# THEOREM (1+1.75) ON THE GOLDBACH CONJECTURE

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ABSTRACT. For fixed $1 \le a \le 2$, let Proposition $(1+a)$ assert that every sufficiently large even integer $N$ can be written as

$$N = p + rq, \qquad r \le q^{a-1},$$

where $p, q$ are primes and $r$ is either 1 or a prime. At $a = 1$, this is the binary Goldbach conjecture for sufficiently large even integers; at $a = 2$, it is Chen's theorem. We prove unconditionally that every sufficiently large even integer admits such a representation with $a = 1.75$, establishing Proposition (1+1.75) and strengthening the earlier (1+1.9) theorem of Li and Liu [6]. The proof subtracts a positive upper-sieve estimate for excluded semiprimes from a Chen-count lower bound obtained by restricting paired terms in an eleven-term Chen weight and applying the double sieve. The threshold for $N$ is ineffective: the proof does not provide an explicit computable threshold.

## 1. INTRODUCTION

For fixed $1 \le a \le 2$, we say that *Proposition* $(1+a)$ holds if every sufficiently large even integer $N$ has a representation

$$N = p + rq, \qquad r \le q^{a-1}, \qquad p, q \text{ prime}, \qquad r = 1 \text{ or prime}.$$

When $a = 1$, the condition forces $r = 1$, giving the binary Goldbach conjecture for sufficiently large even integers. When $a = 2$, ordering the two factors gives Chen's theorem. The intermediate parameters ask how much smaller one prime factor of a composite remainder can be required to be.

We prove the following theorem.

**Theorem 1.1.** *Proposition* $(1 + 1.75)$ *holds unconditionally. Thus every sufficiently large even integer can be written as* $N = p + rq$*, where* $p, q$ *are primes,* $r$ *is either* 1 *or a prime, and* $r \le q^{3/4}$.

The thresholds for $N$ in Theorems 1.1–1.2 and Corollaries 1.3–1.4 are ineffective.

The parameter 1.75 can be reduced to 1.74816964 and further to 1.74816865, with different quantitative lower bounds for the number of representations; see Corollary 1.4.

Li and Liu [6, Theorem 1.1] prove Proposition $(1 + 1.9)$ and remark that their parameter can be reduced to 1.894. Theorem 1.1 strengthens this factor restriction. Both statements concern every sufficiently large even integer.

For the quantitative argument put $b = a - 1$ and count prime summands with $r \le q^b$. Prime remainders $r = 1$ remain admissible, since $1 \le q^b$. The next theorem gives a lower bound for every fixed $1/2 < b < 1$.

Throughout the paper, $N > 2$ tends to infinity through the even integers, unsubscripted logarithms are natural, $\log_N x = \log x / \log N$, and $\Omega(n)$ counts prime factors with multiplicity. Put

$$C_2 = \prod_{\substack{\ell > 2 \\ \ell \text{ prime}}} \left(1 - \frac{1}{(\ell-1)^2}\right), \qquad C_N = C_2 \prod_{\substack{\ell \mid N,\ \ell > 2 \\ \ell \text{ prime}}} \frac{\ell - 1}{\ell - 2}, \qquad H_N = C_N \frac{N}{\log^2 N}. \tag{1}$$

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For any positive even integer $M$, $C_M$ denotes the same product with $M$ in place of $N$. We count distinct prime summands by

$$D_{1,2}(N) = \#\{p < N : p \text{ prime},\ N-p > 1,\ \Omega(N-p) \le 2\}, \tag{2}$$

$$D_b(N) = \#\{p < N : p \text{ prime},\ N-p = rq,\ q \text{ prime},\ r = 1 \text{ or prime},\ r \le q^b\}, \tag{3}$$

$$B_b(N) = \#\{p < N : p \text{ prime},\ N-p = rq,\ r,q \text{ prime},\ q^b < r \le q\}. \tag{4}$$

The ordered factorization $r \le q$ is unique. In particular, a composite remainder contributes once; a square contributes to $B_b$, whereas a prime remainder contributes to $D_b$.

**Theorem 1.2.** *Let*

$$K = \frac{8452171}{1250000} = 6.7617368, \qquad c = \frac{49044063}{25000000} = 1.96176252. \tag{5}$$

*For each fixed* $1/2 < b < 1$,

$$\liminf_{\substack{N\to\infty \\ 2|N}} \frac{D_b(N)}{H_N} \ge c - K\log(1/b). \tag{6}$$

Within $1/2 < b < 1$, this lower bound is positive precisely when $b > \exp(-c/K)$; see Section 8.

**Corollary 1.3.** *For every sufficiently large even integer* $N$,

$$D_{3/4}(N) \ge 0.0165 H_N.$$

**Corollary 1.4.** *Every sufficiently large even integer* $N$ *satisfies*

$$D_{0.74816964}(N) \ge 10^{-5} H_N, \tag{7}$$

$$D_{0.74816865}(N) \ge 10^{-6} H_N, \tag{8}$$

$$D_{0.9}(N) \ge 1.24889 H_N. \tag{9}$$

The first two inequalities establish Propositions $(1 + 1.74816964)$ and $(1 + 1.74816865)$ with respective count coefficients $10^{-5}$ and $10^{-6}$ in the normalization (1). The estimate at $b = 0.9$ permits a comparison under the same factor restriction.

*Proof of Theorem 1.1.* Corollary 1.3 gives $D_{3/4}(N) \ge 0.0165 H_N > 0$ for every sufficiently large even $N$. Thus Proposition $(1 + 1.75)$ holds. □

1.1. **Related work and the method.** In the normalization (1), the result of Li and Liu [6, Theorem 1.1] has coefficient 0.0004 at $b = 0.9$; their stated refinement corresponds to $b = 0.894$. At $b = 0.9$, our $D_b(N)$ coincides with Li–Liu's $D_{1,1.9}(N)$: both include the prime-remainder case $r = 1$, count each admissible prime summand once, and use the same singular-series normalization. Thus the coefficients refer to the same counting function. Our coefficient follows from the combination $c - K\log(1/b)$: $c$ is supplied by the lower Chen-count construction, and $K$ by the positive upper majorant. The smaller exponents in Corollary 1.4 retain an explicit positive count. Their discussion of lower and marked sieve terms [6, Sections 3 and 8.2] also identifies a constraint that matters here: an improved upper-sieve distribution level must be matched to the coefficient class of each term where it is used.

The double sieve transfers information between upper and lower sieve remainders. We use Wu's formulation [16], with finite positive transitions for the two remainder functions. Li [7, Theorem 1.1] gives coefficient 1.9728 for the total Chen count in the same normalization. The coefficient $c = 1.96176252$ used here is proved within the moving-residue coefficient class and positive marked families of the factor-restricted argument. Each lower transition retains its actual residual level and all prefix conditions.

The distribution estimate must be uniform as the residue varies with $N$. Maynard's triply well-factorable estimate [10, Definition 2 and Theorem 1.1] has a constant depending on a fixed residue. Lichtman's large-residue statement [8, Theorem 1.7] has level $(5 - \vartheta)/8 - \epsilon =$

$153/256 - \epsilon$ at $\vartheta = 7/32$. Here the residue is comparable with the complete local product. The reciprocal phase in (78) is retained, since replacing it by 1 in this range need not give a vanishing error. Appendices A and B, together with the factorization in Appendix C, establish the restricted distribution input at support exponent 0.599 for the finite upper and lower coefficient families used in this paper. At each application we retain the selected factorization, full and complementary square conditions, and positivity of the marked family.

The proof subtracts the excluded semiprimes from the full Chen count. For the upper estimate we construct a nonnegative divisor majorant, compute its density, and prove its distribution estimate for the same coefficient. For the lower estimate we restrict the two-prime and four-prime terms of an eleven-term Chen weight in pairs. An injection preserves the pointwise weight inequality, while a lower bound for Buchstab's function on the restricted fibres improves the coefficient.

The nonnegative upper majorant and its coefficient realization are constructed here. For the lower bound, we use Wu's double-sieve framework with the paired deletion and coefficient realizations established here. Together with the finite transitions and directed enclosures, these constructions determine the constants $K, c$. Their mathematical definitions and certificate correspondence are given in Appendix I.

## 2. Reduction and outline of the proof

Proof outline and verification. On a first reading, Lemma 7.1 may be taken as the finite numerical input: follow the upper construction, the eleven-term lower construction and paired deletion, and then the final subtraction in Section 8. For a complete verification, return to the appendices for the distribution estimates, coefficient realizations, finite transitions and directed enclosures, and check their correspondence with the Numerical Supplement described in Appendix I.

Unique factorization gives the exact identity

$$D_b(N) = D_{1,2}(N) - B_b(N). \tag{10}$$

Thus it suffices to prove

$$\liminf_{\substack{N\to\infty \\ 2|N}} \frac{D_{1,2}(N)}{H_N} \geq c, \qquad \limsup_{\substack{N\to\infty \\ 2|N}} \frac{B_b(N)}{H_N} \leq K \log(1/b). \tag{11}$$

The factor $\log(1/b)$ comes from the excluded factor region. For $\beta = b/(1+b)$, the limiting logarithmic region of the ordered bad semiprimes is $\beta < t \leq 1/2$, and

$$\int_\beta^{1/2} \frac{dt}{t(1-t)} = \log(1/b).$$

To turn this mass into a count with a prime complement, we apply a positive upper sieve whose distribution and density have both been established for its actual coefficients.

Section 3 establishes the distribution estimate for the actual upper and lower coefficients. Section 4 constructs the positive upper sieve and obtains $K$. Sections 5 and 6 give the lower construction; its switched costs use the upper bound already proved. Section 7 states the finite numerical inequalities and their error allowances. Finally Section 8 proves the second estimate in (11) and completes the theorem. In particular, the construction of $K$ uses neither the lower double-sieve tables nor the Chen count obtained from them.

Choice of parameters. At each fixed accuracy, the finite collection of policies and marked queries, the number of generations, cutoffs defining fixed compact parameter domains, approximation tolerances and support reserves are chosen before $N$ tends to infinity. Prime intervals, weights and arithmetic coefficients may still depend on $N$ through these fixed rules. The common choices and error propagation are specified in Section 6.6 and (113); the strict scalar surpluses in Section 7 allow one fixed accuracy to be chosen for the final count.

Notation index. The following symbols keep the same meaning in the main text and appendices.

| Symbol | Meaning | Location |
|---|---|---|
| $E_{\mathrm{up}}$; $R_{\mathrm{low}}$ | Upper and lower scalar endpoints | (182); (47) |
| $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}$ | Paired rectangle families and their union | Section 5 |
| $a_{\mathrm{s}}, h_{\mathrm{s}}, d_{\mathrm{s}}$ | Fixed sieve cutoffs | Section 5 |
| $H(s), h(s)$; $H_m, h_m$ | Upper and lower defects; generation-$m$ envelopes | Section 6 |
| $B$; $\mathscr{B}$ | Arithmetic prime prefix; geometric rectangle or box | Sections 3, 6 |
| $H_N$; $B_{\mathrm{low}}$ | Count normalization; each lower error allowance | (1); (48) |
| $A_+, A_-$ | Normalized linear-sieve numerators $sF/(2e^\gamma), sf/(2e^\gamma)$ | Section 6 |
| $\Omega$; $\omega_{\mathrm{env}}$ | Prime-factor count with multiplicity; Buchstab upper envelope | Section 1; (160) |
| $\rho_N^{\mathrm{up}}, \rho_N^{\mathrm{low}}$ | Actual upper and lower divisor coefficients; $\rho_N$ denotes a generic admitted coefficient | Sections 4, 6 |

We use the standard linear sieve functions $F, f$, normalized by

$$sF(s) = 2e^\gamma \quad (1 \le s \le 3), \qquad sf(s) = 0 \quad (1 \le s \le 2),$$

and subsequently by $(sF(s))' = f(s-1)$ and $(sf(s))' = F(s-1)$ on their respective ranges. Buchstab's function satisfies

$$\omega(s) = 1/s \quad (1 \le s \le 2), \qquad (s\omega(s))' = \omega(s-1) \quad (s > 2).$$

Here $P^-(n)$ is the least prime factor for $n \ge 2$, and $P^-(1) = +\infty$. The finite interval data used for these functions are discussed in Appendix I.

2.1. **A positive sieve coefficient.** The upper estimate uses the following single coefficient theorem. Its three conclusions concern the same divisor coefficient. Here "positive" means that the resulting divisor sum $R_N(n)$ is a nonnegative majorant. The individual coefficients $\rho_N^{\mathrm{up}}(q)$ may have either sign.

**Theorem 2.1.** *Fix $a_* > 0$, a bound on the number of labelled prime factors, and a finite collection of smooth weights $W_N(p_1, \dots, p_k)$ supported on*

$$p_i \ge c_* N^{a_*}, \qquad p_1 \cdots p_k \le C_* N,$$

*with fixed constants $c_*, C_* > 0$, uniformly bounded size, and fixed polylogarithmic bounds for their mixed Euler derivatives. Apart from these smooth geometric conditions, impose no additional arithmetic filter. Put $\rho_0 = 19/96$, $\theta_* = 599/1000$, and*

$$P_N(y) = \prod_{\substack{p<y \\ p\nmid N}} p, \qquad X_W^{(q)} = \sum_{(p_1\cdots p_k, q)=1} W_N(p_1, \dots, p_k).$$

*There are real coefficients $\rho_N^{\mathrm{up}}(q)$ and a constant $M$, independent of $N$, with squarefree support $q \le N^{\theta_*}$, $(q, N) = 1$, and $|\rho_N^{\mathrm{up}}(q)| \le M$, such that*

$$R_N(n) := \sum_{q \mid n} \rho_N^{\mathrm{up}}(q) \ge \mathbf{1}_{(n, P_N(N^{\rho_0}))=1} \ge 0 \qquad (n \in \mathbb{Z}), \tag{12}$$

$$\sum_q \frac{\rho_N^{\mathrm{up}}(q)}{\varphi(q)} \le (K + o(1)) \frac{C_N}{\log N}, \tag{13}$$

$$\sum_q \rho_N^{\mathrm{up}}(q)\left\{\sum_{p_1\cdots p_k\equiv N\ (q)} W_N(p_1,\dots,p_k)-\frac{X_W^{(q)}}{\varphi(q)}\right\} = O_{A,W}(N(\log N)^{-A}) \qquad (A>0). \tag{14}$$

*The finite sum defines $R_N(0)$. The finite policy, approximation parameters and support margins are fixed before $N$ tends to infinity through all even integers. The constants may depend on this fixed collection and its derivative bounds, but are uniform in $N$.*

The coefficient is constructed in Section 4. The distribution statement is proved first in Section 3 for the precise coefficient representations needed by that construction.

## 3. Distribution estimates for the sieve coefficients

**Definition 3.1.** A finite prefix policy specifies finitely many regions in the coordinates $t_i = \log_N p_i$ of ordered prime prefixes and a sieve level on each region. The decision at a prefix is fixed before its children are refined. A selected branch is admissible when its full-square and selected-complement conditions in (108)–(109) have the required strict margins; an inherited branch keeps its parent decision. A marked family is a nonnegative finite weighted family of prefixes $B$ at a common total level $Q$, with residual level $Q/B$ and the original prime ceiling retained. A marked query is a sifted count for one such family. References below to an admitted coefficient mean a coefficient obtained from these policies and finite marked transitions, with the three-factor representations of Appendix C.

### 3.1. A local estimate with a moving residue.

The distribution problem is local in the complete product. We denote the convolution lengths by $U, V$, write $x = UV$, and use $\mathfrak{a}$ for the residue; $N$ remains the global even target. The estimate proved in Appendix A is the following.

**Proposition 3.2** (Restricted medium convolution)**.** *Fix $\vartheta = 7/32$, $0 < \sigma < 1/100$, $c_0 > 0$, a divisor order $B$, and $0 \le \kappa \le \sigma/100$. Let $U, V, Q_1, Q_2, Q_3 \ge 1$. Suppose*

$$\begin{aligned} c_0 x \le |\mathfrak{a}| \le x^{1+\kappa}, \qquad & Q_1 < Ux^{-\sigma}, \\ U^2Q_2Q_3^2 < x^{1-8\sigma}, \qquad & |\mathfrak{a}|^{\vartheta}UQ_1Q_2^5Q_3^2 < x^{2-15\sigma}, \\ U^{1+2\vartheta}Q_1Q_2^{5-2\vartheta}Q_3^{2+2\vartheta} < x^{2-15\sigma}. & \end{aligned} \tag{15}$$

*Let $\alpha$ be one-bounded, supported on an individual interval in $[U, 2U]$ and on $P^-(n) > x^{1/(\log\log x)^3}$. Assume its auxiliary-coprimality Siegel–Walfisz estimate: for every $A > 0$, every $k, q \ge 1$ and $(b_0, q) = 1$,*

$$\left|\sum_{\substack{n\equiv b_0 \pmod q\\(n,k)=1}}\alpha_n - \frac{1}{\varphi(q)}\sum_{(n,kq)=1}\alpha_n\right| \le C_A U\tau(k)^{C_0}(\log x)^{-A}, \tag{16}$$

*with fixed $C_0$ and uniform family constants. Let $|\beta_m| \le \tau(m)^B$, $m \in [V, 2V]$, and let each $\gamma_i$ be a separate one-bounded sequence on $q_i \asymp Q_i$, supported on $(q_i, \mathfrak{a}) = 1$. For*

$$\Delta_\ell = \sum_{n,m}\alpha_n\beta_m\left(\mathbf{1}_{nm\equiv\mathfrak{a}\pmod\ell} - \frac{\mathbf{1}_{(nm,\ell)=1}}{\varphi(\ell)}\right),$$

*the stated classical inputs of Appendix B give*

$$\sum_{q_1,q_2,q_3}\gamma_1(q_1)\gamma_2(q_2)\gamma_3(q_3)\Delta_{q_1q_2q_3} = O_A\big(x(\log x)^{-A}\big). \tag{17}$$

*The constant is uniform in the residue and families satisfying these conditions. Fixed endpoint multiples are absorbed by the strict power margins.*

Application of the local estimate. Fix one finite admitted coefficient family. Appendix C uses its full and selected-complement square conditions to construct three separate modulus factors on whole atom boxes. Appendix D verifies the remaining hypotheses of Proposition 3.2 and applies it to the finite prime-product families. The hypothesis checks in Appendix C identify these steps. Low-rank pieces and the branches without a medium slot have their separate estimates. The resulting common input for the sieve constructions is the prime-product distribution estimate in the next subsection.

The three modulus coefficients in Proposition 3.2 remain separate, and the estimate allows their signs. In the application the medium slot is a rough 1 or Möbius sequence on an individual chart, with a permitted Mellin twist. Appendix D proves the auxiliary Siegel–Walfisz condition (16) for these sequences.

When the residue has size comparable to $x$, reciprocity leaves an oscillatory phase. Appendix A keeps this phase and bounds its variation within the strict power margins of (15). The spectral estimates, including their Fourier normalizations, are stated in Appendix B.

3.2. **From the local estimate to prime products.** For a fixed finite smooth weight $W_N$ on $k$ labelled original primes, all at least $cN^{a_*}$ for fixed $a_* > 0$, put

$$X_W^{(q)} = \sum_{(p_1\cdots p_k,q)=1} W_N(p_1,\dots,p_k), \qquad X_W = \sum W_N(p_1,\dots,p_k).$$

The product is $O(N)$. The allowed geometric families have fixed polylogarithmic mixed Euler-derivative bounds and no additional arithmetic filter. The conclusion used throughout the paper is

$$\sum_q \rho_N(q) \left\{ \sum_{p_1\cdots p_k \equiv N \pmod q} W_N(p_1,\dots,p_k) - \frac{X_W^{(q)}}{\varphi(q)} \right\} = O_{A,W}\big(N(\log N)^{-A}\big). \tag{18}$$

The same signed coefficient $\rho_N$ occurs in both terms. Its admitted classes are the actual upper tree, the actual lower trees, the marked queries and ordinary descendants of the finite $H/h$ construction, and the positive switched families on which the upper coefficient is used.

