ are eliminated by the 5 exact R_{10} seal welds (Proposition 3.2(iii), matching the 10-face plus 5 triangles plus Q sharing its 3-edge rim with F_{10} and its two flanking triangles); and clean $d(Q) = 15$ is excluded because Q shares an edge with both flanking triangles ($3 + 15 - 2 = 16$). Hence all 5 gap neighbors Q are frozen or clean with $d(Q) \geq 11$ ($d \neq 15$). Because every vertex of F_{10} is cubic, each gap neighbor Q is simultaneously a neighbor of both flanking triangles, so the no-6/no-9 exclusion on the five gap neighbors directly establishes that none of the 5 triangles touches a clean 6-face or clean 9-face; hence each triangle lies in Stage 1A(T5) and sends +1 to F_{10}. This fills F_{10}'s initial deficit -4 to a net relay surplus of

$-4+5=+1$. In Stage 2, F_{10} relays this $+1$ by sending $1/5$ across each of its 5 non-triangle edges (token R_{10} on each neighbor), leaving $\mathrm{ch}_{\mathrm{fin}}(F_{10}) = -4+5-1=0$. (In the clean $d=10$ cyclic DP at `verify_planar_eg.py:1373`, this exact -1 out-transfer is the subtraction `- W_LCM` when `d == 10 and cnt == 5`.)

Stage 3 — Charged Clean 6-Faces (H_6, $\mathrm{ch}_0 = 0$): Every clean 6-face H_6 receives charge from at most one Stage-1 source:

(H1) *Corannulene 6_c ($+1/5$):* Sends $1/10$ across each of its two valley edges, so each valley face ($d \in \{11\} \cup [13,\infty)$ or frozen) receives $1/10+1/10=1/5$ across its consecutive $6_c\,6_c$ pair. ($\mathrm{ch}_{\mathrm{fin}}(H_6)=0$.)

(H2) *Charged TP-bottom hex TP_6 ($+2/3$):* Its two outer faces F_{67}, F_{89} lie in $\{10,11,13,14\} \cup [17,\infty) \cup \mathcal{F}_{\mathrm{frz}}$ with $(F_{67},F_{89}) \notin \{(10,10),(10,11),(11,10)\}$ (and $F_{78} \notin \{5,9\}$). If one of F_{67}, F_{89} is clean 10, send 0 to it and $2/3$ to the other (≥ 13 or frozen); otherwise send $1/3$ to each (bounded by TP_6 weight $1/3$ at $d=11$ and $2/3$ at $d \in \{13,14\} \cup [17,\infty) \cup \mathcal{F}_{\mathrm{frz}}$). ($\mathrm{ch}_{\mathrm{fin}}(H_6)=0$.)

(H3) *Charged tri-hex H_6 ($c(H_6) \in \{3/8, 1/2, 3/4, 1\}$):* For Mini-Corannulene $(6,6,6)$, at least two valleys are clean ≥ 13 or frozen (each taking $1/2+1/2=1$ from its two flanking 6-faces); if the third valley is ≥ 13/frozen it takes $1/2+1/2$, else it is 6 or 7 and both flanking opposite faces are ≥ 10/frozen, each taking $1/2$ from its incident 6-face (total 3, every outer edge $\leq 1/2$). For every other charged tri-hex H_6 ($c(H_6) \leq 1$) with outer neighbors (V_0, O, V_1), assign outer-edge receiver capacities $\mathrm{cap}(\mathcal{F}_{\mathrm{frz}})=1$, $\mathrm{cap}(6)=0$, $\mathrm{cap}(7)=1/4$ (or $1/2$ in the isolated `Triad-D1`($X=10, Z_1=7, Z_2=6$) patch with base faces $T=(0,1,2)$, $H_6=(2,1,3,4,5,6)$, $P=(4,3,10,11,12)$, $X_{10}=(10,3,1,0,20..25)$, $Z_1=(5,4,12,30,31,32,33)$, $Z_2=(6,5,33,40,41,42)$ [1/50 surviving base weld], where attaching a clean 6-face on each of the 4 remaining free edges $(12,30)$ [sharing $(11,12)$ with P], $(30,31)$, $(31,32)$, and $(32,33)$ [sharing $(33,40)$ with Z_2] of $Z_1=7$ is eliminated across $106+142+140+73=461$ boundary welds with 0 survivors [verified in `cert/verify_parity_and_dp.py`], so $Z_1=7$ receives only that single $1/2$ and ends with $\mathrm{ch}_{\mathrm{fin}}(Z_1) = -1/2 < 0$), $\mathrm{cap}(9)=3/4$ (except 0 if $V_{6,6}$ valley of $(3;6,6,X)$), $\mathrm{cap}(10)=2/3$, $\mathrm{cap}(11)=\mathrm{cap}(12)=0$, and $\mathrm{cap}(\geq 13)=1/2$. Notice that any frozen outer neighbor alone has $\mathrm{cap}(\mathcal{F}_{\mathrm{frz}}) = 1 \geq c(H_6)$; in all-clean neighborhoods (including $(3,6,5)$-triads `Triad-D1`, `Triad-Opp`, `Triad-TPP`), clean boundary-weld exclusions eliminate every low-capacity triple (V_0,O,V_1) with $\mathrm{cap}(V_0)+\mathrm{cap}(O)+\mathrm{cap}(V_1) < c(H_6)$ (in particular, when $c(H_6)=1$, at least two outer faces have capacity $\geq 1/2$, or one has $3/4$ and another $\geq 1/4$, or $Z_1=7$ takes $1/2$ alongside $O=10$ or ≥ 13). Allocate $c(H_6)$ greedily across (V_0,O,V_1) in $\prec$-order up to their capacities ($\mathrm{ch}_{\mathrm{fin}}(H_6)=0$). Finally, for clean $d=7$: every non-triad-D1 clean 7-face receives 0 from all $\{3,5\}$-clusters and at most $1/4$ per adjacent charged 6-face (6_o); because 3 consecutive clean 6-faces on a clean 7-face with any charging triangle placement die across all 7 cubic boundary-weld seeds (0/2..0/5 survivors), at most 4 edges of F_7 carry a charged 6-face (by pigeonhole on C_7, any 5 of 7 edges contain 3 consecutive edges), giving $\mathrm{ch}_{\mathrm{fin}}(F_7) \leq -1+4\cdot\frac{1}{4}=0$ directly (while the $d=7$ DP also checks all mixed chains containing 6_o).

By inspection of (T1)–(T5), (P1)–(P2), (TP1)–(TP2), (TPT1)–(TPT2), Stage 2, and (H1)–(H3), every sender's net outgoing charge equals its Stage-1/2/3 balance (yielding $\mathrm{ch}_{\mathrm{fin}}(f)=0$ for all clean 3-, 5-, 6-faces, and R_{10} relays), and every receiving clean or frozen face records at most the exact rational token weights summarized in Table 1.

Proposition 3.2 (51 Deduplicated 5/6-Contact Chains and 520 Seal Welds)**.** *When maximal consecutive runs of clean 5-face contacts and charged clean 6-face contacts lie along a clean or frozen receiver boundary F_d:*

(i) ***Alphabet Exhaustiveness and Valley Deduplication:*** *Every single-edge clean 5-face contact on ∂F_d is an isolated P or a TP bottom (B_L or B_R, according to whether the triangle of*

TABLE 1. Exact cubic discharging token weights (in exact rationals) received by a clean d-face or frozen face ($d \geq 3$), matching `verify_planar_eg.py:942--1029, 1403--1430, 1494--1535`. A dash (—) indicates the token is absent at that degree; 0 indicates a present zero-weight token.

Token	Width	$d=7$	$d=9$	$d=10$	$d=11$	$d=12$	$d=13$	$d=14$	Clean $d \geq 17$	Frozen
T	1	0	3/4	1	9/8	9/8	9/8	9/8	9/8	9/8
T_{10}	1	—	—	—	1	1	1	1	1	1
P	1	0	1/2	1	1/2	1	1	1	1	1
TP_b	1	0	1/4	1	1	1	1	1	1	1
TP_s (std.)	2	0	3/4	2	2	3	2	2	2	2
TP_{b54} / HH	2/3	—	—	—	—	—	—	5/4, 5/2	5/4, 5/2	5/4, 5/2
TPT_A / TPT_{BC} (std.)	3/2	0, 0	3/2, 3/2	2, 23/10	7/3, 21/8	12/5, 7/3	12/5, 7/3	12/5, 7/3	12/5, 7/3	12/5, 7/3
T_1 / T_2	2/3	—	3/4, 1	1, 2	9/4, 3	—	—	3/2, 2	3/2, 2	3/2, 2
6_o	1	1/4 (≤ 4; sole 1/2)	3/4	2/3	—	—	1/2	1/2	1/2	1
$TP_{6,L/R}$ / $TP_{6,2}$	2/3	—	—	—	1/3, 2/3	—	2/3, 4/3	2/3, 4/3	2/3, 4/3	2/3, 4/3
R_{10}	1	—	—	—	1/5	1/5	1/5	1/5	1/5	1/5
6_c (+1/5 on 6_c6_c)	1	—	—	—	0	—	0	0	0	0
$T_{\mathrm{frz},L/R}$ / `CAP`	1/0	—	—	—	—	—	—	—	—	3/2, 0

Notes: (1) At clean $d = 15$, $T, T_{10}, R_{10}, T_1, T_2, TP_6$ are absent ($3 + 15 - 2 = 16$, $6 + 15 - 2 + 3$-detours); $P, TP_b, TP_s, TP_{b54}, HH, TPT_A, TPT_{BC}$ retain their $d = 14$ weights, 6_o has weight 1/2, and 6_c (+1/5 pair bonus) is present. (2) Special variants: TP_{s3} (width 4, weight 3) at $d \in \{17, 21, 23, 28\}$; $TP_{s3,t89}$ (width 4, weight 3), TP_{s73} (width 2, weight 7/3), and $TP_{b54,t89}$ (width 3, weight 19/8) in the clean $d \geq 33$ and frozen models; super-gadget and auxiliary-TPT tokens at $d \geq 17$/frozen: TPT_{A9} (w3, 11/4), TPT_{A10} (w3, 5/2), TPT_{B9A} (w4, 23/8), TPT_{B9C} (w3, 21/8), S_{YZ} (w3, 13/4), S_X (w5, 9/2). In the frozen model, TPT_{BC} splits into $TPT_{BC,L}$ and $TPT_{BC,R}$.

TP lies on the left or right); every 2-edge $(3,5)$-contact is TP_{sR} or TP_{sL}; every charged tri-hex contact is H_L, H_O, H_R (triangle on left, opposite, or right edge of H_6) or part of T_{1R}, T_{1L}, T_2 (a triangle T on ∂F_d flanked by one or two H_0 hexagons). Other cluster types never enter a 5/6-contact run: TPT and S forbid clean 6-faces on all 7 boundary edges (C_8); TP_6 never touches T or P (Stage 1C(TP2)); Corannulene 6_c is tracked separately; and TP_{b54} has the identical underlying (P,T) face-patch as B_L/B_R. Whenever two consecutive macros (m_k, m_{k+1}) with $m_k \in \{B_R, H_R\}$ and $m_{k+1} \in \{B_L, H_L\}$ sit on adjacent edges of F_d, their exterior wing triangles share both the cubic spoke vertex v_{k+1} between them and the two adjacent outer vertices of m_k and m_{k+1} (so by cubicity and Lemma 2.1(iii) they are physically the same valley triangle). Indexing wing triangles by their unordered vertex triple `frozenset((x,y,z))` in `build_cubic_chain_dedup` deduplicates this shared valley triangle cleanly (and cannot merge across run endpoints, whose exterior faces lie outside the 5/6-run), yielding the exact 4-face patches $(\mathrm{HR}, \mathrm{HL}, P)$ and $(P, \mathrm{HR}, \mathrm{HL})$ with common surviving-weld shift spectrum $\{0, 3, 4, 5, 6, 7, 8\}$.

(ii) ***Termination at Length 5 and Absence of Cyclic Wraparound (Including Short Cyclic Runs):*** *Breadth-first expansion of valid linear 5/6-macro sequences terminates at macro length 5 with zero surviving length-6 mixed chains (across 1,581 total BFS welds tested, with all $1{,}327 = 37 + 406 + 535 + 264 + 85$ pruned length-2..6 welds, including all 85 length-6 welds, kernel-certified to contain simple C_4, C_8, C_{16}, or C_{32}), yielding exactly 51 abstract mixed chains (oriented layer counts $24, 30, 18, 9$ at lengths $2, 3, 4, 5$). On a frozen face f_i, the cap edge ab breaks ∂f_i into a linear path of non-cap width $d - 1$, so no cyclic wraparound exists and every maximal 5/6-run has macro length ≤ 5. On a clean receiver F_d ($d \geq 7$), any pure-5/6 cyclic wraparound of k macros around ∂F_d must obey `RTYPE`$[m_i] \to$ `LTYPE`$[m_{i+1}] \in \{5 \to 6,\ 6 \to 5,\ 6 \to 6\}$ at every step, immediately excluding every macro with `LTYPE` $=$ `'T'` or `RTYPE` $=$ `'T'` ($T_{1R}, T_{1L}, TP_{sR}, TP_{sL}$) and restricting the alphabet to the 7 macros $\{P, B_L, B_R, H_L, H_O, H_R, T_2\}$ (six of width 1 and only T_2 of width 3). If $k \geq 6$, any 6-macro subword is eliminated by the length-6 BFS bound; if $k \in \{2, 3, 4, 5\}$ with total width ≥ 7, exhaustive check over all cyclic sequences from those 7 macros confirms zero survive all linear*

rotations in the BFS. Thus no cyclic wraparound pure-5/6 chain can occur on any clean or frozen face.

(iii) 520 ***Certified Seal Welds:*** *Exhaustive boundary-weld enumeration (Lemma 3.1) eliminates all* $520 = 16 + 3 + 216 + 5 + 258 + 22$ *seal patches:*

- 16 *(*$S_2 = 9, B_2 = 6$*): rules out* $B_2 = 6$ *opposite* $S_2 = 9$ *in Stage 1C(TP2);*
- 3 *(*TP_{b54} *+* $B_1 = 6$ *+ clean 6-face on* S_1*-side): shows* B_2 *cannot receive a second adjacent 6-contact on the* S_1*-flank;*
- 216 *(*TP_{b54} *outer faces of* $B_1 = 6$ *with triangle on* $(6, 7)$ *or* $(7, 8)$*): proves* F_{89} *absorbs* $\geq 1/2$ *from the triad triangle, leaving* $\leq 1/4$ *for* B_2*, so* $TP_{b54} + H \leq 5/4 + 1/4 = 3/2 = B + H$ *on every clean* $d \geq 14$ *(and* $\leq 7/4 < 2 = B + H$ *on frozen faces);*
- 5 *(*R_{10} *with non-triangle neighbor in* $\{5, 6, 7, 9, 10\}$*): proves every non-triangle neighbor of* R_{10} *is frozen or clean* ≥ 11 *(*$d \neq 15$*);*
- 258 *(5* `after_A9` *spoke exclusions;* $TP_{b54}+t_{89}$ *outer faces proving* $F_{78} \in \{11, 14\} \cup [17, \infty) \cup \mathcal{F}_{\text{frz}}$ *and* $F_{67} \geq 10$ *so* $TP_{b54}+T_{1L} \leq 5/4+9/8 = 19/8 < 5/2 = B+T_{1L}$*, encoded as* $TP_{b54,t89}$*; and* $TP_{b54} + H + H$ *kills proving* TP_{b54} *cannot occur inside* $C_{B,H,H}$ *or* $C_{H,H,B}$*);*
- 22 *(*`a9_tpsr` *proving* $TP_s \leq 2$ *on an* A_9 *spoke, plus* THT *and* TT_2 *killing* $B_1 = 6$ *on* $C_{TP_{sR},H,TP_{sL}}$ *and* C_{TP_{sR},T_2} *so they have* 0 *free off-*F_d *pentagon edges).*

3.3. **Graph-to-Token Mapping, Transition/Flag Derivations, Conservative Enumeration, and Closure.** Around any clean d-face F_d ($d \in \{7, 9..15, 17..\}$) of F^+, reading its d boundary edges in cyclic order, grouping every multi-edge cluster contact (TP_s, TP_{b54}, HH, TPT, S, T_1, T_2, TP_6, TP_{s3}, TP_{s73}) into its corresponding token, and grouping every maximal consecutive 5/6-contact run into its unique chain token from Proposition 3.2 produces a cyclic word $W = (\tau_0, \tau_1, \ldots, \tau_{r-1})$ of total width $\sum_i \text{width}(\tau_i) = d$ whose total token weight upper-bounds the Stage-1–3 incoming charge to F_d. Every consecutive pair (τ_i, τ_{i+1}) obeys the transition predicate `pair_ok` (`verify_planar_eg.py:1053--1094, 1441--1480`), whose exact clauses reflect the local clean-patch geometry of 3-connected cubic F^+ (all transitions depend only on the clean cluster faces on the two adjacent edges, so they remain valid unchanged along the non-cap boundary of any frozen face f_i):

(a) *Maximality of* 5/6*-chains and cluster contacts:* No chain token is adjacent to 6_o, 6_s, or another chain (else they merge into one maximal run); no token ending in a $\{3, 5\}$-face (`ends_g`) precedes one starting in a $\{3, 5\}$-face (`starts_g`) (else they share an edge and belong to one cluster); no $6_s, 6_o, TP_6$ edge touches a $\{3, 5\}$-edge outside a chain or T_1/T_2 token.

(b) *Plain gadget-side* N*-edges:* For every token in `g_left_N` ($S_{YZ,R}$, S_X, $TPT_{A9,R}$, $TPT_{A10,R}$, $TPT_{B9A,R}$, $TPT_{B9C,R}$), as well as $TP_{s73,R}$ and the right side of $S_{YZ,R}$ (and their left-right mirrors), the face across the gadget-side edge is a clean ≥ 11/frozen face (or plain 9-face for TP_{s73}, or clean 9-face F_9 for the right side of $S_{YZ,R}$) that sends 0 charge across that edge and is therefore encoded as N.

(c) *Certified spoke,* T_{L9}/T_{R9}*,* R_{10}*,* HH*,* T_2*,* 6_c*, and* $d \in \{9, 10\}$ *adjacencies:*

- *Spoke lists (*`lines 1053--1057, 1073--1082`*):* The spoke-side successor sets are `after_A9` $= \{N, T_{L9}, P, TP_{s,R}, TP_b, TP_{b54,R}, TPT_{A10,L}\}$ (justified by the 5 `after_A9` seals and `a9_tpsr` in Proposition 3.2(iii), ruling out $6_o, 6_s, P + H, TP_{s,L}, TP_{s,R} + H, T_{1R}$ on the A_9 spoke), `after_A10` $= \{N, T, T_{10}, T_{1,R}, TPT_{A9,L}, TPT_{A10,L}, TPT_{B9A,L}, TPT_{B9C,L}, TP_{6,L}\} \cup$ {chains starting with T_{1R}} (ruling out 5-faces and 6_o touching $A = 10$ at the TPT spoke, while permitting triangle-starting tokens), and `after_B9` $= \{N, T_{L9}, TPT_{A10,L}\}$ (since the $B = 9$ flank spoke forbids 5-faces, 6-faces, and 9-faces by $9+9-2 = 16$), with exact left-right mirrors `before_A9`, `before_A10`, `before_B9`. The oriented marker T_{L9} denotes an isolated T sharing a spoke with a 9-face on its left, so T_{L9} may only immediately follow $\{TPT_{A9,R}, TPT_{B9A,R}, TPT_{B9C,R}\}$ and cannot precede $\{TPT_{A9,L}, TPT_{B9A,L}, TPT_{B9C,L}, TP_{6,L}\}$ ($9 + 9 - 2 = 16$ across T_{L9}), with exact mirror for T_{R9}.

