Finite-height Li positivity and the first negative coefficient of a moving quartet

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Abstract

How much of Li's positivity criterion follows from verifying the Riemann hypothesis up to height T, and what could cause the first failure? With $L = \log T$, $h = \log L$ and $Q = (L/h)^{1/3}$, we obtain a positive prefix through $F_a(T) - O(T^2)$, where $F_a(T) = T^2(4L + 8aQ + 4h)$ and a is an admissible zero-free constant. At $T = 3 \cdot 10^{12}$, an integer certificate gives strict positivity through order $9.634 \cdot 10^{26}$ using finite-height inputs. We quantify the deficit at 10^{27} and at the three-term center. If a zeta coefficient is nonpositive below $F_a(T) + kT^2$ with fixed k, one zero must simultaneously lie within $O(T/L)$ above T, lie within $O(1/L)$ of the left zero-free boundary, and have a positive, bounded-below phase cosine. For the actual ordinates projected onto the line and perturbed by m copies of a quartet at abscissa $(aQ + \tau)/L$, we determine the least negative index as $F_a(T) + (8\tau + 2\log(2/m))T^2 + o(T^2)$. A uniform oscillatory formula explains the constant. We give explicit uniform constants and a separate, substantially sharper localization at the published verification height. Lean checks the transfers from the stated analytic inputs, the comparison model and the finite certificates.

Keywords. Li coefficients, Riemann zeta function, finite-height verification, zero-density estimates, Lean 4.

1 What a finite verification determines

Let $\mathrm{RH}(T)$ mean that every nontrivial zero $\rho = \beta + i\gamma$ of the Riemann zeta function with $|\gamma| \le T$ satisfies $\beta = 1/2$. Put $q(z) = 1 - 1/z$ and $K_n(z) = 1 - \mathrm{Re}(q(z)^n)$. In the ordinary Li normalization,

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log \xi(s)\right]\Big|_{s=1} = 2\lim_{X\to\infty}\sum_{0<\gamma<X} K_n(\rho), \qquad n \ge 1. \tag{1}$$

Here $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$. Each zero occurs with its analytic multiplicity. The positive-height sum is real and conjugation accounts for the factor two [1, 2]. We use the same sum to define the coefficients of symmetric comparison spectra.

For $T > \mathrm{e}$ and fixed $k \in \mathbb{R}$ define

$$L = \log T, \qquad h = \log L, \qquad Q = (L/h)^{1/3}, \qquad w(T) = Q/L, \tag{2}$$

$$F_a(T) = T^2(4L + 8aQ + 4h), \qquad W_k = F_a(T) + kT^2, \qquad M_T = T^2L^2. \tag{3}$$

The function $w(t)$ is defined by replacing T by t. A constant $a \ge 0$ is admissible if the large-height zeros satisfy $aw(|\gamma|) < \beta < 1 - aw(|\gamma|)$. A positive choice for ζ is $a = 1/48.0718$ by Bellotti [18, Theorem 1.3].

Theorem 1 (Positive prefixes). *For each fixed $c < 4$ and all sufficiently large T,* RH(T) *implies $\lambda_n > 0$ at every integer $1 \le n \le cT^2 \log T$. For each admissible a there is $C > 0$ such that, for all sufficiently large verified T, this positivity holds throughout $1 \le n \le W_{-C}$, including $n \le 4T^2 \log T$. At the published verification height $3 \cdot 10^{12}$ the explicit bound is*

$$\lambda_n \ge n/(56\,000\,000) > 0 \qquad (1 \le n \le 9.634 \cdot 10^{26}).$$

The first assertion uses Ingham's density estimate. The second also uses the zero-free boundary and a bounded count near its edge. The third uses the finite verification of Platt and Trudgian, the two explicit density estimates of Chourasiya and Simonič, and the classical zero-free region of Bellotti, Trudgian and Yang [11, 13, 14]. Its endpoint is about $3.7259\,T^2 \log T$. Theorems 2 and 8 give a necessary condition for failure and its explicit starting heights.

Theorem 2 (Localization of a nonpositive zeta coefficient). *Fix an admissible a and $k \in \mathbb{R}$. There exist $C > 0, \eta > 0$ such that for all sufficiently large T satisfying* RH(T)*, if $1 \le n \le W_k$ and $\lambda_n \le 0$, one nontrivial zeta zero $\rho = \beta + i\gamma$ satisfies all of*

$$0 < \gamma - T \le CT/L, \qquad 0 < \beta - aw(\gamma) \le C/L, \qquad \beta < 1/2, \tag{4}$$

$$\cos(n \arg q(\rho)) \ge \eta, \qquad K_n(\rho) < 0. \tag{5}$$

The height, real part and phase in this statement belong to the same zero. In particular, a verified height followed by an empty region (4) guarantees positivity throughout this window. The additional phase condition applies separately to each order.

Theorem 3 (First negative order in the moving family). *Fix $a \ge 0$, $\tau > 0$ and an integer $m \ge 1$. Move every nontrivial zeta zero to the critical line at its original height, preserving multiplicity. Adjoin m copies of the two points $b+i(T+1)$, $1-b+i(T+1)$ and their conjugates, where $b = (aQ + \tau)/L$. Let λ_n^{mod} denote the resulting coefficient and $\theta = \arg q(b + i(T + 1))$. Uniformly for every integer n whose residual $s = (n - F_a(T))/T^2$ belongs to a fixed compact interval,*

$$\frac{\lambda_n^{\mathrm{mod}}}{M_T} = 4 - 2m \exp(s/2 - 4\tau) \cos(n\theta) + o(1) \qquad (T \to \infty). \tag{6}$$

For all sufficiently large T the least positive integer $n_-(T)$ with $\lambda_{n_-}^{\mathrm{mod}} < 0$ exists, and

$$n_-(T) = F_a(T) + s_* T^2 + o(T^2), \qquad s_* = 8\tau + 2\log(2/m). \tag{7}$$

For a single quartet with $\tau = 1$, the offset is $8 + 2\log 2 = 9.38629436\ldots$. Doubling its multiplicity reduces the offset by $2\log 2$. The inward displacement τ/L varies the constructed spectrum. Equation (1) uses the same ordinary coefficients throughout.

The comparison spectra lie in the open strip and obey both zeta symmetries. For fixed m their counting constants are uniform in T. In particular, for any $c > 4$ a single quartet produces a negative order below $cT^2 \log T$ for all sufficiently large T. The family in Theorem 3 also supplies negative orders in $[W_{d-1}, W_d]$ for any prescribed real d when its fixed multiplicity may depend on d. Thus a common fourth-order offset would require further control of the counting constants or of the spectrum. Theorems 1 and 2 concern the zeta function itself. The negative coefficients in Theorem 3 concern the specified comparison family, for which no Euler product is imposed.

Analytic inputs and earlier work

Write $N(X)$ for the number of positive ordinates strictly below X, and $N(\sigma, X)$ for the subcount with $\beta \ge \sigma$. Define $F(X) = X(\log(X/(2\pi)) - 1)/(2\pi)$ and $D(\sigma) = 3(1 - \sigma)/(2 - \sigma)$. We use the large-height estimates

$$N(X) = F(X) + O(\log X), \qquad N(\sigma, X) \ll X^{D(\sigma)} (\log X)^5 \quad (1/2 \le \sigma \le 1), \tag{8}$$

$$N(1 - Kw(X), X) = O_K(1) \quad (K > 0 \text{ fixed}). \tag{9}$$

The first line is the Riemann–von Mangoldt formula and Ingham's zero-density theorem [12, 17]. The second follows from Bellotti's near-edge theorem [19, Theorem 1.2 and Corollary 1.4 in the preprint]. Increasing the fixed dilate if necessary accommodates the closed abscissa boundary. Reflection supplies the count at the left edge.

Biane, Pitman and Yor [3, p. 441] record Oesterlé's T^2 observation. Lagarias [8, Theorem 1.1] expresses automorphic Li coefficients using a smooth main term and an incomplete sum over zeros. Chaudhary, Khmiri and Mazhouda [9, Theorem 1] give a finite-height T^2 result. Brown's disk-exclusion theorem involves $2T^2 \log T$ [5, Theorem 2], and Droll [6, §3.2] discusses a gap in its supporting lemma. Palojärvi [7, Theorems 3.3 and 4.1] treats finite intervals under a modulus bound and detects an exceptional zero through an absolute real part. Nguyen's public manuscript [10] reports an explicit zeta prefix of approximately $9 \cdot 10^{24}$ with additional computational inputs.