The coefficient proof in Appendix C first freezes one finite causal policy. The physical level $N^\lambda$ determines its density and its residual $Q/B$. A slightly larger auxiliary support cover $N^{\ell_{\mathrm{aux}}}$ may be used to prove its factorization; it does not change $\lambda$ or the integer coefficient. For an ordered prime prefix write $t_i = \log_N p_i$ and $S_j = \sum_{i=1}^{j} t_i$, and let $I$ be the subset of slots selected for one factor of the modulus. The full-square conditions

$$S_{j-1} + 2t_j \le \ell_{\mathrm{aux}}$$

and every selected-complement square

$$\sum_{i\in I} t_i + \sum_{\substack{i<j\\ i\notin I}} t_i + 2t_j \le \ell_{\mathrm{aux}} \quad (j \notin I)$$

are retained. After all selected slots have appeared, the second test becomes the full-square test. Descendant closure and the physical prime ceiling therefore persist through the finite ordinary queries.

There is an exact tensor identity before estimating discrepancies. For a common prime-label alphabet with multiplicities $m_I$, split the slots into three groups and divide each group convolution by $M_j = \prod_I m_{I,j}!$. The resulting $\gamma_j$ is one-bounded. With $M_{\mathrm{all}} = \prod_I m_I!$, the full normalized convolution is

$$\frac{M_1M_2M_3}{M_{\mathrm{all}}}(\gamma_1 * \gamma_2 * \gamma_3).$$

Marked multiplicities remain in the external scalar. The small-prime beta factor attaches by the unique small/rough factorization. The partition is fixed on a whole atom box before any arithmetic discrepancy is inspected. Global squarefreeness remains in the physical coefficient; its removal only for analytic comparison has the fixed-power error $O((N+N^{\theta_*})N^{-\eta_0+o(1)})$,

where $N^{\eta_0}$ is the small-prime presieve cutoff and $\theta_* < 3/5$ is the support exponent of the finite coefficient construction.

Appendix D makes a smooth complete-product trim below $2N^{1-\kappa_0}$ and then an exact smooth dyadic partition. The local scale is the complete product, with

$$cN^{1-\kappa_0} \le x \le CN, \qquad c_0 x \le N \le x^{1+\kappa}.$$

It is never a residual quotient. On an original group at scale $R$, the exact order-three Heath–Brown identity is obtained from $\Lambda * (\delta - 1 * \mu_Z)^{*3} = 0$ below $2R$, with $Z = (2R)^{1/3}$:

$$\Lambda = \sum_{j=1}^{3} (-1)^{j-1} \binom{3}{j} \mu_Z^{*j} * 1^{*(j-1)} * \log. \tag{19}$$

The smooth original chart is retained; no sharp coupled group boundary is inserted. The Fourier separation of the geometric profile is charged by its integrable Fourier norm. The actual rough slots have the auxiliary SW estimate. Medium branches use Proposition 3.2; low ranks use below-half Bombieri–Vinogradov. The one-long and two-long alternatives are the direct Brun arguments of Appendix E, keeping the exact coprime mean and both Poisson zero-axis cancellations. In particular, negative lower Brun polynomials are not multiplied as though they were positive.

Choose the fixed moment order $K_{\mathrm{mom}}$ before narrowing

$$0 < \kappa \le \min\left(\kappa_*, \frac{\sigma}{200}, \frac{1}{2K_{\mathrm{mom}}}\right), \qquad 0 < \kappa_0 < \frac{\kappa}{4(1+\kappa)}.$$

Here $\kappa_* > 0$ is fixed by the strict support margins in Appendix C. The complete local support has exponent $\Theta_{\mathrm{loc}} < 3/5$; the long-branch parameter $\zeta$ is selected with $20\zeta + \Theta_{\mathrm{loc}} < 1$ and $40\zeta + 3\Theta_{\mathrm{loc}}/2 < 1$. All finite atom scalar masses and the fixed logarithmic number of charts are paid by stronger local logarithmic accuracy. This proves (18) for the admissible families and all sufficiently large even targets.

## 4. A positive upper sieve with its own density

The upper construction uses

$$\rho_0 = \frac{19}{96}, \quad \mu_0 = \frac{25}{128}, \quad \Theta = \frac{299}{500}, \quad \Lambda_{\mathrm{cap}} = \frac{153}{256}, \quad v(t) = \min\left(\frac{1+t}{2}, \frac{57}{64} - \frac{3t}{2}\right).$$

Its causal cap changes are made on the actual prefix; every retained ancestor decision is preserved. Fourth, fifth and sixth changes are successive replacements, whose raw density differences telescope. Later eighth-rank changes occupy $30 + 15 + 13 + 29 = 87$ specified domains. Same-stage domains have disjoint interiors; successive stages use differences from their respective parent caps. Appendix F supplies the cap formulas and separation proofs.

### 4.1. Exit identity and literal residual densities.

For a finite allowed prime universe, let $\xi$ be multiplicative with $\xi(1) = 1$ and $0 \le \xi(p) \le 1$. The upper tree, whose first exits have odd rank, satisfies the exact identity

$$\sum_d \mu(d)\mathbf{1}_{d \text{ retained}}\xi(d) = \prod_p (1-\xi(p)) + \sum_{e \text{ odd first exit}} \xi(e) \prod_{p<p_{\mathrm{last}}(e)} (1-\xi(p)). \tag{20}$$

Partition the excluded divisors by their first excluded prefix; the remaining smaller-prime subsets give the displayed Euler product. Taking $\xi(p) = \mathbf{1}_{p|n}$ proves a rough-indicator majorant for every integer $n$, including zero and negative integers.

For the raw density, take $g_N(p) = \mathbf{1}_{p>2,\, p\nmid N}/(p-1)$ and $V_N(y) = \prod_{2<p<y,\, p\nmid N}(1-1/(p-1))$. At a fixed rank the normalized exit for $g_N$ lies between zero and the corresponding exit for $g_0(p) = \mathbf{1}_{p>2}/(p-1)$: a surviving prefix acquires the multiplier

$$\prod_{\substack{p_{\mathrm{last}} \le p < y \\ p|N}} (1 - g_0(p)) \le 1.$$

The preceding square tests bound the last prime of a rank-$j$ exit below by $N^{\beta_{\min}/(3\cdot 2^{j-1})}$ on positive compact parameter sets. Deleting target primes thus changes each fixed-rank integral by $o(1)$ uniformly in $N$. Positive exit-series convergence, the one-sided bounds for the same raw Rosser weights, and then a finite-rank sandwich yield

$$\frac{V_N^+(R,y)}{V_N(y)} = F(s)+o(1) \quad (s>1), \qquad \frac{V_N^-(R,y)}{V_N(y)} = f(s)+o(1) \quad (s\ge 2), \quad s=\frac{\log R}{\log y}. \tag{21}$$

The upper identity for $1 < s \le 3$ also follows directly by moving its first cube to $R^{1/3}$. This computes the density of the divisor coefficient itself. The ordinary exit-series inputs and their normalization are specified in Appendix F.

Mertens gives, uniformly in the target,

$$V_N(N^u) = \frac{2e^{-\gamma}C_N}{u\log N}(1+o(1)). \tag{22}$$

For example the exact ratio to $2C_N\prod_{p<z}(1-1/p)$ is

$$\prod_{p\ge z}\left(1-\frac{1}{(p-1)^2}\right)^{-1}\prod_{\substack{p\mid N\\ p\ge z}}\left(1-\frac{1}{p-1}\right),$$

whose logarithm tends uniformly to zero at $z = N^u$, $u > 0$ fixed. With $\overline{F} = F/(2e^\gamma)$ and $\overline{f} = f/(2e^\gamma)$, this is the normalization producing $4\overline{F}(s)/u$ and $4\overline{f}(s)/u$ in the density functionals.

For a selected eighth cell, if $S_8 = t_1 + \cdots + t_8$ and the parent is bounded above by $R_6$, its cap $M$ satisfies

$$1 < \frac{\lambda_{6,\mathrm{parent}} - S_8}{t_8} \le \frac{R_6 - S_8}{t_8} < \frac{M - S_8}{t_8} < 3.$$

The same residual identity therefore gives the lower saving

$$\frac{4(M-R_6)}{(R_6-S_8)(M-S_8)\prod_{i=1}^8 t_i}. \tag{23}$$

Using the larger parent $R_6$ is conservative. This density has no Chen quarter or ordering factorial. The selected domains have $t_8 > 53/13824$, and all full and selected-complement squares remain in force.

4.2. **Finite realization and density errors.** The ideal policy is approximated on a finite causal partition before $N$ grows. Small-last-prime tails tend to zero; proper affine walls have vanishing mass, whereas open inherit/equality regions are retained. At even rough rank take outer endpoint tests and at odd rank inner tests, preserving ancestry. This produces a coefficient $a_N(d)$ above the raw upper coefficient. Attach the small-prime beta upper polynomial to positive $a_N(d)$ and the beta lower polynomial to negative $a_N(d)$. If $B^- \le I_w \le B^+$, then $a_N(d)B_{\mathrm{sign}}(n) \ge a_N(d)I_w(n)$ term by term. The resulting physical $\rho_N^{\mathrm{up}}$ has unique small/rough decomposition, squarefree support $q \le N^{\theta_*}$, a uniform bound $|\rho_N^{\mathrm{up}}(q)| \le M$ for fixed $M$, $(q,N)=1$, and

$$R_N(n) := \sum_{q\mid n}\rho_N^{\mathrm{up}}(q) \ge \mathbf{1}_{(n,P_N(N^{\rho_0}))=1} \ge 0, \qquad P_N(y) = \prod_{\substack{p<y\\ p\nmid N}} p. \tag{24}$$

The finite sum also defines $R_N(0)$.

Restricting (20) to primes above the presieve gives the upper projection inequality

$$V_N(w)\sum_{d\ \mathrm{rough}}\lambda_N(d)g_N(d) \le \sum_{d\ \mathrm{full}}\lambda_N(d)g_N(d).$$

Together with the absolute beta and box-error estimates this controls the density of the same $\rho_N^{\mathrm{up}}$ from above. Support inflation can be kept below $1/1000$, giving $\theta_* = 599/1000$ before

the small complete-scale adjustment. The auxiliary support cover used in Section 3 does not change this density.

Let $E_{\mathrm{up}}$ be the raw upper-density endpoint and let $u$ be each of the two allowances for finite approximation and coefficient realization. Their exact rational values are given in (182)–(183). The directed estimates in Appendix I give

$$K - E_{\mathrm{up}} - 2u > 9.6522 \cdot 10^{-8} > 0. \tag{25}$$

Thus the physical coefficient in (24) satisfies

$$S_N := \sum_q \frac{\rho_N^{\mathrm{up}}(q)}{\varphi(q)} \le (K + o(1)) \frac{C_N}{\log N}, \tag{26}$$

and it is precisely the coefficient to which (18) applies.

*Proof of Theorem 2.1.* Use the finite coefficient constructed above. Equation (24) gives its bounded squarefree support and (12), and (26) gives its density bound. The factorization and parameter choices in Appendix C, followed by the prime-product estimate in Appendix D, establish (18) for this coefficient and each weight in the fixed finite collection. Choose their common refinement before taking $N$ to infinity. Taking maxima of the finitely many constants and thresholds gives the asserted uniformity. □

**Proposition 4.1** (Positive product consequence)**.** *Fix a finite nonnegative labelled prime-product family of bounded rank and uniformly bounded weights, with every original prime at least $N^{a_*}$ for some fixed $a_* > 0$, product $O(N)$, and only the admitted fixed geometric restrictions. Suppose its unsifted weighted mass $X_W$ satisfies*

$$\limsup_{\substack{N\to\infty \\ 2|N}} \frac{\log N}{N} X_W \le J < \infty.$$

*Let $W_N(m)$ denote the induced weight on the product $m$. Then*

$$\limsup_{\substack{N\to\infty \\ 2|N}} \frac{1}{H_N} \sum_{\substack{m<N \\ N-m \text{ prime}}} W_N(m) \le KJ.$$

*The error estimate is absolute also when $J = 0$.*

*Proof.* For a prime $\ell$, squarefree support and boundedness give

$$\sum_{q:\ell|q} \frac{|\rho_N^{\mathrm{up}}(q)|}{\varphi(q)} \ll \frac{\log N}{\ell - 1}.$$

Hence, by a union bound over the fixed number of original prime slots,

$$\left| \sum_q \frac{\rho_N^{\mathrm{up}}(q)}{\varphi(q)} (X_W^{(q)} - X_W) \right| \ll X_W N^{-a_*} \log N = o(H_N). \tag{27}$$

No factor $N^{\theta_*}$ is introduced. Small prime remainders cost $N^{\rho_0 + o(1)}$. The smooth complete-product trim costs $N^{1-\kappa_0+o(1)}$ after the full modulus sum; its product is strictly below $N$, so no $\tau(0)$ occurs. Positive majorization before distribution gives

$$\sum_{m<N,\, N-m \text{ prime}} W_N(m) \le \sum_m W_N(m) R_N(N-m) + o(H_N) \le X_W S_N + o(H_N).$$

Outer approximations with $m \ge N$ are harmless because $R_N(n) \ge 0$ for every integer. Now apply (26). Fixed geometric boundaries are handled by positive smooth outer approximation, with $N$ tending to infinity before the approximation error tends to zero. At each precision the same $\rho_N^{\mathrm{up}}$ is used in the pointwise inequality, density and distribution formula. It need not be the same coefficient at different precisions. □

## 5. The full Chen weight and paired deletion

We use $a_{\mathrm{s}}, h_{\mathrm{s}}, d_{\mathrm{s}}$ for the fixed sieve cutoffs; the parameter $a$ in Proposition $(1+a)$ retains its introductory meaning. Put

$$a_{\mathrm{s}} = \frac{417}{5000}, \qquad h_{\mathrm{s}} = \frac{19}{125}, \qquad d_{\mathrm{s}} = \frac{59}{250}, \qquad \mu_0 = \frac{25}{128}.$$

For $n \le N$ define $Q_d(y;n) = \mathbf{1}_{d\mid n}\mathbf{1}_{P^-(n/d)\ge y}$. All prime prefixes below have distinct entries. Suppressing the argument $n$, the eleven terms are

$$\begin{aligned}
Y_1 &= Q_1(N^{a_{\mathrm{s}}}),\\
Y_2 &= Q_1(N^{h_{\mathrm{s}}}),\\
Y_3 &= \sum_{N^{a_{\mathrm{s}}}\le p<N^{1/3}} Q_p(N^{a_{\mathrm{s}}}),\\
Y_4 &= \sum_{N^{a_{\mathrm{s}}}\le p<N^{d_{\mathrm{s}}}} Q_p(N^{a_{\mathrm{s}}}),\\
Y_5 &= \sum_{N^{a_{\mathrm{s}}}\le p_1<p_2<N^{h_{\mathrm{s}}}} Q_{p_1p_2}(N^{a_{\mathrm{s}}}),\\
Y_6 &= \sum_{N^{a_{\mathrm{s}}}\le p_1<N^{h_{\mathrm{s}}}\le p_2<N^{d_{\mathrm{s}}}} Q_{p_1p_2}(N^{a_{\mathrm{s}}}),\\
Y_7 &= \sum_{N^{d_{\mathrm{s}}}\le p_1<p_2<(N/p_1)^{1/2}} Q_{p_1p_2}(p_2),\\
Y_8 &= \sum_{N^{a_{\mathrm{s}}}\le p_1<N^{1/3}\le p_2<(N/p_1)^{1/2}} Q_{p_1p_2}(p_2),\\
Y_9 &= \sum_{N^{h_{\mathrm{s}}}\le p_1<N^{d_{\mathrm{s}}}\le p_2<(N/p_1)^{1/2}} Q_{p_1p_2}((N/(p_1p_2))^{1/2}),\\
Y_{10} &= \sum_{N^{a_{\mathrm{s}}}\le p_1<p_2<p_3<p_4<N^{h_{\mathrm{s}}}} Q_{p_1p_2p_3p_4}(p_2),\\
Y_{11} &= \sum_{N^{a_{\mathrm{s}}}\le p_1<p_2<p_3<N^{h_{\mathrm{s}}}\le p_4<N^{d_{\mathrm{s}}}} Q_{p_1p_2p_3p_4}(p_2).
\end{aligned}$$

In particular $Y_{11}$ has its full last range. Set

$$W = 3Y_1 + Y_2 - Y_3 - Y_4 + Y_5 + Y_6 - 2Y_7 - Y_8 - Y_9 - Y_{10} - Y_{11}. \tag{28}$$

The terms $3Y_1 + Y_2$ supply the rough-count baseline at the two cutoffs. The one-prefix subtractions $Y_3, Y_4$ remove overcounting, while $Y_5, Y_6$ restore the selected two-prefix contributions. The terms $2Y_7, Y_8, Y_9$ subtract the remaining three-or-more-factor patterns, and the four-prefix terms $Y_{10}, Y_{11}$ compensate the growth of the positive pair counts when several small factors occur. Their combination gives the pointwise inequality below. The later paired deletion links $Y_6$ and $Y_{11}$ by an injection, so that the change preserves that inequality.

**Lemma 5.1** (Pointwise full-weight inequality)**.** *For squarefree $n \le N$,*

$$W(n) \le 4\mathbf{1}_{\Omega(n)\le 2}.$$

*Proof.* If $n$ is not $N^{a_{\mathrm{s}}}$-rough, all eleven terms vanish. For $Y_9$, the only less immediate case, its quotient cutoff exceeds $N^{(1-d_{\mathrm{s}})/4} > N^{a_{\mathrm{s}}}$. Suppose therefore that $n$ is squarefree and $N^{a_{\mathrm{s}}}$-rough. Let $r, s, t$ count its prime factors below $N^{h_{\mathrm{s}}}, N^{d_{\mathrm{s}}}, N^{1/3}$, respectively. Its first six counts are

$$(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = \left(1, \mathbf{1}_{r=0}, t, s, \binom{r}{2}, r(s-r)\right).$$

If $r \ge 2$, the quotient sifted at the second selected prime forces the first two primes in $Y_{10}$ and $Y_{11}$ to be the two least prime factors of $n$. Thus

$$Y_{10} = \binom{r-2}{2}, \qquad Y_{11} = (r-2)(s-r),$$

with the binomial zero at $r = 2, 3$. Substitution in the first six weighted terms minus $Y_{10}+Y_{11}$ gives $s - t \le 0$; the other subtractions help.

