A SELF-CONTAINED PROOF OF PROPOSITION $(1 - 1.67539)$ FOR THE TWIN PRIME PROBLEM

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Abstract. For $1 \le a \le 2$, Proposition $(1-a)$ asserts that there exist infinitely many primes p such that $p+2 = rq$ with q prime, $r = 1$ or prime, and $r \le q^{a-1}$. Li and Liu recently established Proposition $(1-1.75)$ unconditionally. By combining Runbo Li's published lower bound $\pi_{1,2}(x) \ge 1.2759\, C_2 x/(\log x)^2$ with a self-contained positive upper-sieve majorant at distribution level $5/8$ (derived from Pascadi's fixed-residue convolution estimate) and a directed pure-integer interval certificate at denominator 2^{80}, we prove Proposition $(1 - 1.67539)$ unconditionally. More precisely, writing $A_b(x)$ for the number of primes $p \le x$ satisfying $p+2 = rq$ with $r \le q^b$, we obtain $A_{67539/100000}(x) > 0.0000466\, C_2 x/(\log x)^2$ for all sufficiently large x. We also record an elementary one-page two-factor shortcut yielding Proposition $(1 - 1.7078)$ and a rank-three baseline yielding Proposition $(1 - 1.6767)$.

1. Introduction and main results

Let

$$C_2 := 2\prod_{\ell>2}\left(1 - \frac{1}{(\ell-1)^2}\right) \approx 1.3203236317, \qquad H(x) := \frac{C_2 x}{(\log x)^2},$$

where ℓ runs over primes. For $0 < b \le 1$, let $A_b(x)$ count the number of primes $p \le x$ admitting a representation

$$p + 2 = rq,$$

where q is prime, r is either 1 or prime, and $r \le q^b$. In the notation of Li and Liu [8], Proposition $(1-a)$ $(1 \le a \le 2)$ states that $A_{a-1}(x) \to \infty$ as $x \to \infty$; thus $a = 2$ corresponds to Chen's theorem [2] and $a = 1$ corresponds to the twin prime conjecture.

In [8, Theorem 1.2], Li and Liu proved Proposition $(1 - 1.75)$ unconditionally, obtaining

$$A_{3/4}(x) > 0.036\, H(x)$$

for sufficiently large x. Under a weighted Elliott–Halberstam conjecture WEH(0.999), they improved the exponent to $(1 - 1.4)$ [8, Theorem 1.4]. Our main result strengthens the unconditional bound from 1.75 to 1.67539.

Theorem 1.1. *Proposition* $(1-1.67539)$ *holds unconditionally. Specifically, for* $b = 67539/100000$,

$$\liminf_{x\to\infty} \frac{A_b(x)}{H(x)} \;\ge\; \frac{563809644083048637 85}{12089258196146291747061 76} \;>\; 0.0000466. \tag{1.1}$$

In particular, $A_{67539/100000}(x) > 0.0000466\, H(x)$ *for every sufficiently large* x. *Without the fourth-rank cap replacement, the rank-three construction alone establishes Proposition* $(1 - 1.6767)$, *while an elementary two-factor shortcut proves Proposition* $(1 - 1.7078)$.

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1.1. **Comparison with related work.** It is helpful to distinguish Theorem 1.1 from related results on shifted primes $p+2$:

- *Chen's theorem density.* Runbo Li [7, Theorem 1.3] established the lower bound $\pi_{1,2}(x) \geq 1.2759\, H(x)$ for $\pi_{1,2}(x) = \#\{p \leq x : p+2 \in P_2\}$, using Pascadi's distribution results [10]. Our proof uses Li's 1.2759 lower bound directly as a modular input.
- *Largest prime factor.* Baker and Harman [1] (discussed in [8, §1.4]) showed that $P^+(p+2) > p^{0.676}$ infinitely often, giving $p+2 = rq$ with q prime and $r \leq p^{0.324} \leq q^{0.48}$. However, the cofactor r in Baker–Harman's theorem is an arbitrary integer that may have three or more prime factors, whereas Theorem 1.1 retains Chen's condition $\Omega(p+2) \leq 2$ ($r = 1$ or r prime).
- *Twin-prime upper bounds and Goldbach analogues.* Building on Lichtman [9] (who obtained $3.229\, H(x)$ at level 66/107), Pascadi [10, Corollary 1.4] proved $\pi_2(x) \leq (3.203 + o(1))H(x)$ using an unconditional 5/8 distribution level [10, Proposition 4.4 and Theorem 5.4]. Here we construct an explicit positive upper-sieve majorant adapted to 5/8, prove its progression distribution estimate on prime pairs rq, and subtract the resulting upper bound for excluded semiprimes $q^b < r \leq q$ from Runbo Li's Chen lower bound.

1.2. **Inputs and organization.** The external analytic inputs are Runbo Li's Theorem 1.3 [7]; Pascadi's fixed nonzero residue convolution estimate [10, Proposition 4.4] (which itself relies on Deshouillers–Iwaniec [3], Watt, and Pascadi's exceptional Maass-form large sieve); Iwaniec's well-factorable linear Rosser sieve and the fundamental lemma [5, 6, 4]; and the classical prime number theorem, Mertens's estimates, prime and Möbius Siegel–Walfisz bounds, the multiplicative large sieve, and the Weil bound for Kloosterman sums. No external distribution theorem or density bound for the constructed coefficient is assumed.

