Projective Moduli of Euler Continued-Fraction Tails: Regrouping Spectra, Schwarzian Invariants, and Bauer–Muir Acceleration

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Abstract

Euler's continued-fraction correspondence converts the partial sums of a series into the convergents of a continued fraction. We study the normalized partial numerators arising from this correspondence when the summands are sampled from a regularly varying, centred asymptotic profile. The main construction identifies these normalized numerators with cross-ratios of four consecutive tail sums. This gives an exact projective symmetry: two nondegenerate tail sequences have the same internal normalized Euler numerator sequence (from index $n = 2$ onward) if and only if they are related by one Möbius transformation. We then introduce a centred block-regrouping semigroup and diagonalize it by exact finite-difference eigenfunctions. For every odd integer exponent $p \geq 3$, the residual formal symmetry is an additive-group action with a slice; its invariant ring is a free polynomial algebra. The cubic case $p = 3$ is distinguished because the slice has weight two, so the first invariant has weight four. The cross-ratio expansion is Möbius invariant, and its small-step limit is the Schwarzian derivative. These facts yield an all-orders graded expansion for the normalized Euler coefficients. The complete coefficient sequence separates formal projective orbits, and its leading new invariant at each order is a Schwarzian coefficient. Finally, we construct a universal Bauer–Muir tail approximation. Under explicit nonvanishing and denominator-growth hypotheses, it changes the convergence rate from $O(n^{-2})$ to $O(n^{-6})$. We give a corrected subleading expansion and an exact Catalan-constant example.

Keywords: Euler continued fraction; block regrouping; cross-ratio; Möbius symmetry; locally nilpotent derivation; Schwarzian derivative; Bauer–Muir transformation.
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1 Introduction

Euler's continued-fraction correspondence expresses the partial sums of a series as the convergents of a continued fraction. In particular, if $d_n \neq 0$ and $r_n = d_n/d_{n-1}$, then

$$\sum_{k=0}^{N} d_k = \frac{d_0}{1 + \mathbf{K}_{n=1}^{N} \dfrac{-r_n}{1 + r_n}}. \tag{1.1}$$

The identity is classical; see, for example, Khrushchev [9, Chapter 4] and the modern account in Chen–Liu [1, Lemma 1]. Our purpose is not to claim a new series-to-continued-fraction correspondence. We study the projective and asymptotic structure of the normalized Euler coefficients under exact regrouping of the underlying series.

The starting observation is that, after an equivalence transformation making the internal partial denominators equal to 2, the normalized partial numerators are cross-ratios of four consecutive tail sums. Cross-ratio invariance then gives an exact Möbius symmetry. In a formal asymptotic class this symmetry reduces to a one-parameter additive action. For cubic odd tails,

the action has a slice and its invariant ring is a polynomial algebra in one generator of each even weight at least four.

A second ingredient is a centred block-regrouping semigroup. Its exact eigenfunctions are finite differences of powers. This gives a diagonal spectral description of regrouping and identifies a universal fixed profile. In the cubic case, the tail of that profile is $g_0(t) = 1/(2t^2)$, and its normalized Euler cross-ratio is an elementary rational function.

The principal results are:

(i) an exact spectral diagonalization of centred block regrouping for every real $p > 1$;

(ii) an exact sampled-sequence Möbius classification of normalized Euler tails;

(iii) an explicit $\mathbb{G}_a$-invariant polynomial coordinate ring for every odd integral formal tail, with the cubic case made explicit;

(iv) an all-orders graded expansion of the normalized Euler coefficients, whose leading new invariant at each order is a Schwarzian coefficient;

(v) a universal Bauer–Muir tail approximation giving an $O(n^{-6})$ convergence theorem under stated analytic hypotheses.

The individual ingredients are classical: Euler's correspondence, cross-ratio invariance, the Schwarzian derivative, the slice theorem for locally nilpotent derivations, and the Bauer–Muir transformation. David [5] studies Euler polynomial continued fractions, their Möbius matrix actions, and algorithmic recognition of the Euler family; this is particularly close prior work on the Euler/Möbius interface. Cross-ratios as discrete projective curvatures and their Schwarzian limits are established in discrete differential geometry; see Müller–Vaxman [10] and the projective-geometric discussion in Ovsienko–Tabachnikov [11]. The contribution claimed here is not the invention of any of these ingredients. It is the exact specialization and synthesis for normalized Euler tails: the finite-difference eigenspectrum of centred summand regrouping, the sampled-tail cross-ratio formula, the formal additive-group quotient, and the resulting complete all-orders coefficient moduli.

2 Euler tails and the cross-ratio representation

We use the generalized continued-fraction convention

$$b_0 + \mathbf{K}_{n=1}^{N} \frac{a_n}{b_n} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_N}{b_N}}}.$$

Lemma 2.1 (Euler's finite-sum identity)**.** *Let $d_0, \dots, d_N$ be nonzero, put $r_n = d_n/d_{n-1}$, and assume the finite continued fraction below is defined (in particular, its displayed partial denominators and continuant denominators are nonzero). Then*

$$\sum_{k=0}^{N} d_k = \frac{d_0}{1 + \mathbf{K}_{n=1}^{N} \dfrac{-r_n}{1 + r_n}}.$$

Proof. This is Euler's finite identity. For completeness, write the nested denominator tails as

$$H_N = 1 + r_N, \qquad H_k = 1 + r_k - \frac{r_{k+1}}{H_{k+1}} \quad (k = N-1, \dots, 1),$$

and set $F_k = 1 - r_k/H_k$. Then $F_N = 1/(1 + r_N)$ and a direct rearrangement gives

$$F_k = \frac{F_{k+1}}{r_k + F_{k+1}}.$$

Backward induction therefore yields

$$F_k = \frac{d_{k-1}}{d_{k-1} + d_k + \cdots + d_N}.$$

The full continued-fraction denominator is F_1, so $d_0/F_1 = \sum_{j=0}^{N} d_j$, as required. See also Euler's finite formula reproduced in [1, Lemma 1] and the general treatment in [9, Chapter 4]. □

Proposition 2.2 (Denominator-two normalization)**.** *Set*

$$A_1 = -\frac{2r_1}{1 + r_1}, \qquad A_n = -\frac{4r_n}{(1 + r_{n-1})(1 + r_n)}, \quad n \geq 2. \tag{2.1}$$

Then

$$\sum_{k=0}^{N} d_k = \frac{d_0}{1 + \mathbf{K}_{n=1}^{N} \frac{A_n}{2}}. \tag{2.2}$$

Proof. Apply the standard equivalence transformation to the continued fraction in Lemma 2.1 with scaling factors $c_0 = 1$ and $c_n = 2/(1+r_n)$. The transformed denominators are $c_n(1+r_n) = 2$. The first numerator becomes $c_0 c_1(-r_1) = A_1$, and for $n \geq 2$ the transformed numerator is

$$c_{n-1}c_n(-r_n) = -\frac{4r_n}{(1 + r_{n-1})(1 + r_n)} = A_n.$$

□

Suppose now that the summands are sampled from a function D at half-integers:

$$d_n = D\left(n + \frac{1}{2}\right), \qquad n \geq 0.$$

Assume the series converges and define its tail sum at integer arguments by

$$g(s) = \sum_{m=0}^{\infty} D\left(s + m + \frac{1}{2}\right), \qquad s \in \mathbb{Z}_{\geq 0}. \tag{2.3}$$

Then

$$D\left(t\right) = g\left(t - \frac{1}{2}\right) - g\left(t + \frac{1}{2}\right) \tag{2.4}$$

whenever $t \in \mathbb{Z} + \frac{1}{2}$. For $n \geq 2$ put $t_n = n - \frac{1}{2}$. From Equation (2.1),

$$A_n = -4\frac{D(t_n - 1)D(t_n + 1)}{(D(t_n - 1) + D(t_n))(D(t_n) + D(t_n + 1))}. \tag{2.5}$$

Lemma 2.3 (Cross-ratio representation)**.** *For $n \geq 2$, let*

$$z_1 = g(n-2), \quad z_2 = g(n-1), \quad z_3 = g(n), \quad z_4 = g(n+1).$$

Then

$$A_n = -4\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \tag{2.6}$$

Proof. By Equation (2.4), the three consecutive summands are $z_1 - z_2$, $z_2 - z_3$, and $z_3 - z_4$. Their adjacent sums are $z_1 - z_3$ and $z_2 - z_4$. Substitution in Equation (2.5) gives Equation (2.6). □

Theorem 2.4 (Exact projective classification of sampled tails)**.** *Let $(g_n)_{n\geq 0}$ and $(\widetilde{g}_n)_{n\geq 0}$ be sequences in $\mathbb{C}$ such that every consecutive triple is pairwise distinct and all cross-ratios in Equation (2.6) are defined. Let (A_n) and $(\widetilde{A}_n)$ be the associated normalized Euler tails. Then*

$$A_n = \widetilde{A}_n \quad (n \geq 2)$$

if and only if there exists a single $\mu \in \mathrm{PGL}_2(\mathbb{C})$ such that

$$\widetilde{g}_n = \mu(g_n) \quad (n \geq 0).$$

Proof. The forward implication from a Möbius map follows from cross-ratio invariance. Conversely, because g_0, g_1, g_2 and $\widetilde{g}_0, \widetilde{g}_1, \widetilde{g}_2$ are triples of distinct finite complex numbers, there is a unique $\mu \in \mathrm{PGL}_2(\mathbb{C})$ mapping the first triple to the second. Equality of the cross-ratio for $n = 2$ uniquely determines the fourth point, so $\widetilde{g}_3 = \mu(g_3)$. Inductively, the same argument gives $\widetilde{g}_n = \mu(g_n)$ for every n. At each step the value is the prescribed finite number $\widetilde{g}_n \in \mathbb{C}$; hence μ has no pole at any sampled point g_n. □

Remark 2.5. The theorem is an exact statement about sampled tail sequences and the internal normalized numerators A_n for $n \geq 2$. The initial numerator $A_1 = -2r_1/(1 + r_1)$ is a boundary coordinate and is not included in this projective classification. The theorem does not say that equality of formal asymptotic expansions forces equality of the underlying functions at finite arguments. We will keep these levels separate throughout.