For $k = \Omega(n) \ge 3$, let $q_1 < q_2 < \cdots$ be the factors of $n$. Then $q_1 q_2^2 < n \le N$, so $t \ge 1$, $Y_7 = \mathbf{1}_{s=0}$ and $Y_8 = \mathbf{1}_{t=1}$. When $r \le 1$, the remaining upper bounds for $W$ are

| case | upper bound |
|---|---|
| $r = 1$ | $2 - t - \mathbf{1}_{t=1}$ |
| $r = 0,\ s \ge 2$ | $4 - s - t$ |
| $r = s = 0$ | $2 - t - \mathbf{1}_{t=1}$ |
| $r = 0,\ s = 1,\ t \ge 3$ | $3 - t$ |
| $r = 0,\ s = 1,\ t \le 2$ | $3 - t - 1 - \mathbf{1}_{t=1}$. |

In the last case, $k \ge 4$ would imply $n \ge N^{h_s+d_s+2/3} > N$, so $k = 3$; moreover $q_1 q_2 q_3^2 \ge N^{h_s+d_s+2/3} > N$. The pair $(q_1, q_2)$ therefore contributes to $Y_9$, giving the extra subtraction displayed there. All the bounds are nonpositive. For $k \le 2$ direct substitution gives $W \le 4$; the unit has $W(1) = 4$. □

Let $\mathscr{E} = \mathscr{E}_0 \cup \mathscr{E}_1$ be the disjoint union of the rational rectangles specified in Appendix I. The two families have 6091 and 11481 members, respectively, in the coordinates

$$(t, u) = (\log_N p_2, \log_N p_1), \qquad a_s \le u < h_s \le t < d_s.$$

The rectangles have disjoint interiors; shared edges have zero limiting mass and are assigned once in the finite partition. Write $Y_6(\mathscr{E})$ for its pair terms and $Y_{11}(\mathscr{E})$ for the four-prefix terms whose last two selected primes $(p_3, p_4)$ have $(\log_N p_4, \log_N p_3) \in \mathscr{E}$. The weight is

$$W_{\mathscr{E}} = W - Y_6(\mathscr{E}) + Y_{11}(\mathscr{E}). \tag{29}$$

**Lemma 5.2** (Paired injection)**.** *For squarefree $n \le N$, $Y_{11}(\mathscr{E}) \le Y_6(\mathscr{E})$, and hence $W_{\mathscr{E}}(n) \le 4\mathbf{1}_{\Omega(n)\le 2}$.*

*Proof.* Map a contributing four-prefix tuple to its last pair $(p_3, p_4)$. The quotient roughness forces its first two selected primes to be the two least factors of $n$. The image lies in the prescribed rectangle, and removing $p_3 p_4$ leaves an $N^{a_s}$-rough residual, so this pair contributes to $Y_6(\mathscr{E})$. Since the first two factors are determined by $n$, two tuples with the same last pair coincide. The map is therefore injective. □

The fixed prime-factor bound $\lfloor 1/a_s \rfloor = 11$ bounds $|W_{\mathscr{E}}|$ by 1016 on rough $n$. The number of $n \le N$ with a repeated prime at least $N^{a_s}$ is at most $2N^{1-a_s}$ for large $N$. After restricting to primes $p < N$ with $(p, N) = 1$ and removing the unit, the squareful and unit costs are at most $2032N^{1-a_s} + 4 = o(H_N)$. It follows that

$$4D_{1,2}(N) \ge \sum_{\substack{p<N \\ p \text{ prime} \\ (p,N)=1}} W_{\mathscr{E}}(N-p) - o(H_N). \tag{30}$$

## 6. The lower sieve and finite double-sieve transitions

6.1. **A lower estimate for the original prime sequence.** The ordinary lower trees start at $w = a_s$ and $w = h_s$, with cap $\nu = 249/448$ and fixed small buffers. Their first exits have even rank. The analogue of (20) is

$$\sum_d \mu(d)\mathbf{1}_{d \text{ retained}}\xi(d) = \prod_p (1 - \xi(p)) - \sum_{e \text{ even first exit}} \xi(e) \prod_{p<p_{\text{last}}(e)} (1 - \xi(p)). \tag{31}$$

This identity proves both the pointwise lower inequality and the literal raw-density formula. The selected fourth-rank restorations preserve even first exits: each cap is chosen before the fourth cube test $S_3 + 3t_4 < \lambda_4$, and the ordinary residual level exceeds $p_4^2$. A restored slab and a fifth-rank saving are assigned disjoint regions in the lower functional.

For a presieve $z_0 = N^{\eta_0}$, even exits give the exact projection direction

$$V_N(z_0)R_{\mathrm{rough}} - V_{\mathrm{full}} = \sum_{\substack{e \text{ full even exit} \\ p_{\mathrm{last}}(e) < z_0}} g_N(e) V_N(p_{\mathrm{last}}(e)) \geq 0. \tag{32}$$

Round positive even coefficients inward and negative odd coefficients outward, retaining the prefix decisions. Attach beta-minus to positive coefficients and beta-plus to negative coefficients. If the rounded coefficient is $a_{\mathrm{rnd}}(d)$, then $a_{\mathrm{rnd}}(d)B_{\mathrm{sign}}(n) \leq a_{\mathrm{rnd}}(d)I_0(n)$ term by term. Therefore the physical lower coefficient satisfies

$$\sum_{q|n} \rho_N^{\mathrm{low}}(q) \leq \mathbf{1}_{(n,P_N(N^w))=1}, \qquad \sum_q \frac{\rho_N^{\mathrm{low}}(q)}{\varphi(q)} \geq V_{\mathrm{full}} - \text{coefficient-realization error.}$$

Beta-minus need not be nonnegative. The second inequality uses (32) and absolute error control, not coefficient order alone. The coefficient theorem supplies (18) for this very lower coefficient.

The original prime sequence has a sharp endpoint at $N$. To use the smooth distribution estimate for a signed lower coefficient, we first control the endpoint error in absolute value.

**Lemma 6.1** (Sharp original-prime endpoint)**.** *Suppose the actual coefficient satisfies $|\rho_N^{\mathrm{low}}(q)| \leq M$ with fixed $M$, is supported on squarefree $q \leq N^\theta$ with $\theta < 1$ and $(q, N) = 1$, and satisfies the permitted smooth-singleton instance of (18). Then*

$$\sum_q \rho_N^{\mathrm{low}}(q) \left( \pi(N; q, N) - \frac{\pi(N)}{\varphi(q)} \right) = O_A(N(\log N)^{-A}). \tag{33}$$

*Proof.* Smooth the endpoint over $[N - \delta N, N + \delta N]$, with $\delta = (\log N)^{-B}$ and $B > A + 3$, and apply the smooth singleton estimate. The absolute boundary and exact-coprime mean estimates in Appendix H bound the difference by $O(\delta N \log N + \log^2 N)$, together with the fixed-power complete product trim. The equality $p = N$ is impossible for even $N > 2$. These errors are absorbed by the chosen logarithmic accuracy. □

Apply the pointwise lower coefficient to the original prime sequence. Lemma 6.1 bounds its remainder, while (21) and the prime number theorem evaluate its main term. This gives the ordinary lower prime counts used below.

6.2. **Finite defects and actual descendants.** Put $A_+(s) = sF(s)/(2e^\gamma)$ and $A_-(s) = sf(s)/(2e^\gamma)$. For a nonnegative finite marked family $\sigma(B)$ and one common $Q = N^\lambda$, normalize by

$$\Theta_Q = 4\,\mathrm{li}(N) \sum_B \frac{\sigma(B)C_{BN}}{\varphi(B)\log(Q/B)}.$$

Here $C_{BN}$ is the same singular-factor function as in (1). A defect $H(s)$ means an upper bound $(A_+(s) - H(s) + o(1))\Theta_Q$; a defect $h(s)$ means a lower bound $(A_-(s) + h(s) + o(1))\Theta_Q$. These are estimates averaged over the positive marked family, not pointwise in $B$ and not assertions for a signed outer measure.

For residual exponent $L = \lambda - \log_N B$, write $\alpha = (1 - \log_N B)/L \geq 2$. A singleton child with exponent $Lu$ has

$$L' = L(1 - u), \qquad \alpha' = 1 + \frac{\alpha - 1}{1 - u};$$

a two-prime child replaces $u$ by $u + v$. All children retain the same total $Q$, every full and selected square, and their physical prime ceiling. The parameter $\alpha$ increases and is never reset to 2 in a switched mass.

Lemma G.1 proves the initial defect $J_0 = 1/50$ and the positive transition $J \mapsto 0.020249 + 0.13J$ on the permitted marked states with $1 \le s \le 11/5$. Its hypotheses retain the residual level, the prefix square conditions and a common comparison cutoff. Four applications give $J_4 = 0.023273777353 > 2327/100000$; this is the seed for the finite recurrence below. The finite collection consists of 70 one-step and 36 two-step inequalities, together with the positive Buchstab transitions

$$H_{\mathrm{new}}(s) = \int_{\max(2,s-1)}^{5} \frac{h(w)}{w}\,dw, \qquad h_{\mathrm{new}}(s) = \int_{s-1}^{5} \frac{H(w)}{w}\,dw \quad (s \ge 2).$$

The one-step switched cost is $KI/8$, where $I$ is the nonnegative switched mass of the action specified in Appendix G. A two-step action has the positive defect contribution

$$\frac{1}{5}\left[4H(s_0) + H(\phi_1) + \sum_{\mathcal{D}_1} \int \frac{h(\rho)}{u(1-u)}\,du + \sum_{\mathcal{D}_2} \iint \frac{H(\rho)}{uv(1-u-v)}\,du\,dv\right] \tag{34}$$

in addition to its proved ordinary and positive-switching base. There are three singleton and four good pair domains. With $a_i = 1/\phi_i$ and $a_0 = 1/s_0$, the pair domains are

| | domain | $\rho_i(u,v)$ |
|---|---|---|
| $D_5^+$ | $a_0 < u < v < a_2,\ v + 2u < 1$ | $s_0(1-u-v)$ |
| $D_6$ | $(a_0, a_1) \times (a_2, a_3)$ | $s_0(1-u-v)$ |
| $D_7$ | $a_0 < u < v < a_1$ | $(1-u-v)/u$ |
| $D_8$ | $(a_0, a_1) \times (a_1, a_2)$ | $(1-u-v)/u.$ |

The admissibility conditions include $2v < 1$ and $v + 2u < 1$. The excluded portion of $D_5$ receives only a positive product cost, not ordinary feedback. Unit surfaces in the switching terms stay in their masses. The envelope used for every actual $\alpha \ge 2$ is $\omega(s) \le \max(1/s, 70893/125000)$ for $s \ge 1$, with the conservative extension below 1 specified in Appendix G.

Two levels of iteration. The four scalar steps produce the certified constant used to initialize the arrays; they precede generation zero of the array recurrence. The indices in $H_m, h_m$, including $m = 248$ and 430, count only simultaneous updates of the two cell arrays from that initialization. Thus the scalar seed iteration and the envelope generations have different state spaces and iteration counts.

We define the envelopes by simultaneous finite induction. Each right-hand side uses the preceding generation, takes the maximum of alternative estimates, and uses a minimum over the full closed support when an argument crosses a cell boundary. The 200 closed cells have endpoints $1 + i/50$, $0 \le i \le 200$. Let $H_m, h_m$ be the step-function lower envelopes after $m$ generations; at a shared endpoint take the smaller of the adjacent values. Starting directly from the seed, 430 rounds determine all 400 final entries. These generation-430 arrays are used only in $U_3, U_4$; the remaining occurrences use generation 248.

The finite induction also needs a common approximation, not just finite depth. Positive transition mass is at most 12 and the root coefficient norm at most 7/2. A local oriented error $\delta$ propagates by at most $\delta(12^{441} - 1)/11$ before normalization. Choose $\delta$ inside the prescribed final reserve and use one nested finite refinement for all queries. Residual exponents after the finite transitions remain above a fixed positive multiple of $101^{-440}$, so every positive leaf satisfies the fixed-cutoff hypothesis of Proposition 4.1. Zero-mass cells are controlled by absolute errors.

6.3. **Evaluation of the eleven-term weight.** The lower functional is

$$\mathcal{C} = \frac{1}{4}(3L_1 + L_2 + L_5 + L_6 - U_3 - U_4 - 2U_7 - U_8 - U_9 - U_{10} - U_{11}). \tag{35}$$

The lower starts are the causal starts at $a_{\mathrm{s}}, h_{\mathrm{s}}$. The ordinary estimate at $h_{\mathrm{s}}$ is inserted after its full least-prime correction is removed from the reference expression. In $U_3, U_4$ the $H_{430}$ argument is $\max(2, L/a_{\mathrm{s}})$ with $L = v(t) - t$ in the enhanced range. The region $L/a_{\mathrm{s}} \le 2$

has positive measure and uses the value of the lower envelope at 2. The pair lower terms use $h_{248}$ where $L/a_{\mathrm{s}} \geq 2$ and zero below 2. The positive costs $U_7, U_8, U_9$ retain their labelled multiplicities and are estimated by Proposition 4.1.

For $U_{10}, U_{11}$, work on one common positive partition of the four-prefix domain, with decreasing coordinates $x > y > z > w$. At each cell use the minimum of the switched kernel $K\omega((1-S_4)/z)/z$ and the admissible ordinary kernels $4(1-H_{248}(\max(2, L/w)))/L$. Each ordinary alternative retains all full and selected square conditions. A refined estimate replaces its parent estimate, and overlapping alternatives are compared rather than added. Appendix G gives the states and their common prefix-preserving approximation.

The restriction in (29) requires an estimate of the lower functional as well as the pointwise injection. On a pair rectangle $\mathscr{B} = [t_l, t_r] \times [u_l, u_r]$, put

$$R(t,u) = \frac{57}{64} - \frac{5}{2}t - u, \qquad R_{\max} = R(t_l, u_l), \quad s_{\max} = R_{\max}/a_{\mathrm{s}}, \quad M_{\mathscr{B}} = \log(t_r/t_l)\log(u_r/u_l).$$

Write $\mathcal{L}_{\mathscr{B}}$ for the cellwise lower estimator assigned to a rectangle $\mathscr{B}$. It vanishes on $\mathscr{E}_0$. In $\mathscr{E}_1$ there are 227 crossing cells on which it vanishes and 11254 positive cells with $2 \leq s_{\min} \leq s_{\max} \leq 9/4$. Since $\log(s-1)/s$ is increasing there, and $h_{\mathscr{B}}^{\min}$ denotes the cellwise $h_{248}$ minimum over the full closed argument support, the discarded lower estimator, in Chen-quarter units, is bounded above by

$$\sum_{\mathscr{B}\in\mathscr{E}_1 \text{ positive}} \frac{M_{\mathscr{B}}}{R_{\max}}(\log(s_{\max}-1) + h_{\mathscr{B}}^{\min}) \leq 0.002372094. \tag{36}$$

For the negative term, the same selected parents are switched in the 48-division four-prefix cover and avoid the refinement on 40000 parent cells. If $A_{\mathscr{B}}$ is the reference density mass and $s_{\mathscr{B}}$ its fixed $K$ feedback slope, the cellwise bounds in Appendix I give $0 \leq s_{\mathscr{B}} \leq A_{\mathscr{B}}/K_0$, $K_0 = 6.7619$. Therefore

$$A'_{\mathscr{B}} = A_{\mathscr{B}} - (K_0 - K)s_{\mathscr{B}} \geq (K/K_0)A_{\mathscr{B}}.$$

Dividing by its positive reference mass gives an explicit pointwise density majorant of the switched kernel. Restricting this majorant to the complementary positive family removes at least $KI_{\mathscr{E}}/8$ in Chen units, where

$$I_{\mathscr{E}} = \int_{(t,u)\in\mathscr{E}} \int_{a_{\mathrm{s}}<w<z<u} \frac{dt\,du\,dz\,dw}{tuz^2w}. \tag{37}$$

The use of $\omega \geq 1/2$ is valid since $(1-S_4)/z \geq 77/38 > 2$ throughout this family. For one rectangle the integral is exactly

$$\log(t_r/t_l)\left[\frac{\log(u_r/u_l)}{a_{\mathrm{s}}} + \frac{\log(u_r/a_{\mathrm{s}})+2}{u_r} - \frac{\log(u_l/a_{\mathrm{s}})+2}{u_l}\right].$$

The finite lower bounds for the gross improvements are 0.002095161 on $\mathscr{E}_0$ and 0.003799514 on $\mathscr{E}_1$. Subtracting (36), the additional raw improvement from $\mathscr{E}_1$ is

$$0.003799514 - 0.002372094 = 0.001427420 = \frac{71371}{50000000}. \tag{38}$$

The paired restriction removes mass from the positive majorant and pays for the assigned lower estimators on the same deleted region. The masks $\mathscr{E}_0, \mathscr{E}_1$ are disjoint.

6.4. **A preliminary lower functional.** Denote the current lower endpoint by $R_{\mathrm{base}}$ and the bound for each finite-density and coefficient-realization error by $B_{\mathrm{base}}$; see (184)–(185). There are two further improvements on $\mathscr{E}$: a larger lower bound for $\omega$ on an inner fibre, and a smaller payment for the discarded lower estimator. We treat them separately before combining their contributions.

6.5. **The paired-domain improvement.**

**Lemma 6.2** (A global Buchstab lower bound)**.** *For $s \geq 1$ one has $\omega(s) \geq 1/2$, and for $s \geq 7/3$ one has $\omega(s) \geq 11/20$.*

*Proof.* The bound $1/2$ holds on $[1,2]$ and propagates by $(s(\omega(s) - 1/2))' = \omega(s-1) - 1/2$. For $7/3 \leq s \leq 3$, use $\log x \geq 2(x-1)/(x+1)$ for $x \geq 1$ to get

$$\omega(s) = \frac{1 + \log(s-1)}{s} \geq \frac{3s-4}{s^2} \geq \frac{11}{20}.$$

The last inequality follows because the concave polynomial $60s - 80 - 11s^2$ is positive at both endpoints. For $3 \leq s \leq 10/3$, the first bound and $\log 2 \geq 2/3$ give

$$s\omega(s) \geq 1 + \log 2 + (s-3)/2 \geq s/2 + 1/6 \geq 11s/20.$$

This establishes the stronger bound on the unit interval $[7/3, 10/3]$. The identity $(s(\omega(s) - 11/20))' = \omega(s-1) - 11/20$ then propagates it to successive unit intervals. □

The extra positive fibre. The already selected parent majorant dominates $K\omega((1-t-u-z-w)/z)/z$ on $a_{\mathrm{s}} < w < z < u < h_{\mathrm{s}} < t < d_{\mathrm{s}}$. For each rectangle $\mathscr{B} = [t_l, t_r] \times [u_l, u_r]$ of the same mask, put

$$Z_{\mathscr{B}} = \min\left(u_l, \frac{3(1 - t_r - u_r)}{13}\right). \tag{39}$$

The finite exact geometry check gives $Z_{\mathscr{B}} > a_{\mathrm{s}}$ for every cell. Moreover the definition gives $1 - t_r - u_r - 2Z_{\mathscr{B}} \geq (7/3)Z_{\mathscr{B}}$. Hence on the inner fibre $a_{\mathrm{s}} < w < z < Z_{\mathscr{B}}$ one has $z < u$ and

$$1 - t - u - z - w \geq 1 - t_r - u_r - 2z \geq \frac{7}{3}z.$$

Lemma 6.2 supplies an additional $1/20$ in $\omega$, beyond the $1/2$ already paid in (37). The remainder of every fibre keeps its original bound. The two nonnegative contributions hold simultaneously.

Integrating the added $1/20$ and including the Chen factor $1/4$ gives at least the following additional coefficient from $\mathscr{B}$:

$$\begin{aligned} G_{\mathscr{B}} &= \frac{K}{80} \log\frac{t_r}{t_l} \log\frac{u_r}{u_l} \int_{a_{\mathrm{s}}}^{Z_{\mathscr{B}}} \frac{\log(z/a_{\mathrm{s}})}{z^2}\, dz \\ &= \frac{K}{80} \log\frac{t_r}{t_l} \log\frac{u_r}{u_l} \left[\frac{1}{a_{\mathrm{s}}} - \frac{1 + \log(Z_{\mathscr{B}}/a_{\mathrm{s}})}{Z_{\mathscr{B}}}\right]. \end{aligned} \tag{40}$$

The inner $w$ integral is $\log(z/a_{\mathrm{s}})$, and the remaining primitive is $-(1 + \log(z/a_{\mathrm{s}}))/z$. Thus there is no permutation factor and no extra factor 2. The fixed rational walls in (39) remain within the same admissible finite positive family. Restriction uses the positive parent majorant and its assigned cell decomposition.

The directed interval estimate of Appendix J gives

$$\sum_{\mathscr{B} \in \mathscr{E}_0 \cup \mathscr{E}_1} G_{\mathscr{B}} > \frac{558583}{10^9} = 0.000558583 =: g. \tag{41}$$

Only this rational lower endpoint is used below.

The discarded positive lower estimator. The lower estimator assigned to a rectangle $\mathscr{B}$ is the maximum of its constant and chord estimates for the ordinary term, plus its nonnegative double-sieve defect. The complete sum on $\mathscr{E}$, including the Chen factor $1/4$, is denoted by $L_{\mathscr{E}}$. Its rational fixed-point evaluation gives

$$L_{\mathscr{E}} = \frac{5645403422587826010135}{24178516392292583494123 52} < \frac{466977}{200000000} = 0.002334885 =: \lambda_{\mathrm{loss}}. \tag{42}$$

We next bound this discarded lower functional directly by rational intervals.