Arias de Reyna [23, Theorem 1] proves a stronger expansion for the projected background using the total count alone. Voros [24, 4] analyzes the exponentially oscillating terms generated by off-line zeros of a fixed function. We use the known background scale in a family where the height, abscissa and order vary together. The integer phase selection and positivity at all smaller orders turn the resulting amplitude balance into the least-index formula (7). A public worked example also computes first negative indices after inserting a quartet at the fixed height 3 [22, §9]. The signed estimate leading to (4) addresses the converse question for the actual zeta zeros.

2 Projecting the zeros

Put $q(z) = 1 - z^{-1}$, $z_0 = 1/2 + it$, $z = \beta + it$, and $d = |\beta - 1/2|$. Write $K_n^0(t) = K_n(z_0)$. All heights in this section are positive. We use the following estimate at every order, including $n = 1$.

Lemma 4. *For $0 \le \beta \le 1$,*

$$|K_n(z) - K_n(z_0)| \le \exp(nd/t^2) - 1.$$

Proof. The distance of the two Cayley images is $d/(|z||z_0|) \le d/t^2$. Moreover,

$$|q(z_0)| = 1, \qquad |q(z)|^2 = 1 + \frac{1 - 2\beta}{\beta^2 + t^2} \le 1 + \frac{2d}{t^2}.$$

Write $v = d/t^2$. If $a = q(z)$, $b = q(z_0)$ and $|a^j - b^j| \le \mathrm{e}^{jv} - 1$, then $|a^{j+1} - b^{j+1}| \le \mathrm{e}^v(\mathrm{e}^{jv} - 1) + v \le \mathrm{e}^{(j+1)v} - 1$. Induction from $j = 0$ and taking real parts prove the assertion. □

Write $P_n(X) = \sum_{0<t<X} K_n(1/2 + it)$, using all positive ordinates of ζ. Every term is nonnegative. Both estimates below hold for every indicated truncation.

Lemma 5. *For every positive integer n and $X \ge 2 \cdot 10^6 n$,*

$$P_n(X) \ge n/(56\,000\,000).$$

If $n \ge 10^6$ and $X \ge 2n$, then $P_n(X) \ge n \log n/384$.

Proof. The polar angle of $q(1/2 + it)$ is $2\arctan(1/(2t))$. For $t \ge 1$ the angle $\theta = 2n \arctan(1/(2t))$ satisfies $4n/(5t) \le \theta \le n/t$. Indeed $\arctan u \ge \sin(\arctan u) = u/\sqrt{1 + u^2} \ge 4u/5$ for $0 \le u \le 1/2$. For $n \le t \le bn$, $b \ge 1$, the inequality $1 - \cos\theta \ge 2\theta^2/\pi^2$ gives $K_n(1/2 + it) \ge 1/(14b^2)$. For $b = 2$, using $\pi^2 < 10$ gives $K_n(1/2 + it) \ge 1/32$.

For $x \ge 10^6$ we use

$$N(x) \le x \log x, \qquad N(2x) - N(x) \ge x \log x/12. \tag{10}$$

For completeness, the error bound $|N(x) - F(x)| \le 10 + 3\log x$ for $x \ge 1000$ follows from the Riemann–von Mangoldt formula and the argument bound, and is independently derived in the formal package. Stronger explicit bounds are available [12]. The smooth difference is $x(\log x+\log(2/\pi)-1)/(2\pi) \ge x(\log x-3)/8$. Subtract the endpoint errors $20+3\log(2x)+3\log x$ and use $\log x \ge 12$. This gives (10). In the shell $[10^6 n, 2\cdot 10^6 n)$ there are at least $10^6 n$ ordinates. Use $b = 2 \cdot 10^6$ to obtain the first bound. The shell $[n, 2n)$ gives the second. □

We shall also use the finite, uniform tail estimate

$$\sum_{U \le t < X} \frac{1}{t^2} \le \frac{8(\log U + 2)}{U} \qquad (U \ge 10^6). \tag{11}$$

On $[U, 2U)$ the sum is at most $N(2U)/U^2$. Adding $8(\log(2U) + 2)/(2U)$ to this upper bound gives at most $8(\log U + 2)/U$ by (10). Induction over dyadic shells proves (11), including for cutoffs inside the final shell. For fixed n, $K_n(1/2 + it) \le n^2/(2t^2)$ at $t \ge 1$. Lemma 4 and (11) then give absolute convergence of the real-part series on positive ordinates.

We write $B_n = \lim_X P_n(X)$. On splitting the positive ordinates at n, equations (10) and (11) give

$$B_n \le 2N(n) + \frac{n^2}{2}\frac{8(\log n + 2)}{n} \le 6n(\log n + 2) \qquad (n \ge 10^6). \tag{12}$$

3 A positive prefix from the unverified spectrum

The range below four

Fix $0 < c < 4$, take $n \le cT^2 \log T$, and impose RH(T). Only $t > T$ can contribute projection error. At $T \le t < 2T$ the number of zeros with $d \ge u$ is bounded by $2N(1/2 + u, 2T)$, by horizontal reflection with multiplicity. For $0 \le u \le 1/2$, (8) gives

$$\#\{\rho : 0 < t < 2T,\ d \ge u\} \le CT^{3(1/2-u)}(\log T)^5. \tag{13}$$

The constant is uniform in u. Indeed $D(1/2 + u) \le 3(1/2 - u)$, and $0 \le D(1/2 + u) \le 1$.

Choose an integer J large enough that

$$\alpha = \frac{\max(3, c)}{2} + \frac{c}{2J} < 2, \qquad u_j = \frac{j}{2J} \quad (0 \le j \le J).$$

For $u_j \le d \le u_{j+1}$, Lemma 4 and $\mathrm{e}^x - 1 \le x\mathrm{e}^x$ bound the error by $(n/T^2)T^{cu_{j+1}}$. Using (13) on each cell gives

$$\frac{1}{n}\sum_{T \le t < 2T} |K_n(\rho) - K_n(1/2 + it)| \le JC\frac{(\log T)^5}{T^{2-\alpha}}. \tag{14}$$

Here $3(1/2 - u_j) + cu_{j+1} \le \alpha$, including the last cell. Counting an endpoint twice only enlarges the estimate.

For $t \ge 2T$ the exponent nd/t^2 is at most $\frac{1}{2}\log T$. Lemma 4 and (11) therefore bound the remaining absolute error, divided by n, by $4(\log(2T) + 2)/\sqrt{T}$. Both normalized errors tend to zero independently of n. For all sufficiently large T their sum is below $1/(112\,000\,000)$. Subtract from Lemma 5 and restore the conjugate ordinates. Every sufficiently high cutoff of (1) is at least $n/(56\,000\,000)$, and the same lower bound holds in the limit. This proves the first assertion of Theorem 1. Values $c \le 0$ have no positive integer orders when $T > 1$.

For a family of symmetric multisets with uniform constants and a common starting height in (8), one can choose $H_0 \ge \mathrm{e}^{12}$ so that (10) holds throughout the family for $x \ge H_0$. The shell $[H_0 n, 2H_0 n)$ has at least $H_0 n$ ordinates, giving $P_n(X) \ge n/(56H_0)$ for $X \ge 2H_0 n$. The near and far errors divided by n still tend uniformly to zero by (14) and (11), with $U \ge H_0$ in the latter. This proves the same positivity guarantee for that family.

Resolving the edge of the strip

Assume T is sufficiently large and set $d_\rho = |\beta - 1/2|$. For a fixed real k put

$$R = 4L + 8aQ + 4h + k, \qquad W_k = T^2R = F_a(T) + kT^2.$$

Eventually $4L \le R \le 5L$. We use $h = o(Q)$, $Q^2 = o(L)$, and

$$Q - Lw(2T) \le 1. \tag{15}$$

Indeed $x \mapsto (x/\log x)^{1/3}$ increases for $x > \mathrm{e}$. With $y = L + \log 2$, the left side is at most $Q(y-L)/y$, where $Q = (L/h)^{1/3}$. The function $w(t)$ itself decreases.

Let $\Delta_n(\rho) = |K_n(\rho) - K_n(1/2 + it)|$. For $T \le t < 2T$ and $n \le W_k$, Lemma 4 gives $\Delta_n(\rho) \le \exp(Rd_\rho)$. The folded count in (13) controls the interior.