- *Other certified adjacencies:* R_{10} is flanked strictly by T_{10} on both sides; HH is flanked by N (or TP_6 facing away, since the 7-face edge of HH forbids adjacent $\{3,5,6\}$-tokens and TP_6 facing in); $TP_{b54,L}$ and T_{10} cannot precede $TP_{6,L}$ (and mirror); T_2 cannot follow T_2 (C_{16} perimeter); 6_c can only touch P or P-ending chains among $\{3,5\}$-tokens (Corannulene forbids adjacent T or TP); at $d=9$, 6_o cannot touch 6_s or 6_o (since $V_{6,6}=9$ gets 0 charge); and at $d=10$, `ends_6s` cannot precede `starts_6s` $\cup$ `starts_6o` $\cup\{TP_{6,R}, TP_{6,2}\}$ (and mirror `ends_6s` $\cup$ `ends_6o` $\cup\{TP_{6,L}, TP_{6,2}\}$ before `starts_6s`), and `ends_6o` cannot precede `starts_6o` ($10+6+6-6=16$).

(d) *Global small-degree flags, `n6`, `usyz`, and `vcor_w` (`verify_planar_eg.py:1246--1374`):* Define token predicates `is_T` (matching T, T_{10}, $T_{1,L}$, $T_{1,R}$, T_2, and chains containing $T_{1R}/T_{1L}/T_2$), `is_6s` ($T_{1,L}$, $T_{1,R}$, T_2, and chains containing them), `is_6o` (6_o and chains containing H), `is_TP6` ($TP_{6,L}$, $TP_{6,R}$, $TP_{6,2}$), `is_TPT_A`, `is_TPT_any` (TPT_A, TPT_{BC}), `is_unfusedP` (P, $TP_{s,L/R}$, TP_b, TPT_A, TPT_{BC}), `is_TPT_or_TPs` (TPT, TP_s, and chains containing $TP_{sR/L}$), `is_triadP` (any chain token), and `is_d11_f6s` ($TP_{s,L/R}$, TP_b, TPT_A, TPT_{BC}, $TP_{6,L/R/2}$).

- *Degree* 9 *(`cnt`, `hP`, `hTri`, and `n6`):* `cnt` counts TPT_A (≤ 1, since two TPT_As form a super-gadget S); `hP` tracks `is_TPT_any`; `hTri` tracks `is_triadP`; and `n6` accumulates (saturated at 2) `add6`, defined as $+1$ for each standalone $6_o, T_{1,L}, T_{1,R}$, or T_2 token and $+1$ for each constituent macro in $\{H, T_{1R}, T_{1L}, T_2\}$ inside a chain token (`lines 1340--1341, 1361--1363`). Physically, T_2 places *two* distinct clean 6-faces on ∂F_9 whereas `add6` conservatively counts T_2 as $+1$; because distinct tokens and chain macros occupy disjoint edges of F_9 and each clean 6-face shares at most one edge with F_9 (Lemma 2.1(iii)), `n6` is a conservative lower bound on the number of distinct charged clean 6-faces on F_9, so `n6` ≥ 2 guarantees at least two distinct clean 6-faces on F_9. Both closing exclusions at `line 1370` are certified by boundary welding (Lemma 3.1, replayed in `cert/verify_parity_and_dp.py`): (i) for `(h6o or h6s) and hP`, neither TPT_A (0/1 on each of the 4 non-adjacent edges of F_9) nor TPT_{BC} (0/1 on each of the 5 non-adjacent edges; 9 welds total, 0 survivors) can coexist with any clean 6-face on F_9 (adjacent edges violate C_8); (ii) for `hTri and n6 >= 2`, every one of the 10 surviving $d=9$ chains (those avoiding shift 7) contains an adjacent clean (5-face, 6-face) pair (P, H_1) along ∂F_9, and attaching any second distinct clean 6-face H_2 on any of the 7 remaining edges of F_9 dies across all $78 = 3+5+5\cdot 14$ boundary welds (0 survivors).
- *Degrees* 10 *and* 11*:* At $d=10$, `cnt` counts `is_T`, `hP` tracks `is_unfusedP` (forbidden alongside `h6s`), `hTPT` tracks `is_TPT_or_TPs` (forbidden alongside `h6o` or `hTP6`), `h6o` forces `cnt` ≤ 1 ($10+6-2+2=16$), `h6s` forces `cnt` ≤ 2, and `cnt` $=5$ triggers the Stage-2 R_{10} out-transfer -1 (`- W_LCM`). At $d=11$, `cnt` counts `is_T`, `hP` tracks `is_d11_f6s`, `h6s` forces `cnt` ≤ 1 ($11+6-2+1=16$) and `hP` $=0$, and `hTP6` forces `cnt` $=0$.
- *Super-gadget bit `usyz` and Corannulene pair bonus `vcor_w`:* In all DPs and conservative enumerations (`lines 1246--1309, 1330--1374`), the state bit `usyz` $\in\{0,1\}$ implements Lemma 3.3(ii): the first S_{YZ} token (`usyz == 0`) uses the full super-gadget shift mask `sh_Y` (routing through w_8) and sets `usyz` $\leftarrow 1$, whereas every subsequent S_{YZ} token (`usyz == 1`) falls back to pure-cluster shifts $(0,1,2,3,4)$ so w_8 is used at most once. Likewise, `6c` carries base weight 0 and adds `vcor_w` $=\frac{1}{5}$`W_LCM` ($+1/5$ in Bellman–Ford) on every consecutive transition `6c` $\to$ `6c`, including the cyclic wrap-around (`last`, `si`) at closing (`lines 1290, 1373`), conservatively upper-bounding the Stage 3(H1) Corannulene valley transfer $1/10+1/10=1/5$.

Similarly, around any frozen face $f_i \in \{f_1, f_2\}$ of degree $d \ge 3$, breaking the facial cycle at the cap edge ab (represented by the anchor token `CAP` of width 0 and weight 0) produces a closed walk `CAP` $\to \tau_1 \to \cdots \to \tau_r \to$ `CAP` of total non-cap width $d-1 \ge 2$ (`verify_planar_eg.py:1495--1550`). The two states `st` $\in\{1,2\}$ enforce total non-cap width ≥ 2: closing directly to `CAP` from $(\tau, 1)$

is allowed only when width$(\tau) \geq 2$, whereas after a second token the state moves to `st` $= 2$ and may close to `CAP`. Because f_1 and f_2 share only the single cap edge ab in 3-connected F^+ (Lemma 2.1(iii)), no internal boundary vertex or edge of f_1 away from ab touches f_2; hence every *internal* transition (τ_j, τ_{j+1}) $(1 \leq j < r)$ sees strictly clean faces across its shared internal spoke and obeys `pair_ok_33`, together with the oriented internal rules (`lines 1509--1516`): $T_{\mathrm{frz},L}$ (which sits on a or b directly after ab) has its opposite edge touching a clean ≥ 11/frozen face (so it may only precede $\{N, 6_c, TP_{6,L}\}$ and never appear internally after another token, with exact mirror for $T_{\mathrm{frz},R}$), and TPT_{BC} splits into oriented $TPT_{BC,R}$ (triangle-end on left, preceded only by $\{N, 6_c, TP_{6,R}\}$) and $TPT_{BC,L}$ (triangle-end on right, followed only by $\{N, 6_c, TP_{6,L}\}$) because its flank spoke borders A (clean ≥ 11/frozen, sending 0, 6_c, or TP_6). Crucially, our primary proof of Theorem 1.1 relies *solely* on the relaxed frozen certificate `cubic_frozen_cap_relaxed_bound` (1,351 reachable internal core arcs), which drops *every* `CAP`-endpoint restriction completely (allowing every token to start and end the walk) and still yields excess at most $1/3$ per frozen face, giving $\mathrm{ch}_{\mathrm{fin}}(f_1) + \mathrm{ch}_{\mathrm{fin}}(f_2) \leq 2(5 + 1/3) = 32/3 < 12$. As a strict-endpoint refinement (`cubic_frozen_cap_bound`, 1,588 arcs), excluding the explicit 11-token sets adjacent to `CAP` (`lines 1505--1508`: after `CAP`, excluding $\{R_{10}, T_{L9}, T_{\mathrm{frz},R}, S_{YZ,R}, S_X, TPT_{A9,R}, TPT_{A10,R}, TPT_{B9A,R}, TPT_{B9C,R}, TP_{s73,R}, TPT_{BC,R}\}$; before `CAP`, excluding their exact left-right mirrors $\{R_{10}, T_{R9}, T_{\mathrm{frz},L}, S_{YZ,L}, S_X, TPT_{A9,L}, TPT_{A10,L}, TPT_{B9A,L}, TPT_{B9C,L}, TP_{s73,L}, TPT_{BC,L}\}$) tightens the bound to $\mathrm{ch}_{\mathrm{fin}}(f_1) + \mathrm{ch}_{\mathrm{fin}}(f_2) \leq 10 < 12$.

Lemma 3.3 (Lossless `starts_N` Rotation, Simple-Cycle Detour Compatibility, and Conservative Shift Soundness)**.** *Let W be a valid cyclic token word around a clean d-face F_d ($d \in \{7, 9..15, 17..31\}$) in F^+ with total incoming charge strictly exceeding $d - 6$.*

(i) ***`starts_N` Finite SCC Decomposition and Physical-Word Coverage:*** *Let $\mathcal{S}_g$ (`starts_g`) be the set of tokens beginning with a $\{3,5\}$-edge and $\mathcal{S}_N := \mathcal{T} \setminus \mathcal{S}_g$ (`starts_N`). On the finite digraph induced by `pair_ok` on $\mathcal{S}_g$, there are 0 internal arcs for $d \in \{7, 9..13\}$, 45 for $d = 14$, 16 for $d = 15$, 69 for $d \in \{17, 21, 23, 28\}$, and 68 for all other tested d, with $\{TP_{b54,R}\}$ (self-loop) as the* unique *cyclic vertex. On any physical clean face F_d, a pure word $(TP_{b54,R})^{d/2}$ is impossible for odd d, has weight $\frac{5}{8}d \leq d - 6$ for all even $d \geq 16$, and at $d = 14$ produces a genuine simple cap-avoiding C_{32} in $F^+ - ab$. Hence every overweight surviving physical cyclic word contains at least one token in $\mathcal{S}_N$, and rotating its minimum-index $\mathcal{S}_N$ token to position 0 is lossless.*

(ii) ***Geometric Disjointness, Allowable Vertex Sets, and Collision Bounds*** *($d \in \{14, 17, 18\}$)**:** *Every shift mask used in the forward-backward prefix filter (`verify_planar_eg.py:977--1030, 1244--1310`) and in `seq_killed_conservative` (`lines 1130--1242`) corresponds to pairwise vertex-disjoint, cap-avoiding simple paths glued onto ∂F_d:*

- Pure $\{3,5\}$-cluster detours: *Every token in $\{T, T_{10}, T_{L9}, T_{R9}, P, TP_s, TP_b, TP_{b54}, HH, TPT_A, TPT_{BC}\}$, as well as the prefix shift $(0,1)$ of T_1/T_2 and $(0..3/4)$ of $TPT_{A9}/TPT_{A10}/TPT_{B9}$, uses strictly vertices of its own maximal clean $\{3,5\}$-cluster(s). By Lemma 2.1(iii) and maximality, distinct maximal clean $\{3,5\}$-clusters are pairwise vertex-disjoint, intersect $V(F_d)$ strictly at their prescribed boundary contact edges, and avoid ab.*
- Super-gadget detours (S_X and first S_{YZ}): *All vertices of $S = TPT_1 + F_9 + TPT_2$ lie in $TPT_1 \cup TPT_2 \cup \{w_8\}$, where the sole non-cluster vertex w_8 has degree 3 with incident faces F_9, Y, Z; hence w_8 cannot belong to any other $\{3,5\}$-cluster, cannot belong to $V(F_d)$ except as an endpoint on Y or Z, and cannot belong to any second 9-face ($9 + 9 - 2 = 16$). Using `sh_Y` at most once (`usyz == 0`) ensures complete disjointness.*
- Degree-17 TP_{s3} non-circular two-stage elimination (`sh_TP699`, `lines 959, 999`): *The 5-face patch $TP(B_1 = 6, S_2 = 9, B_2 = 9)$ has a unique surviving weld out of 21, with exact shift spectrum `sh_TP699` $= (0..3, 5..14)$ along its 4-edge arc on $S_1 = F_{17}$. Its faces P_0, T_0, S_2, B_2 already share an edge with F_{17} and cannot meet $V(F_{17})$ elsewhere*

(Lemma 2.1(iii)), while any $(7,8)$ *self-contact of* $B_1 = 6$ *with* F_{17} *kills all* $+1/+3$ *shifts and leaves* $\mathrm{ch}_{\mathrm{fin}}(F_{17}) \le -2 < 0$. *To avoid any circularity between the* $+14$ *shift and the conservative sieve, we eliminate* TP_{s3} *at* $d = 17$ *in two strictly ordered stages (independently replayed in* `cert/verify_parity_and_dp.py`*): (1) restrict* TP_{s3} *to its pure* $\{3,5\}$*-cluster shift mask* $(0,1,2,3)$ *during the* $d = 17$ *forward-backward enumeration, producing* 7,924 *overweight* TP_{s3}*-containing words (of which* 7,080 *contain an isolated triangle token), and run* `seq_killed_conservative` *(whose soundness uses* only *that pure-cluster* $(0,1,2,3)$ *mask on* TP_{s3}*, never* $+14$*), which eliminates* 7,428 *of them and leaves exactly* 496 *post-sieve residuals containing* TP_{s3} *(plus the* 108 *non-*TP_{s3} *residuals); (2) post-sieve, every single one of those* 496/496 *post-sieve* TP_{s3} *residuals contains at least one isolated triangle token* $T' \in \{T, T_{10}, T_{L9}, T_{R9}\}$*, which is non-adjacent to* TP_{s3} *(since* $TP_{s3} \in$ `starts_g` $\cap$ `ends_g`*). Because* T' *has a single off-*∂F_{17} *vertex* w'*, if* w' *lay on any face* Q *of* TP_{699}*, cubicity at* w' *would force one of the two* ∂F_{17} *endpoints of* T' *to lie on* Q*, contradicting non-adjacency and Lemma 2.1(iii). Hence the* $+14$ *path in* TP_{699} *and the* $+1$ *path through* T' *are completely vertex-disjoint, yielding a simple* C_{32} *(*$17 + 14 + 1 = 32$*) for all* 496 *post-sieve residuals.*

- Hexagon/chain detours, drop branches (`lines 1154--1196`), and Dual Jordan Curve: *Every* 5/6*-chain's shift spectrum is intersected over all surviving planar welds (*`common_weld_shifts`*). When* `eval_rem` *activates a* 6*-face detour on* T_1, T_2 *(*`sh_T1`$= (0,1,3,4)$*), fused adjacent pairs (*`sh_11`*,* `sh_12`*), fused triple (*`sh_121`*), or* $6_o/TP_6$ *(*$(0,4)$*), the detour uses only vertices of its own clean* 6*-face(s) and attached* $\{3,5\}$*-cluster(s), avoiding* $V(F_d)$ *outside its contact arc (Lemma 2.1(iii)). Its off-*∂F_d *vertices can collide with non-adjacent* $\{3,5\}$*-tokens only across free off-*∂F_d *edges of* 5*-faces (*`five_i`*, since a triangle's single apex cannot hit* H_6 *without forcing a* F_d*-vertex onto* H_6*):*
  - *Each* T_1 *has at most* 1 *free off-*F_d *hex edge eligible to meet a non-adjacent* 5*-face (*≤ 1 *drop from* `five_i`*); each* T_2 *and each fused adjacent pair* (T_{1R}, T_{1L})*,* (T_{1R}, T_2)*,* (T_2, T_{1L}) *has* ≤ 2 *such collisions (*≤ 2 *drops); and the fused triple* (T_{1R}, T_2, T_{1L}) *drops* every *token in* `five_i` *(*`line 1182`*). Each* TP_s, TP_b, TP_{b54} *has* ≤ 1 *free off-*F_d *pentagon edge, each* P, HH *has* 2*, and* $C_{TP_{sR},H,TP_{sL}}, C_{TP_{sR},T_2}, C_{T_2,TP_{sL}}$ *have* 0 *(by Proposition 3.2(iii)), so if* $\sum_{k \in$ `t12_kill`$}(2 \text{ or } 1) >$ `tot_five_cap`*, by pigeonhole at least one killing* T_1/T_2 *has zero colliding* 5*-faces.*
  - *For* TP_6 *(*`line 1165`*, and* $TP_6 + 2$ *triads at* `lines 1183--1189`*),* $B_1 = 6$ *has outer neighbors* $F_{67}, F_{89} \ge 10$ *and* $F_{78} \ne 5$ *(Stage 3(H2)), so* $B_1 = 6$ *touches* zero *clean* 5*-faces other than its own off-*F_d *TP pentagon; hence* TP_6 *never collides with* `five_i` *and requires* 0 *drops (and if* TP_6 *collided with one triad in* `tri_i`*, it cannot collide with a second distinct triad since* $B_1 = 6$ *has no triad triangle).*
  - *For direct no-drop* 6_o *(*`line 1158`*, both subaudits replayed in* `cert/verify_parity_and_dp.py`*): at* $d \in \{14, 18\}$*, all overweight words (*357 *at* $d = 14$*,* 5,232 *at* $d = 18$*) remain killed (*0 *survivors) even when* 6_o *drops* 1 *token from* `five_i` *in all three* 6_o *branches of* `eval_rem` *(direct, paired* `ok_tp`*, and alternating-pair* `pair_dj`*); at* $d = 17$*, every single one of the* 113,601 *overweight* $d = 17$ *words containing* 6_o *also contains a rim-incident clean triangle sharing an edge with* ∂F_{17} *(*113,601/113,601*, verified directly on the* $d = 17$ *enumeration in* `cert/verify_parity_and_dp.py`*), and exhaustive contact testing of* $(F_{17}, H_6(6_o), P', T_H)$ *(with optional* T_P *on* P'*) over all* 112 *non-adjacent* 5*-face collisions (*14 *rim positions* $\times 2$ *opposite hex-edge pairs* $\times 2$ P'*-orientations* $\times 2$ T_P *choices) shows* 110/112 *die outright while the sole* 2 *mirror survivors (antipodal* $k \in \{8, 9\}$*,* T_P *absent) kill every cubic rim-incident triangle placement on* ∂F_{17} *(all* 12 *non-adjacent edges* $m \in \{2, \ldots, 15\} \setminus \{k-1, k, k+1\}$ *and the unique cubic spoke-sharing edge adjacent to* P'*, while the opposite edge adjacent to* P' *cannot carry a triangle since its shared spoke vertex already has degree* 3*, and edges adjacent to* $H_6(6_o)$ *cannot carry*

a triangle by definition of 6_o); thus 6_o cannot collide off-∂F_{17} with any `five_i` *token on F_{17}.*