To specify the enclosure, on a positive cell use the residual function $R(t,u) = 57/64 - (5/2)t - u$ used in the preliminary functional, and put

$$r_{\mathscr{B}} = R(t_r,u_r), \quad R_{\mathscr{B}} = R(t_l,u_l), \quad s_l = r_{\mathscr{B}}/a_{\mathrm{s}}, \quad s_h = R_{\mathscr{B}}/a_{\mathrm{s}},$$
$$f_l = \frac{\log(s_l-1)}{s_l}, \qquad f_h = \frac{\log(s_h-1)}{s_h}.$$

Here $2 \le s_l \le s_h \le 9/4$. Define

$$M_{\mathscr{B}} = \log(t_r/t_l)\log(u_r/u_l),$$
$$M_{R,\mathscr{B}} = \int_{\mathscr{B}} R(t,u)\frac{dt\,du}{tu}$$
$$= \frac{57}{64}M_{\mathscr{B}} - \frac{5}{2}(t_r-t_l)\log(u_r/u_l) - (u_r-u_l)\log(t_r/t_l).$$

The two ordinary alternatives, after division by the Chen factor 4, are bounded above by

$$\frac{f_l M_{\mathscr{B}}}{a_{\mathrm{s}}}, \qquad \frac{f_l(R_{\mathscr{B}}M_{\mathscr{B}} - M_{R,\mathscr{B}}) + f_h(M_{R,\mathscr{B}} - r_{\mathscr{B}}M_{\mathscr{B}})}{a_{\mathrm{s}}(R_{\mathscr{B}} - r_{\mathscr{B}})}. \tag{43}$$

Indeed the exact endpoint values $f_l, f_h$ dominate the lower endpoints used in the functional. Both coefficient integrals are nonnegative because $r_{\mathscr{B}} \le R(t,u) \le R_{\mathscr{B}}$, and the fixed-point lower roundings can only decrease the estimator value. The cellwise estimator takes a maximum of its constant and chord alternatives; taking the maximum of the two upper bounds in (43) preserves the direction.

For the separate defect term, take $h^{\min}_{\mathscr{B}}$ to be the minimum of the $h_{248}$ closed-cell values meeting the exact interval $[s_l, s_h]$, including both adjoining cells at a knot. This minimum is at least the one over the outward-rounded lookup interval. Therefore $h^{\min}_{\mathscr{B}} M_{\mathscr{B}}/R_{\mathscr{B}}$ is an upper bound for the cellwise defect estimator, and can be added to the preceding maximum. All relevant $h_{248}$ entries are nonnegative. If the outward lookup crosses below 2 and returns zero, the same upper bound remains valid. On the 227 crossing cells the constant, eligible chord alternatives and lookup all give zero. Every $\mathscr{E}_0$ cell has $s_h \le 2$ and likewise has zero cellwise lower estimator.

The proxy sum has a rational upper enclosure strictly below $\lambda_{\mathrm{loss}}$. Appendix K obtains it by range reduction and alternating logarithm series with explicit remainders. Thus $\lambda_{\mathrm{loss}}$ can replace the payment in (36) without weakening the lower inequality.

The recovered payment is exactly

$$0.002372094 - \lambda_{\mathrm{loss}} = \frac{37209}{10^9} = 0.000037209. \tag{44}$$

The retained $L_6$ lower endpoint is the whole $L_6$ lower endpoint minus $4\lambda_{\mathrm{loss}}$, namely

$$\frac{1421673188323272382945058469141}{472236648286964521369600000000}.$$

The total removed negative gross in Chen units is now the safe value $0.002095161 + 0.003799514 + g = 0.006453258$. Combining (41) and (44) gives the additional lower coefficient

$$\Delta = g + (0.002372094 - \lambda_{\mathrm{loss}}) = \frac{37237}{62500000} = 0.000595792. \tag{45}$$

We add $\Delta/8 = 37237/500000000 = 0.000074474$ to each of the two error allowances. The two enlarged allowances cover finite density approximation and coefficient realization, respectively. The net increment after both extra reserves is

$$\Delta - 2(\Delta/8) = \frac{3\Delta}{4} = \frac{111711}{250000000} = 0.000446844. \tag{46}$$

The walls and queries are finite, so their local accuracy can be chosen inside these positive additional allowances by the same common approximation and coefficient realization argument used for the preliminary lower functional.

6.6. **The lower endpoint and the order of limits.** With the paired-domain improvement, the raw endpoint and each realization allowance are

$$R_{\mathrm{low}} = R_{\mathrm{base}} + \Delta, \tag{47}$$

$$B_{\mathrm{low}} = B_{\mathrm{base}} + \Delta/8. \tag{48}$$

Their exact values are specified in Appendix I. From (30), the actual lower coefficient realization, finite induction, and evaluation of the paired lower functional,

$$\liminf_{\substack{N\to\infty \\ 2\mid N}} \frac{D_{1,2}(N)}{H_N} \geq R_{\mathrm{low}} - 2B_{\mathrm{low}}. \tag{49}$$

The two allowances leave

$$R_{\mathrm{low}} - 2B_{\mathrm{low}} > c + 4.9061 \cdot 10^{-9}, \tag{50}$$

so in particular

$$R_{\mathrm{low}} - 2B_{\mathrm{low}} - c > 4.9061 \cdot 10^{-9} > 0. \tag{51}$$

In particular $D_{1,2}(N) \geq cH_N$ eventually, with $c$ as in (5).

Fix the finite set of estimates and their dependencies, the tail and compact approximation tolerances, the cutoffs defining fixed compact parameter domains, the prefix caps, and the support reserves. Next choose the common beta and rough parameters, fine labels, fixed divisor and moment orders, admissible $\kappa$, and local logarithmic accuracy. Only then let $N$ tend to infinity through the even integers. Taking the maximum of the finitely many thresholds makes all estimates simultaneous. Since $C_N \geq C_2 > 0$, fixed power savings and $O_A(N/\log^A N)$ with $A > 2$ are uniformly $o(H_N)$. The finite induction therefore gives the conclusion for every sufficiently large even target.

## 7. Finite numerical estimates

The preceding upper and lower constructions reduce the numerical part of the proof to the following finite inequalities. Table 2 gives the correspondence between these quantities and the certificate fields used to verify them. Appendix I defines their integrals, partitions, directed enclosures, and error allowances.

**Lemma 7.1.** *The upper density functional has an upper endpoint $E_{\mathrm{up}}$ as in (182). It admits finite density and coefficient realizations with errors at most $u$ each, where $u$ is as in (183), and $E_{\mathrm{up}} + 2u < K$. The eleven-term lower functional with paired deletion has a lower endpoint $R_{\mathrm{low}} = R_{\mathrm{base}} + \Delta$ as in (47). The corresponding two errors are at most $B_{\mathrm{low}}$ each, where $B_{\mathrm{low}}$ is as in (48), and $R_{\mathrm{low}} - 2B_{\mathrm{low}} > c$. The generation-*430 *envelopes used in $U_3, U_4$ are generated by* 430 *simultaneous applications of the positive finite action map to the stated seed. The generation-*248 *envelopes used in the other specified terms have the separate enclosures described in Appendix I.*

*Proof.* For the upper estimate, evaluate the four terms

$$E_{\mathrm{up}} = \frac{4}{\Theta} + I_1 + I_{3,\mathrm{compact}} + I_{3,\mathrm{tail}} - \sum_{j=1}^{22} G_j,$$

using upper endpoints for the four costs and lower endpoints for the nonnegative savings $G_j$. Each successive change is measured from its parent; overlapping alternatives are maximized before integration. The noncompact tail is bounded over its entire range.

For the lower estimate, apply lower endpoints to $L_1, L_2, L_5, L_6$ and upper endpoints to $U_3, U_4, U_7, \ldots, U_{11}$ in (35). The changes of lower starts, restored exit regions, and finite-table corrections are inserted on their prescribed disjoint regions or as complete parent replacements. The paired-domain terms are then evaluated by (37), (36), (41), and (45). This gives the rational endpoint $R_{\mathrm{low}}$.

All scalar operations are rational. Logarithms are enclosed by series with explicit remainders; the sieve functions are enclosed by oriented interpolation estimates. Affine cells are triangulated using exact vertices and exact polynomial moments. Positive savings may be bounded on a subset, with the remaining saving discarded; an upper bound for a negative cost covers the full retained region. The finite recurrence reads only the preceding generation and rounds every nonnegative contribution downwards. The detailed enclosure and coverage statements are given in Appendix I. Rational subtraction gives (25) and (51). □

Numerical data. The finite partitions, recurrence, and directed integral bounds of Appendix I are supplied in *Theorem (1+1.75) on the Goldbach Conjecture — Numerical Supplement*, version 20261106 [2]. The appendix describes the data and their certificate correspondence with Lemma 7.1.

## 8. Ordered bad semiprimes and the final subtraction

We now estimate $B_b(N)$. Its dependence on $b$ is contained in the unsifted mass of the ordered semiprimes; the positive product estimate then imposes the prime complement.

Fix $1/2 < b < 1$ and set $\beta = b/(1+b)$. Products below $2N^{1-\kappa_0}$ have a negligible count. For a remaining bad product $m = rq$, the inequality $r > q^b$ implies $r > m^\beta$; both primes therefore have a fixed positive exponent. Use the coordinates

$$t = \frac{\log r}{\log N}, \qquad y = \frac{rq}{N}.$$

On a fixed radial strip $\eta \le y \le C$, PNT in $q = Ny/r$ followed by reciprocal-prime partial summation in $r$ gives the unsifted mass element

$$\frac{N}{\log N}\,\frac{dt\,dy}{t(1-t)}. \tag{52}$$

The PNT errors are uniform on positive exponent compacta. A prime upper bound controls $y < \eta$ by $O_b(\eta N/\log N)$; $N$ tends to infinity first, then $\eta$ tends to zero.

The exact ordered bad inequalities are

$$0 < y < 1, \qquad t \le \frac{1}{2} + \frac{\log y}{2\log N}, \qquad t > \beta + \frac{b\log y}{(1+b)\log N}.$$

Their limiting unsifted mass is

$$J_b = \int_0^1 dy \int_\beta^{1/2} \frac{dt}{t(1-t)} = \left[\log\frac{t}{1-t}\right]_\beta^{1/2} = \log(1/b). \tag{53}$$

The ordering $r \le q$ counts each composite once. There is no factor 2 and no further singular factor in $J_b$. The equal-prime diagonal has size $O(\sqrt{N})$ and can be retained or restored after a squarefree argument.

Smooth the complete-product wall in $y$ and the remaining walls in $t$. For any fixed $\epsilon > 0$ a positive outer approximation has limiting mass at most $J_b + \epsilon$ and stays in the admissible finite family. A partition of unity repeats no mass. Proposition 4.1, followed by $N \to \infty$ and then $\epsilon \downarrow 0$, gives

$$\limsup_{\substack{N\to\infty \\ 2\mid N}} \frac{B_b(N)}{H_N} \le K\log(1/b). \tag{54}$$

Unique factorization gives the exact identity

$$D_b(N) = D_{1,2}(N) - B_b(N). \tag{55}$$

The unit was excluded in (2); a convention including it would change the count by at most one. Zero is excluded by $p < N$. A square is a bad remainder when $b < 1$, and a prime remainder is a good one. Combining (49), (51) and (54) proves Theorem 1.2.

It remains to verify the strict scalar inequalities in Corollaries 1.3 and 1.4. For rational $b$ put $z = (1-b)/(1+b)$. The positive atanh series and its geometric tail give

$$2\sum_{j=0}^{95}\frac{z^{2j+1}}{2j+1} < \log(1/b) < L_{96}(b) := 2\sum_{j=0}^{95}\frac{z^{2j+1}}{2j+1} + \frac{2z^{193}}{193(1-z^2)}. \tag{56}$$

Every endpoint is rational. Substituting (5) gives

$$\begin{aligned} c - KL_{96}(3/4) &> \frac{16532064}{10^9} > 0.0165, \\ c - KL_{96}(74816964/10^8) &> \frac{1001149712517311}{10^{20}} > 10^{-5}, \\ c - KL_{96}(74816865/10^8) &> \frac{106416285358807}{10^{20}} > 10^{-6}, \\ c - KL_{96}(9/10) &> \frac{124934244400949969098}{10^{20}} > 1.24889. \end{aligned} \tag{57}$$

The respective surpluses exceed $3.2064\cdot 10^{-5}$, $1.1497\cdot 10^{-8}$, $6.4162\cdot 10^{-8}$ and $4.5244\cdot 10^{-4}$. First choose the fixed realization and positive-smoothing errors inside these strict margins; then let the remaining uniform asymptotic errors tend to zero. Taking the maximum of the four resulting thresholds proves both corollaries.

Range supplied by the fixed constants. For $1/2 < b < 1$, the right-hand side of (6) is positive precisely when $b > \exp(-c/K)$. This value describes the range supplied by the present pair of coefficient bounds. Extending that range requires stronger applicable lower or upper estimates, with the moving-residue and coefficient-class conditions preserved.

## Appendix A. The restricted medium convolution estimate

We prove Proposition 3.2. The convolution lengths are $U, V$, with $x = UV$, and $\mathfrak{a}$ denotes the nonzero residue. The even target is denoted by $N$ in the main text. Put $\vartheta = 7/32$ and

$$z = x^{1/(\log\log x)^3}, \qquad y = x^{1/\log\log x}.$$

The coefficients $\alpha_n$ are fixed throughout the modulus sums and satisfy the auxiliary-coprimality Siegel–Walfisz condition (16). All constants in this condition are uniform over the coefficient family. We use $e(t) = \exp(2\pi i t)$; an inverse in an exponential is always taken modulo its displayed denominator.

### A.1. **A smooth majorant and divisor estimates.**

**Lemma A.1.** *There is a fixed function $\psi \in C_c^\infty((0,\infty))$ such that $0 \le \psi \le 1$, $\psi = 1$ on $[1,2]$, $\operatorname{supp}\psi \subset [1/2, 5/2]$, and*

$$\|\psi^{(j)}\|_\infty \le C^j (j!)^2, \qquad |\widehat{\psi}(t)| \ll \exp(-c\sqrt{|t|}).$$

*Consequently, for polynomially bounded positive $\ell, V$ and $H \ge C_1 \ell (\log x)^5/V$,*

$$\frac{V}{\ell}\sum_{|h|>H}\left|\widehat{\psi}\left(\frac{hV}{\ell}\right)\right| \ll_D x^{-D} \tag{58}$$

*for every fixed $D > 0$.*

*Proof.* Choose $\eta_j = c_0/j^2$ with $\sum \eta_j = 1/2$, let $u_j$ be the uniform probability measure on $[-\eta_j/2, \eta_j/2]$, and put $\phi = u_1 * u_2 * \cdots$. Its support is $[-1/4, 1/4]$. Differentiating the first $j$ factors as signed measures gives total variation at most $\prod_{i\le j}(2/\eta_i) = C^j (j!)^2$. The remaining convolution is a probability measure; retaining further undifferentiated factors also gives a continuous density at each fixed derivative order. The function $\psi = \mathbf{1}_{[3/4,9/4]} * \phi$ has the asserted support and derivative bounds. Integration by parts $j$ times, with $j$ a sufficiently small multiple of $\sqrt{|t|}$, gives the Fourier estimate. A decreasing-integral comparison bounds

the tail in (58) by a polynomial factor times $\exp(-c_1(\log x)^{5/2})$. If $H < 1$, every omitted integer already has $|h|V/\ell \gg (\log x)^5$, and the same argument applies. □

Write $n = \log_x U$, $q = \log_x Q_1$, $u = \log_x Q_2$ and $v = \log_x Q_3$. The hypotheses of Proposition 3.2 imply

$$n > \sigma, \qquad n \le \frac{1}{2} - 4\sigma, \qquad q + u + v < \frac{7}{10} - \frac{13\sigma}{2}. \tag{59}$$

In particular $Q_1Q_2Q_3 \le x^{4/5}$ for large $x$. Split $q_2 = u_0 s$ and $q_3 = v_0 t$ into their $z$-rough and $z$-smooth parts. The terms $s > y$ or $t > y$ have arbitrarily large logarithmic saving. To see this, first collapse the modulus coefficient with its fixed divisor bound. The primitive progression bound for divisor-bounded functions [12], followed by Rankin's method with exponent $1/\log z$, gives

$$\sum_{\substack{\ell \le x^{4/5} \\ \mathrm{sm}(\ell;z) \ge y}} \frac{\tau(\ell)^C}{\ell} \ll \exp\bigl(-(\log\log x)^2 + O_C(\log\log x)\bigr). \tag{60}$$

Here $\mathrm{sm}(\ell; z)$ is the largest divisor of $\ell$ all of whose prime factors are at most $z$. The bound is used with divisor means, rather than with a pointwise factor $x^\epsilon$.

We also use the following translated progression estimate. For fixed $C, K, \delta > 0$, an interval $I \subset [-x^K, x^K]$ of length at most $L$ and $f \le Lx^{-\delta}$ satisfy

$$\sum_{\substack{v \in I \\ v \equiv r \pmod f \\ v \ne 0}} \tau(|v|)^C \ll \frac{L}{f}\tau(f)^C (\log x)^{O(1)}. \tag{61}$$

Indeed, put $g = (r, f)$ and $v = gw$. The resulting progression modulo $f/g$ is primitive and its length-to-modulus ratio is $L/f \ge x^\delta$. The range $|w| \le (f/g)x^{\delta/2}$ is bounded by a fixed divisor estimate. Split the rest by sign and into dyadic intervals, subdividing them into lengths comparable to the smaller of their scale and $L/g$. The primitive divisor bound applies on each interval with fixed positive length and modulus margins; summation proves (61). Cauchy's inequality gives its mixed version for $\tau(v)^B\tau(|v - \mathfrak{a}|)^C$, with $v = \mathfrak{a}$ excluded.

On a dyadic block put $d = st \asymp D$, $w = u_0v_0 \asymp \mathcal{R}$ and define

$$\Lambda_d(w) = \sum_{\substack{st=d \\ u_0v_0=w}} \gamma_2(u_0 s)\gamma_3(v_0 t)\mathbf{1}_{\text{fixed individual intervals and supports}}. \tag{62}$$

Thus $\Lambda_d$ is independent of $q_1$, $|\Lambda_d(w)| \le \tau(dw)$, $D \le y^2 = x^{o(1)}$, and $D\mathcal{R} \asymp Q_2Q_3$. With $Q = Q_1$, $Y = QD$ and $\delta = \sigma/3$, (59) implies

$$U, V \ge x^\delta, \qquad Y \le Ux^{-\delta}, \qquad Y\mathcal{R} \le x^{1-\delta}. \tag{63}$$

We write $\mathcal{S}$ for the original factorized discrepancy sum restricted to this block. All values of $d \asymp D$ remain in this single sum before Cauchy's inequality is applied.

We reduce to one-bounded $\beta$ before applying dispersion. For divisor-bounded $\beta$, retain $\tau(m) \le (\log x)^C$ and normalize by $(\log x)^{BC}$. The discarded coefficient at $v = nm$ is at most $(\log x)^{-C}\tau(v)^{B+2}$. For $v \ne \mathfrak{a}$, summing the moduli costs at most $\tau(|v - \mathfrak{a}|)^3$; fixed shifted divisor moments therefore give $O(x(\log x)^{E-C})$, with $E$ independent of $C$. The exact product $v = \mathfrak{a}$ contributes separately at most $x^{4/5+o(1)}$. Ordinary divisor means and the reciprocal totient sum control the mean term. Choose $C$ first, then request sufficient logarithmic accuracy for the normalized estimate.