3 The centred regrouping semigroup and its spectrum

Fix $p > 1$. Let $\mathscr{V}_p$ be the vector space of formal generalized Laurent expansions

$$D(t) = \sum_{m\geq 0} d_m t^{-(p+m)},$$

and write $\mathscr{V}_p^{\times}$ for the subset with nonzero leading coefficient d_0. Let $\mathscr{V}_p^{(2)}$ be the even-step subspace

$$\mathscr{V}_p^{(2)} = \left\{ \sum_{k\geq 0} d_k t^{-(p+2k)} \right\}.$$

This notation avoids assigning a parity to nonintegral p. When p is an integer, it agrees with the usual parity condition $D(-t) = (-1)^p D(t)$.

Define the centred block-regrouping operator

$$(\mathcal{R}_M D)(s) = \sum_{j=0}^{M-1} D\left(Ms + j - \frac{M-1}{2}\right), \qquad M \in \mathbb{Z}_{\geq 1}, \tag{3.1}$$

and the symmetric difference

$$(\Delta g)(t) = g\left(t - \frac{1}{2}\right) - g\left(t + \frac{1}{2}\right). \tag{3.2}$$

Also put $(S_M g)(t) = g(Mt)$. On formal series, every shifted term in Equation (3.1) is interpreted s^{-1}-adically: for example,

$$(Ms + \xi)^{-\alpha} = (Ms)^{-\alpha} \sum_{j\geq 0} \binom{-\alpha}{j} \left(\frac{\xi}{Ms}\right)^j.$$

Since the sum over $j = 0, \ldots, M-1$ is finite, this defines $\mathcal{R}_M$ coefficientwise on the formal asymptotic spaces used below.

Theorem 3.1 (Regrouping semigroup and intertwining)**.** *For all positive integers M, N,*

$$\mathcal{R}_N \mathcal{R}_M = \mathcal{R}_{MN}, \qquad \mathcal{R}_M \Delta = \Delta S_M. \tag{3.3}$$

Proof. The first identity follows by expanding the double sum. The argument of D becomes

$$MNs + (Mi + j) - \frac{MN - 1}{2},$$

and $Mi + j$ runs once through $0, \dots, MN - 1$. For the second identity, the sum telescopes:

$$\begin{aligned}(\mathcal{R}_M \Delta g)(s) &= \sum_{j=0}^{M-1} \left[g\left(Ms + j - \frac{M}{2}\right) - g\left(Ms + j + 1 - \frac{M}{2}\right) \right] \\ &= g\left(Ms - \frac{M}{2}\right) - g\left(Ms + \frac{M}{2}\right) = (\Delta S_M g)(s).\end{aligned}$$

□

For $\alpha > 0$, define the exact difference eigenfunctions

$$e_\alpha(t) = \frac{1}{\alpha} \Delta(t^{-\alpha}) = \frac{1}{\alpha} \left[\left(t - \frac{1}{2}\right)^{-\alpha} - \left(t + \frac{1}{2}\right)^{-\alpha} \right]. \tag{3.4}$$

They satisfy $e_\alpha(t) = t^{-(\alpha+1)} + O(t^{-(\alpha+3)})$.

Corollary 3.2 (Exact eigenbasis and regrouping coordinates)**.** *For every $\alpha > 0$ and every positive integer M,*

$$\mathcal{R}_M e_\alpha = M^{-\alpha} e_\alpha. \tag{3.5}$$

Moreover, every $D \in \mathscr{V}_p^\times \cap \mathscr{V}_p^{(2)}$, with leading coefficient $C \neq 0$, has a unique expansion

$$\frac{D(t)}{C} = e_{p-1}(t) + \sum_{k \geq 1} X_{2k} e_{p-1+2k}(t). \tag{3.6}$$

Consequently,

$$\frac{M^{p-1}}{C} (\mathcal{R}_M D)(t) = e_{p-1}(t) + \sum_{k \geq 1} M^{-2k} X_{2k} e_{p-1+2k}(t). \tag{3.7}$$

Proof. By Theorem 3.1,

$$\mathcal{R}_M e_\alpha = \frac{1}{\alpha} \Delta S_M(t^{-\alpha}) = M^{-\alpha} e_\alpha.$$

The map Δ is triangular with nonzero diagonal in the ordered asymptotic monomial bases, hence is an isomorphism between the corresponding formal spaces. This gives the unique expansion. Applying the eigenvalue formula termwise proves Equation (3.7). □

The inverse difference operator is particularly simple as a formal differential operator. With the zero constant of integration convention,

$$\Delta^{-1} = -\partial_t^{-1} \frac{\partial_t/2}{\sinh(\partial_t/2)} = -\partial_t^{-1} \left(1 - \frac{\partial_t^2}{24} + \frac{7\partial_t^4}{5760} - \frac{31\partial_t^6}{967680} + \cdots\right). \tag{3.8}$$

Remark 3.3 (Formal character of the inverse difference). The inverse in Equation (3.8) is an inverse in the formal t^{-1}-adic category; it need not define a convergent Laurent series. For rational profiles such as D_G and D_5 below, the corresponding tail sums have special-function representations, while their Euler–Maclaurin expansions are generally divergent asymptotic series, with Bernoulli terms that grow factorially. Accordingly, the coordinates obtained from $g = \Delta^{-1}(D/C)$ in Sections 3–5 are formal asymptotic coordinates unless convergence is separately proved. The Bauer–Muir coefficient theorem in Section 6 does not require convergence of this formal inverse: it uses the actual discrete formula Equation (2.5) and the asymptotic expansion of the summand profile D directly.

If $g = \Delta^{-1}(D/C)$, then Equation (3.6) is equivalent to

$$g(t) = \sum_{k\geq 0} \frac{X_{2k}}{p-1+2k} t^{-(p-1+2k)}, \qquad X_0 = 1. \tag{3.9}$$

The leading eigenfunction

$$F_p(t) := e_{p-1}(t) = \frac{1}{p-1}\left[\left(t-\frac{1}{2}\right)^{-(p-1)} - \left(t+\frac{1}{2}\right)^{-(p-1)}\right] \tag{3.10}$$

is the universal fixed profile of the normalized regrouping semigroup. In particular, $F_p(t) \sim t^{-p}$.

Remark 3.4. For $p = 3$,

$$F(t) := F_3(t) = \frac{t}{(t^2-\frac{1}{4})^2}, \qquad g_0(t) = \frac{1}{2t^2}. \tag{3.11}$$

Indeed, $F = \Delta g_0$. The rescaled regrouping operator in Equation (3.7) has eigenvalues M^{-2k}, so the scaling dimensions $2, 4, 6, \ldots$ are exact spectral data, not empirical fitting exponents.

4 The formal projective quotient for odd tails

From now on we take $p = 3$ and work over a field K of characteristic zero (for the displayed rational formulae, one may take $K = \mathbb{Q}$). The formal tail is

$$g(t) = \frac{1}{2}t^{-2} + \sum_{k\geq 1} \frac{X_{2k}}{2k+2} t^{-(2k+2)}. \tag{4.1}$$

The normalized sampled-sequence symmetry of Theorem 2.4, when imposed on this formal asymptotic class, reduces to

$$\mu_\lambda(g) = \frac{g}{1-\lambda g}, \qquad \lambda \in K. \tag{4.2}$$

Indeed, write a Möbius map as $\mu(z) = (az+b)/(cz+d)$. Since $g(t) = \frac{1}{2}t^{-2} + O(t^{-4})$ and the transformed tail is required to have the same leading term, one must have $\mu(0) = 0$ and $\mu'(0) = 1$. Thus $b = 0$ and $a = d \neq 0$; after projective rescaling, $\mu(z) = z/(1-\lambda z)$ with $\lambda = -c/d$. This is an additive-group action because $\mu_\lambda \circ \mu_\nu = \mu_{\lambda+\nu}$. It preserves the even Laurent subspace. For a general odd integer $p = 2q+1$, the same argument starts from $g_p(t) = (p-1)^{-1}t^{-(p-1)} + O(t^{-(p+1)})$: preservation of this normalized leading term again forces $\mu(0) = 0$ and $\mu'(0) = 1$, hence the same $\mathbb{G}_a$ action. Since $p-1$ is even, every power g_p^r has the allowed even-step parity. If p is even, the quadratic term g_p^2 has the wrong parity unless $\lambda = 0$.

Let

$$R = K[X_2, X_4, X_6, \ldots], \qquad \mathrm{wt}(X_{2k}) = 2k.$$

Differentiating Equation (4.2) at $\lambda = 0$ gives the locally nilpotent derivation δ determined by

$$\delta\left(\frac{X_{2k}}{2k+2}\right) = \sum_{i+j=k-1} \frac{X_{2i}}{2i+2}\frac{X_{2j}}{2j+2}, \qquad k \geq 1, \quad X_0 = 1. \tag{4.3}$$

Equivalently,

$$\delta X_{2k} = (2k+2) \sum_{i+j=k-1} \frac{X_{2i}X_{2j}}{(2i+2)(2j+2)}.$$

In particular,

$$\delta X_2 = 1, \quad \delta X_4 = \frac{3}{2}X_2, \quad \delta X_6 = \frac{4}{3}X_4 + \frac{1}{2}X_2^2, \quad \delta X_8 = \frac{5}{4}X_6 + \frac{5}{6}X_2X_4. \tag{4.4}$$

The derivation lowers weight by two and is locally nilpotent on every polynomial.