  - Dual Jordan Curve (`lines 1190--1196`): *For four face-disjoint tri-hex clusters X_0, X_1, X_2, X_3 in cyclic order around ∂F_d, each X_i is a connected closed union of clean faces contained in the closed exterior disk $D_{\text{ext}} := S^2 \setminus F_d$ attaching to $\partial D_{\text{ext}} = \partial F_d$ along a pairwise disjoint boundary interval I_i (appearing in cyclic order I_0, I_1, I_2, I_3). If (X_0, X_2) shared an off-∂F_d vertex $p \in \text{int}(D_{\text{ext}})$ and (X_1, X_3) shared an off-∂F_d vertex $q \in \text{int}(D_{\text{ext}})$ ($p \neq q$ since $X_0, \dots, X_3$ are pairwise face-disjoint in cubic F^+), then connecting I_0 to I_2 through $X_0 \cup X_2$ via p and I_1 to I_3 through $X_1 \cup X_3$ via q would produce two internally disjoint arcs inside D_{ext} joining alternating boundary intervals on ∂D_{ext}, contradicting the Jordan curve theorem. Hence at least one of (X_0, X_2) or (X_1, X_3) is off-rim vertex-disjoint.*

- Auxiliary-face collisions in `hd_strict` (`lines 1199--1240`): *For TPT_{A10} and TPT_{B9}, cubic disjointness implies F_{aux} touches at most two isolated T-tokens and zero 5-face tokens (and if a T-token is already attached at the spoke,* `has_att`*, at most one additional T-token remains). For TPT_{A9}, F_9 touches at most two T-tokens or one $P/TP_s/TP_b$-token (plus at most one 6_o-token for P/TP_b); furthermore, when* `has_att` *holds (`line 1208`), the attached token in* `ATT_R`/`ATT_L` *places an* isolated *clean triangle T_{att} on the spoke edge $(7, 10)$ of F_9, which forbids a 5-face on the adjacent edge $(10, 11)$ (else T_{att} would lie in a TP cluster, contradicting* `ATT_R`/`ATT_L`*) and kills a 5-face on both remaining free edges $(11, 12)$ (0/1 weld) and $(12, 8)$ (0/1 weld), proving zero $P/TP_s/TP_b$-tokens can touch F_9 when* `has_att` *holds.*

Thus any sequence discarded by prefix masks or `seq_killed_conservative` *forces a genuine simple cycle in $\{C_4, C_8, C_{16}, C_{32}\}$ in $F^+ - ab \subseteq G$.*

Proof. (i) Every token in $\mathcal{S}_g$ other than $\{TP_{b54,R}, S_{YZ,R}, TPT_{A9,R}, TPT_{A10,R}, TPT_{B9A,R}, TPT_{B9C,R}\}$ ends in `ends_g` $\cup$ `ends_6s` $\cup$ `ends_6o` $\cup$ `ends_tp6`, none of which may precede any token in $\mathcal{S}_g$ under `pair_ok`. Meanwhile, each of $S_{YZ,R}, TPT_{A9,R}, TPT_{A10,R}, TPT_{B9A,R}, TPT_{B9C,R}$ lies in `g_left_N` and requires predecessor $N \in \mathcal{S}_N$ (and $S_{YZ,R}$ also requires successor $N \in \mathcal{S}_N$), so all five have in-degree 0 inside $\mathcal{S}_g$. Hence in the finite `pair_ok` digraph restricted to $\mathcal{S}_g$, $\{TP_{b54,R}\}$ is the sole nontrivial strongly connected component, with the stated internal arc counts (verified directly on `adj_next`).

To translate this finite SCC result to a physical clean face $F_{14} = (v_0, \dots, v_{13})$ carrying $(TP_{b54,R})^7$: each of the 7 tokens encodes a maximal clean TP cluster $P_i \cup T_i$ ($0 \le i \le 6$) whose pentagon P_i shares the single edge (v_{2i}, v_{2i+1}) with F_{14} (and whose side $S_2 = 9$ shares (v_{2i+1}, v_{2i+2}) with F_{14}). Since P_i shares only (v_{2i}, v_{2i+1}) with F_{14} (Lemma 2.1(iii)), the 7 pentagons $P_0, \dots, P_6$ are pairwise distinct, and because maximal clean $\{3, 5\}$-clusters are pairwise vertex-disjoint, the 7 clusters $P_i \cup T_i$ are pairwise distinct and vertex-disjoint. The triangle T_i is glued along a pentagon edge e_i of P_i vertex-disjoint from (v_{2i}, v_{2i+1}) (specifically opposite $B_2 = F_{14}$ in Stage 1C(TP2)), so the apex of T_i cannot lie on F_{14} (otherwise T_i would share an edge with F_{14}, forcing one endpoint of e_i — a third vertex of P_i — onto F_{14}, contradicting Lemma 2.1(iii)). Choosing the $+4$ detour around $P_i \cup T_i$ on $i \in \{0, 1, 2\}$, the $+3$ detour around P_i on $i \in \{3, 4\}$, and shift 0 along ∂F_{14} on $i \in \{5, 6\}$ produces an explicitly simple, cap-avoiding cycle of length $14 + 4 + 4 + 4 + 3 + 3 = 32$ in $F^+ - ab \subseteq G$.

(ii) Forward-backward maximum pruning retains every valid cyclic word of weight $> d - 6$ whose prefix shift convolution avoids $\{4, 8, 16, 32\}$. By the allowable vertex sets and collision-neutralization bounds established in the statement, every active shift in the convolution is realized by a simple path strictly inside its clean local neighborhood avoiding both $V(F_d)$ outside its prescribed contact arc and ab, and every possible inter-token collision between non-adjacent tokens is neutralized either by quantifying over all drop subsets (forcing shift 0, i.e., staying on ∂F_d) or by the geometric

exclusions verified for TP_{s3}, TP_6, 6_o, Dual Jordan Curve, and `hd_strict`. Hence every surviving shift sum in $\{4, 8, 16, 32\}$ splices into a genuine simple cycle of that length in $F^+ - ab \subseteq G$. □

Proof of Theorem 1.1. We verify that every clean face of F^+ satisfies $\mathrm{ch}_{\mathrm{fin}}(f) \leq 0$ and that the frozen faces satisfy $\sum_{f \in \mathcal{F}_{\mathrm{frz}}(F^+)} \mathrm{ch}_{\mathrm{fin}}(f) \leq 32/3 < 12$ under the zero-`CAP`-assumption relaxed bound (and $\leq 10 < 12$ under the strict-`CAP` refinement):

(1) **Clean** $d \in \{3, 5, 6\}$**:** By Section 3.2, every clean 3-, 5-, and 6-face ends with $\mathrm{ch}_{\mathrm{fin}} = 0$.

(2) **Clean** $d \in \{7, 9..13, 15, 19..31\}$**:** Exact cyclic DP (and the packed shifted-slope BF, cyclic DAGs, and 19-row backward DP in `CubicDischarging.lean`), together with the 4-hex $d = 7$ bound from Stage 3(H3), verifies $\mathrm{ch}_{\mathrm{fin}}(f) \leq 0$ for all $d \in \{7, 9, 10, 11, 12, 13, 15, 19, \ldots, 31\}$.

(3) **Clean** $d \in \{14, 17, 18\}$**:** By Lemma 3.3, the forward-backward conservative enumeration inspects all 357 ($d = 14$), 302,652 ($d = 17$), and 5,232 ($d = 18$) overweight cyclic words. At $d = 14$ and $d = 18$, 0 words survive. At $d = 17$, exactly 108 residuals survive:

- 60 *residuals containing both* $C_{TP_{sR},H,TP_{sL}}$ *and* R_{10} *(*48 *with two* R_{10} *tokens and* 12 *with three* R_{10} *tokens, each containing exactly one* $C_{TP_{sR},H,TP_{sL}}$*):* Mapping the 5-edge arc $(0, 1, 2, 3, 4, 5)$ of $C_{TP_{sR},H,TP_{sL}}$ on $F_{17} = (0, 1, \ldots, 16)$ directly to `build_cubic_chain_dedup(('T','P',H,'P','T'))`, its six clean faces are $T_1 = (1, 0, 50)$, $P_1 = (2, 1, 50, 200, 52)$, $H_6 = (3, 2, 52, 202, 203, 53)$, $P_2 = (4, 3, 53, 201, 54)$, $T_2 = (5, 4, 54)$, and the wing triangle T_H of H_6: for the side-wing orientations H_L ($T_H = (202, 52, 200)$) and H_R ($T_H = (53, 203, 201)$), boundary-weld enumeration (Lemma 3.1) yields 0/1 survivors each (eliminating both side-wing realizations), leaving H_O ($T_H = (203, 202, 204)$) as the unique geometric realization (1/1 weld, shifts $\{0..7\}$). In this surviving patch, the explicit 12-edge (+7) simple path

$$P_{+7} = (0,\ 1,\ 50,\ 200,\ 52,\ 202,\ 204,\ 203,\ 53,\ 3,\ 4,\ 54,\ 5)$$

runs from 0 to 5 using only vertices inside the 5-edge arc $\{0, 1, 3, 4, 5\}$ of $C_{TP_{sR},H_O,TP_{sL}}$ plus its 8 off-∂F_{17} vertices $\{50, 200, 52, 202, 204, 203, 53, 54\}$ — avoiding every vertex of $\{6, \ldots, 16\}$ on ∂F_{17} by Lemma 2.1(iii) (if the apex 204 of T_H lay on F_{17}, T_H would share an edge with F_{17}, forcing 202 or 203 — a third vertex of H_6 — onto F_{17}). Meanwhile, any R_{10} token in the word is a clean 10-face $F_{10} = (u_0, \ldots, u_9)$ sharing one edge (u_0, u_1) strictly inside $\{6, \ldots, 16\}$ on ∂F_{17} (since R_{10} is flanked on both sides by T_{10}). Its 8 off-∂F_{17} vertices $u_2, \ldots, u_9$ avoid $V(F_{17})$ by Lemma 2.1(iii), and by Stage 2 each u_j ($2 \leq j \leq 9$) has its three incident faces in $\{F_{10}, T^*, Q^*\}$, where $Q^* \notin \{5, 6, 7, 9, 10\}$ (by the 5 R_{10} seals) and T^* touches neither a 5-face nor a 6-face. Because every face of $C_{TP_{sR},H_O,TP_{sL}}$ is a 5-face, a 6-face, or a 3-face touching a 5- or 6-face, $\{u_2, \ldots, u_9\}$ is completely disjoint from $V(P_{+7})$. Splicing P_{+7} (length 12, replacing the 5-edge arc 0–5) together with the 9-edge (+8) detour $(u_0, u_9, \ldots, u_2, u_1)$ around F_{10} (replacing edge (u_0, u_1)) onto the remaining 11 edges of the 5–0 arc of ∂F_{17} yields an explicitly simple, cap-avoiding cycle of length $12 + 9 + 11 = 32$ in $F^+ - ab \subseteq G$.

- 48 *remaining residuals mapped to* 12 *explicit orbit seeds (*114 *welds):* Rotating the unique adjacent pair (T_{1R}, T_{1L}) to index 0 puts every one of the 48 words into the exact cyclic form $(T_{1R}, T_{1L}, N, A, N, B, N, C, N)$, where (A, B, C) ranges over the $12 = 3 \times 4$ ordered triples obtained by placing T in one of 3 positions and choosing an ordered pair from $\{LL, LR, RL, RR\}$ in the other two positions, yielding the exact 12 orbit codes

  `XTLL`, `XTLR`, `XTRL`, `XTRR`, `XLTL`, `XLTR`, `XRTL`, `XRTR`, `XLLT`, `XLRT`, `XRLT`, `XRRT`

  (each cyclic orbit appears 4 times according to which of its 4 N-tokens starts the `starts_N` rotation: $12 \times 4 = 48$). Here `X` maps (T_{1R}, T_{1L}, N) to the 5-vertex (4-edge) contact arc $(0, 1, 2, 3, 4)$ of `tri_hex_pair` plus its trailing N-edge $(4, 5)$; `L` maps $(C_{T_{1R},TP_{sL}}, N)$ to the 4-edge arc $(10, 6, 2, 1, 5)$ of (T_1, H_6, P, T_2) plus its trailing N-edge; `R` maps $(C_{TP_{sR},T_{1L}}, N)$ to the exact mirror of `L`; and `T` maps (T, N) to the 1-edge triangle contact $(0, 1)$ plus its trailing

(a) 7-Vertex Planar Gadget $[T_1 + P + T_2]$ **(b) Vertex Inflation in** $G_{168} = C_{24} * [T_1 + P + T_2]$

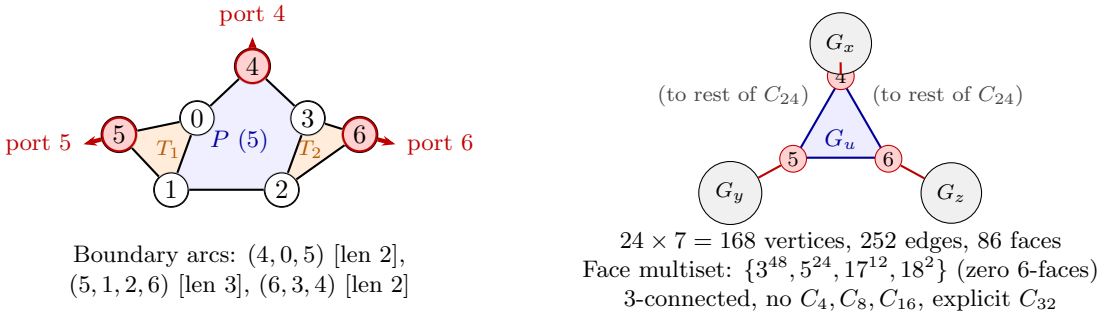


FIGURE 2. Sharpness witness $G_{168} = C_{24} * [T_1 + P + T_2]$ for Theorem 1.1: (a) the 7-vertex planar gadget with external ports 4, 5, 6; (b) local incidence at an inflated vertex G_u of the 24-vertex fullerene C_{24} with its three neighboring gadgets G_x, G_y, G_z, achieving exact 3-connectivity, zero C_4, C_8, C_{16}, and an explicit simple C_{32}.

N-edge ($5 + 5 + 5 + 2 = 17$ edges in `degree17_patch`, `verify_planar_eg.py:789--813`). Full boundary-weld enumeration across all 12 seeds produces 114 total quotients; all 114 carry kernel-certified strict-schema simple cycles in $\{C_8, C_{16}, C_{32}\}$ (Lemma 3.1).

(4) **Clean $d \geq 33$:** On the 667-edge clean transition graph (with conservative $B = 5/4$ inside all 51 chains and slope $b = 9/11$), the exact rational Bellman–Ford dual potential π (common denominator 1,320, kernel-checked in `cubic_clean_d_ge_33_charge_le`) satisfies $\pi(u) + (w(u,v) - \frac{9}{11}\text{width}(v)) \leq \pi(v)$ for every transition (u, v). Telescoping around any closed walk of edge width $d \geq 33$ yields

$$\text{ch}_{\text{fin}}(f) \;\leq\; 6 - d + \frac{9}{11}d \;=\; 6 - \frac{2}{11}d \;\leq\; 6 - 6 \;=\; 0.$$

(5) **Frozen faces f_1, f_2 ($d \geq 3$):** On the 1,351-arc reachable internal core of the frozen transition graph (with conservative $H = 1$ inside all 51 chains, slope $b_f = 1$ per non-cap edge, and *zero* `CAP`-endpoint exclusions), the exact dual potential kernel-checked in `cubic_frozen_cap_relaxed_bound` yields excess at most $1/3$ per frozen face, so $\text{ch}_{\text{fin}}(f_i) \leq (6-d) + (d-1) + 1/3 = 16/3$ and $\text{ch}_{\text{fin}}(f_1) + \text{ch}_{\text{fin}}(f_2) \leq 32/3 < 12$. Adding the explicit 11-token `CAP`-endpoint exclusions (1,588-edge frozen `CAP` graph, scale 120, `cubic_frozen_cap_bound`) refines this bound to $\text{ch}_{\text{fin}}(f_i) \leq 5$ and $\text{ch}_{\text{fin}}(f_1) + \text{ch}_{\text{fin}}(f_2) \leq 10 < 12$.

By charge conservation and (5), $12 = \sum_f \text{ch}_{\text{fin}}(f) \leq 32/3 < 12$ (and $\leq 10 < 12$ under the strict `CAP` refinement), a contradiction.

Sharpness (G_{168}): Construct $G_{168} = C_{24} * [T_1 + P + T_2]$ by inflating each vertex $u \in \{0, \ldots, 23\}$ of the 24-vertex fullerene C_{24} (two hexagons and twelve pentagons) into the 7-vertex planar gadget on $\{0, \ldots, 6\}$ with internal faces $(0, 4, 3, 2, 1)$, $(5, 0, 1)$, and $(6, 2, 3)$ and three external ports $\{4, 5, 6\}$ (Figure 2), oriented so that the boundary arcs $(4, 0, 5)$ (length 2), $(5, 1, 2, 6)$ (length 3), and $(6, 3, 4)$ (length 2) assign each base pentagon two length-3 corners (yielding twelve 17-faces and two 18-faces, plus 48 triangles and 24 pentagons). Thus $V = 168$, $E = 252$, $F = 86$, $V - E + F = 2$. In `SharpWitnesses.lean`, Lean 4 kernel-verifies that G_{168} is cubic, 3-connected (`g168_is3Connected`), has an exact dart bijection and a single cyclic rotation at every vertex (`g168_dartBijection`, `g168_singleVertexRotation`, confirming with connectedness and Euler characteristic 2 a cellular spherical embedding on S^2), contains no simple C_4, C_8, or C_{16} (via internal gadget absence and exact port-path DP over all simple C_{24}-cycles of length ≤ 10, since each port-to-port traversal plus exit edge has length ≥ 3, so any cycle of length ≤ 16 projects to $\leq 5 \leq 10$ base vertices), and contains an explicit simple C_{32} (`g168_has_C32`). □

4. Main Theorem 2: All Planar Induced Claw-Free Graphs ($\delta \geq 3$) Contain $C_4..C_{64}$

We now prove Theorem 1.2 by reducing any C_4-free planar induced claw-free graph with $\delta(G) \geq 3$ to a planar cubic skeleton H and lifting dyadic cycles from Theorem 1.1 via a sharp interval-realization lemma.

Lemma 4.1 (Claw-Free Bowtie Representation and Planar Bowtie-Splitting)**.** *Let G be a finite nonempty simple induced claw-free (induced $K_{1,3}$-free) graph with $\delta(G) \geq 3$ and no C_4.*

(i) Every vertex $v \in V(G)$ has degree 3 or 4; its open neighborhood $N_G(v)$ is isomorphic to $K_2 + K_1$ (if $d_G(v) = 3$) or $K_2 + K_2$ (if $d_G(v) = 4$).

(ii) Consequently, the triangles of G are edge-disjoint, each degree-3 vertex lies in a unique triangle, each degree-4 vertex lies in exactly two triangles (forming a bowtie*), and $G \cong T(H)/B$ for a marked finite nonempty simple cubic skeleton (H, B) ($B \subseteq E(H)$, with at most one edge of B in any triangle of H), unique up to isomorphism of marked skeletons, where $T(H)$ is obtained from H by inflating each vertex $u \in V(H)$ into a triangle T_u whose three vertices correspond to the three incident edges of u in H, and $T(H)/B$ contracts each inter-triangle edge $e \in B$ to a single degree-4 bowtie vertex x_e.*

(iii) ***Planar Bowtie-Splitting:*** *If G is planar, then both $T(H)$ and the cubic skeleton H are planar.*

Proof. (i) Because G is induced $K_{1,3}$-free, the independence number of $N_G(v)$ satisfies $\alpha(N_G(v)) \leq 2$. Because G is C_4-free, $N_G(v)$ contains no 2-edge path $a - b - c$ (regardless of whether $ac \in E(G)$, $v - a - b - c - v$ is a simple 4-cycle in G). Hence every connected component of $N_G(v)$ is K_1 or K_2. With at most two components ($\alpha \leq 2$) and $|N_G(v)| = d_G(v) \geq 3$, the only possibilities are $K_2 + K_1$ ($d = 3$) and $K_2 + K_2$ ($d = 4$).