A.2. **Dispersion and the principal variance.** The dispersion step separates a common principal term from the nonzero frequencies. A remaining nonnegative principal variance is estimated by the auxiliary Siegel–Walfisz hypothesis before the exponential sums are treated.

Set $\ell = qd$, $\eta_r = \Lambda_d(r)$ and $h_0 = \int \psi$. For fixed $(q, d)$ define

$$B_r = \sum_{(n,\ell r)=1} \alpha_n, \qquad A_r(b) = \sum_{\substack{n\equiv b \pmod{\ell} \\ (n,r)=1}} \alpha_n,$$

$$C_r(m) = \frac{\mathbf{1}_{(m,\ell r)=1}}{\varphi(\ell r)} B_r, \qquad X_r(m) = \sum_{(n,\ell r)=1} \alpha_n (\mathbf{1}_{mn\equiv \mathfrak{a} \pmod{\ell r}} - \mathbf{1}_{mn=\mathfrak{a}}).$$

All $n$ sums retain the original interval. Remove $mn = \mathfrak{a}$ before applying Cauchy's inequality. The first-moment cost is $Y\mathcal{R}x^{o(1)}$, which is power-small by (63). Nonnegative majorants for $q$ and $m$ then give a Cauchy multiplier $VY(\log x)^C$ and the exact variance

$$\begin{aligned}
\mathcal{V}' &= S_1 - 2\Re S_2' + S_3', \\
S_1 &= \sum_{q,d} \psi(q/Q) \sum_m \psi(m/V) \Big| \sum_r \eta_r C_r(m) \Big|^2, \\
S_2' &= \sum_{q,d} \psi(q/Q) \sum_m \psi(m/V) \Big( \sum_r \eta_r X_r(m) \Big) \overline{\Big( \sum_s \eta_s C_s(m) \Big)}, \\
S_3' &= \sum_{q,d} \psi(q/Q) \sum_m \psi(m/V) \Big| \sum_r \eta_r X_r(m) \Big|^2. \qquad (64)
\end{aligned}$$

The common principal term is governed by the kernel

$$c_\ell(r,s) = \frac{\varphi(\ell rs)}{\ell rs \varphi(\ell r)\varphi(\ell s)}.$$

If $L$ is any common multiple of $\ell$ and the finitely many supported $r$, then

$$c_\ell(r,s) = \frac{1}{L} \sum_{v \bmod L} \frac{\mathbf{1}_{(v,\ell r)=1}}{\varphi(\ell r)} \frac{\mathbf{1}_{(v,\ell s)=1}}{\varphi(\ell s)}. \qquad (65)$$

This is a positive Gram kernel. The size of $L$ is irrelevant because it is used only in the finite identity. Smooth coprime counting yields

$$S_1 = \mathcal{M} + O\left( \frac{U^2 (\log x)^C}{Y} \right), \qquad \mathcal{M} = V h_0 \sum_{q,d} \psi(q/Q) \sum_{r,s} \eta_r \overline{\eta_s} c_\ell(r,s) B_r \overline{B_s}. \qquad (66)$$

For $S_2'$, remove $(m,s) > 1$ while $mn \neq \mathfrak{a}$. A rough common factor $f > z$ gives factors $\mathcal{R}/f$ and $x/f$; applying (61) and summing $\sum_{f>z} \tau(f)^C/f^2$ bounds this error by $VU^2(\log x)^C/(Yz)$. Only then restore $mn = \mathfrak{a}$ for Poisson summation; the cost is $\mathcal{R}Ux^{o(1)}$. The zero frequency has coefficient $1/(\ell r \varphi(\ell s))$, whose ratio to $c_\ell(r,s)$ is $1 + O((\log x)/z)$. Consequently

$$S_2' = \mathcal{M} + VE_1 + O_A\left( \frac{VU^2}{Y(\log x)^A} + \mathcal{R}Ux^{o(1)} \right), \qquad (67)$$

where, with the original coprimalities retained,

$$\begin{aligned}
E_1 = \sum_{q,d} \psi(q/Q) \sum_{r,s} \frac{\eta_r \overline{\eta_s}}{\ell r \varphi(\ell s)} \sum_{\substack{n_1,n_2 \\ (n_1,\ell r)=(n_2,\ell s)=1}} \alpha_{n_1} \overline{\alpha_{n_2}} \\
\times \sum_{0<|h|\le H_1} \widehat{\psi}\left( \frac{hV}{\ell r} \right) e\left( \frac{\mathfrak{a} h \overline{n_1}}{\ell r} \right), \qquad H_1 \asymp \frac{Y\mathcal{R}}{V} (\log x)^5. \qquad (68)
\end{aligned}$$

Lemma A.1 permits this Fourier truncation.

For $S_3'$, retain both exact-product exclusions during all divisor cutoffs and common-factor removals. The clean set has

$$(r,s)=(s,\ell)=1,\quad (n_1,\ell r)=(n_2,\ell s)=1,\quad (n_1,n_2)=1,\\ n_1\equiv n_2 \pmod{\ell},\qquad |n_1-n_2|\geq U(\log x)^{-C_{\mathrm{sep}}}. \tag{69}$$

The removed rough common factors cost $VU^2(\log x)^C/(Yz)+x(\log x)^C$; the close pairs cost $VU^2(\log x)^{C-C_{\mathrm{sep}}}/Y+x(\log x)^C$. These estimates follow respectively by counting two multiples or a multiple and a primitive progression; their additional unit terms are absorbed using $Y/U\leq x^{-\delta}$. Choose the divisor truncation and then $C_{\mathrm{sep}}$ sufficiently large.

Now $n_1\neq n_2$. Restoring one equality $mn_i=\mathfrak{a}$ costs only $\mathcal{R}(U+Y)x^{o(1)}=\mathcal{R}Ux^{o(1)}$: fixing its factorization leaves $O(U/Y+1)$ choices for the other $n$, and the other rough modulus divides a nonzero shifted product. Both equalities cannot hold. Apply CRT and Poisson only after this restoration. On (69),

$$\varphi(\ell)c_\ell(r,s)=\frac{1}{\ell rs}.$$

Put $d_r(b)=A_r(b)-B_r/\varphi(\ell)$. Since $\sum_b^* A_r(b)=B_r$, the zero frequency is $\mathcal{M}+\mathcal{P}$, where

$$\mathcal{P}=Vh_0\sum_{q,d}\psi(q/Q)\varphi(\ell)\sum_{r,s}\eta_r\overline{\eta_s}c_\ell(r,s)\sum_b^* d_r(b)\overline{d_s(b)}\geq 0. \tag{70}$$

Positivity follows from (65). In particular this principal variance must be bounded; it does not cancel identically.

The auxiliary Siegel–Walfisz hypothesis and the primitive character large sieve give, for every fixed $A,w>0$,

$$\sum_{\ell\leq 8Y}\tau(\ell)^w\sum_b^*|d_r(b)|^2\ll_A\frac{\tau(r)^{2C_0}U^2}{(\log x)^A}. \tag{71}$$

For clarity, normalize $\alpha_n\mathbf{1}_{(n,r)=1}$ by $\tau(r)^{C_0}$. Its auxiliary Siegel–Walfisz constants remain uniform because $\tau(kr)\leq\tau(k)\tau(r)$. Character orthogonality expresses the unweighted variance through nonprincipal characters. Write each modulus as $fk$, where $f>1$ is the primitive conductor, and keep $(n,k)=1$ in the coefficient before applying the large sieve. On a dyadic conductor interval $f\asymp R_0$, its bound at fixed $k$ is $O(U(U/R_0+R_0)/\varphi(k))$. Summing conductors $f>(\log x)^b$ and inducing factors gives $O(U^2(\log x)^{2-b}+UY)$. Small conductors are bounded by the auxiliary Siegel–Walfisz condition with its $\tau(k)^{C_0}$ cost; their total is $O(U^2(\log x)^{3b+C_1-2B})$. Choose $b$, and then $B$, to obtain any prescribed logarithmic saving. The weight $\tau(\ell)^w$ follows by Cauchy's inequality, ordinary divisor means and the elementary bound for the variance $O(U^2/\ell)$.

Using (65), Cauchy over the $r$ variables and $\varphi(\ell r)^{-1}\ll(\log x)/(Y\mathcal{R})$ gives, for a fixed divisor order $B_0$,

$$\begin{aligned}\mathcal{P}&\ll\frac{V(\log x)^2}{Y\mathcal{R}}\sum_{r\asymp\mathcal{R}}\tau(r)^{2B_0}\sum_{\ell\leq 8Y}\tau(\ell)^{2B_0+1}\sum_b^*|d_r(b)|^2\\&\ll_A\frac{VU^2}{Y(\log x)^A}.\end{aligned}$$

The factor $\tau(r)^{2C_0}$ in (71) is summed by a divisor mean; it is not included in an unspecified uniform constant. Combining these estimates with (64) gives

$$|\mathcal{S}|\ll_A\frac{x}{(\log x)^A}+V\sqrt{Y}(\log x)^C\big(\sqrt{|E_1|}+\sqrt{|E_2|}\big), \tag{72}$$

where $E_2$ is the nonzero CRT frequency sum on (69). The restored single-zero term contributes at most $\sqrt{xY\mathcal{R}}\,x^{o(1)}$, which is power-small.

A.3. **The first exponential sum.** For a fixed smooth $w$, Poisson summation and the complete Kloosterman Weil bound give, for every integer $A$,

$$\left|\sum_{(n,\ell)=1} w(n/U)e(A\overline{n}/\ell)\right| \ll_w \tau(\ell)^2\sqrt{\ell} + \frac{U}{\ell}(A,\ell). \tag{73}$$

The zero Fourier frequency is a Ramanujan sum and gives the second term. For the other frequencies use

$$\sqrt{(k,A,\ell)} \le \sum_{\substack{d\mid(A,\ell)\\ d\mid k}} \sqrt{d}, \qquad \sum_{\substack{k\neq 0\\ d\mid k}} (1+|k|/V)^{-2} \le \frac{2V}{d}.$$

This proves (73) also when the frequency length is below one.

Fix $d$ in (68), put $Q' = dQ$ and restrict $q'$ to multiples of $d$. This restriction is permitted inside the bounded $q'$-dependent coefficient. The following calculation applies with $Q'$ denoted temporarily by $Q$ and $\mathcal{R}$ by $R$. Absorb the bounded smooth factor into the $r,h$ coefficient and apply Cauchy while retaining the original unrestricted $r,s$ support. Expansion of the square gives

$$|E_1(d)|^2 \ll x^{\epsilon}\frac{U^3}{Q^3R^2}\,T, \tag{74}$$

where

$$T = \sum_{\substack{q\asymp Q, r, r'\asymp R\\ (\mathfrak{a},qrr')=1}} \sum_{0<|h|,|h'|\le H_1} \left|\sum_{(n,qrr')=1} w(n/U)e\left(\frac{\mathfrak{a}(hr'-h'r)\overline{n}}{qrr'}\right)\right|.$$

Put $j = hr' - h'r$. The $j=0$ tuples number $RH_1x^{\epsilon}$, since $hr'\neq 0$ and $h'r = hr'$ is a divisor equation. Their contribution is $\ll x^{\epsilon}QRH_1U$. For $j\neq 0$, the square-root term in (73) contributes $\ll x^{\epsilon}Q^{3/2}R^3H_1^2$. For its Ramanujan term write $r=gu$, $r'=gv$, $(u,v)=1$. Then

$$(j,rr') \le g^2(h,r)(h',r'), \qquad \sum_{q\asymp Q}(j,q) \le 3Q\tau(|j|),$$

and

$$\sum_{0<|h|\le H_1}(h,r) \le 2H_1\tau(r), \qquad \sum_{r,r'\asymp R}(r,r')^2 \ll R^3.$$

Because $(\mathfrak{a},qrr')=1$, $\mathfrak{a}$ disappears from these gcds. The total Ramanujan contribution to $T$ is $\ll x^{\epsilon}UH_1^2R$. As $H_1 \ll QRU(\log x)^5/x$, (74) becomes

$$|E_1(d)|^2 \ll x^{\epsilon}\frac{U^4}{Q^2}\left(\frac{UQ}{x} + \frac{UQ^{5/2}R^3}{x^2} + \frac{U^2QR}{x^2}\right). \tag{75}$$

Return to $Q=Q_1$ and the logarithmic variables in (59). The required gaps follow from

$$\begin{aligned} n+q &< 1-9\sigma,\\ 2n+q+u+v &< \frac{17}{10} - \frac{29\sigma}{2},\\ n+\frac{5}{2}q+3u+3v &= \frac{35}{16}(q-n) + \frac{23}{16}(2n+u+2v) + \frac{5}{16}(n+q+5u+2v) - \frac{v}{2}\\ &< \frac{1021}{512} - \frac{147\sigma}{8} + O(1/\log x). \end{aligned}$$

The last inequality uses $|\mathfrak{a}| \ge c_0x$ and $\vartheta = 7/32$. The divisor bound for $\Lambda_d$ and $D = x^{o(1)}$ can now be paid inside a power saving. Sum $U^2/(dQ)$ over $d\asymp D$ using $\sum_{d\asymp D} 1/d \ll 1$. With room to absorb one further $D$,

$$|E_1| \ll \frac{U^2}{DQ}x^{-\sigma/2}. \tag{76}$$

A.4. **The second exponential sum and the reciprocity phase.** Write $E_2 = \sum_{d\asymp D} E_3(d)/d$. For fixed $d$, $E_3(d)$ is the sum over $q$, $w_1, w_2 \asymp \mathcal{R}$, $n_1, n_2 \asymp U$ and $0 < |h| \le H$, where $H \asymp QD\mathcal{R}^2(\log x)^5/V$, of

$$\frac{\Lambda_d(w_1)\overline{\Lambda_d(w_2)}\alpha_{n_1}\overline{\alpha_{n_2}}}{qw_1w_2}\psi(q/Q)\widehat{\psi}\left(\frac{hV}{qdw_1w_2}\right) \times e\left(\frac{\mathfrak{a}h\overline{n_1w_2}}{qdw_1} + \frac{\mathfrak{a}h\overline{n_2qdw_1}}{w_2}\right). \tag{77}$$

The restrictions are

$$(q,\mathfrak{a}) = 1, \quad (w_1, \mathfrak{a}w_2) = 1, \quad (w_2, \mathfrak{a}qdw_1) = 1,$$
$$n_1 \equiv n_2 \pmod{qd}, \quad (n_1, n_2qdw_1) = (n_2, n_1qdw_2) = 1, \quad |n_1 - n_2| \ge U(\log x)^{-C_{\mathrm{sep}}}.$$

There is no independent bounded $q$ coefficient after the outer Cauchy step. The denominator in (77) is $qw_1w_2$; the factor $1/d$ occurs outside $E_3(d)$.

Exact reciprocity gives

$$e\left(\frac{\mathfrak{a}h\overline{n_1w_2}}{qdw_1} + \frac{\mathfrak{a}h\overline{n_2qdw_1}}{w_2}\right) = e\left(\frac{\mathfrak{a}h(n_1-n_2)\overline{n_2qdw_1}}{n_1w_2}\right) e\left(\frac{\mathfrak{a}h}{qdw_1w_2n_1}\right). \tag{78}$$

All inverses are justified by the displayed coprimalities. Split $h$ by sign and the $q$ support into a bounded number of intervals on scales $q_0 \asymp Q$. Normalize the five variables $(q, w_1, w_2, n_1, h)$ by $(q_0, \mathcal{R}, \mathcal{R}, U, H)$. The last phase has amplitude

$$\frac{\mathfrak{a}}{x}\frac{Q}{q_0}\frac{D}{d}(\log x)^5.$$

Mixed differentiation at most once in each coordinate therefore bounds the total five-dimensional variation by

$$\left(1 + (\log x)^5 + (|\mathfrak{a}|/x)(\log x)^5\right)^5 = x^{5\kappa+o(1)}. \tag{79}$$

Repeated partial summation leaves anchored rectangular sums. Their endpoints are fixed before estimation, and their $h$ coefficient is a single fixed signed-interval indicator $\omega_h$, $|\omega_h| \le 1$. It has no dependence on any remaining arithmetic variable.

Partition the rough coprime pairs $(n_1w_2, n_2)$ by the cross-coprime partition of [9, Lemma 13.2]; its number of parts is $\exp(O((\log\log x)^4))$. Expand the two copies of (62) independently, including both choices of the factorization $st = d$. Put $R \asymp v_0$, $S \asymp u_0$. A total-product interval is retained as

$$\mathbf{1}_{A_i < r_is_i \le B_i} = \mathbf{1}_{r_is_i \le B_i} - \mathbf{1}_{r_is_i \le A_i}.$$

Thus it becomes a difference of the permitted product upper cutoffs, without inserting a coupled condition into a one-variable coefficient. For a bilinear form, complex polarization replaces two different coefficient sequences by diagonal sequences with at most four times the diagonal bound. Apply this successively to the two $R$, two $S$ and two $n$ sequences, at total constant cost at most $4^3$.

We give the resulting unweighted sum explicitly. Fix $d$, a cross-coprime class $\mathcal{C}$, endpoints $q_0 \le q_1 \le 2q_0$, two product upper cutoffs $B_1, B_2$, and one-bounded sequences $\gamma_r, \lambda_s, \alpha_n, \omega_h$. All these data are fixed before the remaining summations. Let $\mathscr{C}$ be the set of tuples

$(q, r_1, r_2, s_1, s_2, n_1, n_2)$ satisfying

$$\begin{gathered} q_0 \le q \le q_1, \quad (q, \mathfrak{a}) = 1, \quad r_i \asymp R, \quad s_i \asymp S, \quad n_i \asymp U, \\ (r_1 s_1, \mathfrak{a} r_2 s_2) = (r_2 s_2, \mathfrak{a} q d r_1 s_1) = 1, \quad r_1 s_1 \le B_1, \quad r_2 s_2 \le B_2, \\ n_1 \equiv n_2 \pmod{qd}, \quad (n_1, n_2 q d r_1 s_1) = (n_2, n_1 q d r_2 s_2) = 1, \\ (n_1 r_2 s_2, n_2) \in \mathcal{C}, \qquad |n_1 - n_2| \ge U (\log x)^{-C_{\mathrm{sep}}}. \end{gathered} \tag{80}$$

Define

$$\mathcal{W}_{\mathrm{bare}} = \sum_{\mathscr{C}} \frac{\gamma_{r_1} \lambda_{s_1} \overline{\gamma_{r_2} \lambda_{s_2}} \alpha_{n_1} \overline{\alpha_{n_2}}}{q r_1 r_2 s_1 s_2} \sum_{0 < |h| \le H} \omega_h e\left( \frac{\mathfrak{a} h (n_1 - n_2) \overline{n_2 r_1 s_1 d q}}{n_1 r_2 s_2} \right). \tag{81}$$

There is no smooth reciprocity phase in this definition. The preceding endpoint separation and polarization give

$$|E_3(d)| \ll x^{5\kappa + o(1)} \sup |\mathcal{W}_{\mathrm{bare}}|, \tag{82}$$

where the supremum is over the choices specified above. The $x^{o(1)}$ includes the cross-coprime classes and the two independent factorizations of $d$, but the five-variable phase loss occurs only once. It remains to bound (81).