Theorem 4.1 (Slice and invariant ring)**.** *The element X_2 is a slice for δ. The invariant ring is the free polynomial algebra*

$$R^{\mathbb{G}_a} = \ker \delta = K[J_4, J_6, J_8, \ldots], \tag{4.5}$$

where

$$J_{2k} = (2k+2)\,[t^{-(2k+2)}]\left(\frac{g(t)}{1+X_2 g(t)}\right), \qquad k \geq 2. \tag{4.6}$$

Moreover, $R = R^{\mathbb{G}_a}[X_2]$.

Proof. Equation (4.4) gives $\delta X_2 = 1$. The slice theorem for locally nilpotent derivations therefore gives $R = (\ker \delta)[X_2]$. The projection onto the kernel is the slice projection

$$\pi = \exp(-X_2\delta).$$

Since $\exp(\lambda\delta)g = g/(1-\lambda g)$, we have $\pi(g) = g/(1+X_2 g)$. Its coefficients are precisely the J_{2k} in Equation (4.6). The triangularity of the coefficient transformation shows that the J_{2k} are algebraically independent and generate the kernel: recursively, $J_{2k} = X_{2k}$ plus a polynomial in lower-weight coordinates after the slice projection. Equivalently, restriction to the slice $X_2 = 0$ identifies the kernel with the polynomial ring in the remaining coordinates. This proves Equation (4.5). □

The first generators are

$$J_4 = X_4 - \frac{3}{4}X_2^2, \tag{4.7}$$

$$J_6 = X_6 - \frac{4}{3}X_2X_4 + \frac{1}{2}X_2^3, \tag{4.8}$$

$$J_8 = X_8 - \frac{5}{4}X_2X_6 + \frac{5}{12}X_2^2X_4. \tag{4.9}$$

Remark 4.2 (Exact and formal symmetry). For actual sampled tail sequences, Theorem 2.4 gives an exact single Möbius map. For formal tails, the action Equation (4.2) and the invariant ring classify formal orbits. A vanishing sequence of all formal invariants proves formal Möbius equivalence; it does not, without analyticity or an additional uniqueness theorem, prove equality of actual functions at every finite argument.

4.1 The quotient for every odd integral exponent

The slice structure is not restricted to the cubic case. Let $p = 2q+1 \geq 3$ be an odd integer, put $\beta = p-1 = 2q$, and write a normalized even formal tail as

$$g_p(t) = \frac{1}{\beta}t^{-\beta} + \sum_{k\geq 1}\frac{X_{2k}}{\beta+2k}t^{-(\beta+2k)}. \tag{4.10}$$

The action $g_p \mapsto g_p/(1-\lambda g_p)$ preserves this series space. Let $R_p = K[X_2, X_4, \ldots]$, with $\mathrm{wt}(X_{2k}) = 2k$, and let $\delta g_p = g_p^2$. Equating coefficients gives

$$\delta\left(\frac{X_{2k}}{\beta+2k}\right) = \sum_{\substack{i+j=k-q\\ i,j\geq 0}}\frac{X_{2i}X_{2j}}{(\beta+2i)(\beta+2j)}, \qquad X_0 = 1, \tag{4.11}$$

where an empty sum is zero. In particular, $\delta X_{2k} = 0$ for $k < q$, while

$$\delta X_{2q} = \frac{2}{\beta} = \frac{1}{q}. \tag{4.12}$$

Thus $s := qX_{2q}$ is a slice.

Theorem 4.3 (Formal projective quotient for odd p)**.** *For $p = 2q + 1 \geq 3$, define, for $k > q$,*

$$\widehat{J}_{2k}^{(p)} = (\beta + 2k)\,[t^{-(\beta+2k)}]\left(\frac{g_p(t)}{1 + sg_p(t)}\right), \qquad s = qX_{2q}. \tag{4.13}$$

Then

$$R_p^{\mathbb{G}_a} = K\,[X_2, X_4, \ldots, X_{2q-2}, \widehat{J}_{2q+2}^{(p)}, \widehat{J}_{2q+4}^{(p)}, \ldots\,], \tag{4.14}$$

a free polynomial algebra. The displayed generators are homogeneous of their indicated weights, and under normalized regrouping each generator of weight $2k$ scales by M^{-2k}.

Proof. Equation (4.11) shows that the derivation is locally nilpotent, lowers weight by β, annihilates the coordinates below weight β, and satisfies $\delta s = 1$. The slice theorem gives $R_p = (\ker \delta)[s]$ and the projection onto the kernel is $\exp(-s\delta)$. Since the flow is $g_p \mapsto g_p/(1-\lambda g_p)$, this projection sends g_p to $g_p/(1 + sg_p)$. Its coefficients above the slice weight are precisely the $\widehat{J}_{2k}^{(p)}$. The lower coordinates $X_2, \ldots, X_{2q-2}$ are already invariant. At each higher weight, the coefficient $\widehat{J}_{2k}^{(p)}$ equals X_{2k} plus a polynomial in lower-weight coordinates, so the change of generators is triangular and invertible. This proves algebraic independence and generation. Homogeneity and the regrouping eigenvalues give the scaling statement. □

For $p = 3$ one has $q = 1$, so there are no lower-weight invariants: the first generator has weight four. For $p \geq 5$, the invariant coordinates of weights $2, 4, \ldots, p-3$ survive below the slice. This explains algebraically why the cubic case is exceptional, rather than merely a convenient example.

Remark 4.4 (Dropping the even-step parity restriction). The parity restriction is essential to the odd-p theorem above, but it is not essential to the additive-group quotient itself. For an integer $p \geq 2$, put $\beta = p - 1$ and allow the full Laurent tail

$$g_p(t) = \frac{1}{\beta}t^{-\beta} + \sum_{k\geq 1} \frac{X_k}{\beta + k}t^{-(\beta+k)}, \qquad \mathrm{wt}(X_k) = k.$$

The action $g_p \mapsto g_p/(1 - \lambda g_p)$ preserves this full Laurent space. Its infinitesimal derivation satisfies

$$\delta\left(\frac{X_k}{\beta + k}\right) = \sum_{i+j=k-\beta} \frac{X_i X_j}{(\beta + i)(\beta + j)}, \qquad X_0 = 1,$$

with an empty sum interpreted as zero. Thus $\delta X_k = 0$ for $k < \beta$ and $\delta X_\beta = 2/\beta$; the slice is $s = \beta X_\beta/2$. The same slice theorem gives

$$K[X_1, \ldots, X_{\beta-1}, \widehat{J}_{\beta+1}, \widehat{J}_{\beta+2}, \ldots], \quad \widehat{J}_k = (\beta + k)[t^{-(\beta+k)}]\frac{g_p(t)}{1 + sg_p(t)},$$

as the full Laurent invariant ring. In particular, when $p = 2$ the slice has weight one and the first invariant has weight two. The paired Leibniz representation

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\left(\frac{1}{4k+1} - \frac{1}{4k+3}\right), \qquad D_{\pi/4}(t) = \frac{2}{16t^2 - 1}, \quad t = k + \frac{1}{2},$$

is a $p = 2$ block profile. Its even-step summand expansion lies in $\mathscr{V}_2^{(2)}$, but its Möbius orbit generally leaves the even-step subspace and must be considered in the full Laurent space. We do not assert the even-in-t, doubly graded all-orders theorem of Section 5 for this unrestricted space; its coefficient expansion has to be organized with all integer powers.

For example, when $p = 5$ one has $q = 2$: X_2 is an invariant of weight two, whereas X_4 is the slice coordinate. On the canonical slice $X_4 = 0$, direct expansion gives

$$\mathcal{C}_1[g_5](t) = -1 + \frac{15}{4t^2} + \frac{2X_2 - 105/16}{t^4} + O(t^{-6}).$$

Thus the fourth-order coefficient already detects the surviving weight-two invariant; this contrasts with $p = 3$, where the weight-two coordinate is entirely gauge and κ_4 is universal.

5 Schwarzian expansion and the all-orders graded theorem

For a formal tail g define the step-h cross-ratio coefficient

$$\mathcal{C}_h[g](t) := -4\,\frac{\big(g(t-\frac{3}{2}h) - g(t-\frac{1}{2}h)\big)\big(g(t+\frac{1}{2}h) - g(t+\frac{3}{2}h)\big)}{\big(g(t-\frac{3}{2}h) - g(t+\frac{1}{2}h)\big)\big(g(t-\frac{1}{2}h) - g(t+\frac{3}{2}h)\big)}. \tag{5.1}$$

For $h = 1$ this is the normalized Euler numerator with the formal asymptotic interpolation convention. For an actual sequence, the tail sum is initially defined only at integer arguments; throughout this section $\mathcal{C}_h[g]$ means the formal Laurent expansion obtained from its asymptotic tail g. No arbitrary interpolation of finite sample values is assumed. The expression is invariant under post-composition of g by any Möbius map.

Lemma 5.1 (The Schwarzian limit)**.** *If g is three-times formally differentiable and $g'(t)$ is invertible in the Laurent field, then*

$$\mathcal{C}_h[g](t) = -1 - \frac{h^2}{2}\{g, t\} + O(h^4), \tag{5.2}$$

where

$$\{g, t\} = \frac{g'''(t)}{g'(t)} - \frac{3}{2}\left(\frac{g''(t)}{g'(t)}\right)^2$$

is the Schwarzian derivative.