In the remainder of Section 1, we reduce $A_b(x)$ to subtracting the bad semiprimes $B_b(x)$ and present the one-page shortcut to $(1-1.7078)$. Section 2 defines the rank-three and rank-four upper-sieve trees and proves the pointwise first-exit identity. Section 3 verifies the capacity conditions for Pascadi [10, eq. (4.15)] (including the $C < 1$ two-factor repair and three-slot W packing) and realizes a single finite signed coefficient ρ_x. Section 4 establishes the uniform prime-pair distribution estimate for ρ_x at fixed residue 2. Sections 5 and 6 compute the limiting density, bound the small-prime tail, and transfer the majorant to $B_b(x)$. Section 7 describes the standalone integer certificate script `verify_twin_upper_sieve_cert.py` and records the certified rational bounds.

1.3. **The subtraction identity and an elementary shortcut.** Let $B_b(x)$ count the primes $p \leq x$ such that $p+2 = rq$ with r, q prime and $q^b < r \leq q$. Ordering the two prime factors ensures that every composite P_2 remainder has a unique factorization $r \leq q$ (squares contribute to B_b), while prime remainders ($r = 1$) belong to A_b. Allowing for at most one exception at $p = 2$, Runbo Li's lower bound [7, Theorem 1.3] gives

$$\liminf_{x\to\infty} \frac{A_b(x)}{H(x)} \;\geq\; \frac{12759}{10000} - \limsup_{x\to\infty} \frac{B_b(x)}{H(x)}. \tag{1.2}$$

In odd moduli d, the congruence $rq - 2 \equiv 0 \pmod d$ reads $rq \equiv 2 \pmod d$. The two-prime sequence has local density $g(\ell) = 1/(\ell-1)$ for odd primes ℓ, with no factor at 2. Since $(1 - g(\ell))/(1-1/\ell) = 1 - (\ell-1)^{-2}$, putting $P(y) = \prod_{2<\ell<y} \ell$ and $V(y) = \prod_{2<\ell<y}(1-g(\ell))$ yields

$$V(x^t) \sim \frac{e^{-\gamma} C_2}{t \log x} \tag{1.3}$$

uniformly for t in any positive compact interval. The factor 2 from omitting $\ell = 2$ is already contained in C_2, matching [8, (6.6)–(6.10)] and [10, Corollary 1.4].

Proposition 1.2 (Two-factor shortcut)**.** *Proposition* $(1-1.7078)$ *holds unconditionally; more precisely, for* $b = 3539/5000$*,*

$$\liminf_{x\to\infty} \frac{A_b(x)}{H(x)} \;\geq\; \frac{377883443722357593132}{1208925819614629174706176} \;>\; 0.000312.$$

Proof. We apply Li and Liu's fixed-residue bilinear distribution lemma [8, Lemma 3.4] (originating in Bombieri–Friedlander–Iwaniec and Wu) to one-bounded well-factorable modulus weights. Take N in that lemma to be the shorter dyadic scale (corresponding to r); prime indicators satisfy its Siegel–Walfisz and roughness hypotheses. In the region $q^b < r \leq q$ with $rq \asymp X$ and $b > 2/3$, we have $\nu := \log N/\log X \in [2/5, 1/2]$ for large x, allowing bounded dyadic endpoint errors and choosing the smaller scale if r exceeds q near the diagonal. The table in [8, Lemma 3.4] gives distribution exponent $1-\nu-\epsilon$. On each dyadic chart we apply the ordinary one-bounded well-factorable upper linear sieve (Section 3 below shows how to realize this greedily atom-by-atom via a single constant cubic cap below $1-\nu-\epsilon$, reserving the small-prime attachment $x^{\sqrt{\eta}}$ in the larger group). With $t = \log r/\log x$ and sifting threshold at a third of the level exponent, (1.3) and $F(3) = 2e^{\gamma}/3$ yield limiting upper density $2/(1-t)$ in units of $C_2/\log x$ after letting fixed reserves tend to zero. Smooth separation as in Section 4 preserves prime Siegel–Walfisz by partial summation. The region $rq < \sigma x$ contributes $O(\sigma H(x))$ using a fixed level $1/2$ after discarding $rq < x/(\log x)^5$. Integrating against $dt/(t(1-t))$ gives

$$\limsup_{x\to\infty} \frac{B_b(x)}{H(x)} \;\leq\; \int_{b/(1+b)}^{1/2} \frac{2\,dt}{t(1-t)^2} \;=\; 2(1-b-\log b),$$

since an antiderivative of $1/(t(1-t)^2)$ is $\log(t/(1-t)) + (1-t)^{-1}$. Substituting into (1.2) and evaluating at $b = 3539/5000$ via Section 7 completes the proof. □

2. The positive upper-sieve tree and cap rules

Write $L = \log x$, $\rho = 1/5$, $T = 5/8$, and

$$v(u) := \min\left\{\frac{1+u}{2},\, 1-\frac{3u}{2}\right\}.$$

Order the prime factors of any odd squarefree modulus decreasingly as $\ell_1 > \ell_2 > \cdots > \ell_r > 2$, and set $t_i = \log \ell_i/L$ and $S_j = \sum_{i\leq j} t_i$ ($S_0 = 0$). Fix a root cutoff $\rho' < \rho$ arbitrarily close to ρ. An odd-rank prefix $(\ell_1, \ldots, \ell_j)$ is retained precisely when its local cubic test satisfies $S_{j-1} + 3t_j < \lambda$, where λ is the current cap; every even child of a retained odd prefix is retained automatically.

We start at rank one with cap T. At rank three, writing (t_1, t_2, t_3) before testing the third cube, replace the cap by the ideal third-rank cap $J(t_1, t_2, t_3)$, defined as the maximum of:

- $v(t_1)$, $v(t_1 + t_2)$, and $v(t_1 + t_2 + t_3)$;
- $v(t_1 + t_3)$, if $v(t_1 + t_3) > t_1 + t_3 + 2t_2$;
- $v(t_2 + t_3)$, if $v(t_2 + t_3) > t_2 + t_3 + 2t_1$;
- $\min\{5/8 - 3t_3/8,\, 1 - 2t_2\}$.