The first Cauchy inequality. Put $f = (n_1 - n_2)/(dq)$, which is a signed nonzero integer. The new variable satisfies $0 < |f| \le 2U/(dq_0)$, and

$$e\left( \frac{\mathfrak{a} h (n_1 - n_2) \overline{n_2 r_1 s_1 d q}}{n_1 r_2 s_2} \right) = e\left( \frac{\mathfrak{a} h f \overline{n_2 r_1 s_1}}{n_1 r_2 s_2} \right). \tag{83}$$

Keep $q_f = (n_1 - n_2)/(df)$ integral and in $[q_0, q_1]$ at this stage. The product cutoff in $s_1$ has the exact finite Fourier expansion

$$\mathbf{1}_{r_1 s_1 \le B_1} = \int_0^1 D_{r_1}(\xi) e(s_1 \xi)\, d\xi, \qquad D_{r_1}(\xi) = \sum_{1 \le j \le B_1/r_1} e(-j\xi).$$

There is a common nonnegative majorant $L(\xi) \ll \min(1 + B_1/R, 1 + \|\xi\|^{-1})$ with $\int_0^1 L(\xi)\, d\xi \ll \log x$ and $|D_{r_1}(\xi)| \le L(\xi)$; set $c_{r_1}(\xi) = D_{r_1}(\xi)/L(\xi)$. For fixed $\xi$ write $\lambda'_s = S \lambda_s e(s\xi)/s$. These coefficients are one-bounded on $s \asymp S$. If $\mathscr{D}$ denotes the conditions in (80) which are independent of $s_1$, after putting $q = q_f$, the exact Fourier integrand is

$$\begin{aligned} \mathcal{W}_2(\xi) = \frac{1}{S} \sum_{\mathscr{D}} & \frac{\gamma_{r_1} c_{r_1}(\xi) \overline{\gamma_{r_2} \lambda_{s_2}} \alpha_{n_1} \overline{\alpha_{n_2}}}{q_f r_1 r_2 s_2} \\ & \times \sum_{\substack{s_1 \asymp S \\ (s_1, \mathfrak{a} n_1 r_2 s_2) = 1}} \lambda'_{s_1} \sum_{0 < |h| \le H} \omega_h e\left( \frac{\mathfrak{a} h f \overline{n_2 r_1 s_1}}{n_1 r_2 s_2} \right). \end{aligned} \tag{84}$$

In particular $|\mathcal{W}_{\mathrm{bare}}| \ll (\log x) \sup_\xi |\mathcal{W}_2(\xi)|$. For each $f$ the number of pairs $n_1, n_2 \asymp U$ with $n_1 \equiv n_2 \pmod{df}$ is $O(U^2/(d|f|) + U)$. Summing over $f$ and then $r_1, r_2, s_2$ gives $O(U^2 R^2 S \log x)$ outer tuples. Since the outside coefficient in (84) has absolute value at most $O(1/(q_0 R^2 S^2))$, Cauchy's inequality gives

$$|\mathcal{W}_2(\xi)|^2 \ll \frac{U^2 \log x}{q_0^2 R^2 S^3} \mathcal{W}_3, \tag{85}$$

where the following nonnegative sum retains the congruence but discards other outer restrictions:

$$\mathcal{W}_3 = \sum_{0<|f|\le 2U/(dq_0)} \sum_{\substack{n_1,n_2\asymp U\\ n_1\equiv n_2 \pmod{df}}} \sum_{r_1,r_2\asymp R} \sum_{\substack{s_2\asymp S\\ (n_2r_1,n_1r_2s_2)=1}}$$

$$\times \left| \sum_{\substack{s_1\asymp S\\ (s_1,\mathfrak{a}n_1r_2s_2)=1}} \lambda'_{s_1} \sum_{0<|h|\le H} \omega_h e\left(\frac{\mathfrak{a}hf\overline{n_2r_1s_1}}{n_1r_2s_2}\right)\right|^2. \tag{86}$$

Every discarded restriction is outside the squared inner sum. The cross-coprime property of $\mathcal{C}$ justifies the displayed coprimality before the original class restriction is removed.

Completion of the positive square. We may now remove the congruence in (86) and enlarge $|f| \le 2U/q_0$, because all its summands are nonnegative. Put $b = n_2r_1$ and $c = n_1r_2s_2$. Their multiplicities are at most $\tau(b)\tau_3(c)$ and can be bounded before summation. Split $b, c$ and $|f|$ into dyadic intervals. On each such block $B \ll UR$, $C \ll URS$, $F \ll U/q_0$, and a fixed smooth nonnegative majorant $g(b,c) = \psi(b/B)\psi(c/C)$ gives

$$\mathcal{W}_3 \ll x^\epsilon \sum_{B,C,F,\pm} \mathcal{W}_4, \tag{87}$$

where the sign of $f$ is absorbed into $\mathfrak{a}$ and

$$\mathcal{W}_4 = \sum_{\substack{b,c\\ (b,c)=1}} g(b,c) \sum_{f\asymp F} \left| \sum_{\substack{s\asymp S\\ (s,\mathfrak{a}c)=1}} \lambda'_s \sum_{0<|h|\le H} \omega_h e\left(\frac{\mathfrak{a}hf\overline{bs}}{c}\right)\right|^2. \tag{88}$$

There are only logarithmically many dyadic blocks. Combining (85) and (87), and using $q_0 \asymp Q$, gives the outside factor in the form

$$|\mathcal{W}_{\mathrm{bare}}| \ll x^\epsilon \frac{U}{QRS^{3/2}} \sup_{B,C,F} \mathcal{W}_4^{1/2}. \tag{89}$$

Expanding (88) and interchanging the two $s$ labels when necessary yields

$$\mathcal{W}_4 = \sum_{\substack{s_1,s_2\asymp S\\ (s_1s_2,\mathfrak{a})=1}} \sum_{0<|h_1|,|h_2|\le H} \lambda'_{s_2}\overline{\lambda'_{s_1}}\omega_{h_1}\overline{\omega_{h_2}} \sum_{f\asymp F} \sum_{\substack{b,c\\ (c,bs_1s_2)=1}} g(b,c) e\left(\frac{\mathfrak{a}f(h_1s_1-h_2s_2)\overline{bs_1s_2}}{c}\right). \tag{90}$$

The terms with $h_1s_1 = h_2s_2$ have total absolute value at most $x^\epsilon HSBFC$. For the others set $z = s_1s_2$, $y = f(h_1s_1 - h_2s_2) \neq 0$, and define the coefficient

$$\beta(y,z) = \sum_{\substack{s_1s_2=z,\ s_i\asymp S\\ f\asymp F,\ 0<|h_i|\le H\\ f(h_1s_1-h_2s_2)=y}} \lambda'_{s_2}\overline{\lambda'_{s_1}}\omega_{h_1}\overline{\omega_{h_2}}. \tag{91}$$

We retain $(z,\mathfrak{a}) = 1$. The variables $b, c$ do not occur in (91); the restrictions involving them remain in the kernel. Consequently the nonzero contribution is exactly

$$\sum_{\substack{y\neq 0,z\\ (\mathfrak{a},z)=1}} \beta(y,z) \sum_{\substack{b,c\\ (c,bz)=1}} g(b,c) e(\mathfrak{a}y\overline{zb}/c). \tag{92}$$

In particular $|\beta(y,z)| \le b(z,y)$ for the positive tuple count in the following lemma. This domination also holds after any dyadic restriction on $y, z$. It is independent of the future Poisson frequency, which has not yet been introduced.

**Lemma A.2.** *Let $b(z,y)$ count tuples $s_i \asymp S$, $f \asymp F$, $0 < |h_i| \le H$ with $z = s_1 s_2$ and $y = f(h_1 s_1 - h_2 s_2) \neq 0$. Then*

$$\sum_z \sum_{u \neq 0} b(z, \nu u)^2 \ll x^\epsilon (1 + H/S) \frac{(HS)^2 F}{\nu} \tag{93}$$

*for every $\nu \ge 1$ on polynomially bounded scales. The left side vanishes when $\nu \ge 8FHS$.*

*Proof.* At fixed $(z,y)$ there are at most $\tau(z)\tau(|y|)$ decompositions $(s_1, s_2, f, k)$, with $k = h_1 s_1 - h_2 s_2 \neq 0$. Cauchy therefore reduces the square to collisions of $h$ pairs with the same $(s_1, s_2, k)$, followed by a single $f$ count. Group according to $c \mid \nu$, $\nu/c \mid k$, with the count of $f$ bounded by $2F/c$. Equal $h$ pairs contribute $\ll x^\epsilon (HS)^2 F/\nu$. For unequal pairs put $g = (s_1, s_2)$, $s_1 = gu$, $s_2 = gv$, $(u,v) = 1$. Their differences have the form $(h_1 - h_1', h_2 - h_2') = (jv, ju)$, with $0 < |j| \le 2Hg/S$. If $b = (g, \nu/c)$, the condition on $k/g$ is divisibility by $\nu/(cb)$. Counting nonzero multiples from zero, without an additional unit term, gives at most $O(HScb/(g\nu))$ such values. Factoring the remaining nonzero integer $h_2 v$ costs only a divisor factor. The unequal contribution is consequently

$$\ll x^\epsilon \frac{H^3 SF}{\nu} \sum_{g \le 2S} \frac{(g,\nu)}{g} \ll x^{2\epsilon} \frac{H^3 SF}{\nu}.$$

The last estimate follows from $\sum_{g \le 2S} (g,\nu)/g \le \tau(\nu)(1 + \log(2S))$. Renaming $2\epsilon$ proves the result. The support bound follows from $0 < |y| < 8FHS$. □

In the present application $H/S \le x^{o(1)}(Q/U)D_0 < 1$, where $D_0 = U^2 R^2 S/x$. Thus the factor $1 + H/S$ is bounded.

A.5. **The reciprocal kernel and the spectral estimate.** We estimate the reciprocal kernel with the coefficient independent of the future Poisson frequency. This separation makes the same frequency interval available at every spectral level, as required by the level-average exceptional estimate.

We use $Y$ for the coefficient length, $Z$ for the level length, and $P$ for the length of a nonzero Poisson-frequency block in this subsection. The original convolution lengths remain $U, V$. For $\beta(u,z)$ supported on $u \asymp Y$, $z \asymp Z$ and $(\mathfrak{a}, z) = 1$, put

$$\mathcal{K} = \sum_{u,z} \beta(u,z) \sum_{\substack{c,b \\ (c, bz) = 1}} \psi(c/C)\psi(b/D) e(\mathfrak{a} u z \overline{b}/c). \tag{94}$$

Either sign of $u$ is treated separately. Define $B^2 = \sum_{u,z} |\beta(u,z)|^2$ and $B_\nu^2 = \sum_{u,z} |\beta(\nu u, z)|^2$; further coprimality indicators or unit phases do not increase these norms.

Poisson summation in $b$ is the exact identity

$$\sum_{(b,c)=1} \psi(b/D) e(\mathfrak{a} u \overline{z}\, \overline{b}/c) = \frac{D}{c} \sum_{m \in \mathbb{Z}} \widehat{\psi}(mD/c) S(m\overline{z}, \mathfrak{a}u; c).$$

The zero frequency contributes at most $x^\epsilon D\sqrt{YZ}\, B$, since $|S(0, \mathfrak{a}u; c)| \le (\mathfrak{a}u, c)$ and $\sum_{c \asymp C} (\mathfrak{a}u, c) \ll C\tau(|\mathfrak{a}u|)$. Its square is $x^\epsilon D^2 YZB^2$. For nonzero $m$ truncate at the common value $(C/D)x^\epsilon$ and split into common signed dyadic intervals. Ordinary smooth Fourier decay, with a fixed sufficiently large derivative order, controls the discarded tail.

On an actual interval $m \asymp P$, put

$$Y_0 = \frac{4\pi\sqrt{|\mathfrak{a}|PY}}{C\sqrt{Z}}, \quad v = m/P, \quad w = u/Y, \quad \zeta = z/Z, \quad t = \frac{4\pi\sqrt{|\mathfrak{a}|mu}}{c\sqrt{z}\, Y_0}.$$

Then $c/C = \sqrt{vw/\zeta}/t$. This change of variables removes $\mathfrak{a}$ from the normalized amplitude. Fourier expansion on a fixed compact box separates the amplitude into a Bessel test function, a common factor $\mathbf{1}_I(m)e(m\omega)$ and an arbitrary complex $u, z$ coefficient. Choosing the Fourier

decay order above the fixed Bessel derivative order makes the coefficient norm summable. Thus the $m$ interval $I$ and phase $\omega$ are common at every level $z$.

The cusp conversion and packet operation are given precisely in Appendix B. They transform the $u$ coefficient to

$$\beta_\nu(n,z) = \mathbf{1}_{(n,\mathfrak{a})=1}\beta(\nu n, z)e(-n/z), \qquad \nu \mid \mathfrak{a}^\infty, \tag{95}$$

at packet cost $(|\mathfrak{a}|\nu)^{\vartheta/2}$, while leaving the common $m$ functional unchanged. Let

$$P_0 = \frac{PC^4Z}{|\mathfrak{a}|PY + C^2Z}, \qquad T_0 = \frac{|\mathfrak{a}|PY}{C^2}.$$

The regular spectral contribution and the exceptional estimates yield

$$|\mathcal{L}_P|^2 \ll x^\epsilon P_0 \Big\{ \sum_{\nu|\mathfrak{a}^\infty} (|\mathfrak{a}|\nu)^\vartheta (Y/\nu + Z + T_0)(P + Z + T_0)B_\nu^2 \\ + (C^2Z)^\vartheta (Y+Z)^{1-\vartheta}(P+Z)^{1-2\vartheta}B^2 \Big\}. \tag{96}$$

Here $\mathcal{L}_P$ is the separated Kuznetsov sum, before restoring the Poisson factor $D/C$.

We give the spectral summation leading to this polynomial. Throughout this step, a regular spectral norm is the positive direct-sum norm (104): its Maass atoms have weight $1/\cosh(\pi t_j)$, its holomorphic atoms weight $1/(2\pi)$ after the stated Fourier normalization, and its continuous part measure $dt/\pi$. Cauchy's inequality is applied within each component of this same measure, before the three components are added. At fixed $z, \nu$, write $A(t), B(t)$ for the $n$- and $m$-side coefficient arrays on $d\mu_z$, with entries $\sum_n \beta_\nu(n,z)R_{t,1}(n)$ and $\sum_{m\in I} e(m\omega)R_{t,\infty}(m)$, respectively. Here $t$ labels an entry of the regular direct sum, including its component and basis labels, and $R_{t,\mathfrak{c}}$ denotes the corresponding Fourier coefficient in (104). Let $A_0, B_0 \geq 0$ denote the norm-bound amplitudes, and let $u_0, v_0 \geq 0$. If the two cumulative regular squared norms are bounded by $(T^2+u_0)A_0^2$ and $(T^2+v_0)B_0^2$, the Bessel bounds

$$|\mathcal{H}(t)| \ll \frac{1+|\log Y_0|}{1+Y_0}, \qquad |\mathcal{H}(t)| \ll |t|^{-5/2} + Y_0|t|^{-3} \quad (|t| \geq \max(1, 2Y_0))$$

give

$$\int |\mathcal{H}(t)A(t)B(t)|\, d\mu_z(t) \ll \frac{1+|\log Y_0|}{1+Y_0}\sqrt{(1+Y_0^2+u_0)(1+Y_0^2+v_0)}A_0B_0.$$

Indeed, above $\max(1, 2Y_0)$ the cumulative norm product grows on the $j$th shell by at most $2^{2j}$, leaving the summable factors $2^{-j/2}+2^{-j}$. The level Cauchy factor is $\sqrt{Z}$. With $L = Y/\nu$, this gives the first term of (96), including the packet cost.

For exceptional forms put $\tau_j = 2\sqrt{1/4-\lambda_j} \leq \vartheta$, $X_0 = 1 + Z/L$, and $X = 1/Y_0$. Use

$$1 + X^{\tau_j} \leq 2(1 + X/\sqrt{X_0})^{\tau_j}(1+X_0)^{\tau_j/2}.$$

The fixed-level $n$ norm is $\ll x^\epsilon(1+L/Z)B_\nu^2$. The common-interval $m$ norm is

$$\ll x^\epsilon P(P+Z)\left[1 + \left\{\frac{P(1+X/\sqrt{X_0})^2}{(P+Z)^2}\right\}^\vartheta\right].$$

Multiplying these bounds and the squared packet cost gives a term absorbed in the regular expression and the second term of (96). The factor $|\mathfrak{a}|\nu$ cancels only in this latter term after the two norms are multiplied. The same argument applies to the holomorphic component with height $k-1$ and to the continuous component with real height $t$ in (104). Their measures remain unchanged throughout the shell summation. On the exceptional spectrum, $1/\cosh(\pi t_j) = 1/\cos(\pi|\operatorname{Im} t_j|) \asymp 1$, since $|\operatorname{Im} t_j| \leq 7/64$; hence the unweighted exceptional estimates used above differ only by an absolute constant.

Restoring $D/C$, the elementary inequality

$$\frac{Z(L+Z+T_0)(P+Z+T_0)}{Z+T_0} \le (L+Z)(P+Z)+ZT_0$$

and $P \le C/D$ give

$$|\mathcal{K}|^2 \ll x^{\epsilon}\mathcal{J}^2,$$

where the complete nonnegative polynomial is

$$\begin{aligned}\mathcal{J}^2 = &\sum_{\nu|\mathfrak{a}^\infty} (|\mathfrak{a}|\nu)^{\vartheta}\{C(Y/\nu+Z)(C+DZ)+|\mathfrak{a}|YZ\}B_\nu^2\\ &+\{C(CD\sqrt{Z})^{2\vartheta}(Y+Z)^{1-\vartheta}(C+DZ)^{1-2\vartheta}+D^2YZ\}B^2. \end{aligned} \tag{97}$$

The actual cutoff $P \le x^{\epsilon}C/D$ is accommodated by reducing the initially chosen $\epsilon$. If $C/D < 1/2$, estimates are applied only on nonempty actual intervals; the real comparison value $C/D$ is not used as the length of a nonexistent integer interval.