(ii) By (i), no edge lies in two triangles (else a diamond contains C_4). Construct H with one vertex u per triangle T_u of G, adding an edge $uv \in E(H) \setminus B$ for each edge of G joining a degree-3 vertex of T_u to a degree-3 vertex of T_v, and adding a bowtie edge $uv \in B$ for each degree-4 vertex $x_{uv} \in V(T_u) \cap V(T_v)$. Since every triangle has 3 vertices, H is 3-regular and nonempty. Moreover, H is simple: a loop at u is impossible (vertices of T_u have no extra edges inside T_u), two parallel physical edges between T_u and T_v form a C_4, two parallel bowtie vertices form a diamond (hence C_4), and a bowtie vertex $x \in V(T_u) \cap V(T_v)$ together with a physical edge a_1b_1 between $T_u = \{x, a_1, a_2\}$ and $T_v = \{x, b_1, b_2\}$ yields the simple 4-cycle $x - a_1 - b_1 - b_2 - x$. Similarly, if a triangle uvw of H had two edges $uv, vw \in B$ with bowtie vertices x_{uv}, x_{vw}, then letting $c_u \in T_u, c_w \in T_w$ be the endpoints of the third edge uw (whether physical or bowtie), $x_{uv} - x_{vw} - c_w - c_u - x_{uv}$ (if $uw \notin B$) or $x_{uv} - x_{vw} - x_{uw}$ sharing $x_{uv}x_{vw}$ with T_v (if $uw \in B$, forming a diamond hence C_4) yields a C_4. Hence (H, B) is simple, has ≤ 1 edge of B per triangle, and is uniquely determined up to isomorphism of marked skeletons by the triangle decomposition of G.

(iii) Fix a planar embedding of G on S^2. At any degree-4 bowtie vertex $x_{uv} = T_u \cap T_v$, the two incident edges of T_u must appear *consecutively* in the cyclic rotation around x_{uv}: if the edges of T_u alternated with those of T_v, then the third edge of T_v would connect the two endpoints of T_v on opposite sides of the Jordan curve T_u, forcing a crossing. Splitting x_{uv} locally into two degree-3 vertices (separating the two consecutive edges of T_u from those of T_v) joined by a short edge inside a small disk around x_{uv} preserves planarity. Performing this split at every degree-4 vertex yields a planar embedding of $T(H)$. Finally, contracting each of the pairwise vertex-disjoint triangles T_u ($u \in V(H)$) to a single vertex produces a planar embedding of H. □

Lemma 4.2 (Simple Cycle Lift-Window and Bowtie-Forest Exclusion)**.** *Let $G = T(H)/B$ be a C_4-free induced claw-free graph with simple cubic skeleton H and bowtie set $B \subseteq E(H)$, and let C be a simple cycle of length $L \geq 3$ in H containing $k := |E(C) \cap B|$ traversed bowtie edges and $m := |E_{\text{chord}}(C) \cap B|$ internal chordal bowtie edges (edges of $B \setminus E(C)$ with both endpoints on $V(C)$). Then:*

(i) ***Exact Realization of the Lift Window:*** *Every integer* $\ell \in [2L-k,\ 3L-k-m]$ *is the length of a genuine simple cycle in* G.

(ii) ***Bowtie-Forest Exclusion:*** *If* $m=0$ *and* $r := L-k \le 2$, *then* $[2L-k, 3L-k] = [L+r, 2L+r]$ *contains a power of two* $2^a \ge 4$ *(with* $2^a \le 2L+2$*). In particular, if* G *avoids all dyadic cycles, then* $H_B := (V(H), B)$ *is an* induced forest *in* H *(*$r=0,1$ *impossible) and the contraction* H/H_B *is* simple *(*$r=2$ *impossible).*

(iii) ***Universal Dyadic Transfer:*** *Whenever* $H_B[V(C)]$ *is acyclic (a forest on* L *vertices), we have* $k+m \le L-1$, *and therefore both* $2L$ *and* $2L+1$ *lie in* $[2L-k, 3L-k-m]$.

Proof. (i) For each vertex $u \in V(C)$, its two incident cycle edges $e_{\text{in}}(u), e_{\text{out}}(u) \in E(C)$ are distinct and therefore correspond to two distinct port vertices $p_{\text{in}}(u), p_{\text{out}}(u) \in V(T_u)$; let $p_{\text{sp}}(u)$ denote the third (*spare*) vertex of T_u. Each edge $e \in E(C) \setminus B$ contributes 1 physical inter-triangle edge of G ($L-k$ edges in total), whereas each edge $e \in E(C) \cap B$ identifies $p_{\text{out}}(u) = p_{\text{in}}(v) = x_e$ (0 inter-triangle edges). The $2L$ traversed port occurrences along C are pairwise distinct except for the k exact coincidences along traversed bowtie edges $e \in E(C) \cap B$, yielding $2L-k$ distinct traversed port vertices in G (since H is simple).

Inside each triangle T_u ($u \in V(C)$), we may route from $p_{\text{in}}(u)$ to $p_{\text{out}}(u)$ either via the single edge $p_{\text{in}}(u)p_{\text{out}}(u)$ (length 1) or via the 2-edge detour $p_{\text{in}}(u) - p_{\text{sp}}(u) - p_{\text{out}}(u)$ (length 2, contributing $+1$ to the cycle length). A spare vertex $p_{\text{sp}}(u)$ is never a traversed port of any T_v ($v \in V(C)$), because $p_{\text{sp}}(u)$ corresponds to the unique edge of H at u not in $E(C)$. Distinct spare occurrences $p_{\text{sp}}(u), p_{\text{sp}}(v)$ ($u \ne v \in V(C)$) coincide if and only if $uv \in E_{\text{chord}}(C) \cap B$, forming m pairwise disjoint pairs alongside $L-2m$ unshared spare vertices. Choosing any subset S of $|S| = s \in \{0, 1, \dots, L-m\}$ spare detours taking at most one detour from each of the m chordal pairs yields a simple cycle in G of exact length $(L-k) + L + s = 2L - k + s \in [2L-k,\ 3L-k-m]$.

(ii) Let $x := L+r \ge 3$ and let 2^a be the smallest power of two $\ge x$ (so $a \ge 2$). Then $2^{a-1} \le x-1$, hence $2^a \le 2(x-1) = 2L + 2r - 2 \le 2L + r$ whenever $r \le 2$ (proved in Lean 4 as `bowtie_forest_exclusion`). Applying this to: (a) a shortest all-bowtie cycle in H_B ($r=0, m=0$) proves H_B is acyclic; (b) the unique path in a tree component of H_B closed by a single edge of $H \setminus B$ ($r=1, m=0$ by acyclicity) proves H_B is induced; (c) two edges of $H \setminus B$ joining two distinct tree components of H_B closed by the unique paths (possibly of length 0 if endpoints coincide) in those two trees ($r=2, m=0$) proves at most one edge joins any two tree components, so H/H_B is simple.

(iii) A forest on L vertices has at most $L-1$ edges (`forest_edge_bound_on_cycle`), so $k+m = |E(H_B[V(C)])| \le L-1$, giving $2L-k \le 2L < 2L+1 \le 3L-k-m$ (`universal_bowtie_transfer`). □

Proof of Theorem 1.2. Let G be a finite nonempty simple planar induced claw-free graph with $\delta(G) \ge 3$. If G contains a C_4, we are done. Otherwise, by Lemma 4.1, $G \cong T(H)/B$ for a finite nonempty simple cubic planar graph H and bowtie set $B \subseteq E(H)$. By Theorem 1.1, H contains a simple cycle C of length $L \in \{4, 8, 16, 32\}$. Consider the induced bowtie subgraph $H_B[V(C)]$:

- If $H_B[V(C)]$ contains a cycle, let C' be a shortest cycle in $H_B[V(C)]$, of length $3 \le L' \le L \le 32$. Then C' has $k' = L'$ ($r' = 0$) and $m' = 0$ (by minimality of L'), so by Lemma 4.2(i)–(ii), G contains a simple cycle of length $2^a \in [L', 2L'] \cap [4, 64]$.
- Otherwise $H_B[V(C)]$ is a forest on L vertices, so $k+m \le L-1$. By Lemma 4.2(i) and (iii) (`dyadic_cycle_lift_window`), G contains a simple cycle of length $2L \in \{8, 16, 32, 64\}$. □

5. Main Theorem 3: General Planar ($\delta \ge 3$) C_{16} Trilogy

Fix a forbidden cycle length $g_{\text{forb}} \in \{3, 5, 6\}$, and suppose for contradiction that G is an edge-minimal counterexample to Theorem 1.3 among finite nonempty simple planar graphs with $\delta(G) \ge 3$ avoiding $\{C_4, C_8, C_{16}, C_{g_{\text{forb}}}\}$. By Lemma 2.3, $V_{\ge 4}(G)$ is an independent set, and there exists a 3-connected plane graph H^+ ($\delta(H^+) \ge 3$) with at most two frozen faces $\mathcal{F}_{\text{frz}}(H^+) \subseteq \{f_1, f_2\}$ sharing

the cap edge ab such that: every cycle avoiding ab avoids $\{C_4, C_8, C_{16}, C_{g_{\text{forb}}}\}$, and no clean edge joins two vertices of $V_{\geq 4}(H^+)$.

Assign Euler vertex-face initial charges

$$\mathrm{ch}_0(v) := 6 - 2d_{H^+}(v) \quad (v \in V(H^+)), \qquad \mathrm{ch}_0(f) := 6 - d(f) \quad (f \in F(H^+)),$$

so that $\sum_v \mathrm{ch}_0(v) + \sum_f \mathrm{ch}_0(f) = 12$ by (4). Note that $\mathrm{ch}_0(v) = 0$ for cubic vertices ($d = 3$) and $\mathrm{ch}_0(v) = 6 - 2k \leq -2$ for high-degree vertices ($d = k \geq 4$), allowing every $v \in V_{\geq 4}(H^+)$ to act as a charge sink (Figure 3(b)).

Lemma 5.1 (Soundness and Seed Completeness of Noncrossing Outer-Rim Partition Enumeration)**.** *Let $\mathcal{P} = (F_1, \ldots, F_m)$ be an abstract planar 2-complex forming a closed topological disk D_{abs} bounded by a simple outer-rim cycle $R = (r_0, r_1, \ldots, r_{q-1})$, mapped into S^2 by the realization $\Phi : D_{\text{abs}} \to S^2$ of m distinct clean faces of 3-connected H^+ such that open faces map homeomorphically onto disjoint open faces of H^+, interior edges map injectively onto their prescribed shared edges, and every interior vertex of D_{abs} is mapped injectively and disjointly from $\Phi(R)$. Then:*

- *(i)* ***Noncrossing Exterior-Disk Partition:*** *The equivalence relation on $\{0, \ldots, q-1\}$ induced by $\Phi(r_i) = \Phi(r_j)$ is a* noncrossing partition *of the cyclic sequence R.*
- *(ii)* ***Validity Filters:*** *The realized quotient $\Phi(\mathcal{P})$ satisfies: (a) every face is simple, (b) any two distinct faces share at most one edge (Lemma 2.1(ii)), (c) no two vertices of degree ≥ 4 in the edge union of $\Phi(\mathcal{P})$ are adjacent along any clean edge (since patch degree lower-bounds ambient degree in H^+ and $V_{\geq 4}(H^+)$ is independent on clean edges by Lemma 2.3(ii)(b)), and (d) the edge union of $\Phi(\mathcal{P})$ contains no cycle in $\{C_4, C_8, C_{16}, C_{g_{\text{forb}}}\}$.*
- *(iii)* ***Seed Completeness:*** *For every seed class in `cert/verify_planar_eg.py` (Sections 5.1–5.3), the omitted/fixed rim vertices and contact tests in `rim_quotients` and `check_cor_attach` omit zero valid surviving quotients; hence if every valid emitted quotient fails `patch_status`, then $\mathcal{P}$ cannot occur on clean faces of H^+.*

Proof. (i) Choose a slightly shrunken concentric closed disk $D' \subset \mathrm{int}(D_{\text{abs}})$ whose boundary $\Gamma' := \partial D'$ stays strictly inside $\mathrm{int}(D_{\text{abs}})$ (crossing each interior edge incident to R once just before its rim endpoint). Because Φ is injective on $\mathrm{int}(D_{\text{abs}})$, the image $\Phi(\Gamma')$ is a genuine simple closed Jordan curve on S^2. By the Jordan–Schoenflies theorem, $S^2 \setminus \Phi(\Gamma')$ consists of two open topological disks: the interior disk $U_{\text{in}} := \Phi(\mathrm{int}(D'))$ and the exterior open disk $U_{\text{out}} := S^2 \setminus \Phi(D')$. For each $k \in \{0, \ldots, q-1\}$, draw a short radial arc $\gamma_k \subset D_{\text{abs}} \setminus \mathrm{int}(D')$ inside the face corner(s) incident to r_k from a point $p_k \in \Gamma'$ (in cyclic order $p_0, \ldots, p_{q-1}$ around Γ') to r_k, so that the images $\Phi(\gamma_k \setminus \{p_k, r_k\})$ are pairwise disjoint open arcs in U_{out}. If four indices $i_1 < j_1 < i_2 < j_2$ in cyclic order satisfied $\Phi(r_{i_1}) = \Phi(r_{i_2}) = u$ and $\Phi(r_{j_1}) = \Phi(r_{j_2}) = v$ with $u \neq v$, then $\Phi(\gamma_{i_1}) \cup \Phi(\gamma_{i_2})$ and $\Phi(\gamma_{j_1}) \cup \Phi(\gamma_{j_2})$ would be two disjoint simple arcs inside the closed exterior disk $\overline{U_{\text{out}}}$ connecting alternating boundary pairs (p_{i_1}, p_{i_2}) and (p_{j_1}, p_{j_2}) on $\partial U_{\text{out}} = \Phi(\Gamma')$, contradicting the Jordan curve theorem.

(ii) Conditions (a)–(d) hold in $H^+ - ab \subseteq G$ by Lemma 2.1(i)–(ii) and Lemma 2.3(ii), because every edge of $\Phi(\mathcal{P})$ is a clean edge and patch vertex degrees lower-bound ambient degrees in H^+.

(iii) In `rim_quotients`, the exact noncrossing interval guard (`peer[0] < last < peer[-1]`) and hereditary pairwise face-contact guard (`compatible_with_contacts`, which checks only that identifying two rim vertices never forces two faces to share ≥ 3 vertices or 2 non-adjacent vertices) preserve every valid quotient. In every seed below, each shared interior edge maps injectively by facial contact hypotheses (Lemma 2.1(ii)), and when repeated rim incidence is already prescribed in the seed (e.g. `f7_pair_k` at vertex 1, `v9` at vertex 0, or `3-hex+2T` at vertex 0), the noncrossing corner-arc proof of (i) applies directly to the cut-open cyclic rim sequence with the fixed vertex excluded (since its incident wedges form a contiguous sector on ∂U_{out}). We justify interior-vertex injectivity and every omitted/fixed label across all seed classes:

- *All-interior-faces-present seeds* (C_3-free `enum_ring`, C_5-free `4-hex`, `5-hex+2v`, `k4t2`, `f7_3h`, and all 9 C_6-free seeds): Every interior vertex has its full ambient degree already saturated by its incident faces inside $\mathcal{P}$ (either cubic with all 3 faces present, or degree 4/5 with all 4/5 faces present in `4-hex`, `5-hex+2v`, `k4t2`), so its closed local wedge fan leaves zero free wedges and it cannot coincide with any other vertex of $\mathcal{P}$.
- *Corannulene extensions* (`check_cor_attach`, `verify_planar_eg.py:1628--1656`): By `enum_ring((6,6,6,6,6))` ($46 \to 1$ survivor), the 15-rim Corannulene is uniquely unidentified. Hence in `(*cor, new_face)`, no two vertices of `cor` coincide and no two vertices of `new_face` coincide, so every identification is a partial bijection between `new_vs` and the remaining rim vertices of `cor`. By (i), traversing `new_vs` clockwise and the Corannulene rim counterclockwise (`arc_rev`), the bijection must be order-preserving. Vertices of H_0 (in the $+P$ cases) and of $H_0 \cup H_4$ (in the valley cases, leaving `v_arc_rev`$= [17, 16, 8, 15, 14, 7, 13, 12]$) are excluded from `arc_rev` because `new_face` already shares an edge with H_0 (and H_4) and cannot share a third vertex with them by Lemma 2.1(ii).
- *C_5-free `v9` (fixed vertex 0), `3-hex+2T` (fixed vertex 0), and `f7_pair_k` (fixed vertex 1):* In `f7_pair_k` (`lines 1776--1784`), vertex 1 belongs to F_7, H_1, H_2, so by face simplicity and Lemma 2.1(ii) it cannot identify with any vertex of F_7, H_1, H_2 or either triangle apex $30, 31$ (0 extra quotients when included); skipping `len(set(q[1]) & set(q[2])) == 2` excludes only the adjacent-hexagon case already covered by `f7_3h`. In `3-hex+2T` (`lines 1760--1771`), vertex 0 has $d(0) \geq 4$ (so H_1 and H_3 cannot share an edge, enforced by `len(set(q[0]) & set(q[2])) < 2`) and lies on all three hexagons, preventing identification with any vertex of $H_1 \cup H_2 \cup H_3$ or either triangle apex (C_4/C_5). In `v9` (`lines 1724--1752`), vertex 0 lies on H_1 and H_2, and every noncrossing identification of 0 with a remaining rim vertex violates `patch_status` (C_4, C_5, double contact, or adjacent high vertices; direct verification confirms 0 extra valid quotients across all 25 `v9` seeds when 0 is included).
- *C_5-free `aux10` and `H6` (`lines 1817--1854`):* Every vertex of `rim10` is fixed because each triangle and `aux`/`h6` already shares an edge with F_{10} and cannot share a third vertex with F_{10} (Lemma 2.1(ii)). In `aux10`, omitting $(21, 29)$ holds whenever T_1, T_9 have cubic apices (all 3 edges at $21, 29$ are already present in the patch); if 21 or 29 has degree ≥ 4, the corresponding triangle is already non-cubic (sending $\leq 2/3$ to F_{10}; moreover, even including $21, 29$ in `ext` yields only 3 additional valid—and killed—quotients alongside 1,187 invalid ones, with zero survivors).
- *C_5-free clean $T_{\text{frz}} + H_6 + T'$ (`lines 1785--1788`):* Here $T_{\text{frz}} = (0, 1, 2)$ is a clean triangle (avoiding ab, with both frozen faces sharing edges $(0, 1)$ and $(0, 2)$ at the cubic cap endpoint 0), $H_6 = (2, 1, 3, 4, 5, 6)$ is a clean 6-face sharing edge $(1, 2)$, and $T' = (v, u, 7)$ is a clean triangle on one of the 5 remaining edges $uv \in \{(3, 4), (4, 5), (5, 6), (6, 2), (1, 3)\}$ of H_6. In the unmerged 3-face model, each of the 5 placements contains an explicit clean C_4 or C_8 avoiding ab. Could two distinct vertices $x \neq y$ of that C_4 or C_8 coincide in H^+? Any such identification must pair vertex 0 or vertex 7 with a vertex not sharing a face with it (since faces are simple and share at most one edge by Lemma 2.1(ii)): (a) identifying $0 = 7$ creates either a double vertex contact of T_{frz} with T' or a clean C_4 (0–1–3–0 or 0–2–6–0) or clean C_5 (0–1–3–4–0 or 0–2–6–5–0); (b) identifying $0 \in \{3, 4, 5, 6\}$ gives T_{frz} and H_6 a third common vertex, violating Lemma 2.1(ii); and (c) identifying $7 \in \{1, 2, 3, 4, 5, 6\} \setminus \{u, v\}$ gives T' and H_6 a third common vertex, again violating Lemma 2.1(ii). Thus all 5 explicit C_4/C_8 cycles lift simply to $H^+ - ab$. □

5.1. **Part (a): All C_3-Free General Planar Graphs ($\delta \geq 3$).** Clean faces have degrees in $\{5, 6, 7, 9, 10, 11, \dots\}$; only clean 5-faces ($\text{ch}_0 = +1$) are positive, and no two clean 5-faces share an edge (C_8). Fix a canonical total ordering $\prec$ on $V(H^+) \cup E(H^+) \cup F(H^+)$.