Notice the explicit $1-2t_2$ guard in the last branch, which is essential here since $2\rho = 2/5 > 1-5/8$. No unguarded nonleading complementary square is used.

At an admitted rank-four prefix $t = (t_1, t_2, t_3, t_4)$ with $t_4 \geq 3/100$, examine every subset $I \subseteq \{1, 2, 3, 4\}$ containing 4 and at least one other index, and put $u_I := \sum_{i\in I} t_i$. Raise the inherited cap to the largest $M = v(u_I) > J$ over those candidates satisfying

$$u_I + \sum_{\substack{i<j \\ i\notin I}} t_i + 2t_j < M \qquad (j \leq 4,\ j \notin I). \tag{2.1}$$

If no candidate satisfies (2.1), retain J. Since rank four is even, this update occurs without a retention test at rank four. All subsequent odd ranks use the inherited cap. In Section 3, strictly lowered piecewise-constant versions of these ideal caps will be frozen for the arithmetic construction.

Lemma 2.1 (First-exit identity)**.** *For any finite prime set and any multiplicative function g satisfying $0 \le g(\ell) \le 1$, the retained prefix tree $\mathcal{T}$ satisfies*

$$\sum_{d\in\mathcal{T}} \mu(d)g(d) = \prod_{\ell}(1-g(\ell)) + \sum_{e \text{ first omitted prefix}} g(e) \prod_{\ell<\ell_{\mathrm{last}}(e)} (1-g(\ell)). \tag{2.2}$$

In particular, taking $g(\ell) = \mathbf{1}_{\ell\mid n}$ shows that $\sum_{d\mid n,\, d\in\mathcal{T}} \mu(d) \ge \mathbf{1}_{(n,\prod \ell)=1} \ge 0$ for every integer n.

Proof. Every divisor $d \notin \mathcal{T}$ has a unique first omitted prefix e, which necessarily has odd rank so that $\mu(e) = -1$. Summing $\mu(d)g(d)$ over all squarefree continuations of e by primes $\ell < \ell_{\mathrm{last}}(e)$ yields $-g(e)\prod_{\ell<\ell_{\mathrm{last}}(e)}(1-g(\ell))$. Subtracting these disjoint sums from $\prod_\ell(1-g(\ell))$ gives (2.2). Raising a cap after an even prefix or selecting it before its own odd test preserves the odd parity of every first exit. □

3. Capacity packing and finite realization

For a slot index j, the *full square* is $S_{j-1}+2t_j$. For a selected index subset I, the *complementary squares* are $u_I + \sum_{i<j,\, i\notin I} t_i + 2t_j$ for $j \notin I$, as in (2.1); after the largest index of I, each complementary square coincides with the full square. Any passing odd cube $S_{j-1} + 3t_j < \lambda$ implies both its own full square and the next full square $S_j + 2t_{j+1} \le S_{j-1} + 3t_j < \lambda$.

For the initial checks of J: the first two full squares are bounded by $3t_1$, while $v(t_1) = (1+t_1)/2 > 3t_1$ holds strictly for all $t_1 < 1/5$. Moreover, a passing third cube controls both the third and fourth full squares even if the rank-four cap is subsequently raised. For $I = \{1\}$, complementary squares reduce to full squares; for $I = \{1,2\}$, the only complementary square up to rank three is a full square; for $I = \{1,2,3\}$, there is none; for $I = \{1,3\}$ and $I = \{2,3\}$, the sole extra square is explicitly guarded in the definition of J; and for rank four, (2.1) enforces every extra square directly. These inequalities persist under appending smaller primes.

Lemma 3.1 (Greedy modulus packing)**.** *Let X be the complete product scale, and let U be a selected medium-variable scale satisfying $X^{2\zeta} \le U \le X^{1/3+\zeta/2}$, with $0 < \zeta < 10^{-5}$ sufficiently small. Convert exponents to base X, let ℓ be a covering exponent, and put $D = X^{\ell-2\zeta}$ and $A = UX^{-\zeta}$. Suppose all full and complementary squares for a box atom are at most ℓ, and that one of the following holds:*

(a) *$\ell < v(\beta)$ for a selected subset I of upper endpoint $B = X^\beta$; or*
(b) *$(1+t_1)/2 < \ell < \min\{5/8 - 3t_3/8,\, 1 - 2t_2\}$.*

Then the atom admits a partition into three modulus groups with upper endpoints $Q_1, Q_2, Q_3 \ge 1$ satisfying Pascadi's system [10, eq. (4.15)]*:*

$$Q_1 \le UX^{-\epsilon}, \qquad U^2Q_2Q_3^2 \le X^{1-15\epsilon}, \qquad U^2Q_2^5Q_3^2 \le X^{2-40\epsilon}, \tag{3.1}$$

provided $\epsilon > 0$ is sufficiently small in terms of ζ and the covering margins.

Proof. In case (a), the condition $\ell < v(\beta)$ is equivalent to

$$b_- := 2\ell - 1 < \beta < \frac{2-2\ell}{3} =: b_+. \tag{3.2}$$

If $C := D/(AB) \ge 1$, place the selected slots I in group two ($Q_2 = B$) and pack the remaining slots greedily in order into capacities A and C. If some slot of exponent s fits neither, the product of the remaining capacities is $< X^{2s}$, contradicting the complementary square at that slot. With $(Q_1, Q_2, Q_3) = (A, B, C)$, replacing U by AX^ζ bounds the two monomials in (3.1) by $X^{2\zeta}D^2/B$

and $X^{2\zeta}D^2B^3$. By (3.2), $D^2/B \le X^{2\ell-4\zeta-(2\ell-1)} = X^{1-4\zeta}$ and $D^2B^3 \le X^{2\ell-4\zeta+2-2\ell} = X^{2-4\zeta}$, leaving fixed power savings 2ζ in both bounds.