Lemma 6. *Put $K = 8a + 12$. In the band $T \le t < 2T$, the total horizontal error from zeros with $d_\rho \le 1/2 - Kw(2T)$ is $o(T^2)$, uniformly in $n \le W_k$.*

Proof. Divide $[0, 1/2]$ into $J = \lceil L \rceil$ cells of width $\delta = 1/(2J)$. For each occupied cell with lower endpoint z, multiply (13) by $\exp(R(z+\delta))$. Here $z \le 1/2 - Kw(2T)$, $R\delta \le 5/2$, and $R \le 4L + (8a+5)Q$. Hence the combined exponent is bounded by

$$\begin{aligned} 3L(1/2 - z) + R(z + \delta) &\le 2L - (1/2 - z)L + \tfrac{1}{2}(8a+5)Q + 3 \\ &\le 2L - Q + K + 3, \end{aligned}$$

where the last step uses (15). There are at most $2L$ cells, giving $O_{a,k}(T^2L^6\mathrm{e}^{-Q}) = o(T^2)$. Boundary points shared by two cells can be counted twice in this upper bound. □

Only a bounded number $B = B(K)$ of positive ordinates in the band have $d_\rho > 1/2 - Kw(2T)$, by reflection and (9). For these ordinates the zero-free region gives $d_\rho \le 1/2 - aw(2T)$. To prove Theorem 1, set

$$C = 2\big(4a + \log(1000(B+1))\big), \qquad k = -C.$$

For large T, $R - 4L \ge 0$. Equations (2) and (15) give

$$R(1/2 - aw(2T)) \le 2L + 2h - C/2 + 4a\big(Q - Lw(2T)\big) \le 2L + 2h - \log(1000(B+1)). \tag{16}$$

Thus the total error from this strip is at most $T^2L^2/1000$. At $t \ge 2T$, $nd_\rho/t^2 \le 5L/8$. Using $\mathrm{e}^y - 1 \le y\mathrm{e}^y$ and (11), the remaining error is $O(nLT^{-3/8})$. It is $o(T^2L^2)$ uniformly over the window. At heights below T the horizontal error vanishes by $\mathrm{RH}(T)$.

For $3T^2L < n \le W_{-C}$, the positive reference sum eventually exceeds $3T^2L^2/384$. Each of the three error bounds is at most $T^2L^2/1000$ for sufficiently large T. Including conjugate ordinates, we obtain

$$\lambda_n \ge T^2L^2/200.$$

The first assertion of Theorem 1 with $c = 3$ covers every smaller order. This proves its refined prefix. For Bellotti's value of a, the coefficient $8a$ equals $40000/240359$.

4 An explicit interval of positive signs

A first bound from sixteen cells

Set $T = 3 \cdot 10^{12}$, $M = 7 \cdot 10^{26}$ and $U = 2T$. Platt and Trudgian supply $\mathrm{RH}(T)$ [11]. Corollary 1 and Table 1 of [13] give, on enlarging their three constants uniformly,

$$N(\sigma, U) \le 47U^{D(\sigma)}(\log U)^3 + 10(\log U)^2 + 168\log U.$$

Our strict height convention can only lower this upper count. In this section $u_j = j/32$, $\sigma_j = 1/2 + u_j$, and $0 \le j < 16$. Let

$$p_j = \left\lceil \frac{29.5 D(\sigma_j)}{0.693} \right\rceil, \qquad q_j = \left\lceil \frac{M u_{j+1}}{0.693 T^2} \right\rceil, \qquad A_j = \min\{30U,\, 47 \cdot 2^{p_j} 30^3 + 10 \cdot 30^2 + 168 \cdot 30\}.$$

The elementary bounds $27 < \log T$, $\log U < 29.5$ and $\log 2 > 0.693$ suffice. Thus A_j is an upper count on the right side, and 2^{q_j} bounds the largest exponential required on cell j.

j	σ_j	p_j	q_j	j	σ_j	p_j	q_j
0	.50000	43	4	8	.75000	26	32
1	.53125	41	8	9	.78125	23	36
2	.56250	39	11	10	.81250	21	39
3	.59375	37	15	11	.84375	18	43
4	.62500	35	18	12	.87500	15	46
5	.65625	33	22	13	.90625	11	50
6	.68750	31	25	14	.93750	8	53
7	.71875	29	29	15	.96875	4	57

Convexity gives $\mathrm{e}^{nv} - 1 \le (n/M)(\mathrm{e}^{Mv} - 1)$ for $0 \le n \le M$. The sum of the near projection errors divided by n is at most

$$\frac{1}{M} \sum_{j=0}^{15} 2A_j 2^{q_j} = \frac{34\,593\,152\,258\,774\,731\,810\,406\,400}{7 \cdot 10^{26}} < \frac{1}{20}.$$

For $t \ge 2T$, $\mathrm{e}^{nd/t^2} \le \mathrm{e}^{M/(8T^2)} \le 2^{15}$. Thus (11) bounds the far error divided by n by $2^{15}\, 8(\log(2T) + 2)/(2T) < 1/(100\,000)$. When $T < n \le M$, the positive reference contribution divided by n is at least $27/384$. When $1 \le n \le T$, using $\mathrm{e}^{nd/t^2} \le 3$ on the entire range $t \ge T$ instead gives a normalized error at most $24(\log T + 2)/T < 1/(112\,000\,000)$. Both branches give $\lambda_n \ge n/(56\,000\,000) > 0$. This establishes the first part of the finite certificate.

A 128-layer certificate through $9.1 \cdot 10^{26}$

Throughout this subsection $T = 3 \cdot 10^{12}$ and $U = 2T$. At every $n \ge 7 \cdot 10^{26}$ and final height $X \ge 32n$ we have

$$P_n(X) \ge 12n. \tag{17}$$

Here are effective margins for this estimate. Write $u = n/32$, $v = 32n$, $f(t) = K_n^0(t)$ and $p(t) = F'(t) = \log(t/(2\pi))/(2\pi)$. On $[u, v]$, $p(t) \ge 17/2$, since $t/(2\pi) \ge 2^{80}$. The count error satisfies $|N(t) - F(t)| \le 10 + 3\log t \le n/10^7$, using $\log(t/10^{12}) \le t/10^{12} - 1$. Subtract summation by parts for N and integration by parts for F on $[u, v]$. The difference between the sum and $\int_u^v pf$ has absolute value at most $2(n/10^7)(2 + 32768)$, since $\|f'\|_\infty (v-u) \le n(v-u)/u^2 \le 32768$ and $0 \le f \le 2$. The exact integral in Lemma 9 and its two tails give $\int_u^v f \ge \pi n/2 - 5n/64$. Multiplication by $17/2$ and subtraction of the count error prove (17).

The logarithm exponent in [13, Corollary 1] is $2 + D(\sigma)$. Thus the leading term in its upper count is bounded by $47(\log U)^2 \exp(D(\sigma)(\log U + \log\log U))$. We use $\log U + \log\log U < 33$. Set $u_j = j/256$ for $0 \le j \le 128$. For a nonnegative rational x define an integer majorant for e^x by

$$p = \lfloor 16x/0.693 \rfloor + 1 = 16q + r, \quad 0 \le r < 16, \qquad E(x) = \lfloor (16 + r) 2^q / 16 \rfloor + 1.$$

Indeed $x \le p \log 2/16$, and convexity gives $2^{r/16} \le 1 + r/16$. For $N = A\,10^{25}$ and clipping

exponent b_0, define

$$C_j = \min\{30U,\, 2[47 \cdot 30^2 E(33D(1/2+u_j)) + 14040]\},$$
$$G_{-1} = 0, \qquad G_j = \min\{E(Nu_{j+1}/T^2),\, 2^{b_0}\},$$
$$S(A, b_0) = \sum_{j=0}^{127} C_j(G_j - G_{j-1}).$$

The first term in C_j counts all positive ordinates. The second reflects the right-hand density estimate. In particular each C_j bounds the number of points with $d_\rho \ge u_j$. For $n \le N$ the horizontal error per point in cell j is at most $nE(Nu_{j+1}/T^2)/N$, by convexity as above. If a signed deficit $E_n(\rho) = K_n^0(t) - K_n(\rho)$ also satisfies $E_n(\rho) \le (n/N)2^{b_0}$, its contribution is bounded by nG_j/N. Charge each cumulative count C_j only for the increment $G_j - G_{j-1}$. Telescoping on the cell of each point bounds the total near-band deficit by $nS(A, b_0)/N$.