- **Rule V1 (High-Degree Absorption):** Every clean 5-face incident to at least one vertex in $V_{\geq 4}(H^+)$ sends $+1$ to its $\prec$-first such vertex. Every remaining *pure cubic* clean 5-face P has all

5 vertices of degree 3, so pure cubic 5-faces are pairwise vertex-disjoint (sharing a cubic vertex would force sharing an edge, giving C_8).

- **Rule P1 (Complete Ring Classification for Pure Cubic P):** Partition all pure cubic 5-faces P into three exhaustive cases: (1) if P touches at least one frozen face or clean ≥ 10-face, send $+1$ to the $\prec$-first such face; (2) else all 5 neighbors of P are clean in $\{6, 7, 9\}$: if at least two lie in $\{7, 9\}$, send $1/2$ to each of the $\prec$-first two (bounded by $w_P(7) = 1/2$ and $w_P(9) = 3/4$); (3) otherwise at least four neighbors are clean 6-faces, leaving exactly three rings: $(6, 6, 6, 6, 7)$ ($59 \to 0$ survivors), $(6, 6, 6, 6, 9)$ ($105 \to 0$ survivors), and $(6, 6, 6, 6, 6)$ ($46 \to 1$ survivor, the unique unidentified 15-rim *Corannulene*), all certified by Lemma 5.1.
- **Rule C1 (Corannulene Spoke / Valley Discharge):** By Lemma 5.1 (`check_cor_attach`), attaching a clean 5-face P' to either outer edge type of a Corannulene hexagon yields $106 \to 0$ survivors. Consequently: (a) no clean 6-face belongs to two distinct Corannulenes (else the second Corannulene's 5-face would attach along an outer edge of that 6-face), so distinct Corannulenes incident to a spoke vertex $v \in V_{\geq 4}(H^+)$ occupy *pairwise disjoint pairs of 6-face wedges* at v; and (b) no pure cubic 5-face is adjacent to a Corannulene valley edge. If any of the 5 spoke vertices $a_0, \ldots, a_4$ of a Corannulene lies in $V_{\geq 4}(H^+)$, P sends $+1$ to the $\prec$-first such spoke vertex; at any v of degree $k \geq 4$ receiving from c Corannulenes (consuming $2c$ disjoint 6-face wedges) and non-adjacent clean 5-faces in the remaining $k - 2c$ wedges,

$$\mathrm{ch}_{\mathrm{fin}}(v) \;\leq\; 6 - 2k + \left\lfloor \frac{k - 2c + 1}{2} \right\rfloor + c \;=\; 6 - 2k + \left\lfloor \frac{k+1}{2} \right\rfloor \;\leq\; 0 \qquad (\forall k \geq 4,\ 2c \leq k),$$

kernel-proved in `c3free_vertex_absorption`. Otherwise all 5 spoke vertices are cubic, and `check_cor_attach` kills all 5 valley degrees $d \in \{5, 6, 7, 9, 10\}$ ($990 \to 0$ survivors; $1{,}260 = 59 + 105 + 106 + 990$ total C_3-free rim kills), so every valley is frozen or clean ≥ 11 and receives $+1/5$ from P.
- **Graph-to-Token Map and Clean/Frozen Tally:** Around any clean d-face F_d ($d \geq 6$), each boundary edge is encoded as P (width 1, weight $w_P(d) \in \{0, 1/2, 3/4, 1\}$), V (width 2, weight $1/5$ for $d \geq 11$, covering the two consecutive Corannulene 6-face edges at a valley), or N (width 1, weight 0). No two P tokens are adjacent (C_8), and no P is adjacent to V (by the 106 `Cor+P` kills). Because pure cubic 5-faces are pairwise vertex-disjoint and each gives a $+3$ detour avoiding $V(F_d)$ (Lemma 2.1(ii)), m such 5-faces on F_d yield simple cycles of lengths $d+3j$ ($0 \leq j \leq m$), bounding the count of P by `max_p`. Exact cyclic DP (`c3free_clean_small_bounds`) verifies $\mathrm{ch}_{\mathrm{fin}}(f) \leq 0$ for clean $d \in \{6, 7, 9..13\}$; for clean $d \geq 14$, the packed Bellman–Ford bound (`c3free_clean_d_ge_14_bound`, slope 11/20) gives $\mathrm{ch}_{\mathrm{fin}}(f) \leq 6 - d + \frac{11}{20}d \leq 0$. Around any frozen face f_i ($d \geq 3$), breaking at ab produces a length-$(d-1)$ path in $\{N, P, V\}$ (with $w_P = 1, w_V = 1/5$ and no adjacent PP, PV, VP). If the path contains p P-tokens and v V-tokens, then for $p \geq 1$ the p P-tokens require at least $p - 1$ separating N-tokens (since P cannot touch P or V), so $L = d - 1 \geq 2p - 1 + 2v \implies p + v/5 \leq p + v \leq \lfloor d/2 \rfloor$ (and for $p = 0$, $L = d - 1 \geq 2v \implies v/5 \leq v \leq \lfloor (d-1)/2 \rfloor \leq \lfloor d/2 \rfloor$). This unbounded symbolic counting bound (also cross-checked finitely for $d \leq 24$ in Python and via the Bellman–Ford slope-1/2 potential in `c3free_frozen_cap_bound`) gives $\mathrm{ch}_{\mathrm{fin}}(f_i) \leq (6 - d) + \lfloor d/2 \rfloor \leq 4$ for all $d \geq 3$ and $\mathrm{ch}_{\mathrm{fin}}(f_1) + \mathrm{ch}_{\mathrm{fin}}(f_2) \leq 8 < 12$, contradicting (4).

5.2. **Part (b): All C_5-Free General Planar Graphs ($\delta \geq 3$, Zero R_{10}).** Clean faces have degrees in $\{3, 6, 7, 9, 10, 11, \ldots\}$; only clean 3-faces ($\mathrm{ch}_0 = +3$) are positive. Because $V_{\geq 4}(H^+)$ is independent on clean edges, every clean 3-face contains at most one vertex in $V_{\geq 4}(H^+)$, cannot touch any clean $\{3, 4, 5, 7, 8, 15\}$-face, and every clean 6-face touches at most one clean 3-face ($6 + 2 = 8$).

- **Vertex Absorption (Rules V3/V6) and Wedge Bounds:** Every clean 3-face incident to $v \in V_{\geq 4}(H^+)$ sends $+1$ to v if $d(v) = 4$ and $+2$ to v if $d(v) \geq 5$. For charged clean 6-faces H_6 ($c(H_6) \leq 1$), *Rule V6* applies with *non-valley priority*: if H_6 has a *non-valley* outer vertex $w \in V_{\geq 4}(H^+)$ (i.e. $w \notin V(T)$ and w is not shared with a second clean 6-face on the same charging triangle T), then H_6 sends $c(H_6)$ to its $\prec$-first non-valley outer high vertex w; otherwise, if H_6 has a *valley* outer vertex $w \in V_{\geq 4}(H^+)$ (shared with a second clean 6-face H_6' on T), then H_6 sends $\min(c(H_6), 1/2)$ to w (preserving and strictly improving the $1/2+1/2 = 1$ valley-pair wedge ceiling at w, because whenever one of (H_6, H_6') instead discharges to a non-valley outer high vertex, the other still sends at most $1/2$ to w). Around v of degree $k \geq 4$ with t incident clean 3-faces and h outer-charged clean 6-faces: clean 3-faces are non-adjacent (C_4), and the two wedges immediately flanking any clean 3-face T at v cannot send Rule-V6 charge to v (since $v \in V(T)$ is not an outer vertex of those faces). Moreover, for any three consecutive clean 6-faces H_1, H_2, H_3 around $v \in V_{\geq 4}(H^+)$, their shared spoke vertices $2, 3$ are adjacent to v and therefore cubic ($V_{\geq 4}$-independence), so any charging triangle on an inner spoke edge $(12, 2)/(2, 13)$ or $(15, 3)/(3, 16)$ is necessarily a shared valley triangle; across all 21 non-valley placements of two distinct charging triangles on any two of (H_1, H_2, H_3), noncrossing outer-rim enumeration (`3-hex+2T`, `verify_planar_eg.py:1760--1771`, 47 killed quotients, 0 survivors, kernel-verified in `c5VAbsWelds`) eliminates every non-valley pair. Hence:
  - $k = 4$ ($\mathrm{ch}_0 = -2$): $t = 2 \implies h = 0$, in $= 2$; $t = 1 \implies h \leq 1$, in $\leq 1 + 1 = 2$; $t = 0 \implies h \leq 3$ (since 4 clean 6-faces around v are killed by `4-hex`), and if $h = 3$ then by `3-hex+2T` at least two adjacent hexagons among (H_1, H_2, H_3) must share a valley triangle (contributing $1/2+1/2 = 1$ together, plus at most 1 from the third hexagon), giving in ≤ 2.
  - $k = 5$ ($\mathrm{ch}_0 = -4$): $t = 2 \implies h = 0$, in $= 4$; $t = 1 \implies h \leq 2$, in $\leq 2 + 2 = 4$; $t = 0$: if $h \leq 4$ then in ≤ 4; and $h = 5$ is completely impossible, because 0 valley triangles violate `3-hex+2T` on (H_1, H_2, H_3), 1 valley triangle (say on (H_1, H_2)) leaves (H_3, H_4, H_5) valley-free violating `3-hex+2T`, and ≥ 2 valley triangles must be non-adjacent (as each 6-face touches ≤ 1 triangle) and are eliminated by `5-hex+2v`.
  - $k \geq 6$ ($\mathrm{ch}_0 = 6 - 2k$): if $t = 0$, in $\leq k \leq 2k - 6$; if $t = k/2$, $h = 0$ and in $= k \leq 2k - 6$; if $1 \leq t < k/2$, at least $t + 1$ wedges flank the t triangles, so $h \leq k - 2t - 1$ and in $\leq 2t + h \leq k - 1 < 2k - 6$.

  Thus $\mathrm{ch}_{\mathrm{fin}}(v) \leq 0$ for every $v \in V_{\geq 4}(H^+)$. At a $(k = 4, t = 2)$ vertex $v = 0$, the two clean triangles $T_1 = (0, 1, 2)$ and $T_2 = (0, 3, 4)$ alternate in cyclic rotation order with the two flanking wedges F_a (between 02 and 03) and F_b (between 04 and 01); the two `k4t2` seeds (`verify_planar_eg.py:1758--1759`, giving $51 = 4 + 47$ total vertex-absorption rim kills in `c5VAbsWelds`) exclude both $F_a = F_b = 9$ and $F_a = F_{\mathrm{opp}}(T_1) = 9$ (encoded as vertex type `4m`, vs. `4s` for $k = 4, t = 1$ and `3` for cubic).
- **Complete Mutually Exclusive Stage-1 and Stage-3 Routing (Zero R_{10}):** Note first that at any cubic vertex incident to a clean triangle T, two clean 9-faces cannot meet ($9+9-2 = 16$), a clean 9- and 10-face cannot meet ($3+9+10-6 = 16$), and a clean 6-face cannot touch a clean 12-face ($6+12-2 = 16$) or 13-face ($3+6+13-6 = 16$). Every clean triangle T falls into one of the following mutually exclusive, exact charge-conserving branches (all matched edge-by-edge to `verify_planar_eg.py:1853--2000`):
  (i) *Pure cubic T ($+3$):* (a) Two frozen neighbors $(f_1, f_2, Z$; token $T_{\mathrm{frz}})$: send $(3/2, 3/2, 0)$ (and T_{frz} cannot be followed by a charged 6-face by the 5 direct $T_{\mathrm{frz}} + H_6 + T'$ kills at $\{C_4, C_8\}$). (b) Three clean 6-faces $(6, 6, 6)$: send $(1, 1, 1)$ to the three 6-faces (each with an outer high vertex uses Rule V6; pure cubic Mini-Corannulene routes across its valleys/opposite faces as in Stage 3(H3)). (c) Two clean 6-faces (H_L, H_R, X) (token $T_{2,L,R}$ on X, where $L, R \in \{3, 4s, 4m\}$ are the outer endpoints of H_L, H_R on X): whenever L (resp. R) is high, L (resp. R) is an outer high vertex of H_L (resp. H_R) absorbing up to $+1$ by Rule V6! Send $(1, 1, 1)$ if $X = 9$; if $X = 10$ or $X \in \{14\} \cup [17, \infty) \cup \mathcal{F}_{\mathrm{frz}}$,

send $(1/2, 1/2, 2)$ when $(L, R) = (\texttt{3}, \texttt{3})$, $(1, 1/2, 3/2)$ when L alone is high (symmetric for R), and $(1, 1, 1)$ (for $X = 10$) or $(5/6, 5/6, 4/3)$ (for $X \geq 14/\mathcal{F}_{\text{frz}}$) when both L, R are high; if $X = 11$, send $(0, 0, 3)$ when $(L, R) = (\texttt{3}, \texttt{3})$, $(3/4, 0, 9/4)$ when L alone is high, and $(5/6, 5/6, 4/3)$ when both L, R are high. (d) One clean 6-face $(6, X, Y)$ (token $T_{1,L/R}$ on X, Y): define $w_1(9) = 3/4, w_1(10) = 1, w_1(\geq 11/\mathcal{F}_{\text{frz}}) = 3/2$; if $X = 11$ (choosing $\prec$-first if both are 11), send 0 to 6, $w_1(Y)$ to Y, and $3 - w_1(Y) \leq 9/4$ to X; else send $w_1(X), w_1(Y)$ to X, Y and $3 - w_1(X) - w_1(Y) \in \{0, 1/2, 3/4, 1\}$ to 6. (e) Zero clean 6-faces (X, Y, Z) (token T): send $(2/3, 7/6, 7/6)$ on $(9, \geq 11/\mathcal{F}_{\text{frz}}, \geq 11/\mathcal{F}_{\text{frz}})$ and on $(10, \geq 11/\mathcal{F}_{\text{frz}}, \geq 11/\mathcal{F}_{\text{frz}})$; $(11/12, 11/12, 7/6)$ on $(10, 10, \geq 11/\mathcal{F}_{\text{frz}})$; and $(1, 1, 1)$ on $(10, 10, 10)$ or $(\geq 11/\mathcal{F}_{\text{frz}})^3$ (`c5free_budget_identities`).

(ii) *Non-cubic T at $v \in V_{\geq 4}$ (+2 left if $k = 4$; +1 left if $k \geq 5$; ordered $(F_{\text{opp}}, F_a, F_b)$ with F_a, F_b at v):* Observe first that no special two-frozen branch is needed: if a clean non-cubic T touches both frozen faces f_1, f_2, then f_1, f_2, T cannot close around the high vertex v (forcing $d(v) = 3$), so the cap endpoint in $V(T)$ is cubic and one of $\{f_1, f_2\}$ is F_{opp} while the other is F_a or F_b, covered directly by the general rules below (with $w_T(\mathcal{F}_{\text{frz}}) = 7/6, w_{Ts}(\mathcal{F}_{\text{frz}}) = 2/3, w_{T1}(\mathcal{F}_{\text{frz}}) = 3/2, w_{Ts6}(\mathcal{F}_{\text{frz}}) = 5/4$).