Proof. Expand the two short differences in the numerator around $t - h$ and $t + h$, and the two long differences around $t - h/2$ and $t + h/2$. Writing $a = g'(t)$, $b = g''(t)$, and $c = g'''(t)$, the numerator product, after removing the common factor h^2, is

$$a^2 + h^2\left(\frac{13}{12}ac - b^2\right) + O(h^4),$$

whereas the denominator product, after removing $4h^2$, is

$$a^2 + h^2\left(\frac{7}{12}ac - \frac{1}{4}b^2\right) + O(h^4).$$

Their quotient is $\frac{1}{4}[1 + \frac{h^2}{2}\{g, t\} + O(h^4)]$. Multiplication by -4 proves Equation (5.2). □

Corollary 5.2 (Universal second coefficient for general p)**.** *If $g(t) = (p-1)^{-1}t^{-(p-1)} + O(t^{-p-1})$, then the coefficient of t^{-2} in $\mathcal{C}_1[g](t) + 1$ is*

$$\kappa_2(p) = \frac{p(p-2)}{4}. \tag{5.3}$$

Proof. For $g_0(t) = (p-1)^{-1}t^{-(p-1)}$, direct calculation gives

$$-\frac{1}{2}\{g_0, t\} = \frac{p(p-2)}{4}t^{-2}.$$

The next terms in the tail expansion contribute only to the coefficient of t^{-4} and lower asymptotic orders. □

Proposition 5.3 (Sensitivity to the first even-step mode)**.** *Let $p > 1$ and perturb the universal profile by its first higher even-step eigenmode:*

$$D_\varepsilon(t) = F_p(t) + \varepsilon e_{p+1}(t), \qquad q(t) := \frac{e_{p+1}(t)}{F_p(t)} = t^{-2} + O(t^{-4}).$$

For any quantity F_ε depending on this perturbation, write

$$\delta F := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon.$$

Then the first variation of the normalized cross-ratio satisfies

$$\delta \log(-\mathcal{C}_1[\Delta^{-1} D_\varepsilon])(t) = (3-p)t^{-4} + O(t^{-5}).$$

Consequently, with $\mathcal{C}_1[\Delta^{-1} D_\varepsilon] = -1 + \kappa_2 t^{-2} + \kappa_4 t^{-4} + \cdots$,

$$\frac{\partial \kappa_4}{\partial X_2} = p - 3.$$

Among odd integers $p \geq 3$, the fourth coefficient is therefore independent of the first even-step coordinate only for $p = 3$.

Proof. More generally, if a perturbation has relative profile $q(t) = t^{-k} + O(t^{-k-2})$, the first variation of the logarithmic cross-ratio is

$$\frac{F_p(t)[q(t-1) - q(t)]}{F_p(t-1) + F_p(t)} + \frac{F_p(t)[q(t+1) - q(t)]}{F_p(t) + F_p(t+1)}.$$

Since $F_p(t) \sim t^{-p}$,

$$\frac{F_p(t)}{F_p(t-1) + F_p(t)} = \frac{1}{2} - \frac{p}{4t} + O(t^{-2}), \quad \frac{F_p(t)}{F_p(t) + F_p(t+1)} = \frac{1}{2} + \frac{p}{4t} + O(t^{-2}),$$

while

$$q(t-1) - q(t) = kt^{-k-1} + \frac{k(k+1)}{2} t^{-k-2} + O(t^{-k-3}),$$
$$q(t+1) - q(t) = -kt^{-k-1} + \frac{k(k+1)}{2} t^{-k-2} + O(t^{-k-3}).$$

Thus the derivative is $\delta \log(-\mathcal{C}_1) = \frac{k(k+1-p)}{2} t^{-k-2} + O(t^{-k-3})$. Taking $k = 2$ gives $(3-p)t^{-4}$. Since $-\mathcal{C}_1 = 1 - \kappa_2 t^{-2} - \kappa_4 t^{-4} + \cdots$, it follows that $\delta\kappa_4 = p - 3$. The first eigenmode coefficient is X_2, so this is precisely $\partial\kappa_4/\partial X_2$, proving the result. □

For $p = 3$, the normalized reference tail is $g_0(t) = 1/(2t^2)$. A direct calculation gives

$$\mathcal{C}_h[g_0](t) = -\frac{t^2 - h^2}{t^2 - h^2/4} = -1 + \sum_{m \geq 1} \frac{3}{4^m} h^{2m} t^{-2m}. \tag{5.4}$$

Theorem 5.4 (All-orders graded invariant theorem for $p = 3$)**.** *Let g be the formal tail in Equation (4.1), and let $J_4, J_6, \ldots$ be its projective invariants from Theorem 4.1. There exist weighted-homogeneous polynomials*

$$P_{m,r} \in K[J_4, J_6, \ldots, J_{2r}], \qquad \mathrm{wt}(P_{m,r}) = 2r,$$

for $m \geq 3$ and $2 \leq r \leq m-1$, such that

$$\mathcal{C}_h[g](t) = -1 + \sum_{m \geq 1} \left[\frac{3}{4^m} h^{2m} + \sum_{r=2}^{m-1} P_{m,r}(J_4, \ldots, J_{2r}) h^{2(m-r)} \right] t^{-2m}. \tag{5.5}$$

The polynomials are defined constructively by the coefficient extraction

$$P_{m,r} = [h^{2(m-r)}t^{-2m}](\mathcal{C}_h[g^\circ](t) + 1), \tag{5.6}$$

where the canonical slice representative is

$$g^\circ(t) = \frac{1}{2t^2} + \sum_{k\geq 2} \frac{J_{2k}}{2k+2} t^{-(2k+2)}. \tag{5.7}$$

For a fixed pair (m, r), this prescription is a finite algebraic algorithm: truncate g° after the term t^{-2m}, evaluate the four shifted series in Equation (5.1) modulo terms of total degree above the requested (h, t^{-1}) order, and perform formal division. Thus every $P_{m,r}$ is effectively computable using finitely many rational operations in $J_4, \ldots, J_{2r}$. For $h = 1$ and after regrouping by M, the coefficient $\kappa_{2m}^{(M)}$ of t^{-2m} is

$$\kappa_{2m}^{(M)} = \frac{3}{4^m} + \sum_{r=2}^{m-1} \frac{P_{m,r}(J_4, \ldots, J_{2r})}{M^{2r}}. \tag{5.8}$$

The leading representation-dependent coefficient is the Schwarzian coefficient

$$P_{m,m-1} = [t^{-2m}] \left(-\frac{1}{2}\{g, t\} \right). \tag{5.9}$$

For every $m \geq 3$ it has triangular form

$$P_{m,m-1} = -2(m-1)(m-2)J_{2m-2} + Q_m(J_4, \ldots, J_{2m-4}), \tag{5.10}$$

where Q_m is weighted homogeneous of weight $2m-2$. Consequently, the sequence of leading Schwarzian invariants $P_{m,m-1}$, $m \geq 3$, generates the full invariant ring $K[J_4, J_6, \ldots]$ and separates formal $\mathbb{G}_a$-orbits.

Proof. First, because $\mathcal{C}_h$ is a cross-ratio, it is invariant under $g \mapsto g/(1-\lambda g)$. Each coefficient in its Laurent expansion is therefore in $R^{\mathbb{G}_a} = K[J_4, J_6, \ldots]$ by Theorem 4.1.

Assign degree 1 to t and h, and degree $2k$ to X_{2k}. Then every term of g in Equation (4.1) has degree -2, and $\mathcal{C}_h[g]$ has degree zero. Thus the coefficient of t^{-2m} is a sum of terms $h^{2(m-r)}$ times invariant polynomials of weight $2r$. The expression is even in h and, since g is even, even in t. At $h = 0$ the limiting cross-ratio is -1, so the term with $r = m$ vanishes. There is no nonzero invariant of weight 2, because the invariant ring is generated in weights $4, 6, 8, \ldots$. Hence only $r = 0$ and $2 \leq r \leq m-1$ occur.

At $J_4 = J_6 = \cdots = 0$, the canonical representative is $g_0 = 1/(2t^2)$, and Equation (5.4) gives the universal term $3h^{2m}/4^m$. This proves Equation (5.5) and the coefficient-extraction formula Equation (5.6). Regrouping sends $J_{2r} \mapsto M^{-2r}J_{2r}$ by Equation (3.7), proving Equation (5.8).

The coefficient of h^2 in Equation (5.5) is $-\frac{1}{2}\{g, t\}$ by Lemma 5.1, proving Equation (5.9). To obtain its dependence on the top coordinate, put all lower coordinates equal to zero and perturb

$$g_0(t) = \frac{1}{2t^2} \quad \text{by} \quad \varepsilon \frac{J_{2m-2}}{2m} t^{-2m}.$$

Writing $w = \delta g'/g_0' = J_{2m-2}t^{-2m+2}$, the first variation of the Schwarzian is

$$\delta\{g, t\} = w'' + \frac{3}{t}w' = 4(m-1)(m-2)J_{2m-2}t^{-2m}.$$

Multiplication by $-1/2$ gives the coefficient in Equation (5.10). Since it is nonzero for $m \geq 3$, the leading invariants recursively recover $J_4, J_6, \ldots$. The slice theorem then gives orbit separation. □

Theorem 5.5 (All-orders graded moduli for odd integral p)**.** *Let $p = 2q + 1 \geq 3$ and let g_p be as in Equation (4.10). Expand the step-h cross-ratio in powers of t^{-1} and h. For every $m \geq 1$ there is a finite decomposition*

$$\mathcal{C}_h[g_p](t) = -1 + \sum_{m\geq 1} \left[c_m(p) h^{2m} + \sum_{1\leq r\leq m-1} P^{(p)}_{m,r} h^{2(m-r)} \right] t^{-2m}, \tag{5.11}$$

where $c_m(p) = [t^{-2m}](\mathcal{C}_1[g_0] + 1)$ for $g_0(t) = \beta^{-1}t^{-\beta}$, and $P^{(p)}_{m,r}$ is a weighted-homogeneous polynomial of weight $2r$ in the generators of $R_p^{\mathbb{G}_a}$ of weights at most $2r$. The coefficient $P^{(p)}_{m,m-1}$ is the t^{-2m} coefficient of $-\frac{1}{2}\{g_p, t\}$.

At the slice weight $2q = p - 1$, the polynomial $P^{(p)}_{m,q}$ involves only the lower-weight invariants $X_2, \ldots, X_{2q-2}$ and contains no slice coordinate. For $p = 3$ one has $q = 1$ and there are no lower-weight invariants, so $P^{(3)}_{m,1} \equiv 0$; this is why the sum in Equation (5.5) starts at $r = 2$ in the cubic theorem.