A	b_0	Exact $S(A, b_0)$	Consequence if $\lambda_n \le 0,\ n \le N$
91	100	9 564 269 736 266 989 226 601 431 040	Strict positivity throughout
93	67	10 104 075 369 399 742 078 975 476 120	$\gamma < 1.07T, \quad \beta < 0.06, \quad \cos(n\theta) \ge 2^{-10}$
116	48	13 073 705 088 520 303 598 870 824 920	$\gamma < 1.41T, \quad \beta < 0.25, \quad \cos(n\theta) \ge 2^{-47}$
120	47	13 690 532 787 725 291 109 510 645 720	$\gamma < 1.45T, \quad \beta < 0.27, \quad \cos(n\theta) \ge 2^{-52}$

Here $\theta = \arg q(\rho)$. Each row satisfies $2S(A, b_0) < 23N$. Above $2T$, $\exp(n/(8T^2)) \le 2^{25}$ throughout the listed ranges. The normalized far error is at most $2^{25}8(\log(2T)+2)/(2T) < 1/100$. Consequently if every near deficit obeys the clipping bound and $n \ge 7 \cdot 10^{26}$, then

$$\lambda_n \ge 2(12 - 23/2 - 1/100)n = 49n/50.$$

In the first row the cap 2^{100} exceeds the largest radial bound, so every point meets it. This proves the intermediate prefix used in Section 5. In the other rows a nonpositive coefficient forces $n > 91 \cdot 10^{25}$ and one near-band point with deficit larger than $P = (n/N)2^{b_0}$.

For this point, $K_n(\rho) < 0$, $\beta < 1/2$ and $\gamma > T$ by RH(T). More quantitatively, $\mathrm{Re}(q(\rho)^n) > P - 1$, and $\exp(n(1/2-\beta)/\gamma^2) > P$. Since $P \ge 2^{b_0-1}$, put $J = 0.693(b_0 - 1)$ and $R = N/T^2 = 10A/9$. Then

$$J(\gamma/T)^2 < R(1/2-\beta), \qquad |q(\rho)|^n \le \exp(R/2).$$

In the last three rows $\exp(R/2) \le 2^p$ for $p = 75, 93, 97$, respectively. Thus $\cos(n\theta) \ge 2^{b_0-2-p}$. These inequalities give the three rectangles and phase floors in the table. The classical zero-free strip of [20, Theorem 1.3] and reflection also give $\beta > 1/(6\log\gamma)$ in each rectangle.

For the original choice $a_b = 1/48.0718$, rational log bounds give $F_{a_b}(T) \le 116 \cdot 10^{25}$. In detail, $27 \le L \le 115/4,\ 3 \le h \le 27/8,\ Q \le 9/4$. The upper log bounds use $T^{32} \le 2^{1327}$ and $29^{64} \le 2^{311}$. On $[T, 2T]$ one also has $a_b w(t) < 1/(6\log t)$. Hence the definition (4)–(5) at this particular T and $k = 0$ holds with

$$C = 12.3, \qquad \eta = 2^{-47}. \tag{18}$$

The negative summand, the two coordinates and the phase constraint all concern the same actual zero. The last two rows give necessary phase conditions beyond the prefix of Theorem 1.

5 One-sided counting at the published height

Put $T = 3 \cdot 10^{12}$ and $V = 17T/16 = 3\,187\,500\,000\,000$. The contribution of the contracting half of the critical strip is inexpensive. For $\beta \ge 1/2$,

$$|q(\beta + it)|^2 = \frac{(1-\beta)^2 + t^2}{\beta^2 + t^2} \le 1, \qquad E_n(\rho) = K_n^0(t) - K_n(\rho) \le 2.$$

For $\beta < 1/2$, reflection sends each zero, with its multiplicity, to a zero with real part $1-\beta$ at the same height. Thus the number of these left-hand zeros with $1/2-\beta \ge w$ below H is at most $N(1/2+w, H)$. This saves the factor of two in the earlier folded count. In the two intervals $[T, V)$ and $[V, 2T)$, the total contracting-side loss is at most $4N(2T) \le 7.2 \cdot 10^{14}$.

Proposition 7 (Finite-height certificate and local alternatives)**.** *The inputs* RH(T)*, the explicit density estimates of Chourasiya and Simonič, and the classical region of Bellotti, Trudgian and Yang imply* $\lambda_n \ge n/(56\,000\,000)$ *for every* $1 \le n \le 9.634 \cdot 10^{26}$*. Any of the following additional finite exclusions suffices for the indicated longer prefix. An entry* d *means that all left-hand zeros in the indicated height interval satisfy* $\beta \ge d$*.*

All orders through	$T < t < V$	$V \le t < 2T$
10^{27}	0.084	*published region*
$1.01 \cdot 10^{27}$	0.102	*published region*
$1.1583 \cdot 10^{27}$	0.24	0.19

In particular, a nonpositive coefficient of order at most 10^{27} *forces a zero with* $T < t < V$ *and* $\beta < 0.084$*. The last row covers the exact center* $F_{1/48.0718}(T)$*.*

Positive contribution. For $n \ge 9.1 \cdot 10^{26}$, let $u = n/128$, $z = n/8$, $v = 128n$. On $[u, v]$ the smooth density $p(t) = \log(t/(2\pi))/(2\pi)$ is at least $35/4$. Above z it is at least $46/5$. The two lower bounds follow from $u/(2\pi) \ge 3 \cdot 2^{78}$ and $z/(2\pi) \ge 3 \cdot 2^{82}$. Since $0 \le K_n^0 \le 2$ and its truncated integral is at least $(\pi/2 - 5/256)n$, the integral of pK_n^0 is at least

$$\frac{46}{5}\left(\frac{\pi}{2} - \frac{5}{256}\right) n - \frac{9}{10}(z-u).$$

The count remainder on this interval satisfies $|N(t) - F(t)| \le n/10^{11}$, using $\log(t/10^{16}) \le t/10^{16} - 1$. Summation by parts as in (17) costs at most $2(2 + 2097152)n/10^{11}$. Consequently $P_n(X) \ge 14.165n$ for $X \ge 128n$. Above $2T$ the normalized loss from the preceding section is at most $2^{25}\, 8(29.5+2)/(2T) < 1/700$.

An integer upper bound for the expanding side. Write $s_j = 1/2 + w_j$, $w_j = j/1024$, $0 \le j \le 512$. Equation (119) and Table 4 of [13] give, for $5/8 \le s \le 1$ and $H \ge T$,

$$N(s, H) \le (\log H)^2 \{A(s) \exp([\tfrac{8}{3}\log H + 4\log\log H](1-s)) + 6\}.$$

The successive upper endpoints for s are $25/32$, $7/8$, $15/16$, $31/32$ and 1. We round the leading constants on those intervals up to $6, 10, 13, 15, 17$, respectively. For the slabs $[u_b, H_b) = [T, V), [V, 2T)$ take

$$(D_b, L_b) = (91, 830),\ (93, 871), \qquad \tfrac{8}{3}\log H_b + 4\log\log H_b \le D_b, \quad (\log H_b)^2 \le L_b.$$

Use $C_{bj} = 30H_b$ below $s_j = 5/8$. Elsewhere use the minimum of $30H_b$ and the displayed density bound with D_b, L_b, rounded up. The strict zero-free region in Theorem 1 of [14] gives by reflection $\beta \ge 1/(4.896 \log t) > 0.006923$ throughout both slabs. For any available lower bound $\beta \ge d_b$ and order ceiling M, define

$$G_{b,-1} = 0, \qquad G_{bj} = \left\lfloor \mathcal{E}\left(\frac{M}{u_b^2}\min\{w_{j+1}, 1/2 - d_b\}\right)\right\rfloor + 1,$$

$$S_b(M, d_b) = \sum_{j=0}^{511} C_{bj}(G_{bj} - G_{b,j-1}).$$

For reproducibility put $p = \lfloor 64000x/693 \rfloor + 1 = 64q + r$, $0 \le r < 64$, and $\mathcal{E}(x) = 2^q m_r/65536$, where m_r is the least integer with $m_r^{64} \ge 2^r 65536^{64}$. Then $\exp x \le \mathcal{E}(x)$. The rounded density

entry is $\lfloor A(s_j)L_b\mathcal{E}(D_b(1-s_j))\rfloor+1+6L_b$. The multiplicities of the expanding zeros enter each layer once through reflection. The convexity inequality $\exp(ny)-1\le(n/M)(\exp(My)-1)$ for $0\le n\le M$ makes their entire loss at order n at most $n(S_0+S_1)/M$.