- *Exact T_{s6} caps:* Whenever $F_{\text{opp}} = 6$ and $F_i \neq 6$ ($i \in \{a, b\}$), let $L_i \in \{\texttt{3}, \texttt{4s}, \texttt{4m}\}$ be the outer endpoint of the shared edge between F_{opp} and F_i (if $L_i \neq \texttt{3}$, then L_i is a *non-valley* outer high vertex of F_{opp} absorbing up to +1 via Rule V6). Define the exact token cap

$$c_i^{\max} := \begin{cases} w_{Ts6,3}(F_i) & \text{if } L_i = \texttt{3} \quad \big(3/4 \text{ for 9, } 1 \text{ for 10, } 5/4 \text{ for } \geq 11/\mathcal{F}_{\text{frz}}\big), \\ w_{Ts,v}(F_i) & \text{if } L_i \neq \texttt{3} \quad \big(1/2 \text{ if } F_i = 9 \text{ and } v = \texttt{4m}, \text{ else } 2/3\big). \end{cases}$$

- $k \geq 5$ *(+1 left):* If at least two neighbors of T are not clean 6-faces, allocate +1 deterministically in $\prec$-order across the non-6 neighbors up to their token caps ($w_T \geq 2/3$, $w_{T1} \geq 3/4$, $w_{Ts} \geq 1/2$, $c_i^{\max} \geq 1/2$), so every clean 6-face receives 0. If $F_{\text{opp}} \neq 6$ and $F_a = F_b = 6$, send $+1 \leq w_{T2}(F_{\text{opp}}, L, R)$ to F_{opp} and 0 to F_a, F_b. If $F_{\text{opp}} = F_a = 6$ and $F_b \neq 6$ (symmetric for $F_b = 6$), send $c_b := \min(1, c_b^{\max})$ to F_b, $1 - c_b \leq 1/2$ to F_{opp}, and 0 to F_a (if $L_b \neq \texttt{3}$, L_b absorbs $1 - c_b \leq 1/2$ via Rule V6; if $L_b = \texttt{3}$, $1 - c_b \leq 1/4$ is absorbed via Rule V6 or Stage 3 by the universal $(T, H_6, X = 6)$ capacity bound $\geq 1/2$ below). If $F_{\text{opp}} = F_a = F_b = 6$, send $(1, 0, 0)$ to $(F_{\text{opp}}, F_a, F_b)$ (covered by the non-cubic Mini-Coranulene Stage-3 capacity ≥ 1 on F_{opp} below).
- $k = 4$ *(+2 left), 0 clean 6-faces:* Note that if $F_{\text{opp}} = 9$, then at the two cubic endpoints of F_{opp} both F_a, F_b are $\geq 11/\mathcal{F}_{\text{frz}}$ ($9+9-2 = 16$, $3+9+10-6 = 16$), so $(2/3, 2/3, 2/3)$ already balances +2 within $(w_T(9) = 2/3, w_{Ts}(\geq 11) = 2/3, 2/3)$. Exhaustively: send $(1, 1/2, 1/2)$ if $F_a = F_b = 9$ (only at `4s`); $(5/6, 1/2, 2/3)$ if $\{F_a, F_b\} = \{9, \geq 10/\mathcal{F}_{\text{frz}}\}$ (since $F_{\text{opp}} \geq 11/\mathcal{F}_{\text{frz}}$); and $(2/3, 2/3, 2/3)$ in all remaining cases (including $F_{\text{opp}} = 9$ and all $\geq 10/\mathcal{F}_{\text{frz}}$). Every triple sums to 2 and sends $\leq 2/3$ to every clean 10-face.
- $k = 4$ *(+2 left), ≥ 1 clean 6-face:* Order the two wedges at v with $F_a \prec F_b$; every branch below is exact, capacity-respecting, and verified against Stage-3 noncrossing outer-rim exclusions in `cert/verify_parity_and_dp.py`: (1) $F_{\text{opp}} = 6$ *and $F_a, F_b \neq 6$:* Set

$$c_a := \min(2, c_a^{\max}), \qquad c_b := \min(2 - c_a, c_b^{\max}), \qquad c_{\text{opp}} := 2 - c_a - c_b \geq 0.$$

If $L_a \neq \texttt{3}$ or $L_b \neq \texttt{3}$, then $c_a^{\max} + c_b^{\max} \geq 1/2 + 1/2 = 1$, so $c_{\text{opp}} \leq 1$ is absorbed directly by the non-valley outer high vertex L_a or L_b via Rule V6 (since F_{opp} is the unique clean 6-face on T). If $L_a = L_b = \texttt{3}$, then $c_{\text{opp}} = 0$ unless $\{F_a, F_b\} = \{9, 10\}$ ($c_{\text{opp}} = 1/4$) or $F_a = F_b = 9$ (only at `4s`; $c_{\text{opp}} = 1/2$): in either case, if F_{opp} has no outer high vertex then all 4 outer vertices are cubic; for $\{F_a, F_b\} = \{9, 10\}$ (w.l.o.g. $F_b = 10$), the outer face V_b flanking $F_b = 10$ at cubic L_b avoids $\{6, 7, 11, 12\}$ (0 survivors) and hence satisfies

$w_{6o}(V_b) \geq 1/2 > 1/4 = c_{\text{opp}}$; for $F_a = F_b = 9$, the outer face V_a flanking $F_a = 9$ at cubic L_a avoids $\{6, 7, 9, 11, 12\}$ (0 survivors) and hence satisfies $w_{6o}(V_a) \geq 1/2 = c_{\text{opp}}$. (2) $F_a = 6$ *and* $F_{\text{opp}}, F_b \neq 6$ *(symmetric for* $F_b = 6$*):* Set $c_{\text{opp}} := \min(2, w_{T1}(F_{\text{opp}}))$, $c_b := \min(2 - c_{\text{opp}}, w_{Ts,v}(F_b))$, and $c_a := 2 - c_{\text{opp}} - c_b \in [0, 7/12]$. Whenever $c_a > 0$, we have $w_{T1}(F_{\text{opp}}) < 3/2$, forcing $F_{\text{opp}} \in \{9, 10\}$: if $F_{\text{opp}} = 10$, then $c_a \leq 1/2$ and (when all 4 outer vertices of F_a are cubic) the outer face V_1 flanking $F_{\text{opp}} = 10$ avoids $\{6, 7, 11, 12\}$, giving $w_{6o}(V_1) \geq 1/2$; if $F_{\text{opp}} = 9$, then $c_a \leq 7/12 < 3/4$ and $(T, F_a, F_{\text{opp}} = 9)$ kills every cubic outer triple (V_1, O, V_2) with Stage-3 capacity $< 3/4$ (via $V_1 \notin \{7, 9, 11, 12\}$, $O \notin \{11, 12\}$, $V_2 \neq 12$, and all low-capacity pairs/triples killed). (3) $F_a = F_b = 6$ *and* $F_{\text{opp}} \neq 6$*:* Send $c_{\text{opp}} := \min(2, w_{T2}(F_{\text{opp}}, L, R))$ to F_{opp}, where L, R are the outer endpoints of F_a, F_b on F_{opp}. If $F_{\text{opp}} = 9$ ($c_{\text{opp}} = 1$), send $1/2$ to each of F_a, F_b (if all outer vertices are cubic, each has $X = F_{\text{opp}} = 9$, guaranteeing Stage-3 capacity $\geq 3/4 > 1/2$ as in (2)). If $F_{\text{opp}} \geq 10/\mathcal{F}_{\text{frz}}$: when $(L, R) = (\mathtt{3}, \mathtt{3})$, $w_{T2} \geq 2$ so both F_a, F_b receive 0; when $L \neq \mathtt{3}, R = \mathtt{3}$, $w_{T2} \geq 3/2$ so send $2 - c_{\text{opp}} \leq 1/2$ to F_a (absorbed by outer high vertex L via Rule V6) and 0 to F_b (symmetric for $R \neq \mathtt{3}$); when both $L, R \neq \mathtt{3}$, $w_{T2} \geq 1$ so send $(2 - c_{\text{opp}})/2 \leq 1/2$ to each of F_a, F_b (absorbed by L and R via Rule V6). (4) $F_{\text{opp}} = F_a = 6$ *and* $F_b \neq 6$ *(symmetric for* $F_b = 6$*):* Split into three exhaustive subcases according to (L_b, F_b): (4a) $L_b \neq \mathtt{3}$: send $c_b := w_{Ts,v}(F_b) \in \{1/2, 2/3\}$ to F_b, 1 to F_{opp} (completely absorbed by the non-valley outer high vertex L_b via Rule V6, leaving 0 Stage-3 outflow from F_{opp}), and $1 - c_b \in \{1/3, 1/2\}$ to F_a (absorbed via Rule V6 if F_a has any outer high vertex, or else by Stage 3 since every cubic outer triple (V, H_1, H_2) on $(T, F_a, F_{\text{opp}} = 6)$ with capacity $< 1/2$ dies across noncrossing rim quotients); (4b) $L_b = \mathtt{3}$ and $F_b \geq 10/\mathcal{F}_{\text{frz}}$: send $c_b := w_{Ts6,3}(F_b) \in \{1, 5/4\}$ to F_b and $b := (2 - c_b)/2 \in \{3/8, 1/2\}$ to each of F_{opp}, F_a. Any hexagon $H \in \{F_{\text{opp}}, F_a\}$ with an outer high vertex discharges $b \leq 1/2$ via Rule V6; if H is all-cubic (so the shared valley vertex is cubic with shared valley face V), list the three cubic-endpoint outer edges $e_1 \prec e_2 \prec e_3$ of H in the fixed edge order $\prec$ with effective per-edge caps $c_i := 0$ if e_i borders $V = 9$ and $c_i := w_{6o}(F(e_i))$ otherwise (so $c_i = 1/4$ when $F(e_i) = 7$), and route $b \leq 1/2$ greedily in order e_1, e_2, e_3 up to c_1, c_2, c_3, leaving residual $\max(0, b - \sum_{i=1}^{3} c_i)$. By the universal $(T, H_6, X = 6)$ replay on both orientations (`cert/verify_parity_and_dp.py`), $V \notin \{10, 11, 12\}$ (and $V \neq 3$ by the C_8 perimeter of $T \cup F_{\text{opp}} \cup F_a \cup V$), and $\sum_{i=1}^{3} c_i \geq 1/2 \geq b$ holds in all four exhaustive valley cases, forcing residual 0: (i) $V \in [13, \infty) \cup \mathcal{F}_{\text{frz}}$, where the valley edge alone has $w_{6o}(V) \geq 1/2$; (ii) $V = 7$, where every all-cubic outer triple with $V = 7$ and $\sum_{i=1}^{3} w_{6o}(F(e_i)) < 1/2$ is killed on both orientations (so $\sum_{i=1}^{3} c_i \geq 1/2 \geq b$, leaving residual 0 while sending at most $w_{6o}(7) = 1/4$ across the valley edge into $V = 7$, respecting the global F_7 per-edge ceiling $1/4$ bounded by `f7_3h` + `f7_pair_k` to ≤ 4 charged 6-face edges); (iii) $V = 6$, where $w_{6o}(6) = 0$ and every all-cubic triple with $V = 6$ and total capacity $< 1/2$ is killed on both orientations; and (iv) $V = 9$, where setting $c_i = 0$ on the valley edge prevents consecutive 6_o contacts on F_9 (matching `ok_c5` at $d = 9$), while the dedicated $V = 9$ replay on both orientations guarantees $\sum_{i=1}^{3} c_i \geq 1/2 \geq b$ strictly off the valley across the other two outer edges; (4c) $L_b = \mathtt{3}$ and $F_b = 9$: send $w_{Ts6,3}(9) = 3/4$ to F_b, leaving $5/4$ for (F_{opp}, F_a). Here the outer face G_1 of F_{opp} flanking $F_b = 9$ at cubic L_b avoids $\{6, 7, 9, 11, 12\}$ (hence $w_{6o}(G_1) \geq 1/2$), and the 4th wedge F_4 at the degree-4 vertex v avoids $\{6, 9, 11, 12\}$ (hence $w_{6o,\text{nc}}(F_4) \geq 1/4$, and $\geq 1/2$ whenever $F_4 \neq 7$). Route the $5/4$ deterministically: if F_{opp} has a non-valley outer high vertex, assign $(1, 1/4)$ to (F_{opp}, F_a) (1 absorbed via Rule V6, $1/4$ from F_a across its edge into F_4); else if F_a has a non-valley outer high vertex, assign $(1/4, 1)$ (1 absorbed via Rule V6, $1/4$ from F_{opp} across its edge into G_1); else if the shared valley vertex y_1 is high, assign $(3/4, 1/2)$

(1/2 each to y_1 via Rule V6, plus 1/4 from F_{opp} across its edge into G_1); else all outer vertices of both F_{opp} and F_a are cubic, so assign $(3/4, 1/2)$ and route across distinct outer boundary edges obeying their exact per-edge caps (w_{6o} on cubic-endpoint edges, $w_{6o,\text{nc}}$ on F_4): F_{opp} sends 1/2 across its edge into G_1 and 1/4 across its edge into G_2 (if $G_2 \neq 6$, since $G_2 \notin \{9, 11, 12\}$) or across its valley edge into V (if $G_2 = 6$, since $(V = 6, G_2 = 6)$ is killed and $V \notin \{10, 11, 12\}$), while F_a sends 1/2 across its edge into F_4 (if $F_4 \neq 7$) or 1/4 across its edge into $F_4 = 7$ and 1/4 across its edge into its outer face H_2 flanking F_4 (since $(F_4 = 7, H_2 = 6)$ is killed and $H_2 \notin \{9, 11, 12\}$). Every global receiving-face restriction holds unconditionally without assuming $\{G_1, G_2, V\}$ and $\{F_4, H_2\}$ are pairwise distinct faces: F_a never sends positive charge across the shared valley edge into V (so $V = 9$ receives charge from at most one of F_{opp}, F_a, avoiding consecutive 6_o contacts on F_9), $G_1 \notin \{7, 9\}$, $G_2 \neq 9$, $H_2 \neq 9$, $F_4 \neq 9$, and every edge into any clean 7-face pays at most 1/4, bounded globally by `f7_3h` + `f7_pair_k` ($\leq 4 \cdot \frac{1}{4} = 1$). (5) $F_{\text{opp}} = F_a = F_b = 6$ *(non-cubic Mini-Corannulene):* Send $(1, 1/2, 1/2)$ to $(F_{\text{opp}}, F_a, F_b)$ (both valleys of F_{opp} avoid $\{9, 10, 11, 12\}$ and cannot both lie in $\{6, 7\}$; if a valley lies in $\{6, 7\}$, the opposite face of F_{opp} avoids $\{6, 7, 11, 12\}$ giving $\geq 1/2$, while F_a, F_b each satisfy the universal $(T, F_a, F_{\text{opp}} = 6)$ capacity bound $\geq 1/2$).

In Stage 3, every charged 6-face that did not fully discharge via Rule V6 routes its remaining balance across its cubic-endpoint outer faces within w_{6o} (1/4 on 7, 3/4 on 9, 2/3 on 10, 0 on 11, 12, 1/2 on ≥ 13, 1 on $\mathcal{F}_{\text{frz}}$) and (when adjacent to $v \in V_{\geq 4}$) its 4th wedge F_4 within $w_{6o,\text{nc}}$ (1/4 on 7, 0 on 9, 11, 12, 1/2 on $10, \geq 13$, 1 on $\mathcal{F}_{\text{frz}}$). Pure cubic branches (b)–(d) are covered by the exact same universal noncrossing outer-rim capacity bounds verified in `cert/verify_parity_and_dp.py` (14,962 killed quotients, 0 survivors across all Stage-3 exclusions: $(T, H_6, X = 10)$ gives $w_{6o}(V_b) \geq 1/2$, covering $(6, 10, 10)$ and $(6, 10, \geq 11)$; $(T, H_6, X = 9)$ gives capacity $\geq 3/4$, covering $(6, 9, \geq 11)$; $(6, 6, 9)$ gives $w_{6o}(V_b) \geq 1/2$ and forces $w_{6o}(O) \geq 1/2$ whenever $V_a \in \{6, 7, 9\}$; and pure cubic Mini-Corannulene $(6, 6, 6)$ kills all $(6, 6)/(6, 7)/(7, 7)$ valley pairs and forces opposite face $\geq 1/2$ next to any $\{6, 7\}$-valley).

- **Valley-9 Routing (25 Seeds, 2,887 Rim Kills) and Exact Case Correspondence:** On a valley triangle $T(6, 6, 9)$ at $v_0 = 0 \in V_{\geq 4}(H^+)$, hexagon $H_1 = (0, 1, 10, 11, 12, 2)$ sends 1/2 to 0 and discharges its remaining 1/2 across its outer neighbors F_A (on $(1, 10)$) and F_B (on $(10, 11)$). If either F_A or F_B is frozen or clean with degree in $\{10\} \cup [13, \infty)$, then H_1 immediately sends its remaining 1/2 to that face (bounded by $w_{6o}(10) = 2/3, w_{6o}(\geq 13) = 1/2, w_{6o,\text{nc}}(10, \geq 13) = 1/2$, and $w_{6o}(\mathcal{F}_{\text{frz}}) = w_{6o,\text{nc}}(\mathcal{F}_{\text{frz}}) = 1$). Otherwise both F_A and F_B are clean in $\{6, 7, 9, 11, 12\}$. Because 10 and 11 cannot both lie in $V_{\geq 4}(H^+)$, the 27 ordered configurations $(d_B, d_A) \in \{6, 7, 9\}^2 \times \{\text{cubic}, \ d(10) \geq 4, \ d(11) \geq 4\}$ plus $d_A, d_B \in \{11, 12\}$ are exhausted by the exact 25 rim-quotient checks in `verify_planar_eg.py:1736--1751` (2,887 killed, 0 survivors):
  (1) 11 seeds (`lines 1736--1741`) kill $(6, 6), (6, 7), (7, 6)$ in all 3 degree cases plus $(7, 7)$ cubic and $(7, 7)$ $d(10) \geq 4$;
  (2) 6 seeds (`lines 1742--1745`) kill $d_A \in \{11, 12\}$ (unmerged at 10, subsuming both $d(10) = 3$ and $d(10) \geq 4$) and $d_B \in \{11, 12\}$ (both $d(11) = 3$ and $d(11) \geq 4$);
  (3) 8 seeds (`lines 1746--1751`) kill $(6, 9), (7, 9), (9, 6), (9, 7)$ for both $d(10) \geq 4$ and $d(11) \geq 4$ — note that each $d(10) \geq 4$ seed leaves the rim neighbors of 10 on F_A and F_B unmerged, so `rim_quotients` enumerates their identification and strictly subsumes the corresponding cubic seed;
  (4) all 3 $(9, 9)$ cases immediately form C_{16} ($F_B = 9$ shares an edge with $V_9 = 9$ in cubic/$d(10) \geq 4$, and $F_B = 9$ shares an edge with $F_A = 9$ in $d(11) \geq 4$);
  (5) the sole remaining pair $(7, 7)$ with $d(11) \geq 4$ places a second high-degree vertex $11 \in V_{\geq 4}$ directly on H_1 alongside two neutral spacer wedges $F_B = 7$ and $V_9 = 9$, leaving $k - 3$ remaining wedges at vertex 11 outside the three wedges $\{H_1, F_B, V_9\}$. Sending the

remaining $1/2$ of H_1 directly to vertex 11 cannot double-count on wedge H_1: H_1 is itself one of the three excluded wedges (so no other source charges vertex 11 through wedge H_1), and H_1 touches only the single valley triangle $T(6,6,9)$ (sending $1/2$ to 0, $1/2$ to 11, and 0 elsewhere, never sending additional Rule-V6 charge to 11). Thus when $k=4$, the single remaining wedge sends at most $+1$, giving $\mathrm{ch}_{\mathrm{fin}}(11) \le 6-8+\frac{1}{2}+1 = -\frac{1}{2} < 0$; when $k \ge 5$, at most $\lceil (k-3)/2 \rceil$ of the $k-3$ remaining wedges are non-adjacent clean 3-faces (each sending $+2$) while the other wedges send at most $+1$ each, giving $\mathrm{ch}_{\mathrm{fin}}(11) \le 6-2k+\frac{1}{2}+(k-3)+\lceil (k-3)/2 \rceil \le 6-2k+\frac{3k-7}{2} = \frac{5-k}{2} \le 0$.

Hence H_1 never sends charge to a clean $\{9,11,12\}$-face, certifying $w_{6o,\mathrm{nc}}(d) = 0$ for $d \in \{9,11,12\}$. On clean $d=7$, 71 cubic 3-hex rim kills (`f7_3h`) plus 31 high-vertex 2-hex kills (`f7_pair_k`; $102 = 71+31$ total F_7 kills) ensure at most 4 charged 6-faces touch F_7, so $\mathrm{ch}_{\mathrm{fin}}(F_7) \le -1+4\cdot\frac{1}{4} = 0$. Reconciling all counts: $2{,}962 = 2{,}887$ (v9) $+ 4$ (core vabs) $+ 71$ (cubic F_7), and adding 47 (`3-hex+2T`) $+ 71$ (aux_{10}) $+ 385$ ($F_{10}H_6$) $+ 31$ (F_7-pair) $+ 5$ (T_{frz}) gives 3,501 total C_5-free kills (in Lean: $71+385+51+102+5+2{,}887 = 3{,}501$).