If instead $C = D/(AB) < 1$, repack *all* slots greedily into $(A, D/A)$ using the full squares and leave group three empty ($Q_3 = 1$). Note that $A \ge X^{\zeta} \ge 1$ and $D/A \ge X^{\ell-1/3-7\zeta/2} \ge 1$ since $\ell > 1/2$. Because $Q_2 = D/A < B$, the second monomial $X^{2\zeta}D^2Q_2^3$ strictly decreases from $X^{2\zeta}D^2B^3$, while the first becomes $U^2(D/A) = X^{2\zeta}AD \le X^{1/3+\ell-5\zeta/2}$, which stays strictly below $X^{1-4\zeta}$ since $\ell \le 5/8 + o(1)$ and $1/3 + 5/8 = 23/24 < 1$. (This is the same two-factor repair as [10, Lemma 5.6].)

In case (b), initialize capacities $(A, X^{b_-}, D/(AX^{b_-}))$. If the third capacity is below 1, apply the same two-group repair into $(A, D/A)$. Otherwise, place the largest prime in group two (since $t_1 < 2\ell - 1 = b_-$). The second prime fits into group one or three because the product of their capacities is $X^{1-\ell-2\zeta}$ and $2t_2 < 1-\ell$ with a strict reserve. Continue inserting slots greedily in order. If all fit, (3.2) gives (3.1). At the first slot ℓ_j of exponent $s \le t_3$ that fits no group, adjoin ℓ_j to group two, obtaining a new product B' with exponent in $[b_-, b_- + t_3]$. Since $b_+ - b_- = (5-8\ell)/3 > t_3$, (3.2) still holds for B'. Moreover, if (a_1, a_2, a_3) were the partial products before ℓ_j, then $A < a_1X^s$ and the full square gives $a_1a_2a_3X^{2s} \le D$, whence $AB'a_3 < D$ for $B' = a_2X^s$. Thus a_3 fits inside $D/(AB')$, and freezing group two lets the remaining slots be packed into groups one and three via the full squares. Both monomial bounds in (3.1) follow as before. □

Remark 3.2. In the finite realization below, small-prime weights up to $x^{\sqrt{\eta}}$ are attached to group three; choosing η sufficiently small relative to the covering reserves absorbs an extra $x^{\sqrt{\eta}}$ factor in Q_3 into the strict margins of (3.1).

3.1. **Construction of a single finite coefficient.** Fix a small density allowance $u_0 > 0$, a root cutoff $\rho' < \rho$, and a tail cutoff $\delta_0 > 0$ for t_3. On the compact third-rank domain, partition into sufficiently small grid cells and freeze branch labels and lower caps to constants with strict reserves away from switching walls, near-equality of guards, and near-ties with the fallback. At omitted walls and for $t_3 < \delta_0$ use the fallback $v(t_1)$ lowered by a small fixed amount; near $t_1 = 0$ use an ordinary cap strictly below 1/2 so that all moduli there stay below $x^{1/2-\epsilon_0}$. This maintains a uniform positive gap above $t_1 + t_2$, since the ideal baseline satisfies $J - t_1 - t_2 \ge (1-3\rho)/2 = 1/5$. At rank four, restrict to finitely many cells with strict admissibility, trim near equalities, and freeze the lowered new cap where it improves the parent. Every retained branch selects a covering exponent ℓ strictly between its physical cap and ideal bound; in all branches the physical cap never exceeds $T = 5/8$.

Next fix $\eta > 0$ sufficiently small and restrict the tree to primes $\ell \ge x^{\eta}$; its rank is bounded by a constant J_η. Refine a common finite alphabet of half-open exponent intervals so that all covering reserves hold on upper box endpoints, deciding odd cubes on upper endpoints and retaining even children automatically. For each rough atom of sign + or −, attach the ordinary upper (λ^+) or lower (λ^-) small-prime weights on $\ell < x^{\eta}$ of support at most $x^{\sqrt{\eta}}$ and $|\lambda^{\pm}| \le 1$. By the fundamental lemma, small-prime density equals $V(x^{\eta})(1 + O(e^{-c/\sqrt{\eta}}))$ uniformly in x, while the absolute rough density is $O(1/\eta)$, giving attachment error $O(e^{-c/\sqrt{\eta}}\eta^{-2}C_2/L)$. Moreover, since

$$\lambda^-(n) \le \mathbf{1}_{(n,P(x^{\eta}))=1} \le \lambda^+(n) \qquad (\forall n \in \mathbb{Z}),$$

multiplying by the atom's sign and rough indicator produces a pointwise majorant of the rough-tree sum times $\mathbf{1}_{(n,P(x^{\eta}))=1}$, hence majorizes $\mathbf{1}_{(n,P(x^{\rho'}))=1}$ by Lemma 2.1. Unique prime factorization across x^{η} ensures $|\rho_x(d)| \le 1$, supported on odd squarefree $d \le x^{5/8+\sqrt{\eta}}$, while rough pieces of rank ≤ 2 (and the small-t_1 branch) have support $\le x^{2\rho+\sqrt{\eta}} \le x^{1/2-\epsilon_0}$ for small η, ϵ_0.

By Lemma 2.1, multiplying the unrounded rough-tree density by $V(x^{\eta})$ isolates the subset of full exits whose last prime is $\ge x^{\eta}$, which is at most the full unrounded density. For fixed η, refining the common alphabet reduces the discrepancy to $O_{\text{mesh}\to 0}(C_2/L)$ via Mertens's harmonic measure dt/t on finitely many compact cells.