For $r = m - 1 \neq q$, its dependence on the highest-weight generator of weight $2r$ is triangular, with coefficient

$$-r(2r + 1 - p). \tag{5.12}$$

Consequently the complete sequence of normalized asymptotic Euler coefficients separates formal $\mathbb{G}_a$-orbits for every odd integer $p \geq 3$.

Proof. The proof follows the grading and projective-invariance argument of Theorem 5.4. The cross-ratio is invariant under post-composition of g_p by a Möbius map, so every coefficient lies in $R_p^{\mathbb{G}_a}$. Since g_p is even in t, the expansion is even in t and h. Assigning degree one to t, h and weight $2k$ to X_{2k} shows that the coefficient of $t^{-2m}h^{2(m-r)}$ is an invariant polynomial of weight $2r$. The h^0 term is zero because the coincident-step cross-ratio equals -1. The invariant ring has no new generator at weight $\beta = 2q$, because that coordinate is the slice; however its homogeneous component of weight $2q$ can contain polynomials in the lower-weight invariants. Such a contribution is already determined by previously recovered coordinates. At the invariant origin, $g_p = g_0$, giving the universal coefficient $c_m(p)$. The coefficient of h^2 is the Schwarzian term by Lemma 5.1.

For the triangular coefficient, perturb the summand profile by its weight-$2r$ eigenmode,

$$D_\varepsilon = F_p + \varepsilon e_{p-1+2r}.$$

By the eigenfunction normalization, the relative summand perturbation is $e_{p-1+2r}/F_p = t^{-2r} + O(t^{-2r-2})$. The first-variation calculation in Proposition 5.3, with $k = 2r$, therefore gives

$$\delta \log(-\mathcal{C}_1) = r(2r + 1 - p)t^{-2r-2} + O(t^{-2r-3}).$$

Since $-\mathcal{C}_1 = 1 - \kappa_{2r+2}t^{-2r-2} + \cdots$, the coefficient of the new coordinate in the cross-ratio coefficient is $-r(2r + 1 - p)$. It vanishes exactly at $r = q$, the slice coordinate, and is nonzero for every other r. Below the slice the generator is X_{2r}; above it the projected generator $\widehat{J}^{(p)}_{2r}$ is triangularly equal to X_{2r} plus lower-weight terms, so the same diagonal coefficient applies. At $r = q$ the coefficient is already determined by lower-weight invariants and introduces no new coordinate. Induction over the generator weights therefore recovers all invariant coordinates from the complete coefficient sequence. The invariant-ring theorem then gives separation of formal orbits. □

Corollary 5.6 (Completeness of the normalized Euler-tail coefficients)**.** *Let g_1 and g_2 be two formal cubic even tails of the form Equation (4.1), and write*

$$\mathcal{C}_1[g_j](t) = -1 + \sum_{m\geq 1} \kappa_{2m}(g_j)t^{-2m}, \qquad j = 1, 2.$$

Then the following are equivalent:

(i) $\kappa_{2m}(g_1) = \kappa_{2m}(g_2)$ *for every* $m \geq 1$*;*

(ii) $J_{2k}(g_1) = J_{2k}(g_2)$ *for every* $k \geq 2$*;*

(iii) there is a unique $\lambda \in K$ *such that*

$$g_2 = \frac{g_1}{1 - \lambda g_1}.$$

Thus the complete asymptotic coefficient sequence of the normalized Euler tail is a complete set of formal moduli for the residual projective symmetry.

Proof. Möbius invariance gives (iii)⇒(i). The slice theorem gives (ii)⇔(iii). It remains to prove (i)⇒(ii). Proceed inductively. At order t^{-2m}, the coefficient κ_{2m} is the sum of terms of weights $4, 6, \ldots, 2m-2$. Once $J_4, \ldots, J_{2m-4}$ have been recovered from the preceding coefficients, subtracting their known contribution leaves the leading invariant $P_{m,m-1}$. By Equation (5.10), this is a nonzero linear function of J_{2m-2} plus a polynomial in the already recovered lower generators. Hence J_{2m-2} is uniquely determined. Induction proves (ii). □

Remark 5.7. The all-orders theorem is a theorem in the formal asymptotic category. Its coefficient-extraction prescription is constructive: to compute a fixed $P_{m,r}$, only finitely many J-coordinates and finitely many terms of the cross-ratio expansion are required. It does not assert that two actual interpolating functions with the same asymptotic expansion agree pointwise.

5.1 The first three invariants

Write the block expansion

$$D(t) = Ct^{-3}\left(1 + \frac{u}{t^2} + \frac{v}{t^4} + \frac{w}{t^6} + \frac{r}{t^8} + O(t^{-10})\right). \tag{5.13}$$

Using Equation (3.8),

$$X := X_2 = u - \frac{1}{2}, \tag{5.14}$$

$$Y := X_4 = v - \frac{5}{4}u + \frac{7}{16}, \tag{5.15}$$

$$Z := X_6 = w + \frac{49}{24}u - \frac{7}{3}v - \frac{31}{48}, \tag{5.16}$$

$$T := X_8 = r - \frac{15}{4}w + \frac{49}{8}v - \frac{155}{32}u + \frac{381}{256}. \tag{5.17}$$

Define

$$\mathcal{I} = 3X^2 - 4Y, \tag{5.18}$$

$$\mathcal{J} = 16XY - 6X^3 - 12Z, \tag{5.19}$$

$$\mathcal{K} = -24T + 30XZ + 16Y^2 - 34X^2Y + 9X^4. \tag{5.20}$$

Then

$$\mathcal{I} = -4J_4, \qquad \mathcal{J} = -12J_6, \qquad \mathcal{K} = -24J_8 + \mathcal{I}^2.$$

The first three nontrivial regrouping laws are

$$\kappa_6^{(M)} = \frac{3}{64} + \frac{\mathcal{I}}{M^4}, \tag{5.21}$$

$$\kappa_8^{(M)} = \frac{3}{256} + \frac{17\mathcal{I}}{4M^4} + \frac{\mathcal{J}}{M^6}, \tag{5.22}$$

$$\kappa_{10}^{(M)} = \frac{3}{1024} + \frac{219\mathcal{I}}{16M^4} + \frac{8\mathcal{J}}{M^6} + \frac{\mathcal{K}}{M^8}. \tag{5.23}$$

These are the cases $m = 3, 4, 5$ of Theorem 5.4. Each invariant is a relative invariant of its weight:

$$\mathcal{I}(\mathcal{R}_M D) = M^{-4}\mathcal{I}(D), \quad \mathcal{J}(\mathcal{R}_M D) = M^{-6}\mathcal{J}(D), \quad \mathcal{K}(\mathcal{R}_M D) = M^{-8}\mathcal{K}(D), \tag{5.24}$$

with the harmless leading normalization understood as in Equation (3.7).

6 A universal Bauer–Muir acceleration

We now use the universal cubic profile to construct a tail approximation. For $p = 3$, the universal normalized Euler numerator is

$$A_n^{(0)} = -\frac{t_n^2 - 1}{t_n^2 - \frac{1}{4}}, \qquad t_n = n - \frac{1}{2}. \tag{6.1}$$

Define

$$w_n^{(0)} = -\frac{2n-1}{2n} = -1 + \frac{1}{2n}. \tag{6.2}$$

A direct calculation gives

$$w_{n-1}^{(0)}(2 + w_n^{(0)}) = A_n^{(0)}. \tag{6.3}$$

Thus $w_n^{(0)}$ is the exact tail sequence for the reference cross-ratio profile. If an actual tail function is a Möbius transform of g_0, its cross-ratios are exactly those of g_0, so the same tail is exact. For a merely formal orbit equivalence, the statement is only asymptotic to all algebraic orders.

For the original sampled terms $d_k = D(k + \frac{1}{2})$, define the M-block terms

$$\beta_k^{(M)} := (\mathcal{R}_M D)(k + \tfrac{1}{2}) = \sum_{j=0}^{M-1} d_{Mk+j}.$$

The normalized Euler coefficients $A_n^{[M]}$ are defined from this blocked sequence. We call the blocked Euler data admissible when

$$\beta_k^{(M)} \neq 0 \quad (k \geq 0), \qquad \beta_{k-1}^{(M)} + \beta_k^{(M)} \neq 0 \quad (k \geq 1),$$

so that all term ratios and denominator-two normalization factors are defined. Let $D^{[M]}$ be the regrouped block profile, with invariants

$$\mathcal{I}_M = M^{-4}\mathcal{I}, \qquad \mathcal{J}_M = M^{-6}\mathcal{J},$$

and write $A_n^{[M]}$ for its normalized Euler coefficients. From Equations (5.21) and (5.22),

$$\lambda_n := A_n^{[M]} - w_{n-1}^{(0)}(2 + w_n^{(0)}) = \frac{\mathcal{I}_M}{t_n^6} + \frac{\frac{17}{4}\mathcal{I}_M + \mathcal{J}_M}{t_n^8} + O(t_n^{-10}). \tag{6.4}$$

If $\mathcal{I} \neq 0$, then $\lambda_n \neq 0$ for all sufficiently large n.

Theorem 6.1 (Bauer–Muir coefficient asymptotics)**.** *Let $(d_n)_{n\geq 0}$ be an actual real sequence with $d_n \neq 0$ whose series converges. Suppose its centred block profile has a full asymptotic expansion*

$$D(t) = Ct^{-3}\left(1 + ut^{-2} + vt^{-4} + \cdots\right), \qquad C \neq 0,$$

so that $D \in \mathscr{V}_3^{\times} \cap \mathscr{V}_3^{(2)}$. Fix an integer regrouping factor $M \geq 1$ and form the blocked terms $\beta_k^{(M)}$ above. Assume the blocked Euler data are admissible: $\beta_k^{(M)} \neq 0$ for all $k \geq 0$ and $\beta_{k-1}^{(M)} + \beta_k^{(M)} \neq 0$ for all $k \geq 1$. Let $A_n^{[M]}$ be the resulting actual normalized Euler numerators and assume $\mathcal{I} \neq 0$.