Proof of Proposition 7. A sufficient integer inequality is $25(S_0+S_1)\le 354M$, or $(S_0+S_1)/M\le 14.16$. For $n>9.1\cdot10^{26}$ the resulting lower bound for λ_n is $2[(14.165-14.16-1/700)n-7.2\cdot10^{14}]>n/1000$. The 128-layer certificate covers every smaller order. For the published width the charges at $M=9.634\cdot10^{26}$ and $9.635\cdot10^{26}$, divided by their respective M, lie in $(14.154,14.155)$ and $(14.203,14.204)$. With the former classical rounding $1/(6\log t)$, the same calculation proves the prefix through $9.628\cdot10^{26}$. For the three local alternatives the normalized charges lie in $(14.015,14.016)$, $(13.766,13.767)$ and $(13.889,13.890)$, respectively. Dyadic comparisons give $28.7296<L<28.7297$, $3.3578<h<3.3580$, $2.045<Q<2.046$. Substitution gives

$$1.1582\cdot10^{27}<F_{1/48.0718}(T)<1.1583\cdot10^{27}.$$

In fact the last integer in the exact window is

$$M_*=\lfloor F_{1/48.0718}(T)\rfloor=1\,158\,215\,588\,964\,762\,074\,525\,024\,105.$$

To resolve the integer, evaluate $\log y$ using $z=(y-1)/(y+1)$ and the first 45 terms of $2\sum_{k\ge0}z^{2k+1}/(2k+1)$. For $0\le z<1$ the remaining terms sum to at most $2z^{91}/(1-z^2)$. Reduce T by 2^{41} and the logarithm L by 16 before evaluating. Outward rational bounds for L,h and cubing the two bounds for Q place F strictly between the displayed integer and its successor. □

What exhausts the finite-height estimate. With just the published width, the two slabs contribute the following amounts. Entries are normalized by M and rounded to the nearest hundredth.

M	S_0/M	S_1/M	$(S_0+S_1)/M$
10^{27}	87.35	0.37	87.72
$1.1582\cdot10^{27}$	298588.79	329.04	298917.83
$M_*=\lfloor F_{1/48.0718}(T)\rfloor$	298678.14	329.24	299007.38
$1.1583\cdot10^{27}$	299571.33	331.09	299902.42

The layers with $s_j\ge15/16$ account for over 74% of the first total and over 90% at the exact center and its two enclosing decimal orders. At M_* the full integer charge is 346315005917986800762613086509568. With a common width in both slabs, $d=0.082901$ pays for 10^{27}, and $d=0.235818$ pays for $1.1583\cdot10^{27}$. Reducing either by 10^{-6} makes its corresponding integer test fail.

One can also test the density majorant before rounding or choosing a mesh. Up to $s=0.9925$ its unrounded Table 4 envelope is at least 26000 at every $H\in[T,2T]$. Indeed $\log H\ge28.7$ and $\frac{8}{3}\log H+4\log\log H\ge88$. The leading coefficients are at least 2.9 through $s=0.96$, at least 14 on $0.96<s\le0.9875$, and at least 16.46 thereafter. The additive coefficient is at least 2.1. In the last interval even $s=0.993$ gives an exponential factor exceeding 1.8. Below the domain of Table 4 the total-count majorant is larger still. The continuum layer integral in the first slab therefore assigns at least

$$26000\big[\exp(0.4925M/T^2)-1\big]$$

to $0\le w\le0.4925$ alone. This exceeds $15M$ at $M=10^{27}$ and $55000M$ at $M=1.1582\cdot10^{27}$. Refining the horizontal mesh cannot pay the $14.16M$ budget with this cumulative majorant and radial envelope. These numbers measure the loss allowed by the inequalities. Actual zero counts and phases can give much smaller losses.

The separate consequence of global RH. Alpöge's global theorem and its independent verification [15, 16] apply to the actual zeta function. Every positive-height zero then equals its projection. The zero formula gives $\lambda_n = 2\lim_{X\to\infty} P_n(X)$, so the projected lower bounds yield $\lambda_n \ge n/(28\,000\,000) > 0$ at every integer $n \ge 1$ and $\lambda_n \ge 28.33n$ for $n \ge 9.1 \cdot 10^{26}$. Both requested thresholds follow on this input. The finite-height certificate in Proposition 7 uses only its listed height, density, zero-free and zero-formula inputs.

6 Which zero could cause a failure?

Proof of Theorem 2. Fix k. By Theorem 1 a nonpositive coefficient in this window has $n > 3T^2L$. Write $K_n^0(t) = K_n(1/2+it)$ and use the signed deficit

$$E_n(\rho) = K_n^0(t) - K_n(\rho), \qquad t = \operatorname{Im}\rho > 0.$$

For any $X \ge 2T$, half the truncated coefficient is $P_n(X)$ minus the sum of these deficits in $[T, 2T)$ and $[2T, X)$. The identity includes zeros at T, whose deficit vanishes. Once $X \ge 2n$ the reference contribution is at least $3M_T/384$. For sufficiently large T, the interior of the near band and the entire far band each have absolute error at most $M_T/1000$, uniformly in X. These are the estimates of Section 3 with the present k.

The remaining near-edge set contains at most B points counted with multiplicity. Put $D_0 = 1000(B+1)$. If each of their signed deficits were smaller than M_T/D_0, their sum would be at most $M_T/1000$. Every sufficiently high cutoff would then give a coefficient at least $M_T/200 > 0$, which persists in the limit. Consequently some single zero in this edge set has $E_n(\rho) \ge M_T/D_0$. Since $0 \le K_n^0(t) \le 2$,

$$K_n(\rho) < 0, \qquad \operatorname{Re}(q(\rho)^n) \ge M_T/(2D_0) \tag{19}$$

for large T. The modulus of q is at most one on the right half of the strip, so $\beta < 1/2$. The verified height then gives $t > T$.

Put $d = 1/2-\beta$, $d_{\max} = 1/2 - aw(2T)$, $v = nd/t^2$ and $R = W_k/T^2$. The horizontal estimate gives $M_T/D_0 \le \mathrm{e}^v$. The zero-free boundary and the monotonicity of w give $d \le d_{\max}$. The calculation in (16) with a general k yields

$$v \ge 2L + 2h - \log D_0, \qquad Rd_{\max} \le 2L + 2h + k/2 + 4a, \qquad v(t/T)^2 \le Rd.$$

Let $C = |k/2 + 4a| + \log D_0 + 1$. We have $v, R \ge L$ for large T. Thus both $v((t/T)^2 - 1)$ and $R(d_{\max} - d)$ are at most C. These inequalities imply $L(t/T - 1) \le C$ and $L(\beta - aw(2T)) \le C$. Since $w(t) \ge w(2T)$, the second gives the desired upper bound in (4). The strict lower bound follows from the zero-free region at t.

Finally $|q(\rho)|^n \le \mathrm{e}^v \le M_T \exp(k/2 + 4a)$. Dividing (19) by this positive amplitude gives (5) with $\eta = (2D_0 \exp(k/2 + 4a))^{-1}$. All the estimates apply to the same point selected by its signed deficit. □

Uniformity in the spectrum. For a single fixed zeta function, the premise RH(T) ceases to hold once T passes a hypothetical first off-line zero. The proof gives more information about the quantifiers. Take any family of locally finite symmetric multisets in the open strip, without real points, with common implied constants and starting heights in the total count, horizontal density, zero-free strip and edge bound for the fixed $K = 8a + 12$. The shell counts of Lemma 5 then hold uniformly for sufficiently large shells. The estimates of Section 3 and the signed-deficit proof above give a common T_0, C, η for the entire family. The bounds are selected from the counting data before one chooses a spectrum satisfying RH(T). The moving spectra below give nonvacuous examples at arbitrarily large T.

7 Explicit constants at all large verified heights

A bounded near-edge count with an unspecified implied constant cannot by itself supply a numerical starting height. We use a small, explicit difference between two zero-free widths to absorb an explicit density estimate. This also gives a stronger conclusion for the actual zeta function far beyond the heights used in Section 4.

Theorem 8. *Let $a_b = 1/48.0718$. For any $0 \le a \le a_b$ and fixed $k \in \mathbb{R}$, take*

$$g = a_b + 10^{-9}, \qquad D = 10^6 + |k| + (g-a)^{-1}, \qquad V = \exp(D^8).$$

In Theorem 2 the following choices suffice for every verified $T \ge T_0$:

$$T_0 = \exp(\exp(10^{30})), \qquad C = 10V^2, \qquad \eta = \exp(-5V). \tag{20}$$

For every verified $T \ge \exp(V)$ the whole prefix $1 \le n \le W_k$ satisfies $\lambda_n \ge n/(56\,000\,000)$. Alternatively, for any $0 \le a < 1/54.004$, one may take $g = 1/54.004$ in D, V, C, η and $T_0 = 3 \cdot 10^{12}$ in the localization assertion. The positive-prefix assertion at $T \ge \exp(V)$ holds with this choice as well.