Finally, for each rough atom with interval multiplicities m_I, its ordered squarefree coefficient equals $\mathbf{1}_{\mathrm{sf}}\,\mathcal{N}/\prod_I m_I!$, where $\mathcal{N}$ is the unrestricted interval convolution. Dividing each of the three packed groups by its own $\prod_I m_{I,j}!$ makes every group coefficient 1-bounded even on repeated primes, leaving a fixed external scalar. Attaching $\lambda^{\pm}$ to group three preserves 1-boundedness by disjoint prime universes. Removing the global squarefree restriction only during analytic comparison introduces a progression error at most

$$x^{o(1)} \sum_{\substack{d\le x^{5/8+\sqrt{\eta}}\\ \exists \ell\ge x^{\eta}:\, \ell^2\mid d}} \left(\frac{x}{d}+1+\frac{x}{\varphi(d)}\right) \;\ll\; x^{1-\eta+o(1)}, \tag{3.3}$$

which handles $rq-2=0$ as well (no divisor bound at zero is needed).

4. Distribution on prime pairs

Set $\Delta_d(n) := \mathbf{1}_{n\equiv 2 \pmod d} - \mathbf{1}_{(n,d)=1}/\varphi(d)$.

Proposition 4.1. *Let ρ_x be the finite signed coefficient family constructed in Section 3 with fixed reserves, and let $a_*>0$. For any smooth weight $w(r,q)$ supported on $r,q\gg x^{a_*}$ and $rq=O(x)$ whose mixed Euler derivatives are bounded by fixed powers of $L=\log x$, we have*

$$\sum_d \rho_x(d) \sum_{r,q \text{ prime}} w(r,q)\Delta_d(rq) \;=\; O_A(xL^{-A}) \qquad (\forall A>0). \tag{4.1}$$

Proof. Smoothly discard products $rq<2x^{1-\kappa}$ for a very small $\kappa>0$; this costs $x^{1-\kappa+o(1)}$ using $\tau(rq-2)$ ($rq-2\neq 0$ for large x) and $\sum_{d\ll x} 1/\varphi(d)\ll L$. On each remaining dyadic chart of product scale $X\gg x^{1-\kappa}$, choose κ small relative to all reserves so that base-X exponents preserve Lemma 3.1, and total support is $D_0\le X^{5/8+\xi}$ for an arbitrarily small $\xi>0$. By Cauchy–Schwarz and $\tau^h\le d_{2^h}$, shifted divisor moments satisfy

$$\sum_{n\asymp X} \tau(n)^j \tau(|n-2|)^k \ll XL^{O_{j,k}(1)}, \tag{4.2}$$

while coprime means add at most $L^{O_J(1)}$ under $\tau(d)^J/\varphi(d)$.

For pieces with support $D_0\le X^{1/2-\epsilon_0}$, separate the smooth weight in logarithmic coordinates and apply the multiplicative large sieve: writing $d=gh$ with primitive conductor $g>1$ and retaining $(n,h)=1$ on both prime factors, Cauchy–Schwarz bounds the $G<g\le 2G$ contribution with weight $1/\varphi(g)$ by $L^{O(1)}(X/G+\sqrt{X}(\sqrt{R}+\sqrt{U})+G\sqrt{X})$. Summing $G>L^B$ over $h\le D_0/g$ via $1/\varphi(gh)\le 1/(\varphi(g)\varphi(h))$ and $\sum_h 1/(h\varphi(h))<\infty$, and handling $g\le L^B$ by prime Siegel–Walfisz (removing $\ell\mid h$ at cost $O(\omega(h)L^{O(1)})$), gives $O_A(XL^{-A})$.

For the high-support pieces, fix a common roughness cutoff $z_*=(CX_0)^{1/(\log\log(CX_0))^3}$ across all subcharts of each original dyadic chart of scale X_0; for large X_0 this exceeds $X^{1/(\log\log X)^3}$. Replace each prime indicator by $\Lambda(m)/\log m$ with $P^-(m)>z_*$ (proper prime powers $\gg x^{a_*}$ cost $O(x^{-a_*/2+o(1)})$). At each original scale R_i, put $Z_i=(2R_i)^{1/3}$ and $m_i=\mu\mathbf{1}_{[1,Z_i]}$; expanding $(\delta-1*m_i)^{*3}*\Lambda=0$ below $2R_i$ yields the order-three Heath–Brown identity

$$\Lambda=\sum_{j=1}^{3}(-1)^{j-1}\binom{3}{j} m_i^{*j}*1^{*(j-1)}*\log \qquad (n\le 2R_i). \tag{4.3}$$

Factor roughness over the at most six variables per prime, dyadically subdivide each slot, and separate the smooth profile and log weights by Fourier inversion in logarithmic coordinates with two integrations by parts per variable (arbitrarily many to truncate frequencies at L^B, controlling tails via (4.2)).

Case 1: A medium slot exists. Suppose some dyadic slot has length $U \in [X^{2\zeta}, X^{1/3+\zeta/2}]$. Take it for Pascadi's short sequence α_n (a bounded rough 1 or rough μ times a logarithmic twist on an interval). To verify Siegel–Walfisz [10, Definition 4.3]:

- For rough 1, use the even and odd Brun polynomials U_z, L_z below, expanding $d \le y_* = X^{o(1)}$ and counting multiples in each class modulo polylogarithmic q; the two bounds match up to $O_A(UL^{-A}) + O(y_* L^{O(1)})$.
- For rough μ, expand $\mu(n)\mathbf{1}_{P^-(n)>z_*} = \sum_{d|n,\, P^+(d)\le z_*} \mu(n/d)$. Terms with $d > U^{1/2}$ contribute at most $U\exp(-\log U/(2\log z_*))\prod_{\ell\le z_*}(1-\ell^{-1+1/\log z_*})^{-1} \ll_A UL^{-A}$ by Rankin's trick and Mertens; terms with $d \le U^{1/2}$ satisfy Möbius Siegel–Walfisz at length $U/d \ge U^{1/2}$.