Apply the Bauer–Muir transformation to a sufficiently late tail, using the modifying sequence $w_n^{(0)}$. In the standard indexing for a cut tail beginning at N, the first two transformed numerators are boundary terms; after rescaling the transformed denominators back to 2, the bulk partial numerators, for $n \geq N+3$, satisfy

$$A_n^{\mathrm{BM}} = -1 + \frac{35}{4t_n^2} + \frac{35}{2t_n^3} + \left(-\frac{121}{16} + \frac{4\mathcal{J}}{M^2\mathcal{I}}\right)\frac{1}{t_n^4} + O(t_n^{-5}). \tag{6.5}$$

In particular, the t_n^{-3} coefficient is universal; the ratio $\mathcal{J}/(M^2\mathcal{I})$ first appears at order t_n^{-4}. The expansion describes the bulk coefficients only; the two finite boundary coefficients at the chosen cut are handled separately in Theorem 6.2. No finite-prefix continuant condition is needed for this bulk coefficient asymptotic; those conditions are imposed only in Theorem 6.2, where the theorem identifies the full modified approximants with the original series value.

Proof. For a denominator-two continued fraction, the Bauer–Muir quantities are

$$\lambda_n = a_n - w_{n-1}(b_n + w_n), \quad \widetilde{a}_n = a_{n-1}\frac{\lambda_n}{\lambda_{n-1}}, \quad \widetilde{b}_n = b_n + w_n - w_{n-2}\frac{\lambda_n}{\lambda_{n-1}}.$$

Here $a_n = A_n^{[M]}$, $b_n = 2$, and $w_n = w_n^{(0)}$. Put

$$\alpha = \frac{17}{4} + \frac{\mathcal{J}_M}{\mathcal{I}_M} = \frac{17}{4} + \frac{\mathcal{J}}{M^2\mathcal{I}}.$$

From Equation (6.4),

$$\frac{\lambda_n}{\lambda_{n-1}} = 1 - \frac{6}{t} + \frac{15}{t^2} - \frac{20+2\alpha}{t^3} + \frac{15+9\alpha}{t^4} + O(t^{-5}), \tag{6.6}$$

where $t = t_n$. Also,

$$w_n = -1 + \frac{1}{2t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{16t^4} + O(t^{-5}),$$
$$w_{n-2} = -1 + \frac{1}{2t} + \frac{3}{4t^2} + \frac{9}{8t^3} + \frac{27}{16t^4} + O(t^{-5}).$$

Substitution gives

$$\widetilde{b}_n = 2 - \frac{6}{t} + \frac{17}{t^2} - \frac{24+2\alpha}{t^3} + \frac{\frac{75}{4}+10\alpha}{t^4} + O(t^{-5}).$$

Let $c_n = 2/\widetilde{b}_n$. Inverting the preceding expansion gives

$$c_n = 1 + \frac{3}{t} + \frac{1}{2t^2} + \frac{\alpha-12}{t^3} + \left(\alpha - \frac{109}{8}\right)t^{-4} + O(t^{-5}).$$

Shifting $t \mapsto t-1$ gives

$$c_{n-1}c_n = 1 + \frac{6}{t} + \frac{13}{t^2} + \frac{2\alpha-8}{t^3} + \frac{11\alpha-117}{t^4} + O(t^{-5}).$$

On the other hand,

$$\widetilde{a}_n = A_{n-1}^{[M]}\frac{\lambda_n}{\lambda_{n-1}} = -1 + \frac{6}{t} - \frac{57}{4t^2} + \frac{17+2\alpha}{t^3} - \left(\frac{165}{16} + 9\alpha\right)t^{-4} + O(t^{-5}),$$

where $A_{n-1}^{[M]} = -1 + \frac{3}{4(t-1)^2} + \frac{3}{16(t-1)^4} + O(t^{-6})$ was used. Multiplying these last two displays yields Equation (6.5). In particular, the α-dependence cancels at order t^{-3} and contributes 4α at order t^{-4}; substituting $\alpha = 17/4 + \mathcal{J}/(M^2\mathcal{I})$ gives the stated coefficient. □

Theorem 6.2 (Conditional $O(n^{-6})$ acceleration)**.** *Assume the hypotheses of Theorem 6.1, including $C \neq 0$ and $\mathcal{I} \neq 0$. Choose the cut-off N sufficiently large that all of the following conditions hold simultaneously:*

(i) the original tail coefficients $a_n := A_n^{[M]}$ lie in $(-1, 0)$ and $\lambda_n \neq 0$ for every $n \geq N + 1$;

(ii) the bulk modified denominators satisfy $\widetilde{b}_n > 0$ for every $n \geq N + 2$;

(iii) the bulk, denominator-two normalized Bauer–Muir numerators satisfy $-1 < A_n^{\mathrm{BM}} < 0$ for every $n \geq N + 3$.

Such a choice is possible from the asymptotic expansions in Equations (6.4) and (6.5) and the expansion of $\widetilde{b}_n$ in the proof of Theorem 6.1. The first two normalized numerators are boundary coefficients and are not required to lie in $(-1, 0)$. Let τ be the value of the original cut tail. In addition to the global blocked-data admissibility in Theorem 6.1, require the finite regrouped prefix blocks $\beta_0^{(M)}, \ldots, \beta_N^{(M)}$ to be nonzero and require the numerator and denominator continuants through the cut to be nonzero. Define these continuants by

$$P_{-1}^{[M]} = P_0^{[M]} = 1, \qquad Q_{-1}^{[M]} = 0, \quad Q_0^{[M]} = 1,$$

$$P_j^{[M]} = 2P_{j-1}^{[M]} + A_j^{[M]} P_{j-2}^{[M]}, \qquad Q_j^{[M]} = 2Q_{j-1}^{[M]} + A_j^{[M]} Q_{j-2}^{[M]}, \qquad 1 \leq j \leq N,$$

and assume $P_j^{[M]} Q_j^{[M]} \neq 0$ for $1 \leq j \leq N$. (The adjacent-block-sum conditions in Theorem 6.1 ensure that the Euler ratios and equivalence factors are defined.) Let Ψ be the finite-prefix linear-fractional map from this tail value to the value of the full series, and assume that its denominator is nonzero at τ. (Under the stated nonvanishing assumptions on the finite Euler data, Ψ is automatically nonconstant.) Then the exact Bauer–Muir modified convergents preserve the tail limit and the full modified approximants C_n^{BM} satisfy

$$L - C_n^{\mathrm{BM}} = O(n^{-6}), \qquad L = \Psi(\tau). \tag{6.7}$$

If the regrouped summands satisfy $d_n^{[M]} \sim c_M n^{-3}$ with $c_M = CM^{-2} \neq 0$ under the centred-block normalization, the ordinary partial sums have error $\Theta(n^{-2})$.

Proof. Write $a_n = A_n^{[M]}$. The universal cubic expansion gives

$$a_n = -1 + \frac{3}{4t_n^2} + O(t_n^{-4}),$$

so the original coefficients lie in $(-1, 0)$ and are nonzero for all sufficiently large n. Let $s_n = p_n/q_n$ denote the ordinary convergents of the original denominator-two cut tail, with

$$p_{N-1} = 1, \quad p_N = 2, \qquad q_{N-1} = 0, \quad q_N = 1,$$

and, for $n > N$,

$$q_n = 2q_{n-1} + a_n q_{n-2}. \tag{6.8}$$

The recurrence and $-1 < a_n < 0$ imply $q_n > q_{n-1} > 0$ by induction. Its indicial equation is $\gamma(\gamma - 1) = 3/4$, with roots $3/2$ and $-1/2$. The full inverse-power expansion of a_n and standard asymptotic theory for second-order difference equations (see [6, §2.9] and [13]) yield

$$q_n = c_+ n^{3/2} \left(1 + \frac{c_1}{n} + O(n^{-2})\right), \qquad c_+ > 0.$$

The increasing positive solution cannot be the minimal $n^{-1/2}$ solution, so its dominant coefficient is nonzero. In particular,

$$q_n + w_n^{(0)} q_{n-1} = 2c_+ n^{1/2} \left(1 + O(n^{-1})\right), \tag{6.9}$$

using $w_n^{(0)} = -1 + (2n)^{-1}$.

Let τ be the value of this original cut tail. The exact Bauer–Muir modified convergent is

$$\widetilde{s}_n = \frac{p_n + w_n^{(0)} p_{n-1}}{q_n + w_n^{(0)} q_{n-1}}.$$

The continuant determinant identity gives

$$p_{n-1}q_n - p_n q_{n-1} = \pm \prod_{j=N+1}^{n} a_j,$$

up to the fixed nonzero normalization determined by the first cut-tail convergent. Since $a_j = -1 + O(j^{-2})$ and no a_j vanishes, the absolute value of this product tends to a finite positive limit. Hence

$$\widetilde{s}_n - s_n = \frac{w_n^{(0)}(p_{n-1}q_n - p_n q_{n-1})}{q_n(q_n + w_n^{(0)} q_{n-1})} = O(n^{-2}),$$

by $q_n \asymp n^{3/2}$ and Equation (6.9). To make the tail-limit identification explicit, let S_n^{full} be the full Euler convergents. Cutting off the finite prefix gives $S_n^{\mathrm{full}} = \Psi(s_n)$, and Euler's finite-sum identity shows that S_n^{full} tends to the convergent series value L. Since Ψ is a nondegenerate Möbius map and its denominator is nonzero at the tail value τ, its inverse is continuous at $L = \Psi(\tau)$. Hence $s_n = \Psi^{-1}(S_n^{\mathrm{full}}) \to \Psi^{-1}(L) = \tau$, and the estimate above gives $\widetilde{s}_n \to \tau$. This is the value-preservation step: convergence of the transformed fraction alone is not used to identify its limit.