- **Elimination of R_{10} via the F_{10} 5-Triangle Bound ($47/12 < 4$):** On a clean 10-face F_{10} with 5 adjacent clean triangles, two triangles on consecutive edges share a common vertex in $V_{\ge 4}(H^+)$ (else C_4), and $V_{\ge 4}(H^+)$ is independent along C_{10}. For any clean 6-face H_6 on a non-triangle gap edge $g=(a,b)$ of $F_{10}+5T$, switching independently whether H_6 shares an edge with a flanking triangle at a (`sh_L`) and at b (`sh_R`) maps every realized attachment in cyclic order to one of the 102 canonical models in `verify_planar_eg.py:1827--1854`, all of which die across 385 full-rim quotients; hence incoming hexagon charge to F_{10} is 0. Moreover, for any non-triangle gap edge a flanked by two triangles with both *rim endpoints* $a, (a+1)$ mod 10 cubic, its adjacent face F_{aux} shares edges with both flanking triangles and cannot be $3,5,6,7,8,9$ (C_4, C_5, H_6 kills, C_8, $3+9+10-6=16$) and cannot be 10 (all 10 canonical $F_{10}+5T+\mathrm{aux}_{10}$ configurations die across 71 full-rim quotients, `lines 1814--1826`, plus the 3 additional valid killed quotients when apices $21, 29$ are included in `ext`); hence F_{aux} is frozen or clean ≥ 11. By Stage 1 above: every non-cubic triangle on F_{10} (whether its high vertex is on C_{10} or at its off-rim apex) sends at most $2/3$ to F_{10} (since $F_{\mathrm{opp}} = 9$ and $F_a = 9$ at cubic vertices of F_{10} are forbidden by $3+9+10-6=16$); and every pure cubic triangle on F_{10} adjacent to $s \in \{0,1,2\}$ safe gaps has s neighbors certified ≥ 11/frozen and sends at most 1 ($s=0$), $11/12$ ($s=1$), or $2/3$ ($s=2$) to F_{10} (if a triangle has cubic rim endpoints but a high apex, its charge is $\le 2/3 \le \min(1, 11/12, 2/3)$ and its cubic rim endpoints still enforce the adjacent safe-gap wing contact). Checking all $\binom{10}{5} = 252$ placements against all 123 independent high-vertex subsets of C_{10} (30,996 pairs, 2,136 admissible; kernel-proved in `c5free_f10_5tri_bound`) yields maximum incoming charge $\mathbf{47/12 < 4}$. Thus F_{10} never overflows, eliminating R_{10} completely.
- **Graph-to-Token Map and Clean/Frozen Tally:** Each boundary edge/arc of a clean d-face ($d \ge 9$) or frozen face ($d \ge 3$) maps to tokens in `verify_planar_eg.py:1872--2033` recording endpoint vertex types in $\{\mathtt{3}, \mathtt{4s}, \mathtt{4m}\}$ and contact tags in $\{\mathtt{N}, \mathtt{T}, \mathtt{Ts}, \mathtt{6s}, \mathtt{Ts6}, \mathtt{6o}, \mathtt{6o_nc}\}$ (with verified T_1, T_2 shifts $(0,1,3,4)$). Exact non-cubic cyclic DP ($d \in \{9..14\}$), plain BF ($d=15$, 44e, slope 3/5), and 17-row backward DP ($d=17$) certify $\mathrm{ch}_{\mathrm{fin}}(f) \le 0$ (`c5free_clean_small_bounds`; all 7 weighted DP-collapse simulations to the packed Lean automata are verified in `cert/verify_parity_and_dp.py`). Exact rational Bellman–Ford certifies clean $d \ge 18$ (452 edges, $b = 2/3$, `c5free_clean_d_ge_18_bound`) and frozen `CAP` $d \ge 3$ (1,029 edges, $b_f = 1$, $\mathrm{ch}_{\mathrm{fin}}(f_1) + \mathrm{ch}_{\mathrm{fin}}(f_2) \le 10 < 12$, `c5free_frozen_cap_bound`), contradicting (4).

5.3. **Part (c): All C_6-Free General Planar Graphs ($\delta \ge 3$).** Since $H^+ - ab$ is C_6-free, there are *zero* clean 6-faces, and no two faces in {clean 3-faces} $\cup$ {clean 5-faces} share an edge (C_4, C_6, C_8). Every clean face touching a clean 3-face has degree ≥ 9 ($d \ne 15, 16$), and every clean face touching a clean 5-face has degree ≥ 7 ($d \ne 8, 13, 16$). No Stage-2 or Stage-3 relays are needed:

- **One-Line Vertex Absorption:** Every clean 5-face incident to at least one vertex in $V_{\geq 4}(H^+)$ sends $+1$ deterministically to its $\prec$-first such vertex (preventing overspending when a 5-face has two non-adjacent high vertices), and every clean 3-face incident to its unique vertex $v \in V_{\geq 4}(H^+)$ sends $+1$ to v if $d(v) = 4$ and $+2$ if $d(v) \geq 5$. Around any vertex v of degree $k \geq 4$ incident to t clean 3-faces and p clean 5-faces, non-adjacency implies $t + p \leq \lfloor k/2 \rfloor$, yielding

$$\mathrm{ch}_{\mathrm{fin}}(v) \;\leq\; 6 - 2k + \begin{cases} t+p & (k=4), \\ 2t+p & (k \geq 5) \end{cases} \;\leq\; 6 - 2k + 2\lfloor k/2 \rfloor \;\leq\; 0 \qquad (\forall k \geq 4),$$

  kernel-proved in `c6free_vertex_absorption`. Every remaining pure cubic 5-face sends $1/5$ across each of its 5 edges.
- **Deterministic Stage-1 Clean 3-Face Rules:** Note that two clean 9-faces cannot share an edge ($9 + 9 - 2 = 16$), a clean 3-, 9-, and 10-face cannot meet at a cubic vertex u (by Lemma 2.1(ii), the three faces share only their pairwise edges at u, so the outer boundary of $(F_{10} \cup F_9 \cup T) \setminus \{u\}$ is an explicit simple cycle of length $8 + 7 + 1 = 16$), and `k4t2` Seed #1 (`verify_planar_eg.py:2079`) kills $F_a = F_b = 9$ at a $(k = 4, t = 2)$ vertex (`4m`).
  - *Pure cubic T ($+3$):* Send $(3/4, 9/8, 9/8)$ on $(9, \geq 11/\mathcal{F}_{\mathrm{frz}}, \geq 11/\mathcal{F}_{\mathrm{frz}})$; $(4/5, 11/10, 11/10)$ on $(10, \geq 11/\mathcal{F}_{\mathrm{frz}}, \geq 11/\mathcal{F}_{\mathrm{frz}})$; $(9/10, 9/10, 6/5)$ on $(10, 10, \geq 11/\mathcal{F}_{\mathrm{frz}})$; and $(1, 1, 1)$ on $(10, 10, 10)$ or $(\geq 11/\mathcal{F}_{\mathrm{frz}})^3$. (Every triple sums to 3.)
  - *Non-cubic T at $k = 4$ ($+2$ left; ordered $(F_{\mathrm{opp}}, F_a, F_b)$ with F_a, F_b at v):* (1) if $F_{\mathrm{opp}} = 9$, send $(1/2, 3/4, 3/4)$; (2) else if $F_a = F_b = 9$ (possible only at `4s`, $k = 4, t = 1$), send $(1, 1/2, 1/2)$; (3) else if $\{F_a, F_b\} = \{9, 10\}$ (so $F_{\mathrm{opp}} \geq 11/\mathcal{F}_{\mathrm{frz}}$), send $(13/12, 1/4, 2/3)$ (with $1/4$ to 9 and $2/3$ to 10; note $13/12 \leq 6/5$); (4) else if one of F_a, F_b is 9 and the other is $\geq 11/\mathcal{F}_{\mathrm{frz}}$, send $(1, 1/4, 3/4)$ (with $1/4$ to 9 and $3/4$ to $\geq 11/\mathcal{F}_{\mathrm{frz}}$); (5) else all three are $\geq 10/\mathcal{F}_{\mathrm{frz}}$: send $(2/3, 2/3, 2/3)$. (Every triple sums to 2.)
  - *Non-cubic T at $k \geq 5$ ($+1$ left):* Send the fixed allocation $(1/2, 1/4, 1/4)$ to $(F_{\mathrm{opp}}, F_a, F_b)$ (sum = 1), which is dominated edge-by-edge by every `4m` receiver weight ($1/2 \leq w_T(d)$ and $1/4 \leq w_{Ts,m}(d)$ for all $d \geq 9$ and frozen).

  Crucial F_{10} non-cubic ceiling: Every non-cubic triangle T adjacent to a clean 10-face F_{10} (whether its high vertex lies on ∂F_{10} or at its off-rim apex) sends at most $2/3$ to F_{10}. Indeed, at $k \geq 5$ every payout is $\leq 1/2 < 2/3$; at $k = 4$, if $F_{10} = F_{\mathrm{opp}}$ then both endpoints of $T \cap F_{10}$ are cubic so neither F_a nor F_b can be 9 (by the $8 + 7 + 1 = 16$ cubic $(3, 9, 10)$-exclusion above), forcing branch (5) ($2/3$); and if $F_{10} \in \{F_a, F_b\}$ then F_{opp} meets F_{10} and T at a cubic vertex so $F_{\mathrm{opp}} \neq 9$, ruling out branch (1) and forcing branch (3) or (5) ($2/3$).
- **Degree-7 Bound and Complete Reduction of Clean Decagons to the 9 Checked Seeds (45 Full-Rim Kills):** A clean 7-face cannot touch any clean 3-face ($7 + 3 - 2 = 8$) and its adjacent pure cubic 5-faces are non-adjacent on C_7 (every pure cubic 5-face has both rim endpoints cubic, so sharing a vertex forces an edge contact $\implies C_8$), so at most $\lfloor 7/2 \rfloor = 3$ pure cubic 5-faces touch F_7, giving $\mathrm{ch}_{\mathrm{fin}}(F_7) \leq -1 + 3 \cdot \frac{1}{5} = -\frac{2}{5} < 0$. On a clean 10-face F_{10} ($\mathrm{ch}_0 = -4$) with t adjacent clean 3-faces and p pure cubic 5-faces: $t \geq 6$ creates C_{16} ($10 + 6 = 16$), so $t \leq 5$. Because every pure cubic 5-face has both endpoints cubic, no pure cubic 5-face can share a vertex on C_{10} with another pure cubic 5-face (C_8) or with any triangle (C_6); only two triangles may share a high vertex on C_{10}.
  - If $t \leq 3$: since each 5-face requires two flanking non-adjacent edges, $t + p \leq 5$ (even if two triangles share a high vertex, $p \leq 2$ unless $t \leq 2$, in all cases $t + p/5 \leq 3 + 2/5 = 17/5 < 4$).
  - If $t = 4$ and $p = 0$: incoming charge $\leq 4(1) = 4$. If $t = 4$ and $p \geq 1$: if any two of the 4 triangles are consecutive (sharing a high vertex on C_{10}), then at least 2 triangles are non-cubic (sending $\leq 2/3$ each), and when $p = 2$ the 2 separated 5-faces leave at most 6 edges for 4 triangles forcing ≥ 2 consecutive pairs (≥ 3 non-cubic triangles); hence incoming charge is at most $2(2/3) + 2(1) + 1/5 = 53/15 < 4$ ($p = 1$) or $3(2/3) + 1 + 2(1/5) = 17/5 < 4$ ($p = 2$).

Thus the *only* layout with $t = 4, p \geq 1$ that can exceed 4 has all 4 triangles non-adjacent, forcing the unique alternating $4T + 1P$ layout (Seed #2, `line 2081`, killed).

– If $t = 5$ ($p = 0$): call a non-triangle gap edge *safe* if it is flanked by two triangles and *both* of its rim endpoints on C_{10} are cubic (so its adjacent face F_{aux} is forced to share edges with both flanking triangles; note $F_{\text{aux}} \notin \{3, 5, 6, 7, 8, 9\}$ by $C_4, C_6, C_8, 3 + 9 + 10 - 6 = 16$, and if $F_{\text{aux}} = 10$ is killed, $F_{\text{aux}} \geq 11/\mathcal{F}_{\text{frz}}$). A triangle with a high vertex on C_{10} sends $\leq 2/3$, and a triangle with both rim endpoints cubic adjacent to $r \in \{0, 1, 2\}$ safe gaps sends at most 1 ($r = 0$), 9/10 ($r = 1$), or 4/5 ($r = 2$) (even if its off-rim apex is high, its payout is $\leq 2/3 \leq 4/5$ and its cubic rim endpoints still enforce safe-gap wing contacts!). Let s_{rim} be the number of triangles incident to a high vertex on C_{10}: (a) if $s_{\text{rim}} \geq 3$, incoming charge $\leq 2(1) + 3(2/3) = 4$; (b) if $s_{\text{rim}} = 0$, all 5 triangles alternate with all 10 rim vertices cubic, so all 5 gaps are safe and map to Seed #3 (`f10_5t + aux10`, `line 2083`, killed), giving $\leq 5(4/5) = 4$; (c) if $s_{\text{rim}} = 1$, all 5 triangles alternate and the single high rim vertex destroys only 1 gap, leaving 4 safe gaps (each with both endpoints cubic, so Seed #3 applies to all 4 safe gaps!), giving incoming charge $\leq \frac{2}{3} + \frac{9}{10} + 3 \cdot \frac{4}{5} = \frac{119}{30} < 4$ (`c6free_f10_and_budget_identities`); (d) if $s_{\text{rim}} = 2$ with no two triangles sharing a high rim vertex, all 5 triangles alternate and the two non-adjacent high rim vertices destroy 2 gaps, leaving 3 safe gaps (Seed #3 applies to all 3!), so the 3 remaining triangles either have safe-gap counts $(0, 2, 2)$ or $(1, 1, 2)$, giving incoming charge $\leq \max\big(2 \cdot \frac{2}{3} + 1 + 2 \cdot \frac{4}{5},\ 2 \cdot \frac{2}{3} + 2 \cdot \frac{9}{10} + \frac{4}{5}\big) = \max\big(\frac{59}{15}, \frac{118}{30}\big) = \frac{59}{15} < 4$; (e) if $s_{\text{rim}} = 2$ with two triangles sharing one high rim vertex v (both receiving $\leq 2/3$), placing the remaining 3 non-adjacent triangles on the remaining 8-edge path yields exactly two symmetry classes (`b_L3` and `b_L4`) with safe-gap counts $(2, 2, 1)$ and $(2, 1, 1)$ on those three triangles, and because $s_{\text{rim}} = 2$, all 6 endpoints of the 3 flanked gaps in each class are cubic! Hence Seeds #4–6 (`lines 2085--2088`) and Seeds #7–9 (`lines 2090--2093`) apply with both wing contacts forced at every flanked gap, giving incoming charge at most

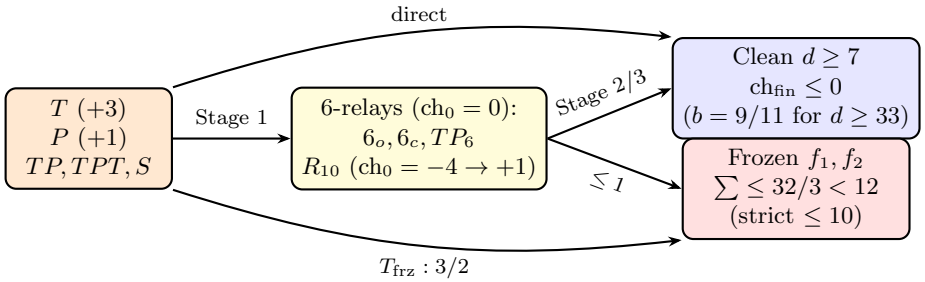
$$\max\Big(2 \cdot \frac{2}{3} + 2 \cdot \frac{4}{5} + \frac{9}{10},\ \ 2 \cdot \frac{2}{3} + \frac{4}{5} + 2 \cdot \frac{9}{10}\Big) \ = \ \max\Big(\frac{23}{6},\ \frac{59}{15}\Big) \ = \ \frac{59}{15} \ < \ 4$$

(`c6free_f10_and_budget_identities`).

Together with Seed #1 (`k4t2`), all 45 full-rim quotients across the 9 seeds carry kernel-certified C_8/C_{16} cycles (`c6free_outer_rim_all_killed`).

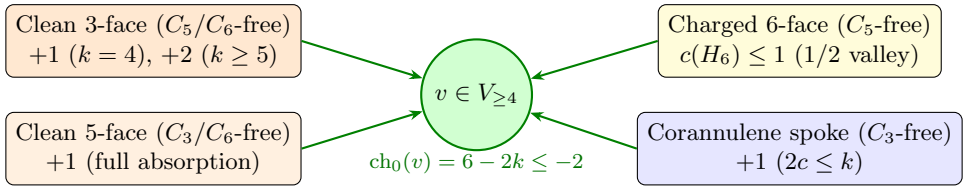
- **State-Graph Mappings, $(0, 1)$ Detour Disjointness, and Clean/Frozen Tally:** Around clean $d \in \{9, 11..14\}$, each edge maps to $\{N_{33}, N_{34s}, N_{4s3}, N_{34m}, N_{4m3}, T, T_{s,34s/4s3}, T_{s,34m/4m3}, P\}$ (omitting T/T_s at $d = 15$ and P at $d = 13$, with T/T_s shift $(0, 1)$, non-adjacent T/P at cubic vertices, and non-adjacent T_s at `4s`). To justify the $(0, 1)$ shift mask across $d \in \{9, 11..14\}$: every triangle's off-rim apex avoids $V(F_d)$ by Lemma 2.1(ii), and *no two distinct clean triangles $T_1 = (u_1, v_1, w)$, $T_2 = (u_2, v_2, w)$ on F_d ($d \leq 14$) can share a common off-rim high apex $w \in V_{\geq 4}(H^+)$*. Indeed, by $V_{\geq 4}$-independence all four rim endpoints u_1, v_1, u_2, v_2 are cubic and therefore pairwise distinct (sharing a cubic vertex forces an edge contact $\implies C_4$), splitting ∂F_d into two arcs of lengths $a, b \geq 1$ with $a + b = d - 2$. Traversing either arc together with the two edges of T_1, T_2 at w produces simple cycles of all lengths $\{a+2, a+3, a+4, b+2, b+3, b+4\}$ in $H^+ - ab$. Since $d \leq 14$, $\min(a, b) \leq \lfloor (d-2)/2 \rfloor \leq 6$, so $\{a+2, a+3, a+4\} \cup \{b+2, b+3, b+4\}$ always intersects $\{4, 6, 8\}$, contradiction. Hence all triangle detours on F_d ($d \in \{9, 11..14\}$) have pairwise disjoint off-rim vertices (consecutive triangles at a high rim vertex share only that rim endpoint, so substituting their detours onto consecutive rim edges still produces a simple cycle)! Exact cyclic DP (`c6free_clean_small_bounds`; all 5 weighted DP-collapse simulations to the packed Lean automata are verified in `cert/verify_parity_and_dp.py`) certifies $\text{ch}_{\text{fin}}(f) \leq 0$. For clean $d \geq 15$, merging `4s`/`4m` into a single high-vertex type `4` (using upper weights $w_T = 6/5, w_{Ts} = 3/4, w_P = 1/5$) gives the 20-edge Bellman–Ford potential ($b = 3/5$, `c6free_clean_d_ge_15_bound`), and breaking frozen faces ($d \geq 3$) at ab gives the 54-edge `CAP` potential ($b_f = 1$,

(a) Cubic Planar Face Discharging (Theorem 1.1)



Euler: $\sum_f (6 - d(f)) = 12$ (667e clean $d \geq 33$ BF; 1,351e relaxed / 1,588e strict frozen BF)

(b) General Planar Vertex-Face Absorption (Theorem 1.3)



Euler: $\sum_v (6 - 2d(v)) + \sum_f (6 - d(f)) = 12$; $\mathrm{ch}_{\mathrm{fin}}(v) \leq 0$ ($\forall k \geq 4$)

FIGURE 3. Discharging architecture: (a) three-stage cubic face discharging for Theorem 1.1 (distinguishing zero-initial-charge 6-face relays from the Stage-2 10-face relay R_{10} whose -4 deficit is filled to $+1$ by five triangles, and qualifying the 9/11 slope for clean $d \geq 33$); (b) general planar vertex-face absorption at independent high-degree sinks $v \in V_{\geq 4}(H^+)$ for Theorem 1.3, labeled by forbidden-cycle regime.

zero `CAP`-endpoint assumptions, $\mathrm{ch}_{\mathrm{fin}}(f_1) + \mathrm{ch}_{\mathrm{fin}}(f_2) \leq 10 < 12$, `c6free_frozen_cap_bound`), contradicting (4).