Auxiliary coprimality $(n,h)=1$ deletes at most $O((U/z_*+1)\omega(h))$ integers by roughness; $q > L^B$ is trivial by 1-boundedness. Truncating the complementary divisor-bounded sequence at L^B via (4.2) and applying Lemma 3.1, Pascadi [10, Proposition 4.4] (after dyadically subdividing the three modulus groups, since all powers in (3.1) are positive) yields (4.1).

Case 2: No medium slot exists. With $\zeta \ll a_*$, each of the two original primes contains a long slot of length $> X^{1/3+\zeta/2}$ (since six short slots have product $O(X^{12\zeta})$). There are exactly two long slots (three would exceed X), neither can be truncated Möbius, and the short complementary product has scale $M \le X^{24\zeta+o(1)}$ and L^1-norm $O(ML^C)$. We prove an absolute modulus average.

For fixed short product m with $(m,d)=1$, set $b_0 \equiv 2\bar{m} \pmod d$. For smooth compact long profiles f_0, g_0 at scales U, V, double Poisson summation gives the unrestricted discrepancy

$$\frac{UV}{d^2}\sum_{j,k\in\mathbb{Z}} \widehat{f}_0(jU/d)\widehat{g}_0(kV/d)\left(S(b_0 j,k;d) - \frac{c_d(j)c_d(k)}{\varphi(d)}\right). \tag{4.4}$$

Both axes $j=0$ and $k=0$ vanish because $(b_0,d)=1$. Using $|\widehat{f}_0(t)| \ll (\|f_0\|_1 + \|f_0''\|_1)(1+|t|)^{-2}$, the Weil bound $|S(b_0 j,k;d)| \le \tau(d)\sqrt{d}\,(j,k,d)^{1/2}$, and $\sum_{j\ne 0,\, h|j}(1+|j|/H)^{-2} \le 2H/h$, summing over $d \le D_0$ yields $O(MD_0^{3/2}L^{O(1)})$ (and $O(\tau(d)^2/\varphi(d))$ from Ramanujan sums), which has a fixed power saving since $24\zeta + \frac{3}{2}(5/8+\xi) < 1$.

To restore roughness without multiplying negative lower-sieve polynomials, set h_0 to the even integer $2\lfloor(\log\log X)^2\rfloor$ and put $y_* = z_*^{h_0+1} = X^{o(1)}$. Let $r_z := \mathbf{1}_{P^-(n)>z_*}$ denote the rough indicator. Form Brun polynomials L_z and U_z satisfying $L_z \le r_z \le U_z$, truncated at h_0+1 and h_0 prime factors, with $G_z = U_z - L_z \ge 0$ and coefficients bounded by 1. Their coprime densities satisfy $a_d \ll 1$ and $0 \le g_d \le (\sum_{\ell\le z_*} 1/\ell)^{h_0+1}/(h_0+1)! \ll_A L^{-A}$. Using

$$0 \le U_1U_2 - r_1r_2 = (U_1-r_1)U_2 + r_1(U_2-r_2) \le G_1U_2 + U_1G_2, \tag{4.5}$$

expanding Brun divisors shifts (U,V,m) to $(U/e, V/e', mee')$ with at most a y_*^2 loss in (4.4), while the coprime means from

$$\sum_{(n,d)=1} f_0(n/U)U_z(n) = U\frac{\varphi(d)}{d}a_d\int f_0 + O(y_*\tau(d)(\|f_0\|_1 + \|f_0'\|_1))$$

and its G_z analogue give total mean error $L^{O(1)}(MUV\varepsilon_X + M(U+V)y_* + My_*^2) \ll_A XL^{-A}$. Sharp interval indicators are retained via smooth error strips of relative width L^{-B}. This completes the proof of (4.1). □

5. Limiting density and small-prime tail

Let $\widetilde{F}(s) := F(s)/(2e^\gamma)$ and $\widetilde{f}(s) := f(s)/(2e^\gamma)$ be the normalized linear-sieve functions, extended by $\widetilde{f}(s) = 0$ for $s \le 2$. They satisfy

$$\begin{aligned} \widetilde{F}(s) &= \frac{1}{s}\ (1 \le s \le 3), \qquad \widetilde{f}(s) = 0\ (s \le 2), \\ (s\widetilde{F}(s))' &= \widetilde{f}(s-1)\ (s \ge 3), \qquad (s\widetilde{f}(s))' = \widetilde{F}(s-1)\ (s \ge 2). \end{aligned} \tag{5.1}$$

Here $\widetilde{F}$ decreases and $\widetilde{f}$ increases to the same limit, $\widetilde{F}$ is convex on $s \ge 1$ (as $-\widetilde{F}' = (\widetilde{F} - \widetilde{f}(s-1))/s$ decreases), and $\widetilde{f}$ is concave on $s \ge 2$ (as $\widetilde{f}' = (\widetilde{F}(s-1) - \widetilde{f})/s$ decreases), with an upward derivative jump of $1/2$ at $s = 2$.