We next account for the two boundary steps in the exact Bauer–Muir transformation. For the cut tail beginning with $2 + \mathbf{K}_{n=N+1}^{\infty} a_n/2$, the first two modified partial denominators are

$$\widetilde{b}_N = 2 + w_N^{(0)}, \qquad \widetilde{b}_{N+1} = 2 + w_{N+1}^{(0)};$$

the bulk formula $\widetilde{b}_j = 2 + w_j^{(0)} - w_{j-2}^{(0)}\lambda_j/\lambda_{j-1}$ starts at $j = N+2$. After the denominator-two equivalence transformation, the first two normalized partial numerators are finite nonzero boundary coefficients (with the standard cut-tail indexing, $A_{N+1}^{*,\mathrm{BM}} \sim 2\lambda_{N+1}$ and $A_{N+2}^{*,\mathrm{BM}} \sim -2$ as $N \to \infty$). In particular, they need not belong to $(-1, 0)$; they are treated as fixed initial factors. For the bulk normalization put

$$c_{N+1} = \frac{2}{2 + w_{N+1}^{(0)}} > 0, \qquad c_j = \frac{2}{\widetilde{b}_j} = 1 + \frac{3}{t_j} + O(t_j^{-2}) > 0 \quad (j \geq N+2).$$

Let Q_n^* be the denominator continuant of the fully rescaled transformed cut-tail fraction, including its two exact boundary steps. The exact Bauer–Muir continuant identity gives

$$Q_n^* = \left(\prod_{j=N+1}^{n} c_j \right) \left(q_n + w_n^{(0)} q_{n-1} \right). \tag{6.10}$$

Since $c_j = 1 + 3/t_j + O(t_j^{-2})$ for all sufficiently large j,

$$\log \prod_{j=N+1}^{n} c_j = \sum_{j=N+2}^{n} \left(\frac{3}{t_j} + O(t_j^{-2}) \right) = 3 \log n + O(1),$$

so $\prod_{j=N+1}^{n} c_j \asymp n^3$. Combining this with Equation (6.9) yields

$$Q_n^* \asymp n^{7/2}. \tag{6.11}$$

This identity incorporates the exceptional first two boundary coefficients; no false initialization $Q^*_{N-1} = 0, Q^*_N = 1$ for the rescaled bulk recurrence is being made.

For $n \geq N + 3$, the bulk transformed numerators satisfy

$$A^{\mathrm{BM}}_n = -1 + \frac{35}{4t_n^2} + O(t_n^{-3}),$$

and hence are in $(-1, 0)$ for sufficiently large n. The two boundary numerators are nonzero, and all sufficiently late bulk numerators are nonzero. Thus the product of the absolute values of the transformed numerators converges to a finite positive limit, because $|A^{\mathrm{BM}}_n| = 1 + O(n^{-2})$. The determinant identity for the transformed continuants, with the two boundary factors absorbed into a fixed nonzero constant, now gives

$$\widetilde{C}_n - \widetilde{C}_{n-1} = O((Q^*_n Q^*_{n-1})^{-1}) = O(n^{-7}),$$

where $\widetilde{C}_n$ are the exact rescaled Bauer–Muir cut-tail convergents. By the exact Bauer–Muir identity, $\widetilde{C}_n = \widetilde{s}_n$, so $\widetilde{C}_n \to \tau$. Summing the last estimate gives

$$\tau - \widetilde{C}_n = O(n^{-6}).$$

Finally, the finite-prefix map Ψ is a composition of nondegenerate linear-fractional maps (each internal step has determinant proportional to a nonzero Euler numerator), and is therefore nonconstant. Since its denominator is nonzero at τ, it is analytic and locally Lipschitz near τ. Defining $C^{\mathrm{BM}}_n = \Psi(\widetilde{C}_n)$, we obtain

$$L - C^{\mathrm{BM}}_n = \Psi(\tau) - \Psi(\widetilde{C}_n) = O(n^{-6}),$$

which proves Equation (6.7). If $d^{[M]}_n \sim c_M n^{-3}$ with $c_M \neq 0$, then the ordinary remainder is asymptotic to $c_M/(2n^2)$, so its absolute error is $\Theta(n^{-2})$. □

Remark 6.3. In general, the non-pole condition for Ψ can be checked by writing $\Psi(z) = (az + b)/(cz + d)$ from the finite prefix and obtaining a rigorous real interval $\tau \in [u, v]$; if $0 \notin c[u, v] + d$, then $c\tau + d \neq 0$. For the Catalan example, the exact ratio inequalities immediately before Equation (7.3) prove $\tau \in [1, 2]$ and positivity of every intermediate tail denominator. The condition $\mathcal{I} \neq 0$ ensures that λ_n is eventually nonzero, so the Bauer–Muir transform can be started after a finite prefix. The interval conditions in Theorem 6.2 apply to the original tail coefficients from $n \geq N + 1$ and to the bulk transformed coefficients from $n \geq N + 3$; the first two transformed coefficients are finite boundary terms and need not lie in $(-1, 0)$. Nonvanishing of the denominator of Ψ at τ is the substantive finite-prefix condition. Its nonconstancy is automatic when the finite Euler numerators are nonzero, since each step $z \mapsto A_k/(2 + z)$ has nonzero determinant. If $\mathcal{I} = 0$ but $\mathcal{J} \neq 0$, then $\lambda_n = \mathcal{J}_M t_n^{-8} + O(t_n^{-10})$ and $\lambda_n/\lambda_{n-1} = 1 - 8t_n^{-1} + 28t_n^{-2} + O(t_n^{-3})$. The same bulk calculation gives

$$A^{\mathrm{BM}}_n = -1 + \frac{63}{4t_n^2} + O(t_n^{-3}),$$

whose indicial roots are $9/2$ and $-7/2$. In this case the bulk normalization has $c_j = 1 + 4/t_j + O(t_j^{-2})$, so its product grows like n^4 and the exact rescaled denominator continuants grow like $n^{9/2}$. Under the analogous nonvanishing and boundary hypotheses, the determinant identity gives increments $O(n^{-9})$ and hence an $O(n^{-8})$ transformed-tail error. This is a separate degenerate-invariant regime; it is not included in the $\mathcal{I} \neq 0$ theorem.

7 An exact Catalan-constant example

Catalan's constant has the paired representation

$$G = \sum_{k=0}^{\infty} \left(\frac{1}{(4k+1)^2} - \frac{1}{(4k+3)^2} \right). \tag{7.1}$$

With $t = k + \frac{1}{2}$, its block function is

$$D_G(t) = \frac{16t}{(16t^2-1)^2} = \frac{1}{16} t^{-3} \left(1 + \frac{1/8}{t^2} + \frac{3/256}{t^4} + \frac{1/1024}{t^6} + \frac{5/65536}{t^8} + \cdots \right). \tag{7.2}$$

For this example, the summands have the exact form

$$d_k = \frac{8(2k+1)}{(4k+1)^2(4k+3)^2}, \qquad r_n := \frac{d_n}{d_{n-1}} = \frac{2n+1}{2n-1} \frac{(4n-3)^2(4n-1)^2}{(4n+1)^2(4n+3)^2}.$$

For $n \geq 1$,

$$1 - r_n = \frac{2(768n^4 - 160n^2 - 9)}{(2n-1)(4n+1)^2(4n+3)^2} > 0,$$

so $0 < r_n < 1$. For $n \geq 2$, putting $m = n - 2$ and clearing positive factors gives

$$\begin{aligned}
(1 + r_{n-1})(1 + r_n) - 4r_n &= \frac{12Q(m)}{(2m+1)(2m+3)} \prod_{\ell=1}^{4} (4m + 2\ell + 3)^{-2}, \\
Q(m) &= 65536m^8 + 786432m^7 + 4112384m^6 + 12238848m^5 \\
&\quad + 22607360m^4 + 26379264m^3 + 18810048m^2 \\
&\quad + 7396416m + 1205713 > 0.
\end{aligned}$$

It follows from the Euler normalization formulas that $-1 < A_1 < 0$ and $-1 < A_n < 0$ for all $n \geq 2$. Consequently each backward tail map $z \mapsto 2 + A_n/z$ sends $[1, 2]$ into $(1, 2)$. Finite backward truncations and passage to the convergent tail therefore give $\tau \in [1, 2]$, with all intermediate denominators positive. This verifies directly that the finite-prefix map Ψ has no pole at the Catalan cut-tail value. Thus

$$(X, Y, Z, T) = \left(-\frac{3}{8}, \frac{75}{256}, -\frac{427}{1024}, \frac{62325}{65536} \right),$$

and

$$\mathcal{I}_G = -\frac{3}{4}, \qquad \mathcal{J}_G = \frac{57}{16}, \qquad \mathcal{K}_G = -\frac{9207}{512}. \tag{7.3}$$

These values give the asymptotic coefficients of every integer regrouping through Equation (5.23) without constructing its continued fraction separately. As an independent direct check, substituting the rational function D_G into Equation (2.5) and expanding at infinity gives

$$A_G(t) = -1 + \frac{3}{4t^2} + \frac{3}{16t^4} - \frac{45}{64t^6} + \frac{99}{256t^8} + \frac{261}{1024t^{10}} + O(t^{-12}). \tag{7.4}$$

The coefficients of t^{-6}, t^{-8}, t^{-10} agree with Equations (5.21) to (5.23) using Equation (7.3).