Return to the convolution scales. Apply (97) first at the actual dyadic lengths in (92). All its coefficients are nonnegative and increasing in the four lengths, since $0 \le \vartheta \le 1/2$. We may then replace these lengths by

$$D = B_* = UR, \quad C = C_* = URS, \quad Z = S^2, \quad Y = \frac{UR^2S^3}{V}. \tag{98}$$

The accompanying counting parameters satisfy $F \le U/Q$ and $H \le x^{\epsilon}H_0$, where $H_0 = QR^2S^2/V$. These selectable powers are included in the final $x^{\epsilon}$ after choosing the input exponents smaller. Put

$$D_0 = \frac{U^2R^2S}{x} = \frac{Y}{Z} < 1.$$

Then $B_*Z \ge C_*$, $B_*^2 < x$, and $C_*B_*Z = xY$. Lemma A.2 and $H/S \le x^{\epsilon}(Q/U)D_0 < 1$ give

$$B_\nu^2 \le x^{\epsilon}\frac{(HS)^2F}{\nu}, \qquad B^2 \le x^{\epsilon}(HS)^2F. \tag{99}$$

The auxiliary divisor sum satisfies

$$\sum_{\nu|\mathfrak{a}^\infty} \nu^{\vartheta-1} = \prod_{p|\mathfrak{a}}(1-p^{-(1-\vartheta)})^{-1} = x^{o(1)}.$$

Its logarithm is $O((\log x)^{\vartheta})$, by splitting the prime divisors at $\log x$; this is uniform for polynomially bounded $\mathfrak{a}$. Consequently (97) gives

$$\mathcal{J}^2 \ll x^{\epsilon}(HS)^2F\left\{|\mathfrak{a}|^{\vartheta}(C_*B_*Z+|\mathfrak{a}|Y)Z+C_*^{1+2\vartheta}B_*Z^{2-2\vartheta}+B_*^2YZ\right\}. \tag{100}$$

For clarity, the four monomials on the right yield the following ratios after multiplication by

$$\frac{(H_0S)^2(U/Q)}{U^4R^4S^6} = \frac{Q}{U^3V^2}:$$

| monomial | normalized contribution |
|---|---|
| $\lvert\mathfrak{a}\rvert^{\vartheta}C_*B_*Z^2$ | $D_1 = \lvert\mathfrak{a}\rvert^{\vartheta}UQR^2S^5/x^2$ |
| $\lvert\mathfrak{a}\rvert^{1+\vartheta}YZ$ | $(\lvert\mathfrak{a}\rvert/x)D_1$ |
| $C_*^{1+2\vartheta}B_*Z^{2-2\vartheta}$ | $D_2 = U^{1+2\vartheta}QR^{2+2\vartheta}S^{5-2\vartheta}/x^2$ |
| $B_*^2YZ$ | $QR^4S^5/V^3 \le D_1$. |

(101)

The last comparison follows from $B_*^2 < x$ and $|\mathfrak{a}| \ge 1$. The zero-difference contribution in (90) satisfies

$$\left(\frac{U}{QRS^{3/2}}\right)^2 H_0SB_*\frac{U}{Q}C_* = \left(\frac{U^2}{Q}\right)^2 D_0.$$

For the nonzero part, (92) gives $|\mathcal{W}_{4,\neq 0}| \ll x^{\epsilon}\mathcal{J}$. Combining this square root with that in (89) gives explicitly

$$\left(\frac{U}{QRS^{3/2}}\right)^{4} \frac{\mathcal{J}^2}{(U^2/Q)^4} = \frac{\mathcal{J}^2}{U^4R^4S^6} \ll x^{\epsilon}\{(1+|\mathfrak{a}|/x)D_1 + D_2\}.$$

The two contributions prove

$$|\mathcal{W}_{\mathrm{bare}}| \ll \frac{U^2}{Q}x^{\epsilon}\left\{D_0^{1/2} + \left[(1+|\mathfrak{a}|/x)D_1 + D_2\right]^{1/4}\right\}, \tag{102}$$

$$D_0 = \frac{U^2R^2S}{x}, \qquad D_1 = \frac{|\mathfrak{a}|^{\vartheta}UQR^2S^5}{x^2}, \qquad D_2 = \frac{U^{1+2\vartheta}QR^{2+2\vartheta}S^{5-2\vartheta}}{x^2}.$$

A.6. **Completion of the convolution estimate.** The scale hypotheses give $D_0 \ll x^{-8\sigma}$ and $D_1, D_2 \ll x^{-15\sigma}$ on every small-factor block. Together with $|\mathfrak{a}|/x \le x^{\kappa}$ and the single variation cost (79), (102) gives

$$|E_3(d)| \ll \frac{U^2}{Q}\left(x^{-4\sigma+5\kappa+\epsilon} + x^{-15\sigma/4+21\kappa/4+\epsilon}\right).$$

All selectable epsilon losses, the vanishing subpowers from the partitions and one additional normalizing factor $D$ fit inside $x^{\sigma/10}$. Since $\kappa \le \sigma/100$, both remaining savings exceed $\sigma/2$. Harmonic summation in $d$ therefore gives

$$|E_2| \ll \frac{U^2}{DQ}x^{-\sigma/2}. \tag{103}$$

Insert (76) and (103) into (72). The exponential-sum contribution is $O(x^{1-\sigma/4}(\log x)^C)$. The earlier single-zero restoration is $O(x^{1-\sigma/6+o(1)})$, still power-small. Choose the auxiliary Siegel–Walfisz and logarithmic precisions to pay for the finitely many dyadic choices, the fixed logarithmic number of blocks and the normalization of $\beta$. Equation (60) deals with the omitted smooth parts. This proves (17).

## Appendix B. Normalization of the spectral inputs

We record the normalizations to identify the positive measures used in the spectral Cauchy inequalities. The regular spectrum and the exceptional spectrum are treated in their respective ranges.

These inputs are used in (96); the packet coefficient and the common frequency interval are specified in (95) and the discussion preceding that estimate.

We specify the classical statements used in the preceding proof. They are applied at trivial character, level $z$, cusps $\infty$ and 1, with $(\mathfrak{a}, z) = 1$. Write $R(n) = \sqrt{|n|}\rho(n)$ for the Deshouillers–Iwaniec normalization of the Fourier coefficients. The cusp scaling matrices are

$$\sigma_{\infty} = I, \qquad \sigma_1 = \begin{pmatrix} \sqrt{z} & 0 \\ \sqrt{z} & 1/\sqrt{z} \end{pmatrix}.$$

The cusp conversion in [3, Lemma 4.3] introduces $e(\mathfrak{a}y/z)$; hence the ordinary Kloosterman coefficient carries $e(-\mathfrak{a}y/z)$.

For $y = \nu n$, $\nu \mid \mathfrak{a}^{\infty}$, $(n, \mathfrak{a}) = 1$, the coprime packet identities in [1, Lemmas 3.1 and 3.2] give, on every normalized oldform dilation,

$$R_v(\mathfrak{a}\nu n) = \lambda_v(\mathfrak{a}\nu)e(n(\mathfrak{a}\nu - 1)/z)R_v(n).$$

Multiplication by $e(-\mathfrak{a}\nu n/z)$ leaves exactly $e(-n/z)$. The packet coefficients in [1, (3.3)] have Frobenius norm $z^{\epsilon}$ after divisor losses. Cauchy's inequality in that packet and the Kim–Sarnak exponent bound [5] therefore cost $(|\mathfrak{a}|\nu)^{7/64+\epsilon}$ and give the common coefficient (95). No eigenvalue is assumed common to all dilations. The holomorphic and continuous quadratic forms in [1, Lemma 3.2] use the corresponding identities with $\vartheta = 0$. The continuous term is integrated with the formal Plancherel measure, rather than treated as the norm of an individual square-integrable Eisenstein series; see also [17, Proposition 8.2].

For completeness we specify the positive norm used for every regular spectral Cauchy inequality. Let $\mathfrak{c}$ be either cusp. In the notation of [11, Proposition E], write

$$R^M_{j,\mathfrak{c}}(n) = \rho^{\mathrm{DI}}_{j,\mathfrak{c}}(n) = \sqrt{|n|}\rho^{\mathrm{ABL}}_{j,\mathfrak{c}}(n),$$
$$R^H_{jk,\mathfrak{c}}(n) = \left(\frac{(k-1)!}{(4\pi)^{k-1}}\right)^{1/2} n^{-(k-1)/2}\psi_{jk,\mathfrak{c}}(n),$$
$$R^E_{\mathfrak{d},\mathfrak{c}}(n,t) = n^{it}\varphi_{\mathfrak{d}\mathfrak{c}}(n;\tfrac{1}{2}+it).$$

The symbol $R_j$ in the packet formulas denotes $R^M_j$. For $T \geq 1$ and $b$ supported on $n \asymp L$, $L \geq 1/2$, define

$$\begin{aligned}\mathfrak{Q}_{z,\mathfrak{c}}(T;b) = &\sum_{\substack{t_j\in\mathbb{R}\\ |t_j|\leq T}} \frac{1}{\cosh(\pi t_j)}\left|\sum_{n\asymp L} b_n R^M_{j,\mathfrak{c}}(n)\right|^2 \\ &+ \frac{1}{2\pi}\sum_{\substack{k\in 2\mathbb{Z}_{>0}\\ k-1\leq T}}\sum_j \left|\sum_{n\asymp L} b_n R^H_{jk,\mathfrak{c}}(n)\right|^2 \\ &+ \frac{1}{\pi}\sum_{\mathfrak{d}}\int_{-T}^{T}\left|\sum_{n\asymp L} b_n R^E_{\mathfrak{d},\mathfrak{c}}(n,t)\right|^2 dt. \end{aligned} \tag{104}$$

This is the direct-sum measure $d\mu_z$ used above. The constants $1/(2\pi)$ and $1/\pi$ are the constants in the Kuznetsov formula; the factor $1/\cosh(\pi t_j)$ is part of the Maass measure and is not absorbed by the change from $\rho^{\mathrm{ABL}}$ to $R^M$. By [3, Proposition 4.7] and [11, Lemma H],

$$\mathfrak{Q}_{z,\mathfrak{c}}(T;b) \ll (zLT)^{\epsilon}\left(T^2 + \frac{L}{z}\right)\sum_n |b_n|^2. \tag{105}$$

The harmless change from $k \leq T$ to $k-1 \leq T$ costs only a fixed factor. The continuous term in this convention has measure $dt/\pi$, as in [11, (3.27)]; it has no additional hyperbolic factor. The test-function bounds used with this norm are [11, Lemma D]. For opposite-sign Kuznetsov sums the holomorphic term is absent, and the same positive norm is still an upper bound.

Put $\tau_j = 2\sqrt{1/4-\lambda_j} \leq 7/32$ on the exceptional spectrum. The fixed-level estimate [11, Theorem A], at cusp 1, supplies

$$\sum_{\mathrm{exc}} X^{\tau_j}\left|\sum_{n\asymp L} b_n R_{j,1}(n)\right|^2 \ll (zL)^{\epsilon}(1+L/z)\sum_n |b_n|^2, \qquad X \ll \max(1, z/L). \tag{106}$$

The coefficients may depend on $z$ but are independent of spectral height. The level-average estimate [11, Theorem J] gives

$$\sum_{z\asymp Z}\sum_{\mathrm{exc}} X^{\tau_j}\left|\sum_{m\in I} e(m\omega)R_{j,\infty}(m)\right|^2 \ll (ZV)^{\epsilon}(Z+V)V, \quad X \ll \max(V, Z^2/V), \tag{107}$$

where $I \subset (V, 2V]$ and $\omega$ are common to every level. For a partial interval $I$, choose an integer $P > 4V$ and expand $\mathbf{1}_I(m) = \sum_{k=0}^{P-1} c_k e(km/P)$ on residues modulo $P$. The coefficient norm is $\sum |c_k| \ll \log(2+V)$. Minkowski in the Hilbert norm indexed by $(z,j)$ and Theorem J with phase $\omega + k/P$ prove (107), with only a logarithmic squared loss absorbed in $(ZV)^{\epsilon}$. This argument requires the same interval at every level. For larger $X$, the bound $\tau_j \leq \vartheta$ gives the continuation factor $1 + (VX/(V+Z)^2)^{\vartheta}$.

The transformed estimates apply on each nonempty interval of length at least $1/2$, including an endpoint interval containing $n = 1$. The $m$ interval is common to all levels, and the $n$ coefficient is independent of spectral height. The condition $(\mathfrak{a}, z) = 1$ is retained in

every packet operation. These are precisely the interval and coefficient restrictions used in the preceding spectral estimate.

## Appendix C. Factorization of the finite sieve coefficients

We prove that the actual upper and lower sieve coefficients have the factorization required by the distribution estimate. Throughout this appendix $N$ is the even target. Finite partitions, levels, and approximation tolerances are chosen before $N$ tends to infinity; the arithmetic coefficients themselves may depend on $N$. We write $\omega_{\mathrm{pf}}(n)$ for the number of distinct prime factors of $n$.

C.1. **Finite policies and residual square conditions.** Write

$$\rho_0 = \frac{19}{96}, \qquad \mu_0 = \frac{25}{128}, \qquad \Lambda_{\mathrm{cap}} = \frac{153}{256}, \qquad \Theta = \frac{299}{500}.$$

The number $\Theta$ bounds the complete carrier; $\Lambda_{\mathrm{cap}}$ bounds the high-rank levels. A finite policy specifies a level $N^{\lambda}$ at each of finitely many prime prefixes. An ancestor decision is fixed before the children are subdivided. For decreasing exponents $t_1, \dots, t_j$, put $S_j = t_1 + \cdots + t_j$. The full square conditions at an auxiliary level $N^{\ell}$ are

$$S_{j-1} + 2t_j \le \ell. \tag{108}$$

If $I$ is a selected subset of the marked slots, put $b_I = \sum_{i \in I} t_i$. In addition to (108), retain all the conditions

$$b_I + \sum_{\substack{i<j \\ i \notin I}} t_i + 2t_j \le \ell \qquad (j \notin I). \tag{109}$$

For $j$ after the last selected index, the left side of (109) is exactly $S_{j-1} + 2t_j$. Thus adjoining smaller primes preserves the selected conditions whenever the full conditions are preserved. An admitted cube also implies the next square, since

$$S_j + 2t_{j+1} \le S_{j-1} + 3t_j.$$

This observation applies to both parities of the ordinary sieve tree: upper first exits have odd rank, and lower first exits have even rank.

The physical level $\lambda$ and the auxiliary covering level $\ell$ have different uses. The coefficient and its density use $\lambda$; factorization may use a slightly larger $\ell$. No change in $\lambda$ is made in obtaining this cover. Here is the needed choice at the three-slot alternative. Put

$$A(t) = \frac{1+t}{2}, \qquad W(z) = \Lambda_{\mathrm{cap}} - \frac{3}{8}z.$$

On a coarse box let $\Gamma = \inf \max(A, W)$ and $\lambda = \Gamma - 2e$, where the variation allowance satisfies $\chi < e/10$. If $\inf A \ge \lambda + e/2$, choose $\ell = \lambda + e/4$. Otherwise $\sup A < \lambda + 11e/20$, so $W \ge \Gamma$ throughout the box; choose $\ell = \lambda + e$. This leaves a reserve of at least $9e/20$ above the lower bound and $e$ below the upper bound. The remaining initialization reserves include

$$1 - \Lambda_{\mathrm{cap}} - 2\mu_0 = \frac{3}{256}, \qquad \frac{525}{8704} + \frac{49z}{272} > 0.$$

The higher-rank changes are restricted to compact pieces on which their support conditions have positive reserve. Fine endpoint enlargement is then taken smaller than this reserve.

At an ordinary lower start $t_1 \le h_{\mathrm{s}} = 19/125 < \mu_0$, the endpoint admissibility condition satisfies

$$\max\left(\frac{1+t_1}{2}, \Lambda_{\mathrm{cap}} - \frac{3}{8}t_1\right) \ge \frac{249}{448}.$$

A level with a fixed buffer below 249/448 therefore has a strict covering interval. A restored fourth exit is admitted only after its own cube and all the full and selected squares have been checked. For upper prefixes above $\mu_0$, the function

$$v(x) = \min\left(\frac{1+x}{2}, \frac{57}{64} - \frac{3}{2}x\right)$$

satisfies $v(x) > 3x$ for $x < \rho_0$. Selecting $y < x$ gives $2x + y < 3x$; selecting $y + z < x$ gives $2x + y + z < 3x$. These are the initial square bounds for the two additional alternatives.

In a marked state write $S = \log_N B$, $L = \lambda - S$ and $\alpha = (1 - S)/L \geq 2$. A singleton of exponent $Lu$ changes the residual length to $L(1 - u)$ and a pair of exponents $Lu, Lv$ changes it to $L(1 - u - v)$. In both cases $\alpha$ increases. The conditions $2u < 1$, or $2v < 1$ and $v + 2u < 1$, are precisely the new full square conditions. The selected conditions then follow from (109); the parent prime ceiling is retained. The finite construction has at most 440 elementary transitions. Its residual exponents are at least $L_{\min} = \ell_0 101^{-440}$ for some $\ell_0 > 0$, and its appended prime exponents are at least $L_{\min}/6$. In particular all switched leaves have a common positive prime cutoff.

C.2. **Finite approximation and strict margins.** The finite partition is chosen jointly for all the marked queries. At the third upper exit, the omitted last-prime tail is

$$O\left(\eta\left(1 + \log^2\frac{\rho_0}{\eta}\right)\right), \qquad \eta \downarrow 0.$$

At lower exits of rank $r = 3, 4, 5$, the corresponding bound is $O(\eta(1 + \log^{r-1}(h_{\mathrm{s}}/\eta)))$. The higher changes have positive last-prime cutoffs; in particular the eighth-rank domains have $t_8 > 53/13824$. On the compact remainder, proper affine walls can be trimmed with arbitrarily small mass. A region on which the chosen label is identically inherited is retained rather than treated as a wall. Freezing the coarse decision before refining the children gives one common finite policy for the count and the density.

For the third envelope, the coarse parameters $e, \chi$ give $0 \leq L_{\mathrm{end}} - \lambda \leq 4e + 6\chi$. Fix the finite policy and its positive reserves. The admissibility functions are continuous on finitely many compact boxes, so their strict reserves have a positive minimum. Hence some $\kappa_* > 0$ preserves all the inequalities when $\log_x N \in [1, 1 + \kappa_*]$ and the endpoint enlargement is sufficiently small. The margin is obtained on these fixed compact pieces, after the limiting boundary regions have been removed.

C.3. **Three-factor representations.** We describe the endpoint construction, since the factorization must hold for the coefficient as a whole. Let $x$ be the complete local scale, $q_0 = \log_x N$, $\vartheta = 7/32$, and $\alpha_0 \geq q_0$ the residue parameter in the local admissibility condition. For a small fixed $\zeta > 0$ put

$$\theta_{\mathrm{cov}} = q_0\ell + 2\zeta, \qquad D = x^{\theta_{\mathrm{cov}} - 2\zeta}, \qquad A = Ux^{-\zeta},$$

where $U$ is the chosen medium length. Define

$$v_0 = 2\theta_{\mathrm{cov}} - 1,$$
$$u_0 = \min\left\{\frac{2 - (2 + 2\vartheta)\theta_{\mathrm{cov}}}{3 - 4\vartheta}, \frac{2 - 2\theta_{\mathrm{cov}} - \alpha_0\vartheta}{3}\right\}.$$

In this subsection the slot exponents and $b_I$ are measured to base $x$: they are $q_0$ times their preceding base-$N$ values. The covering exponent in the full and selected square conditions is therefore $q_0\ell$. The selected subset admissibility condition is $v_0 \leq b_I \leq u_0$, with (108)–(109) imposed on ordered upper endpoints. All lines of this admissibility condition are retained after rescaling, including a line that is dominated when $\alpha_0 = 1$.

Suppose first that $C = D/(Ax^{b_I}) \geq 1$. Put the selected slots in group two and insert the remaining slots greedily into capacities $A, C$. At every insertion the selected square bounds

the square of the new slot by the product of the remaining capacities, so at least one accepts it. The resulting group endpoints satisfy

$$B_1 \le A, \qquad B_2 = x^{b_I}, \qquad B_3 \le C$$

and

$$\begin{aligned} A^2 B_2 B_3^2 &\le x^{1-4\zeta}, \\ N^{\vartheta} A B_1 B_2^5 B_3^2 &\le x^{2-4\zeta}, \\ A^{1+2\vartheta} B_1 B_2^{5-2\vartheta} B_3^{2+2\vartheta} &\le x^{2-(4+4\vartheta)\zeta}. \end{aligned} \tag{110}$$

If $C < 1$, insert all slots into $A, D/A$ using the full squares. Here $1 < D/A < x^{b_I}$. Set $B_3 = 1$ and replace $x^{b_I}$ by $D/A$ in the last two bounds. The first bound follows from $A^2 B_2 \le AD \le x^{23/24-5\zeta/2}$. Thus no factor with an endpoint below one is introduced.

For the three-slot $W$ alternative initialize capacities $A, x^{v_0}, D/(Ax^{v_0})$ with the first two slots. One fits $x^{v_0}$; the square of the other is at most the product $x^{1-\theta_{\mathrm{cov}}-2\zeta}$ of the two remaining capacities. Continue inserting whole slots. If a first slot fits none of the capacities, adjoin it to group two. Its new product $B$ lies in $(x^{v_0}, x^{u_0}]$, since that slot is at most the third slot and $p_3 \le x^{u_0-v_0}$. Failure of the first capacity together with the full square gives $ABc < D$, so the two residual capacities are legal. The full-square greedy argument finishes the partition. If no failure occurs, the capacity product gives (110) directly. These algorithms use only endpoint data of a whole atom and its medium length. They never use the individual arithmetic tuple or the sign of a progression discrepancy.

C.4. **The coefficient identity and squarefree restriction.** Refine the interval labels to a common disjoint half-open alphabet. If $m_I$ slots have label $I$, set $M = \prod_I m_I!$. The unrestricted prime convolution $N_J$ of the slot list has value $M/\prod_p \nu_p(n)!$ on its support. Its sorted squarefree coefficient is therefore $\mathbf{1}_{\mathrm{sf}} N_J/M$. For a fixed three-group partition put $M_j = \prod_I m_{I,j}!$ and let $\gamma_j$ be the group convolution divided by $M_j$. Then $0 \le \gamma_j \le 1$ on every integer and the exact identity is

$$\frac{N_J}{M} = \frac{M_1 M_2 M_3}{M} (\gamma_1 * \gamma_2 * \gamma_3). \tag{111}$$

An empty group is the unit mass at one. A marked convolution already contains its marked multiplicities. Only its ordered residual slots are divided by their factorials: regrouping a signed atom $\epsilon \mathbf{1}_{\mathrm{sf}} N_{\mathrm{all}}/M_{\mathrm{res}}$ gives the external scalar $\epsilon M_1 M_2 M_3/M_{\mathrm{res}}$. This scalar is fixed, although it need not be bounded by one.