In units of C_2/L, the ordinary reference cubic Rosser tree of cap $T = 5/8$ and root $\rho = 1/5$ has density $2\widetilde{F}(T/\rho)/\rho = 2/T + I_1$, where with $h = T/\rho - 3 = 1/8$,

$$I_1 = \frac{2}{T}\int_0^h \frac{\log(1+t)}{2+t}\,dt \le \frac{h^2}{2T}. \tag{5.2}$$

By the exit-series form of the linear sieve (with uniform prime-harmonic convergence on compact positive parameter ranges, noting that at any fixed exit rank the last exponent is bounded below by a positive multiple of ρ [5, 4]), every third-rank subtree is a negative copy of an ordinary lower cubic sieve of remaining level $J - S_3$ and threshold t_3. Hence the rank-three replacement adds

$$I_3 = 2\int_{0<z<y<u<\rho} \frac{\widetilde{f}((T-u-y-z)/z) - \widetilde{f}((J-u-y-z)/z)}{uyz^2}\,du\,dy\,dz. \tag{5.3}$$

On $z < \delta'$, all finite approximations keep $J - u - y \ge c > 0$, so by the fundamental lemma the residual density difference is $O(V(x^z)e^{-c'/z}) = O(C_2 z/L)$; integrating against the first two harmonic measures $\ll (1 + \log(\rho/z))^2$ and dyadic z gives $O(\delta'(1 + \log^2(\rho/\delta'))C_2/L)$. Thus the compact limit at $z \ge \delta'$ and dominated convergence yield (5.3) and show that the finite construction and attachment errors in Section 3 can be kept at most $2u_0C_2/L$ for any chosen $u_0 > 0$. There is no separate single-prime top interval above μ_0 because $\rho = 1/5 < 1/4$.

To bound the small-z tail of (5.3) explicitly, let $\widetilde{D} := \widetilde{F} - \widetilde{f} \ge 0$ and $\delta = 19/1000$. By (5.1),

$$\widetilde{D}'(s) = \frac{-\widetilde{D}(s) - \widetilde{D}(s-1)}{s} \le -\frac{2\widetilde{D}(s)}{s} \qquad (s \ge 3), \tag{5.4}$$

so $\widetilde{D}(s) \le \widetilde{D}(9)(9/s)^2$ for $s \ge 9$. Since $J - u - y \ge (1 - 3\rho)/2 = \rho$, the lower argument in (5.3) exceeds $(\rho - \delta)/z > 9$ for $z < \delta$. Bounding the numerator by $\widetilde{D}((J-u-y-z)/z)$ and integrating $z < y < u < \rho$ gives

$$I_{3,t} \le \frac{81\widetilde{D}(9)\delta}{(\rho-\delta)^2}\left(\log^2(\rho/\delta) + 2\log(\rho/\delta) + 2\right). \tag{5.5}$$

Letting $I_{3,c}$ be (5.3) restricted to $z \ge \delta$, the rank-three baseline is $U_0 = 2/T + I_1 + I_{3,c} + I_{3,t}$.

At rank four ($t_4 \ge 3/100$, already compact), the residual subtree is a positive upper sieve, so increasing the cap from J to M subtracts the nonnegative gain

$$G_1 = 2\int_{\mathcal{A}} \frac{\widetilde{F}((J-S_4)/t_4) - \widetilde{F}((M-S_4)/t_4)}{t_1t_2t_3t_4^2}\,dt_1\,dt_2\,dt_3\,dt_4, \tag{5.6}$$

where $\mathcal{A}$ consists of ordered rank-four prefixes passing the third cube and admitting a strict improvement (2.1). Note $(J - S_4)/t_4 > 1$ because $J > S_2 + 3t_3$ implies $J - S_4 > 2t_3 - t_4 \ge t_4$.

6. Application to $B_b(x)$ and proof of Theorem 1.1

Let K be any upper bound for $U_0 + 2u_0$ (rank three) or $U_0 + 2u_0 - G_1$ (ranks three and four). Discard $p < x/L^B$ ($B > 4$) from $B_b(x)$ at cost $O(x/L^B)$; for the remaining pairs $p = rq - 2$, both primes satisfy $r, q \gg x^{a_*}$ for any fixed $a_* < b/(1+b)$, and $p > x^{\rho'}$. Applying Lemma 2.1 pointwise to $rq - 2$ and majorizing the pair region by smooth outer weights of polylogarithmic Euler norms, Proposition 4.1 yields an upper bound controlled by the unrestricted pair mass times $\sum_d \rho_x(d)/\varphi(d)$: replacing $\mathbf{1}_{(rq,d)=1}$ by 1 costs $O(Lx^{-a_*})$ relative to the pair mass, because $\sum_{d \le 2x,\, \ell \mid d} 1/\varphi(d) \ll L/\ell$ for each $\ell \in \{r, q\}$.

In coordinates $t = \log r/L$ and $y = rq/x$, on $y \ge \sigma > 0$ the inequalities $q^b < r \le q$ and $rq \le x+2$ match $\beta_b := b/(1+b) < t \le 1/2$ up to $O_\sigma(1/L)$, while $y \le \sigma$ is covered by a smooth majorant on $y < 2\sigma$, $t \ge \beta_b - o(1)$, $r \le 2\sqrt{x}$ of mass $O(\sigma x/L)$ via the prime number theorem. Sending the smoothing width and then σ to zero gives limiting pair mass $\int_{\beta_b}^{1/2} dt/(t(1-t)) = \log(1/b)$ in units of x/L. Combining with (1.2) gives

$$\liminf_{x\to\infty} \frac{A_b(x)}{H(x)} \;\ge\; \frac{12759}{10000} - K\log(1/b). \tag{6.1}$$

7. Directed integer certificate

The companion script `verify_twin_upper_sieve_cert.py` certifies all numerical inequalities in pure integer arithmetic at denominator $Q = 2^{80} = 1208925819614629174706176$ (using `Fraction`, integer `isqrt` rounded upward, and floor/ceiling integer divisions, with zero floating-point operations in any bound).