At $a = a_b$, $k = 0$, $D = 1\,001\,000\,000$ in the first choice. The uniform constants are deliberately conservative. The small constants (18) hold directly at the published numerical verification height.

An explicit width above a verified height

We give the analytic input and the arithmetic of the small difference $g - a_b$, since rounding at this step affects the conclusion. Bellotti's exponential-sum estimate [18, Theorem 1.5 and Lemma 2.12] gives a Richert exponent $2\sqrt{3 \cdot 132.94357}/9 < 4.437944$. In the range $\sigma \ge 15/16$, $t \ge 10^{108}$, her Lemma 2.12 with $C = 8.7979$ gives a prefactor below 100. Sections 5.2–5.3 of the same paper use the smaller exponent 4 in the complementary ranges. Thus the constants $A = 100$, $B = 4.437944$ are admissible in the zeta bound used by Mossinghoff, Trudgian and Yang (MTY). Using the rounded exponent 4.43795 from the headline bound would lose the margin in the computation below.

Use the MTY degree-46 polynomial, normalized by $b_0 = 1$, and write $b = \sum_{j=1}^{46} b_j$, $c_\theta = \cos^2\theta$. Their (3.7) gives $3.5744094302 \le b \le b^+ = 3.574409431$. Their (7.2) then implies

$$c_\theta^3 \ge 0.05617776^3\,\frac{9}{16}b^2(b+1), \qquad c_\theta \ge c = 0.179964344.$$

Set $E = 0.5288$. Direct rational comparisons give

$$\sqrt{E} \le 0.72718636, \qquad \frac{b^+(2/3 + BE\sqrt{E}) + 2/3}{2E} \le 8.651205, \qquad 8.651205\,g + 10^{-11} < c. \tag{21}$$

We use the following explicit form of the MTY inequality [20, Lemma 4.7]. Let $\beta + it$ be a zero, $t \ge 10000$, $U = 46t + 1$, $L_1 = \log U$, $h_1 = \log L_1$, and $d = 1 - \beta$. Suppose $0 < y \le 1/4$, $d \le y/2$, $0 < \ell \le d$, $101\ell \le y$, and the rectangle $1 - \ell < \sigma \le 1$, $t - 1 \le u \le U$ contains no zero. With $s = \ell/y$, multiplication of Lemma 4.7 by ℓ and outward rounding give

$$\begin{aligned} c - 10^8(d/\ell - 1) \;&\le 2(d/\ell)s^2 + \frac{s}{2}\{b^+(2h_1/3 + By\sqrt{y}\,L_1 + 5) + \log(1 + 1/y)\} \\ &\quad + 10^8\{1000\,\ell^2 L_1/\sqrt{y} + s^2(1000 + 10h_1 + 10\log(1/y))\}. \end{aligned} \tag{22}$$

For clarity, $0.087\pi^2 b_1 < 2$, $\log A < 5$ and $\zeta(1+y) \le 1 + 1/y$ by integral comparison. For $R = 100$, the weights in MTY (4.3)–(4.7) satisfy $C_5(R) < 10^6$ and $|W'(0)b_1/w(0)| < 10^6$. These loose bounds follow already from $1.13 < \theta < 1.14$, $2 < \tan\theta < 3$, $\sin\theta > 0.8$, $\cos\theta > 0.4$. The

printed full inequality and these weight bounds are recorded in the Lean input. The passage to (22) and the three comparisons (21) have kernel proofs. A rational program independently checks the weights and brackets the polynomial angle using alternating sine and cosine series.

Suppose now $T \ge T_0$ is verified. Over zeros with $t \ge T$, let m be the infimum of $(1-\beta)/w(t)$. The all-height strip in [18, Theorem 1.2] gives $m \ge 1/54.004 > 0$. If any point violates $1-\beta > gw(t)$, the set is nonempty and $m \le g$. Select a zero with normalized distance at most $m(1+10^{-25})$. Put

$$w_1 = w(U), \qquad \ell = mw_1, \qquad y = Eh_1w_1.$$

Monotonicity of $x \mapsto (x/\log x)^{1/3}$ and $\log U \le \log t + 46$ give $w_1 \le w(t) \le (1+10^{-25})w_1$. Hence $1 \le d/\ell \le 1+10^{-22}$. The identity $w_1^3 L_1^2 h_1 = 1$, together with $h_1 \ge 10^{30}$ and $L_1 \ge 100h_1$, gives $0 < y \le 1/4,\ 101\ell \le y,\ d \le y/2$. The strip in (22) is empty. Below height T this follows from the verification, and above it the definition of m gives $1-\sigma \ge mw(u) \ge \ell$.

Now $w_1 L_1 \sqrt{y} = \sqrt{E}$, $s = m/(Eh_1)$, $\ell^2 L_1/\sqrt{y} \le 2/h_1$, and $\log(1+1/y) \le 2h_1/3 + 4$ by cubing $1/y \le 2\exp(2h_1/3)$. The three terms on the right of (22) are bounded by $16/h_1^2$, $8.651205m + 30/h_1$, and $10^{12}/h_1$, respectively. Since $m \le g$, this contradicts (21). Reflection gives the other side. We have proved the strictly stronger strip $gw(t) < \beta < 1 - gw(t)$ for all $t \ge T$ under the stated finite verification. Its role is to supply a quantified separation from the specific coefficient a_b.

Paying for the number of edge zeros

The following argument applies to either indicated width g. Assume $T \ge \exp(V)$ and the strip with that width throughout $[T, 2T)$. Put $x = \log T$, $h = \log x$, $Q = (x/h)^{1/3}$, and $\epsilon = 1/D$. Then $\epsilon \le g - a$. From $h \ge D^8$, $\exp(h) \ge h^5/120$, and $Q^3 h = x$, we get

$$D^2 h \le Q, \qquad 10^8 D^2 Q \le x, \qquad 4x \le R = 4x + 8aQ + 4h + k \le 5x. \tag{23}$$

Take $J = \lceil x \rceil$ and cells of width $\Delta = 1/(2J)$ in d_ρ. For a cell with lower endpoint z, write $f = 1/2 - z$. The folded count below $2T$ at this displacement is at most

$$10^{32} x^{14} \exp(A_f), \qquad A_f = \begin{cases} \epsilon x f, & f \le \epsilon^2/10^6, \\ 3xf, & f > \epsilon^2/10^6. \end{cases} \tag{24}$$

For the second bound use (8) with the explicit constants of [13], $D(1-f) \le 3f$, and $\log(2T) \le 2x$. For the first use Bellotti's explicit density theorem near unity [21, Theorems 1.1–1.2]. For $H \ge 3 \cdot 10^{12}$ and $0 \le f \le 0.01$, its rounded form is $N(1-f, H) \le 10^{26} (\log H)^{14} \exp(60 \log H\, f\sqrt{f})$. The hypothesis on f gives $\sqrt{f} \le \epsilon/1000$. Multiplication by two for reflection and replacement of $\log(2T)$ by $2x$ give (24).

In any occupied cell $f \ge gw(2T)$, so $xf \ge g(Q-1)$ by (15). Adding $R(z+\Delta)$ to either exponent in (24) gives at most $2x - 40h$. Indeed $R\Delta \le 3$. In the first case the saving $(4-\epsilon)xf$ is at least $4aQ + \epsilon Q - 4$, since $g \le 1$ and $g - a \ge \epsilon$. In the second case $xf \ge 100Q$ by (23). The remaining terms $2h + k/2 + 3$ are absorbed in both cases.

With $M = T^2 R$, convexity supplies n/M times the exponential majorant for each $n \le M$. There are at most $2x$ cells, so the near-band error is bounded by

$$\frac{n}{M}\, 2x\, 10^{32} x^{14} \exp(2x - 40h) = n\, \frac{2 \cdot 10^{32}}{R x^{25}} \le n\, 10^{-10}.$$

Beyond $2T$ the normalized error is at most $\exp(5x/8)\, 8(\log(2T)+2)/(2T) \le 10^{-10}$. To see the last inequality, use $\exp(3x/8) \ge (3x/8)^2/2$ and $x \ge 10^{18}$. Lemma 5 proves the positive-prefix assertion of Theorem 8. It holds from the first integer onward.