5.4. **Sharpness Witnesses: C_{60}, TD_{60}, and 119-Vertex $\Delta = 6$ 1-Sums.** The Lean 4 module `SharpWitnesses.lean` kernel-verifies that the truncated icosahedron C_{60} (60 vertices, 90 edges, 12 pentagons + 20 hexagons) and truncated dodecahedron TD_{60} (60 vertices, 90 edges, 20 triangles + 12 decagons) are nonempty cubic plane maps with exact dart bijections and single cyclic rotations at every vertex, explicit simple C_{16} cycles, and no C_3, C_4, C_8 (for C_{60}) or no $C_4, C_5, C_6, C_7, C_8, C_9$ (for TD_{60}).

Proposition 5.2 (119-Vertex $\Delta = 6$ Planar 1-Sums)**.** *For $H \in \{C_{60}, TD_{60}\}$, let $H \cdot H$ be the 1-sum obtained by taking two disjoint copies $H^{(1)}, H^{(2)}$ of H and identifying a single vertex $v_0^{(1)} \in V(H^{(1)})$ with a single vertex $v_0^{(2)} \in V(H^{(2)})$ into a cut-vertex v_0. Then $H \cdot H$ is a finite simple planar graph on 119 vertices with $\delta(H \cdot H) = 3$, $\Delta(H \cdot H) = 6$, an explicit simple C_{16}, and the exact same forbidden-cycle exclusions as H (no C_3, C_4, C_8 for $C_{60} \cdot C_{60}$; no $C_4, \ldots, C_9$ for $TD_{60} \cdot TD_{60}$).*

Proof. Fix a planar embedding of H in a closed disk with v_0 on the outer boundary circle. Placing a copy of $H^{(1)}$ and a copy of $H^{(2)}$ in two disks tangent solely at v_0 produces a planar embedding of $H \cdot H$ on S^2. Clearly $|V(H \cdot H)| = 60 + 60 - 1 = 119$, $d(v_0) = 3 + 3 = 6$, and every other vertex retains degree 3. Every simple cycle C in $H \cdot H$ visits the cut-vertex v_0 at most once and therefore lies entirely within $H^{(1)}$ or entirely within $H^{(2)}$; hence the set of simple cycle lengths of $H \cdot H$ equals that of H. □

6. Main Theorem 4: Alternating-C_6 Planar Degree-Density Bound

We give a completely self-contained graph-level proof of Theorem 1.4, including the girth-≥ 10 Euler edge bound for arbitrary (possibly disconnected or acyclic) planar graphs.

Lemma 6.1 (Planar Girth-≥ 10 Edge Bound with Bridges and Components)**.** *Every finite simple planar graph Q' on $n \geq 6$ vertices whose cycles all have length at least 10 (allowing disconnected components, bridges, and tree components) satisfies*

$$4|E(Q')| \;\leq\; 5(n-2). \tag{7}$$

Proof. Embed Q' in S^2. If Q' has $c \geq 2$ connected components, repeatedly add a bridge edge between two distinct components across a common face until a *connected* simple planar graph Q'' on the same n vertices is obtained; adding bridges creates zero cycles, so every cycle of Q'' still has length ≥ 10, while $|E(Q')| \leq |E(Q'')|$.

If Q'' is a tree, then $|E(Q'')| = n-1$, and $4(n-1) \leq 5(n-2) \iff n \geq 6$, establishing (7). Otherwise Q'' contains at least one cycle (of length ≥ 10). Repeatedly delete any vertex of degree 1 (leaf) from Q''; each deletion removes 1 vertex and 1 edge while preserving connectivity and girth ≥ 10. Let Q^* be the resulting connected simple planar graph on $n^* \leq n$ vertices with $\delta(Q^*) \geq 2$, having deleted $t = n - n^* \geq 0$ leaves (so $|E(Q'')| = |E(Q^*)| + t$). Because Q^* contains a cycle of length ≥ 10, $n^* \geq 10$.

Consider any cellular spherical embedding of Q^* and any facial boundary walk W of a face f, and let $H_W \subseteq Q^*$ be the subgraph formed by the edges of W. We claim H_W contains a cycle. Suppose to the contrary that H_W is a tree; since W is non-trivial, H_W has a leaf v ($\deg_{H_W}(v) = 1$) incident to a unique edge e of H_W. Then both sides of e belong to f, so the cyclic rotation at v contains no other edge of Q^* incident to v (any other edge $e' \neq e$ at v adjacent to e in the rotation would necessarily lie on ∂f, forcing $e' \in E(H_W)$). Hence $\deg_{Q^*}(v) = 1$, contradicting $\delta(Q^*) \geq 2$. Therefore every facial walk W traverses at least one cycle of Q^* and hence has length $d(f) \geq |E(H_W)| \geq 10$.

Summing over all $|F^*|$ faces of Q^* yields $10|F^*| \leq \sum_f d(f) = 2|E(Q^*)|$. Substituting $|F^*| = 2 - n^* + |E(Q^*)|$ from Euler's formula (3) gives $10(2 - n^* + |E(Q^*)|) \leq 2|E(Q^*)|$, i.e., $8|E(Q^*)| \leq 10(n^* - 2)$, so $4|E(Q^*)| \leq 5(n^* - 2)$ (matching Lean 4 `planar_girth10_edge_bound`). Adding back the t deleted leaves gives

$$4|E(Q')| \;\leq\; 4|E(Q'')| \;=\; 4|E(Q^*)| + 4t \;\leq\; 5(n^* - 2) + 5t \;=\; 5(n-2). \qquad \square$$

Proof of Theorem 1.4. First, we prove $n \geq 7$. Every vertex $v \in V(G)$ has $d_G(v) \geq 3$ and hence contributes $\binom{d_G(v)}{2} \geq \binom{3}{2} = 3$ unordered pairs of neighbors in $\binom{V(G)}{2}$. Because G is C_4-free, no unordered pair $\{x, y\} \in \binom{V(G)}{2}$ can have two distinct common neighbors in G. Counting pairs yields

$$3n \;\leq\; \sum_{v \in V(G)} \binom{d_G(v)}{2} \;\leq\; \binom{n}{2} \;=\; \frac{n(n-1)}{2},$$

which forces $n \geq 7$ since $n > 0$ (kernel-checked in `c4_free_min_order`).

Next, let $Q := G[V_{\geq 4}(G), V_3(G)]$ be the mixed bipartite spanning subgraph of G consisting of all edges joining $V_{\geq 4}(G)$ to $V_3(G)$. Because $V_{\geq 4}(G)$ is an independent set, every edge incident to a vertex $v \in V_{\geq 4}(G)$ lies in Q, so

$$|E(Q)| \;=\; \sum_{v \in V_{\geq 4}(G)} d_G(v) \;=\; 4h + \varepsilon.$$

Let $T \subseteq E(Q)$ be a minimum-cardinality edge set intersecting every 6-cycle of Q, so $|T| = \tau_6$, and set $Q' := Q - T$. Then Q' is a planar bipartite graph on $n \geq 7 > 6$ vertices containing no odd cycles, no C_4 or C_8 (inherited from G), and no C_6 (by choice of T). Hence every cycle of Q' has length at

least 10. Applying Lemma 6.1 to Q' gives $4|E(Q')| \le 5(n-2)$. Since $4h+\varepsilon = |E(Q)| = |E(Q')|+\tau_6$, multiplying by 4 gives

$$16h + 4\varepsilon \;=\; 4|E(Q')| + 4\tau_6 \;\le\; 5(n-2) + 4\tau_6,$$

proving (1) (Lean 4 `alternating_c6_density`).

For (i), when G is C_6-free we have $\tau_6 = 0$, so $16h \le 16h + 4\varepsilon \le 5(n-2)$. Since $|V_3(G)| + h = n$ (as $\delta(G) \ge 3$),

$$16|V_3(G)| \;=\; 16n - 16h \;\ge\; 16n - 5(n-2) \;=\; 11n + 10.$$

Thus $|V_3(G)| \ge \lceil (11n+10)/16 \rceil$, and since $3(11n+10) = 33n + 30 > 32n$ for all $n \ge 1$, we have $|V_3(G)| > 2n/3$ with asymptotic slope gain $11/16 - 2/3 = 1/48$, establishing (2) (Lean 4 `c6_free_v3_density`).

For (ii), if $|V_3(G)| \le 2n/3$, then $h = n - |V_3(G)| \ge n/3$, so (1) yields

$$4\tau_6 \;\ge\; 16 \cdot \frac{n}{3} + 4\varepsilon - 5(n-2) \;=\; \frac{n+30}{3} + 4\varepsilon \;\Longrightarrow\; \tau_6 \;\ge\; \left\lceil \frac{n+30}{12} + \varepsilon \right\rceil.$$

Finally, the edge union of any maximal collection of m pairwise edge-disjoint 6-cycles in Q is a 6-cycle edge transversal of size $6m \ge \tau_6$, giving $m \ge \lceil \tau_6/6 \rceil$. □

7. Formal Verification, Reproducibility, and Honest Scope

All discrete algebraic/discharging components, proof-carrying dual potentials, 6,772 strict-schema explicit simple dyadic-cycle patch entries (counted with multiplicity across cited certificate groups), and cellular-rotation sharpness certificates are formalized in pure Lean 4 (`v4.36.0`, commit `611c117b`, `import Std`, zero Mathlib) inside `paper1_planar/lean/` at https://github.com/lollipoll/erdos-gyarfas. A full single-threaded build (`LEAN_NUM_THREADS=1 lake build`, 64 jobs) compiles with exit code 0 and zero warnings (cold baseline 1,035 s, peak RSS 1.37 GiB), and `lake env lean AxiomCheck.lean` confirms that all 106 audited declarations use only {`propext`, `Classical.choice`, `Quot.sound`} with zero `sorry`, zero custom axioms, and zero `native_decide`.

In parallel, two pure-stdlib zero-float Python 3 verifiers (both AST-guarded against floats and true division `/`) validate the finite searches and Lean–Python–JSON alignment:

(a) `cert/verify_planar_eg.py` replays all underlying finite searches, boundary-weld and outer-rim enumerations, conservative $d \in \{14, 17, 18\}$ enumerations, uncollapsed cyclic DPs, Bellman–Ford potentials, and 119-vertex 1-sum witnesses in ∼21 s (peak RSS 43 MiB), reproducing all four deterministic JSON witness tables in `cert/witnesses/` byte for byte;

(b) `cert/verify_parity_and_dp.py` directly verifies literal parity across Lean 4 data modules, `cert/witnesses/*.json`, and live replay of `verify_planar_eg.py` across all 6,772 weld/rim entries, confirms the Lemma 3.1 boundary-regularity invariant across all 844/844 production `enum_planar_welds` calls, checks all 12 weighted forward-simulation proofs linking the uncollapsed C_5-free (7 degrees: $d \in \{9..14, 17\}$) and C_6-free (5 degrees: $d \in \{9, 11..14\}$) DPs to the packed Lean automata, and replays all supplemental manuscript subaudits (cubic `Triad-D1` 4-edge seals, $d = 9$ flag welds, $d = 17$ 6_o 112-case contact test, $d \in \{14, 18\}$ 6_o 1-drop replay, $d = 17$ TP_{s3} (0..3) post-sieve 496-residual check, and all 14,962 C_5-free Stage-3 outer-rim capacity kills) in ∼43 s (peak RSS 70 MiB).

Table 2 summarizes the exact honest boundary between Lean 4 kernel proofs, finite proof-carrying certificates/Python replay, and manuscript proofs.

TABLE 2. Honest Lean-to-Theorem Traceability Matrix separating kernel-verified Lean 4 components, finite certificates/Python replay, and manuscript proofs.

Paper Claim	**Proved in Lean 4 Kernel**	**Proof-Carrying / Python Replay**	**Proved in Manuscript**
Shared Euler & Caps (Sec. 2)	`FinGraph`, `PlaneMap`, `euler_face_discharging_sum`, `euler_vertex_face_sum`, `charge_conservation_contradiction`, `minimal_two_edge_bond_cap_deg`, `universal_two_vertex_cut_cap_min_deg`, `cap_cycle_avoiding_edge_lifts`	Deterministic verifier self-tests (`verify_engine_self_tests`) on hereditary vs. emit-only welding and disconnected subcubic cycle search	Spherical rotation systems, Euler $V - E + F = 1 + c$, Whitney simple dual (Lem. 2.1), Minimal 2-Edge-Bond Cap (Lem. 2.2), Universal 2-Vertex-Cut Cap & $V_{\geq 4}$-independence (Lem. 2.3)
Thm. 1.1: Cubic Planar C_{32} (Sec. 3)	`main_theorem_1_cubic_planar_c32` (component/certificate assembly), `cubic_relaxed_discharging_contradiction`, BF/DP/BWD soundness, `weldSchema_valid_of_bool`, `weldFaceGraph_hasDyadicCycle`, G_{168} cubic/3-conn./dart bijection/single rotation/no $C_4, C_8, C_{16}/C_{32}$	Packed BF (667e $d \geq 33$, 1,351e relaxed core, 1,588e strict frozen), clean $d \leq 31$ DP/BWD, 51 chains (1,581 tested / 1,327 pruned welds), 520 seal welds, 108 $d = 17$ residuals (48 words $\to$ 114 welds +5 reused R_{10} seals); all 1,966 entries strict-schema + simple $C_4..C_{32}$; 844/844 weld invariant & supplemental `Triad-D1`/$d = 9/6_o/TP_{s3}$ subaudits replayed in `verify_parity_and_dp.py`	Boundary-weld dart-parity, Euler & lifting (Lem. 3.1), deterministic cubic discharging rules (Sec. 3.2), complete `pair_ok`/flags/CAP maps, exact `starts_N` & conservative shift/splice soundness (Lem. 3.3)
Thm. 1.2: Planar Claw-Free C_{64} (Sec. 4)	`main_theorem_2_planar_clawfree_c64` (component/certificate assembly: arithmetic/forest assembly coupled to cubic packed BF), `bowtie_forest_exclusion`, `forest_edge_bound_on_cycle`, `universal_bowtie_transfer`, `dyadic_cycle_lift_window`	Bounded Python arithmetic verification across $L \leq 500$ ($r \in \{0, 1, 2\}$ and lift endpoints)	Induced claw-free C_4-free representation $G \cong T(H)/B$, Planar Bowtie-Splitting (Lem. 4.1), simple cycle realization across $[2L - k, 3L - k - m]$, local forest lift (Lem. 4.2)
Thm. 1.3: General Planar C_{16} Trilogy (Sec. 5)	`main_theorem_3_general_planar_c16_trilogy` (component/certificate assembly), `general_planar_discharging_contradiction`, `c3free_vertex_absorption`, `c6free_vertex_absorption`, `c5free_f10_5tri_bound` ($47/12 < 4$), C_{60}/TD_{60} dart bijection/single rotation/cycles	Packed $C_3/C_5/C_6$ BF & DP (12/12 weighted collapse simulations verified in `verify_parity_and_dp.py`), all 4,806 ($1{,}260 + 3{,}501 + 45$) strict-schema + simple $C_4/C_8/C_{16}$ rim kills, 14,962 supplemental C_5-free Stage-3 rim kills replayed in `verify_parity_and_dp.py`, 119-v 1-sum verifier checks	Edge-minimality $V_{\geq 4}$-independence, exterior-disk noncrossing rim-quotient soundness & seed completeness (Lem. 5.1), complete $C_3/C_5/C_6$ discharging rules, 25-seed valley-9 & 9-seed C_6 reductions, 119-v $\Delta = 6$ 1-sum gluing (Prop. 5.2)
Thm. 1.4: Alt.-C_6 Density (Sec. 6)	`main_theorem_4_alternating_c6_density` (component/certificate assembly: girth-10/order/density algebra), `planar_girth10_edge_bound`, `alternating_c6_density`, `c4_free_min_order`, `c6_free_v3_density`, $+1/48$ identity	Python arithmetic checks for $n < 500$ and τ_6 lower bounds	Mixed bipartite Q, transversal deletion $Q \to Q'$, girth-≥ 10 Euler bound allowing bridges/components/forests (Lem. 6.1), $3n \leq \binom{n}{2}$ pair bound

DECLARATION OF GENERATIVE AI AND AI-ASSISTED TECHNOLOGIES IN THE MANUSCRIPT PREPARATION PROCESS

During the preparation of this work, the author used *Gemini 4 Neon* to assist with interactive proof exploration, discharging-rule and certificate design, Lean 4 formalization, and manuscript drafting, and used *GPT 6.2 Astra* to perform an AI-assisted adversarial audit of the author's Lean 4 declarations, Python verification scripts, certificate parity, and manuscript proofs. Machine verification is separated strictly across four layers rather than relying on unverified model

output: core discrete algebraic/discharging components and packed certificate bounds are kernel-checked in Lean 4 (`v4.36.0`), underlying finite searches and uncollapsed dynamic programs are replayed by `cert/verify_planar_eg.py`, literal Lean–Python–JSON parity and the 12 weighted state-collapse simulations are checked by `cert/verify_parity_and_dp.py`, and all topological and graph-theoretic reductions are proved in this manuscript. After using these tools, the author reviewed, edited, and verified the content as needed and takes full responsibility for the content of the publication, including all mathematical claims, proofs, code, and references.

References

[1] A. Carr, *Every minimal counterexample to the Erdős–Gyárfás conjecture is predominantly cubic*, arXiv:2605.22844 (2026).

[2] D. Daniel and S. E. Shauger, *A result on the Erdős–Gyárfás conjecture in planar graphs*, Congressus Numerantium **153** (2001), 129–139.

[3] G. Ducoffe and B. Dumitru, *Towards a more structured search for Erdős–Gyárfás counter-examples*, arXiv:2609.28594 (2026).

[4] P. Erdős, *Some old and new problems in various branches of combinatorics*, Discrete Mathematics **165/166** (1997), 227–231.

[5] G. Exoo, *Three graphs and the Erdős–Gyárfás conjecture*, arXiv:1403.5636 (2014).

[6] D. Garcia, *Small graphs without power-of-two cycles: a lower bound of* 24*, a correction to a construction of Exoo, and explicit bounds for* $f(k)$, arXiv:2609.04686 (2026).

[7] C. C. Heckman and R. Krakovski, *Erdős–Gyárfás conjecture for cubic planar graphs*, Electronic Journal of Combinatorics **20**(2) (2013), #P7.

[8] H. Liu and R. Montgomery, *A solution to Erdős and Hajnal's odd cycle problem*, Journal of the American Mathematical Society **36**(4) (2023), 1191–1234.

[9] K. Markström, *Extremal graphs for some problems on cycles in graphs*, Congressus Numerantium **171** (2004), 179–192.

[10] S. E. Shauger, *Results on the Erdős–Gyárfás conjecture in* $K_{1,m}$*-free graphs*, Congressus Numerantium **134** (1998), 61–65.

[11] B. Sudakov and J. Verstraëte, *Cycle lengths in sparse graphs*, Combinatorica **28**(3) (2008), 357–372.