(1) *Logarithms and delay tables.* For $1 \le z \le 2$, $\log z = 2\sum_{j=0}^{47} v^{2j+1}/(2j+1) + R_{48}$ ($v = (z-1)/(z+1)$) with $0 \le R_{48} \le 2v^{97}/(97(1-v^2))$; larger arguments reduce by powers of 2. The tables for $s\widetilde{F}$ and $s\widetilde{f}$ on a $1/4096$ mesh integrate the preceding numerator chord against ds/s using alternating fourth/fifth partial sums of the weights $(j+1)\log(1+1/j) - 1$ and $1 - j\log(1+1/j)$. Since $s\widetilde{F}$ is convex and $s\widetilde{f}$ is concave (each with curvature bounded by 1 on its nonconstant cells), the chord error $\le 1/(12n^2 j)$ is subtracted in the lower bound for $s\widetilde{f}$ (for $j \ge 3n$) and added in the upper bound for $s\widetilde{F}$. Beyond $s = 12$, both functions lie in $[\widetilde{f}(12), \widetilde{F}(12)]$. An independent 44-term quarter-interval Taylor solver (`reference_check`) verifies containment of both branches at $s = 4, \dots, 12$ within 10^{-12}.

(2) *Ordered simplex geometry and kinks.* On any grid box intersected with $t_1 \ge t_2 \ge \cdots \ge t_d$, each run of m coincident grid intervals forms an ordered simplex of volume fraction $1/m!$, order-statistic centroid $l + j(h-l)/(m+1)$, and coordinate variance $\le (h-l)^2/12$. For an affine form Y, $\mathbb{E}Y_+ \le (\sqrt{\mathrm{Var}(Y) + (\mathbb{E}Y)^2} + \mathbb{E}Y)/2$, using Cauchy–Schwarz within each run, independence across runs, and $\sqrt{12} > 17/5$. At rank three, only branches whose strict guards hold throughout the cell enter the lower minorant of J; at rank four, any unrefuted parent branch enters the upper majorant of J, while a candidate branch (2.1) enters the lower gain only when both planes satisfy all guards throughout the cell (using the single highest certified gain per cell, with $\widetilde{F}(s) = 1/s$ extended to $0 < s \le 1$).

(3) *Cell Taylor bounds.* For a proxy affine cap P with residual ratio $s = (P - S_d)/t_d$ and centered deviation $g = P - S_d - s(\mathbf{c})t_d$, the weighted linear Taylor error on $\phi \in \{\widetilde{F}, \widetilde{f}\}$ is bounded by

$$B(\mathbf{l})\frac{|\phi'(s(\mathbf{c}))|\,\sigma_g}{l_d}\sum_i (a_i + \mathbf{1}_{i=d})\frac{\sigma_i}{l_i},$$

where $B(\mathbf{t}) = 2\prod_i t_i^{-a_i}$ ($a_i = 1 + \mathbf{1}_{i=d}$), tightened on differences of same-sign slopes via $d_1\sigma_{g_1 - g_2} + |d_1 - d_2|\sigma_{g_2}$ and its symmetric counterpart. Quadratic curvature corrections $\le B(\mathbf{l}) \sup|\phi''|\,\sigma_g^2/(2l_d^2)$ are charged in the non-convex/non-concave direction, with an extra

$B(\mathbf{l})\sigma_g/(4l_d)$ whenever the upper estimate for $\widetilde{f}$ straddles the $s = 2$ kink. The common weight B satisfies $B(\mathbf{c}) \le \mathbb{E}B \le B(\mathbf{c}) + \frac{1}{2}B(\mathbf{l})[(\sum_i a_i\sigma_i/l_i)^2 + \sum_i a_i(\sigma_i/l_i)^2]$.

7.1. **Certified rational bounds.** Taking $u_0 = 1/10^6$, running the verifier evaluates 435001 third-rank and 700001 fourth-rank cells, exits with code 0, and certifies the following exact numerators at $Q = 2^{80}$:

Quantity	Direction	Exact numerator at $Q = 2^{80}$
I_1	$\le$	13943390692550619486377
$I_{3,c}$	$\le$	66925635371004488689726
$I_{3,t}$	$\le$	3409294078124169319
G_1	$\ge$	19374509949983140898664
K_3 (for $U_0 + 2u_0$)	$\le$	39494374759760858206635 35
K_4 (for $U_0 + 2u_0 - G_1$)	$\le$	39300629660261026797648 71

TABLE 1. Certified rational density components ($K_4 \approx 3.2508717$).

$b = a - 1$	Construction	Lower bound numerator on margin at $Q = 2^{80}$
3539/5000 $(1 - 1.7078)$	Two-factor shortcut	377883443722357593132
6767/10000 $(1 - 1.6767)$	Rank three (K_3)	105553009773095358394
17/25 $(1 - 1.68)$	Ranks three and four (K_4)	26790620021371983365686
169/250 $(1 - 1.676)$	Ranks three and four (K_4)	3604340579464536079328
3377/5000 $(1 - 1.6754)$	Ranks three and four (K_4)	114570068060210771664
67539/100000 $(1 - 1.67539)$	Ranks three and four (K_4)	56380964408304863785

TABLE 2. Certified positive lower margins for $\frac{12759}{10000} - K\log(1/b)$ in (6.1).

All six margins are strictly positive; in particular, the final row gives (1.1) and completes the proof of Theorem 1.1.

REPRODUCIBILITY AND ACKNOWLEDGMENTS

The standalone Python verifier `verify_twin_upper_sieve_cert.py` runs with the Python standard library alone (no external packages required) and verifies every bound in Tables 1 and 2 in approximately three minutes. The analytic derivations and certificate implementation were developed in an AI-assisted human-in-the-loop research workflow, with independent line-by-line mathematical auditing and full re-execution of the integer verifier.

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