A fixed witness in the remaining height range

For any $n \ge 7 \cdot 10^{26}$ with $\lambda_n \le 0$, some left-hand zero has $0 < \gamma < \sqrt{n}$ and $K_n(\rho) < 0$. To prove this take $Y = \sqrt{n}$. Then $Y \ge 10^{12}$ and $\log Y + 2 \le Y/100$. If all signed deficits below Y were at most 4, their sum would be at most $4N(Y) \le n$. Above Y, the total absolute error is at most $3n\,8(\log Y + 2)/Y \le n$, since $\exp(n/(2Y^2)) < 3$. This contradicts (17) at sufficiently high cutoffs. A deficit greater than 4 gives the asserted negative summand.

For the remaining verified heights $T < \exp(V)$, any nonpositive order exceeds $91 \cdot 10^{25}$ by the finite theorem. We have $L \le V$ and $R \le 10V$ for the parameters of Theorem 8. The preceding witness lies above T and below $\sqrt{n} \le 10VT$. The effective strip at that height gives $aw(\gamma) < \beta < 1/2$. Therefore $(\gamma - T)L \le 10V^2T$ and $(\beta - aw(\gamma))L \le V$. Finally $|q(\rho)|^n \le \exp(n/(2\gamma^2)) \le \exp(5V)$, while its real part after taking the n-th power exceeds one. This proves the phase floor, completing Theorem 8. In the alternative with $g = 1/54.004$, the whole argument uses the published all-height strip directly.

8 The size of the projected background

Let $B_n = \lim_{X\to\infty} P_n(X)$, counting positive heights only. We need the leading constant for a family of orders varying with T. Arias de Reyna [23, Theorem 1] gives, in the present normalization,

$$\frac{2B_n}{n} = \tfrac{1}{2}\big(\log n + \gamma_{\mathrm{E}} - 1 - \log(2\pi)\big) + y_n, \qquad (y_n) \in \ell^2.$$

His theorem applies to any positive ordinate sequence with the stated counting law, including the projection of the actual ordinates without assuming RH. Here is an independent derivation of the leading term used by the proof checker.

Lemma 9. *For every $n \ge 0$, $\int_0^\infty K_n^0(t)\,dt = \pi n/2$. For $D \ge 2$, $n/D \ge 10^6$ and $\log n \ge 2$,*

$$\left|\frac{B_n}{n\log n} - \frac{1}{4}\right| \le \frac{20}{D} + \frac{600D^5}{\log n}. \tag{25}$$

In particular $B_n/(n\log n) \to 1/4$.

Proof. Set $z = 1/2 + it$ and $p_{n,j}(t) = \mathrm{Re}(q(z)^n z^{-j-1})$. Multiplication by $q = 1 - z^{-1}$ gives $p_{n+1,j} = p_{n,j} - p_{n,j+1}$ and $K_{n+1}^0 - K_n^0 = p_{n,0}$. Now $p_{0,0}$ has primitive $\arctan(2t)$. For $j \ge 1$ a primitive of $p_{0,j}$ is $-\mathrm{Im}(z^{-j})/j$, which vanishes at both endpoints. Integrate first on $[0, V]$ and send V to infinity. Induction gives $\int_0^\infty p_{n,0} = \pi/2$ and $\int_0^\infty p_{n,j} = 0$ for $j \ge 1$. Summing the differences for K_n^0 proves the integral identity.

For the discrete comparison put $u = n/D$, $v = Dn$, $c_n = \log n/(2\pi)$ and $E = 100D^2n$. On $[u, v]$ we have $|N(t) - c_n t| \le E$. Indeed the main-term difference equals $t(\log(t/n) - \log(2\pi) - 1)/(2\pi)$, with $|\log(t/n)| \le \log D$, and the counting error is at most $10 + 3\log t$. For any C^1 function f, the strict-height staircase gives

$$\sum_{u\le t<v} f(t) = [N(v) - N(u)]f(v) - \int_u^v [N(t) - N(u)]f'(t)\,dt.$$

Subtract the identity for the linear count $c_n t$. The error is at most $2E(|f(v)| + (v-u)\sup|f'|)$. For $f = K_n^0$, the angular formula gives $|f| \le 2$ and $|f'(t)| \le n/u^2$. The middle comparison therefore costs at most $600D^5n$.

The omitted head and tail of the discrete sum cost at most $2u\log u + 4n^2(\log v + 2)/v \le 14n\log n/D$, by (11). In the integral the analogous cost is at most $2u + n^2/(2v) \le 3n/D$. Multiply the latter by c_n. As $c_n\pi n/2 = n\log n/4$, the three errors imply (25). First choose D large, then let n grow. □

If $s = (n - F_a(T))/T^2$ stays bounded, then $n/(T^2L) \to 4$ and $\log n/L \to 2$, uniformly. The two factors and Lemma 9 give

$$\frac{2B_n}{M_T} = 2\frac{B_n}{n\log n}\frac{n}{T^2L}\frac{\log n}{L} \longrightarrow 4. \tag{26}$$

9 Phase and the first negative integer

For Theorem 3, fix m, τ, a before taking $T \to \infty$. Let $t = T+1$, $b = (aQ+\tau)/L$, $r = |q(b+it)|$, and retain the multiplicities of all projected ordinates. Eventually $0 < b < 1/2$ and $b > aw(t)$, since w decreases and $\tau > 0$. The projected points also obey the zero-free strip at all sufficiently large heights. Adding the quartet changes the positive count below X by $2m\mathbf{1}_{\{t<X\}}$. For every $\sigma > 1/2$ at most m added points lie to its right and no projected points do. At $\sigma = 1/2$ the total count supplies (8). This proves (8) and (9) with constants uniform in T for each fixed m. Finite cutoffs are restrictions of one multiset and the conjugate-completed coefficient is

$$\lambda_n^{\mathrm{mod}} = 2B_n + 2m\{K_n(b+it) + K_n(1-b+it)\}.$$

The oscillatory law

The expansion needed at order T^2 is sensitive to the $1/L$ inward offset. With $d = 1/2 - b$,

$$\frac{d}{t^2+(1-b)^2} \le \log r \le \frac{d}{t^2+b^2}, \qquad \tan\theta = \frac{t}{t^2+b^2-b}. \tag{27}$$

Indeed $r^2 = 1+2d/(t^2+b^2)$, so the two logarithmic bounds follow from $x/(1+x) \le \log(1+x) \le x$. In particular $T^2\log r = d + O(1/T)$. For bounded residual s, set $R_s = 4L + 8aQ + 4h + s$. Uniformly there,

$$R_s b = 4aQ + 4\tau + \frac{(8aQ+4h+s)(aQ+\tau)}{L} = 4aQ + 4\tau + o(1)$$

because $Q^2/L \to 0$ and $h = o(Q)$. It follows that

$$n\log r = 2L + 2h + s/2 - 4\tau + o(1), \qquad r^n/M_T = \exp(s/2 - 4\tau) + o(1). \tag{28}$$

The reflected positive-height point has Cayley modulus at most one. Consequently

$$\left|\lambda_n^{\mathrm{mod}} - \big(2B_n - 2mr^n\cos(n\theta)\big)\right| \le 6m.$$

Equations (26) and (28) prove (6) uniformly over all the indicated integer orders.

Passing from an amplitude to a first index

Write $s_* = 8\tau + 2\log(2/m)$, so that $2m\exp(s_*/2 - 4\tau) = 4$. For every $k < s_*$ the right side of (6) is bounded below by $4 - 2m\exp(k/2 - 4\tau) > 0$ when $s \le k$ in a fixed compact interval. To extend this to the entire prefix, compare both positive-height additions with their projections. Lemma 4 gives

$$\lambda_n^{\mathrm{mod}} \ge 2B_n - 4m\big(\exp(nd/t^2) - 1\big).$$

For $n \le 3T^2L$ the loss is $O(mn/\sqrt{T})$, and Lemma 5 gives $B_n \gg n$. For $3T^2L < n \le W_{-A}$, take A fixed, positive and sufficiently large. Since $R_{-A} \ge 4L$ eventually,

$$nd/t^2 \le R_{-A}(1/2 - b) \le 2L + 2h - A/2 - 4\tau.$$

The projected sum is at least $3M_T/384$, so the choice of A absorbs the loss. Taking $A > |k|$, equation (6) covers the remaining orders with $-A \le s \le k$. Thus every coefficient through W_k is strictly positive for large T.

For an upper bound on the least negative, fix $k > s_*$ and choose $s_* < u < k$. The integer staircase

$$j = \lceil W_u\theta/(2\pi)\rceil\,, \qquad n = \lceil 2\pi j/\theta\rceil$$

gives $0 \le n - W_u \le 2\pi/\theta + 1$ and $0 \le n\theta - 2\pi j \le \theta$. Equation (27) and the elementary arctangent inequalities give $1/(2t) \le \theta \le 2/T$ for large T. The chosen n has $u \le s \le u + 40/T$ and $\cos(n\theta) \to 1$. Its normalized coefficient tends to $4 - 2m\exp(u/2 - 4\tau) < 0$. It lies below W_k eventually. Well-ordering gives a least negative index, and letting k approach s_* from either side proves (7).