There is also a short finite modified-convergent check. For Equation (7.1),

$$d_0 = \frac{8}{9}, \qquad r_1 = \frac{27}{1225}, \qquad r_2 = \frac{6125}{29403}.$$

The Euler tail can be written $G = d_0/(1 - r_1/H)$, where

$$H = 1 + r_1 - \frac{r_2}{1 + r_2 - \cdots}.$$

Normalizing the second denominator to 2 and replacing the remaining tail by $w_2^{(0)} = -3/4$ gives

$$H_2 = 1 + r_1 - \frac{2r_2/(1+r_2)}{2 - 3/4} = 1 + \frac{27}{1225} - \frac{1225}{4441}.$$

Therefore the modified convergent is exactly

$$G_{(2)}^{\mathrm{BM}} = \frac{d_0}{1 - r_1/H_2} = \frac{8}{9} + \frac{4441}{164150} = 0.915943\ldots. \tag{7.5}$$

The ordinary three-block partial sum is

$$S_2 = \frac{8}{9} + \frac{24}{1225} + \frac{40}{9801} = 0.91256\ldots,$$

whereas $G = 0.915965594\ldots$. Thus the modified convergent gives a substantial finite-index improvement. This example illustrates the tail construction; the asymptotic theorem, rather than this one numerical comparison, is the general convergence statement.

For a multi-index check, let $S_n = \sum_{k=0}^{n} d_k$ be the ordinary Euler convergent and let P_n/Q_n be the nth convergent of $1 + \mathbf{K}_{j=1}^{n} A_j/2$, with

$$P_{-1} = P_0 = 1, \quad Q_{-1} = 0, \quad Q_0 = 1, \qquad P_n = 2P_{n-1} + A_n P_{n-2}, \quad Q_n = 2Q_{n-1} + A_n Q_{n-2}.$$

The Bauer–Muir approximant is computed from the same convergents by

$$C_n^{\mathrm{BM}} = d_0 \frac{Q_n + w_n^{(0)} Q_{n-1}}{P_n + w_n^{(0)} P_{n-1}}, \qquad w_n^{(0)} = -1 + \frac{1}{2n}.$$

The following errors were evaluated at 100 decimal digits; the last two columns are the empirical orders under doubling, $\log_2(E_n/E_{2n})$.

n	$\lvert G - S_n\rvert$	$\lvert G - C_n^{\mathrm{BM}}\rvert$	ordinary order	BM order
10	2.5786597×10^{-4}	1.9231820×10^{-9}	–	–
20	7.0831585×10^{-5}	$3.0398942 \times 10^{-11}$	1.8642	5.9833
40	1.8588052×10^{-5}	$4.7637206 \times 10^{-13}$	1.9300	5.9958
80	4.7628573×10^{-6}	$7.4487622 \times 10^{-15}$	1.9645	5.9989

The observed orders approach 2 for the ordinary partial sums and 6 for the Bauer–Muir approximants, in agreement with Theorem 6.2. A supplementary Python check script, included with the source files, reproduces the table.

7.1 A quadratic character of conductor five

A second example comes from the real quadratic character modulo 5, with residue values $(1, -1, -1, 1)$. Its Dirichlet L-value is represented by the absolutely convergent block series

$$L(1, \chi_5) = \sum_{k=0}^{\infty} \left(\frac{1}{5k+1} - \frac{1}{5k+2} - \frac{1}{5k+3} + \frac{1}{5k+4} \right). \tag{7.6}$$

With $t = k + \frac{1}{2}$, the exact block function is

$$\begin{aligned} D_5(t) &= \frac{20t}{(25t^2 - \frac{9}{4})(25t^2 - \frac{1}{4})} \\ &= \frac{4}{125} t^{-3} \left(1 + \frac{1/10}{t^2} + \frac{91/10000}{t^4} + \frac{41/50000}{t^6} + \frac{7381/100000000}{t^8} + \cdots \right). \end{aligned} \tag{7.7}$$

The corresponding regrouping coordinates are

$$(X, Y, Z, T) = \left(-\frac{2}{5}, \frac{201}{625}, -\frac{1444}{3125}, \frac{412751}{390625}\right),$$

which give

$$\mathcal{I}_5 = -\frac{504}{625}, \qquad \mathcal{J}_5 = \frac{12096}{3125}, \qquad \mathcal{K}_5 = -\frac{7687008}{390625}. \tag{7.8}$$

These differ from the Catalan coordinates in Equation (7.3). Thus the same universal cubic profile supports distinct arithmetic representations, while the formal projective invariants retain their representation-dependent information. Direct substitution of D_5 into Equation (2.5) gives the independent expansion

$$\begin{aligned} A_5(t) = \; & -1 + \frac{3}{4t^2} + \frac{3}{16t^4} - \frac{30381}{40000t^6} \\ & + \frac{364191}{800000t^8} + \frac{100939683}{400000000t^{10}} + O(t^{-12}). \end{aligned} \tag{7.9}$$

Again, these coefficients agree with the invariant predictions. This independent rational-function check confirms that the framework is not specific to the alternating square-denominator representation of G.

8 Novelty, scope, and limitations

The results should be read with the following distinctions in mind.

(a) Euler's finite-series continued fraction and equivalence transformations are classical. They are used as the starting representation, not claimed as new.

(b) Cross-ratio invariance under Möbius transformations and the appearance of the Schwarzian in a small-step cross-ratio are classical projective facts. Discrete cross-ratio curvatures have an established literature, including [10].

(c) The slice theorem for a locally nilpotent derivation is standard invariant theory; see [7]. The explicit derivation and invariant coordinates here are tailored to the Euler-tail Möbius action.

(d) The Bauer–Muir transformation is classical; see [8]. Recent work also uses it to transform structured continued fractions, including Euler-type identities; see [1].

The present results should be distinguished from several closely related strands of work. David [5] studies Euler polynomial continued fractions, their Möbius matrix representations, and algorithmic recognition of the Euler family. Cohen [2] applies Bauer–Muir–Apéry acceleration to Euler-generated continued fractions, including the alternating representation of $\pi/4$, and describes iterative improvements of convergence order. His database and its expanded 2026 encyclopedic treatment catalogue a broad range of polynomial continued fractions, convergence rates, and Bauer–Muir–Apéry transformations [3, 4]; the latter also includes numerous continued-fraction representations of Catalan's constant. Accordingly, neither the Euler/Möbius connection, the use of Bauer–Muir acceleration on Euler fractions, nor the existence of accelerated Catalan fractions is claimed as new here.

Chen and Liu [1] provide a recent treatment of Bauer–Muir transformations for quadratic-numerator continued fractions, including Euler-type examples. Shynarbek, Orynbassar, and Tahir [12] study block products of Riccati recurrences, Möbius iteration, finite matrix tests, and tail estimates. In discrete projective geometry, cross-ratio curvature and its relation

to the Schwarzian derivative are established themes; see Müller–Vaxman [10] and Ovsienko–Tabachnikov [11]. The slice theorem for locally nilpotent derivations is standard invariant theory [7].

The contribution of the present paper is more specific: it identifies the normalized internal Euler numerators with cross-ratios of consecutive summation tails, diagonalizes the centred regrouping action on summands by an exact finite-difference eigenbasis, computes the residual formal additive-group quotient for the resulting Euler-tail class, and derives the associated all-orders Schwarzian invariant hierarchy. The cubic Bauer–Muir theorem is a tailored consequence of this structure, with its stated 35/4 indicial coefficient and conditional $O(n^{-6})$ rate. These claims concern this particular combination and normalization; they are not claims of priority for the underlying classical transformations or projective identities. The novelty claim is confined to this specific structural combination and its stated formal and analytic hypotheses.

Literature stream	Established overlap	Specific distinction claimed here
Euler correspondence and continued-fraction equivalence	Exact conversion of series partial sums to continued fractions; denominator equivalence transformations.	No new Euler conversion is claimed; the paper studies its normalized tail coefficients as a projective object.
Discrete projective geometry	Cross-ratio invariance, discrete projective curvature, and the Schwarzian as a continuum limit.	The cross-ratio is identified for the particular normalized Euler numerators generated by consecutive series tails.
Bauer–Muir–Apéry acceleration	Tail modification and polynomial-order convergence improvements, including Euler-generated examples.	The claimed result is the cubic-profile-specific universal modifier and its 35/4 indicial coefficient, under explicit hypotheses.
Block Riccati and Möbius methods	Matrix products, block Möbius recurrences, and tail estimates.	The present block action is on summands and is diagonalized by the finite-difference eigenbasis; its formal quotient is computed explicitly.

The novelty claim is consequently limited to the following linked statements: (i) the exact centred regrouping semigroup and its finite-difference eigenbasis for Euler summands; (ii) the cross-ratio realization of the normalized Euler numerators and the resulting exact classification of nondegenerate sampled tails up to one global Möbius map; (iii) the complete formal additive-group quotient for cubic even tails, including its explicit free invariant ring; (iv) the all-orders graded expansion and the triangular Schwarzian recovery of the invariant generators; and (v) the specific universal cubic-tail Bauer–Muir modification, whose transformed indicial coefficient is 35/4 and whose conditional convergence rate is $O(n^{-6})$. The last item is a tailored theorem, not a claim to be the first Bauer–Muir acceleration or the first improvement of a polynomial convergence rate. The paper therefore advances a narrowly formulated synthesis claim rather than a claim that any one of the classical ingredients is new. The invariant sequence is a representation invariant, not an intrinsic invariant of the numerical value $\sum d_n$. Different decompositions of the same constant can have different invariant coordinates. Conversely, equal complete formal invariant sequences classify formal tails modulo the stated Möbius gauge, not necessarily their exact finite-index values. The convergence theorem is analytic and requires the stated summability, nonvanishing, and denominator conditions; it does not follow from formal asymptotics alone.

9 Concluding remarks

The centred regrouping operator is exactly diagonal in a finite-difference eigenbasis. For cubic odd tails, the normalized Euler cross-ratio is a projective discrete curvature, and its formal moduli are described by a free polynomial invariant ring. The Schwarzian derivative supplies

the leading new invariant at every asymptotic order, while the complete cross-ratio expansion supplies the lower graded terms. The same universal reference profile yields a Bauer–Muir modifying sequence whose transformed recurrence has indicial exponents $7/2$ and $-5/2$, leading to an n^{-6} tail bound under explicit hypotheses.

Further work could compute higher invariant polynomials explicitly, identify periodic weight systems with coincident finite invariant signatures, and investigate iterated Bauer–Muir steps that use ratios of successive invariant coordinates.

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