With $m = \tau = 1$, the second precursor manuscript's enclosure $W_{-20} < n_-(T) \le W_{41}$ follows directly. The same phase selection at $u = 40$ puts a negative order in $[W_{40}, W_{41}]$. Given $d_0 \in \mathbb{R}$, choose m so that $s_* < d_0 - 1$ and align the phase at $u = d_0 - 1$. This proves the asserted negative interval for arbitrary fixed offsets. For fixed $c > 4$ a stronger first-order construction can even keep b independent of T. Choose $4 < k < c$ and $2/k < d < 1/2$, put $b = 1/2 - d$ and align near kT^2L. Then $r^n = T^{kd+o(1)}$, whereas $B_n = O(T^2L^2)$, by (12). This yields a negative coefficient in the required prefix using just one quartet.

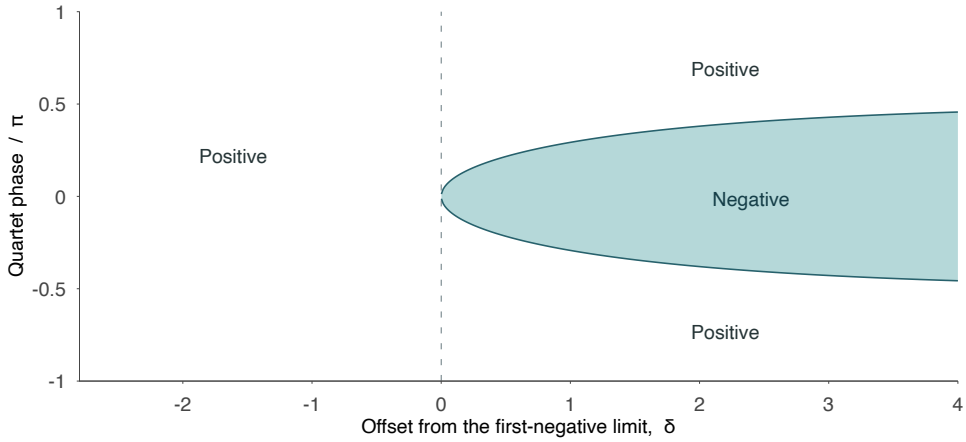

Figure 1: Sign of the limiting profile $4(1 - \mathrm{e}^{\delta/2}\cos\varphi)$, with $\delta = s - s_*$ and $\varphi = n\theta \bmod 2\pi$ represented in $[-\pi, \pi]$. The boundary is $\cos\varphi = \mathrm{e}^{-\delta/2}$. Integer phase alignment approaches its tip from the right.

10 Finite checks of the model

Two integer witnesses test the sign directly at fixed heights. The estimates use the B_n bound (12) and enclose the phase with rational arithmetic.

A fixed abscissa. At $T = 3 \cdot 10^{12}$ choose $b = 1/1000,\ m = 1,\ t = T + 1$. The integer

$$n = 420\,973\,415\,581\,032\,t = 1\,262\,920\,246\,743\,516\,973\,415\,581\,032$$

is at most $5T^2 \log T$. For $j = 67 \cdot 10^{12}$ and $v = t/(t^2 + b^2 - b)$, the inequalities $v/(1 + v^2) \le \arctan v \le v$ imply $|n\theta - 2\pi j| \le 1$ by rational comparison. Equation (27) gives $n \log r > 70 > 100 \log 2$. Since $n < 2^{91}$, $12n(64 + 2) + 6 < 2^{100}$, giving $2B_n + 6 - r^n < 0$.

A moving abscissa. At $T = 2^{32}$ take $a = 1/48.0718,\ \tau = m = 1$. The order

$$n = 609\,629\,535\,106\,(T + 1) = 2\,618\,338\,916\,565\,583\,428\,482$$

has $40.3 < (n - F_a(T))/T^2 < 40.9$. With $j = 97\,025\,553\,967$, the same arctangent inequalities give $-0.416 < n\theta - 2\pi j < -0.413$. Now $b < 0.048$, so $n \log r \ge 0.452n/(1 + t^2) > 64 > 90 \log 2$.

The inequality $n < 2^{72}$ implies $12n \cdot 53 + 6 < 2^{90}$. The coefficient is strictly negative. Both computations certify the displayed orders as witnesses.

Isolating the radial balance. One can also check the slow approach of the amplitude in (28). Let s_{amp} be the real residual at which the exact expanding amplitude $2mr^{F_a(T)+sT^2}$ equals $4M_T$. Thus

$$s_{\mathrm{amp}} = \frac{2L + 2h + \log(2/m)}{T^2 \log r} - (4L + 8aQ + 4h).$$

The uniform gain formula and monotonicity give $s_{\mathrm{amp}} \to s_*$. For $a = 1/48.0718$, $\tau = 1$, the table encloses this scalar root. The integer phase and the finite value of the projected background enter λ_n^{mod} separately.

$\log_2 T$	s_{amp}, one quartet	s_{amp}, four copies
32	$[11.673, 11.674]$	$[8.614, 8.615]$
128	$[10.063, 10.064]$	$[7.223, 7.224]$
512	$[9.591, 9.592]$	$[6.802, 6.803]$
2048	$[9.448, 9.449]$	$[6.671, 6.672]$
Limit	$8 + 2\log 2$	$8 - 2\log 2$

The rational check uses the two endpoints in (27). It encloses logarithms by range reduction and the positive series for $2\operatorname{arctanh} x$, with a geometric tail bound for $0 \le x \le 1/3$. Monotone bisection encloses each cube root.

11 Scope of the formal verification

The Lean 4.35.0 formalization indexes each actual zero by all natural numbers below its analytic order. It proves convergence of the paired sums, the total count estimates used here and the projected asymptotic. For the zeta conclusions the external analytic statements are visible parameters. These are Ingham's horizontal estimate, Bellotti's zero-free region and near-edge density where used, and the classical zero formula for the derivative definition. The finite positive interval takes the verified height, both published density estimates and the stated classical zero-free strip. The transfer in the final paragraph of Section 5 takes Mathlib's separate global RH proposition and the classical Li identity. The archive verifies this transfer in Lean 4.35.0. The independently checked global RH proof uses Lean 4.33.1. For the actual zeta spectrum the machine proof derives the signed loss, localization in both coordinates, the phase bound and the empty-region corollary. The uniform-family observation follows by reading the same estimates with common constants. The model constructions themselves have separate Lean proofs. The quartet profile and the least-index limit use the formalized total count of the true ordinates and require no horizontal zeta hypothesis.

The two precursor manuscripts treated the leading constant and the three-term window, respectively. The present version retains their two finite model witnesses and their finite zeta prefix. The joint localization, oscillatory profile and exact fourth term are retained here. This revision adds the 128-layer certificate, the finite signed rectangles and the completely explicit height thresholds.

The supplement contains source, 295 exported theorem bodies with their transitive dependencies, and independent rational checks. Both height slices, the one-sided count, exact target budgets, local alternatives, continuum obstruction, every row of the 128-layer certificate, the signed boxes and the quantified uniform constants have Lean proofs. The high-height results take the explicit density inputs, the all-height zero-free strip and the MTY rectangle inequality as named published inputs. The new strip follows inside Lean from the latter inequality and the stated finite verification. The radial-balance table has a separate exact-rational verifier.

The proof export was replayed by the Lean kernel in a directory containing only the certificate and its verification files.

AI and human contributions

OpenAI Codex contributed to the mathematical derivations in the merged manuscript, implemented the Lean formalization and numerical checks, ran verification, and drafted and revised the English manuscript. In particular, it developed and formalized the one-sided finite-height estimate and the resulting integer certificate. The human collaborator directed the research, requested the merger of the two precursor manuscripts, and set the finite-height refinement targets.

The corresponding human is the point of contact for questions and corrections.

Artifact availability

The companion Lean 4.35.0 artifact includes pinned dependencies, a theorem list, proof certificates, checksums and numerical data. From its root, run the following commands to replay the certificate, rebuild the source and reproduce the numerical checks, respectively.

```
python3 verify.py
python3 verify.py --source
python3 examples/check.py
```

Lean artifact v1.0.0 is archived on Zenodo at DOI 10.5281/zenodo.24457511. Download all 15 parts and follow `README-reassemble.txt` to reconstruct and verify the ZIP.

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