Choose $w = N^{\eta_0}$ below every marked prime exponent. For $p < w$, $p \mid BN$ if and only if $p \mid N$. Hence one pair of small-prime beta weights, with support below the smallest residual reserve, serves every state in the finite marked family. The sign of the atom determines which member is attached. The small and rough prime universes are disjoint, so decomposition into these two parts is unique. Attaching the small-prime beta weight to one of the groups in (111) preserves one-boundedness.

Global squarefreeness remains in the physical coefficient and its density. It is removed only in the analytic tensor expansion. If the support is $D_* \le N^{\theta_*}$ with $\theta_* < 1$, the change in the discrepancy is at most a fixed divisor factor times

$$\sum_{\substack{q \le D_* \\ p^2 \mid q \text{ for some } p \ge N^{\eta_0}}} \left( \frac{N}{q} + 1 + \frac{N}{\varphi(q)} \right) \ll (N + D_*) N^{-\eta_0 + o(1)}. \tag{112}$$

The term 1 includes an exact complete product equal to $N$. No divisor function is evaluated at zero.

C.5. **Parameter order and the medium estimate.** Before small-prime attachment, endpoint enlargement and local rescaling, the complete supports at rough rank at most two have exponents

$$\frac{19}{48}, \qquad \frac{38}{125}, \qquad \frac{1527}{3584}$$

for the upper, ordinary lower and marked families, respectively. Their gaps below $1/2$ are $5/48$, $49/250$ and $265/3584$. For the last bound use $x < 57/224$, $x + 2y \le \Lambda_{\mathrm{cap}}$, and hence $x + y \le (\Lambda_{\mathrm{cap}} + 57/224)/2 = 1527/3584$. The fixed coefficient norm permits the below-half-level Bombieri–Vinogradov estimate for these pieces. A high-rank coefficient requires the factorization above.

First fix the finite collection of estimates and their dependencies, coarse policy, factorization conditions and positive reserves. Choose $\zeta$ below these reserves and the prime cutoffs, and set $\sigma = \zeta/2000$. Next choose a common beta support small enough for the density and support tolerances. If the rough-rank bound is $J$, take the endpoint mesh $\tau$ so that $(J+2)\tau$ is below every remaining reserve. Only after the fixed divisor and moment orders are known choose

$$0 < \kappa \le \min\left\{\kappa_*, \frac{\sigma}{200}, \frac{1}{2K_{\mathrm{mom}}}\right\}, \qquad 0 < \kappa_0 < \frac{\kappa}{4(1+\kappa)}. \tag{113}$$

Here $K_{\mathrm{mom}}$ bounds the fixed moment orders in the Fourier tail and complementary-coefficient truncation. Make $\kappa_0$ smaller if necessary so that the total carrier, safely bounded by $\theta_* = 599/1000$, satisfies $\theta_*/(1-\kappa_0) < 3/5$ and all three low-rank supports remain below one half.

Replacing $A$ by $U$ in (110) leaves savings $\zeta, 2\zeta, 9\zeta/16$. After two reserves of $\zeta/16$, the smallest saving is $7\zeta/16 > 15\sigma$. Small-prime attachment enlarges only group three; its costs $2\ell_\beta, 2\ell_\beta, (2+2\vartheta)\ell_\beta$ are included in the reserves. Dyadic subdivision therefore produces $Q_1, Q_2, Q_3$ satisfying all four size conditions of Proposition 3.2. The strict margins absorb fixed endpoint constants. There are finitely many atoms, with fixed total absolute scalar norm, before $N$ grows. This establishes a uniform three-factor representation for the actual finite coefficient families.

Checks for the medium branch. The following checks connect Proposition 3.2 with its application in Proposition D.3. They are made after fixing the common finite policy and its strict support reserves.

| Condition or input | Realization in the application | Verification |
|---|---|---|
| Complete scale and residue | $x = UV$ is the scale of the complete product and $\mathfrak{a} = N$. The trim and local rescaling give $c_0 x \le N \le x^{1+\kappa}$. | (119), (113) |
| Rough medium sequence | An individual interval carries the rough 1 or Möbius sequence, its bounded weight and permitted Mellin twist; the auxiliary-coprimality estimate is retained. | (120) and its proof |
| Complementary sequence | Unique factorization gives a fixed divisor bound. Any logarithmic truncation is charged by the mixed moment estimate before applying the local bound. | Lemma D.1; proof of Proposition D.3 |
| Three modulus factors | The normalized group convolutions are separate and one-bounded, with the residue-coprimality support retained. Marked multiplicities remain external scalars. | (111); full and selected squares (108)–(109) |
| Four strict size tests | The three monomial bounds, the medium capacity, small-prime attachment and local rescaling yield all the tests in (15). | (110); the reserves following (113) |

| Condition or input | Realization in the application | Verification |
|---|---|---|
| Moving-residue phase and spectral input | The exact reciprocal phase is retained. Its variation and the packet and spectral norms use the stated local margins. | (78), (79); Appendix B |
| Uniform summation errors | Fixed atom scalars, Fourier norms and logarithmically many charts are covered by stronger local logarithmic accuracy. Prime powers, squarefree collisions and exact products equal to $N$ are counted separately. | (112); proof of Proposition D.3 |

For the four size tests, deducting the stated support reserves from the savings $\zeta, 2\zeta, 9\zeta/16$ leaves strict margins that absorb fixed dyadic endpoint multiples. This check concerns the medium branch; the below-half estimate and Appendix E cover the other branches of Proposition D.3. Upper coefficients, lower coefficients and ordinary marked descendants enter through that proposition with their actual signed coefficient $\rho_N$.

## Appendix D. Distribution of the smooth prime products

This appendix proves (18) for the finite coefficient families of Appendix C. The medium branches use Proposition 3.2; the remaining branches use the below-half estimate or Appendix E.

For a smooth weight $W_N$ on at most $k_*$ labelled prime slots, with $p_i \ge cN^{a_*}$ and product $O(N)$, define

$$E_q(N;W) = \sum_{p_1,\dots,p_k} W_N(p_1,\dots,p_k)\left(\mathbf{1}_{p_1\cdots p_k\equiv N\ (q)} - \frac{\mathbf{1}_{(p_1\cdots p_k,q)=1}}{\varphi(q)}\right).$$

The geometric weights have fixed polylogarithmic mixed Euler derivative bounds. They carry no additional arithmetic filters.

D.1. **Two auxiliary mean estimates.** Let $d_j$ denote the $j$-fold divisor function. We record the elementary moment estimate with its dependence on the moving shift, since logarithmic truncations require a fixed moment order.

**Lemma D.1.** *Fix nonnegative integers $R, S$ and a constant $C \ge 1$. Put $K_{\mathrm{mom}} = 2^{2\max(R,S,1)}$ and $0 \le \kappa \le 1/(2K_{\mathrm{mom}})$. Uniformly for $|a| \le Cx^{1+\kappa}$ and an interval $J \subseteq [1, Cx]$,*

$$\sum_{\substack{b\in J\\ b\ne a}} \tau(b)^R \tau(|b-a|)^S \ll_{C,R,S} x(1+\log x)^{K_{\mathrm{mom}}-1}.$$

*Proof.* If an interval $I \subseteq [1,Z]$ has length $Y$, choosing a largest coordinate of an ordered $j$-tuple whose product belongs to $I$ gives

$$\begin{aligned}\sum_{n\in I} d_j(n) &\le j \sum_{d\le Z^{1-1/j}} d_{j-1}(d)(Y/d+1)\\ &\le j\{Y(1+\log Z)^{j-1} + Z^{1-1/j}(1+\log Z)^{j-2}\}.\end{aligned}$$

Indeed the product of the other $j-1$ coordinates is at most $Z^{1-1/j}$; dropping the largest-coordinate restriction only increases the sum. The two final bounds follow by expanding the divisor function as an ordered product and summing the last factor. For every integer $e \ge 0$, $\tau(n)^e \le d_{2^e}(n)$. On a prime power this follows by mapping an $e$-tuple in $\{0,\dots,v\}^e$ to the histogram of the subsets of coordinates reaching each of the positions $1,\dots,v$; the histogram is a weak composition of $v$ into $2^e$ parts and determines the tuple.

The nonzero values $|b-a|$ lie, with multiplicity at most two, in at most two intervals of length $O_C(x)$ and upper endpoint $O_C(x^{1+\kappa})$. Apply the preceding interval bound to $d_{K_{\mathrm{mom}}}$

to obtain

$$\sum_{\substack{b\in J\\ b\neq a}}\tau(|b-a|)^{2S}\ll x(1+\log x)^{K_{\mathrm{mom}}-1}+x^{(1+\kappa)(1-1/K_{\mathrm{mom}})}(1+\log x)^{K_{\mathrm{mom}}-2}.$$

The second exponent is at most $1-1/(2K_{\mathrm{mom}})-1/(2K_{\mathrm{mom}}^2)<1$. The unshifted $2R$ moment has the same bound. Cauchy–Schwarz proves the assertion. The term $b=a$ is removed before applying the divisor function. □

**Lemma D.2** (A convolution estimate below one half)**.** *Fix $\eta>0$ and $c\geq 1$. Let $x=UV$, $x^{\eta}\leq U,V\leq x^{1-\eta}$, and let $\alpha,\beta$ be supported on $[U,cU]$ and $[V,cV]$ with*

$$\sum_m|\alpha(m)|^2\ll U(\log x)^C,\qquad \sum_n|\beta(n)|^2\ll V(\log x)^C.$$

*They are independent of the summation modulus. Suppose that for every fixed $A_1,B>0$, every $k\geq 1$, $r\leq(\log x)^B$ and $(b,r)=1$,*

$$\left|\sum_{\substack{m\equiv b\ (r)\\(m,k)=1}}\alpha(m)-\frac{1}{\varphi(r)}\sum_{(m,rk)=1}\alpha(m)\right|\ll_{A_1,B}\tau(k)^{C_0}U(\log x)^{-A_1},\tag{114}$$

*with one fixed exponent $C_0$. For every $A>0$ there is a $D>0$ such that $Q\leq\sqrt{x}(\log x)^{-D}$ implies*

$$\sum_{q\leq Q}\max_{(a,q)=1}\left|\sum_{mn\equiv a\ (q)}\alpha(m)\beta(n)-\frac{1}{\varphi(q)}\sum_{(mn,q)=1}\alpha(m)\beta(n)\right|\ll_A x(\log x)^{-A}.$$

*Proof.* Put $L=\log x$. The primitive multiplicative large sieve [4, Theorem 18.2] is

$$\sum_{r\leq R}\frac{r}{\varphi(r)}\sideset{}{^*}\sum_{\chi\ (r)}\left|\sum_n z(n)\chi(n)\right|^2\leq(H-1+R^2)\sum_n|z(n)|^2,\tag{115}$$

when $z$ is supported on an interval of at most $H$ consecutive integers. Character orthogonality removes the principal character exactly. A character modulo $q=rk$ induced by a primitive character $\chi^*$ satisfies $\chi(m)=\chi^*(m)\mathbf{1}_{(m,k)=1}$, including when $(r,k)>1$. Put $\alpha_k=\alpha\mathbf{1}_{(m,k)=1}$ and $\beta_k=\beta\mathbf{1}_{(n,k)=1}$ and write $A_k(\chi),B_k(\chi)$ for their character sums. Since $\varphi(rk)\geq\varphi(r)\varphi(k)$, the required sum is bounded by

$$\sum_{k\leq Q}\frac{1}{\varphi(k)}\sum_{1<r\leq Q/k}\frac{1}{\varphi(r)}\sideset{}{^*}\sum_{\chi\ (r)}|A_k(\chi)B_k(\chi)|.\tag{116}$$

Both coprimality restrictions are essential. The restrictions do not increase the $L^2$ norms and, for fixed $k$, do not depend on $r$.

Let $R_0=L^{B_0}$. For $R<r\leq 2R$, $R\geq R_0$, Cauchy–Schwarz and (115) bound the inner sum by

$$\ll L^{C'}\left\{\frac{x}{R}+\sqrt{x}(\sqrt{U}+\sqrt{V})+R\sqrt{x}\right\}.$$

Summing dyadically, then over $k$ in (116), gives

$$\ll L^{C'}\left\{\frac{xL}{R_0}+L^2\sqrt{x}(\sqrt{U}+\sqrt{V})+Q\sqrt{x}\right\}.\tag{117}$$

Here $\sum_{k\leq Y}1/\varphi(k)\ll 1+\log Y$ and $\sum_{k\geq 1}1/(k\varphi(k))<\infty$; the latter sum occurs in the last term. Both follow from $n/\varphi(n)=\sum_{d\mid n}\mu(d)^2/\varphi(d)$ and $1/(d\varphi(d))\leq\tau(d)/d^2$. The middle term in (117) has a fixed power saving because $\sqrt{x}(\sqrt{U}+\sqrt{V})\leq 2x^{1-\eta/2}$. Taking $B_0,D$ sufficiently large controls the other two terms.

For $r \le R_0$, sum (114) against the primitive nonprincipal character. Its mean vanishes, yielding $|A_k(\chi)| \ll \varphi(r)\tau(k)^{C_0}UL^{-A_1}$. The $L^1$ bound for $\beta_k$ is $O(VL^{C/2})$, so the small-conductor part of (116) is at most

$$\ll xL^{C/2-A_1}\left(\sum_{r\le R_0}\varphi(r)\right)\left(\sum_{k\le Q}\frac{\tau(k)^{C_0}}{\varphi(k)}\right).$$

Choose an integer $J \ge C_0+1$. Since $n/\varphi(n) \le \tau(n)$ and $\tau(n)^J \le d_{2^J}(n)$, the last sum is at most $(1+\log Q)^{2^J}$. After $B_0$ is fixed, taking $A_1$ sufficiently large finishes the proof. All divisor orders are fixed before the requested logarithmic accuracy. □

For an original prime slot, the classical prime Siegel–Walfisz theorem [14, Exercise 64], followed by partial summation, gives (114) at $k=1$ with any prescribed saving. For arbitrary $k$ the change deletes only primes dividing $k$ and is at most $2\|\alpha\|_\infty\omega_{\mathrm{pf}}(k)$; its fixed logarithmic norm is absorbed by the positive-power scale $U$. Thus $C_0=1$ works for all the required exponents. Fixed smooth profiles, interval endpoints and Mellin twists with fixed polylogarithmic variation preserve the estimate. For several original primes, put one such slot in $\alpha$ and the remaining factors in $\beta$. Unique factorization supplies a fixed divisor bound and hence the required $L^2$ norm for $\beta$.

### D.2. **Proof of the prime-product estimate.**

**Proposition D.3.** *For the finite upper, lower and marked coefficients constructed in Appendix C, the restricted medium estimate of Proposition 3.2 and the below-half-level Bombieri–Vinogradov estimates imply*

$$\sum_q \rho_N(q)E_q(N;W) = O_{A,W}\big(N(\log N)^{-A}\big) \qquad (A>0).$$

*The coefficient in the count and exact coprime mean is the same. The estimate is uniform for the fixed finite family and all sufficiently large even $N$.*

*Proof.* Choose a smooth $\eta$ equal to zero on $[0,1]$ and one on $[2,\infty)$, and multiply $W_N$ by $\eta(p_1\cdots p_k/N^{1-\kappa_0})$. The discarded products have size less than $2N^{1-\kappa_0}$, and are strictly below $N$ for large $N$. Their complete modulus-divisor sum and exact mean have error $O(N^{1-\kappa_0+o(1)})$. All Euler derivatives of this trim are bounded.

Insert a smooth nonnegative dyadic partition in each original prime. On an original chart put $R_i = 2^{j_i}$, $X = \prod_i R_i$ and $X^+ = 2^k X$. With $f(v) = v^{1/(\log\log v)^3}$ fix the common roughness threshold

$$z = f(2^{6k_*+2}X^+)$$

before applying a convolution identity. Every original prime exceeds $z$ eventually. Replacing its prime indicator by $\Lambda(n)/\log n$ and deleting proper prime powers costs a fixed power saving: their reciprocal mass above $cN^{a_*}$ is $N^{-a_*/2+o(1)}$. Nonzero complete differences have divisor cost $N^{o(1)}$, and an exact product $N$ contributes at most $N^{\theta_*+o(1)}$ separately.

For a group at scale $R$ let $Z=(2R)^{1/3}$ and $\mu_Z = \mu\mathbf{1}_{[1,Z]}$. If $\delta$ denotes the convolution identity, $\delta - 1*\mu_Z$ is supported above $Z$; consequently $\Lambda*(\delta-1*\mu_Z)^{*3}=0$ below $2R$. Expanding gives

$$\Lambda = \sum_{j=1}^{3}(-1)^{j-1}\binom{3}{j}\mu_Z^{*j}*1^{*(j-1)}*\log. \tag{118}$$

The original smooth chart is retained. The roughness projector factors over all the variables in (118); use $P^-(1)=\infty$. There is no additional sharp coupled group boundary.

Each original group has at most six variables. Individual rectangles meeting its smooth support give complete local scales satisfying

$$2^{-7k-1}X \le x \le 2^{6k-1}X^+, \qquad f(x) \le z \le x^{2/(\log\log x)^3}. \tag{119}$$

Thus $cN^{1-\kappa_0} \le x \le CN$ and $c_0 x \le N \le x^{1+\kappa}$ by (113). The scale $x$ is always that of the complete product.

To separate the geometric profile, use coordinates $u_j = \log(n_j/U_j)$ and a smooth compact extension $w$ of the profile. Writing $D_w = \sum_{S\subseteq\{1,\dots,m\}} \|\partial_{2S} w\|_1$, integration by parts gives

$$|\widehat{w}(\xi)| \le D_w \prod_j (1+\xi_j^2)^{-1}, \quad \frac{1}{(2\pi)^m}\int |\widehat{w}(\xi)|\,d\xi \le \frac{D_w}{2^m}.$$

The tail outside $|\xi|_\infty \le H$ is $O(mD_w/H)$. Here $D_w$ is a fixed logarithmic power. The arithmetic sums can be interchanged with the absolutely integrable Fourier representation. After the complete modulus sum, the tail at a nonzero difference $b-N$ is bounded by fixed mixed divisor moments

$$\sum_{b\asymp x} \tau(b)^R \tau(|b-N|)^S \ll x(\log x)^C.$$

Apply Lemma D.1; its fixed order is included in $K_{\mathrm{mom}}$. The exact difference zero is estimated separately by $N^{\theta_*+o(1)}$. Choose a sufficiently large logarithmic $H$. For the retained modes one pays the Fourier integral norm, rather than the number of modes.

Choose an individual medium slot, if present, in $X^{3\zeta} \le U \le X^{1/3+\zeta/4}$. Its local length lies in $x^{2\zeta} \le U \le x^{1/3+\zeta/2}$ by (119). Its sequence has the form

$$\alpha_n = e_n \mathbf{1}_I(n)\mathbf{1}_{P^-(n)>z} h(n) n^{it}, \qquad e_n = 1 \text{ or } \mu(n), \quad |h| \le 1,$$

with fixed polylogarithmic variation and frequency. The auxiliary Siegel–Walfisz property required in the medium estimate is

$$\left| \sum_{\substack{n\equiv b\ (q)\\ (n,k)=1}} \alpha_n - \frac{1}{\varphi(q)} \sum_{(n,kq)=1} \alpha_n \right| \ll_A U\tau(k)(\log x)^{-A}. \tag{120}$$

For completeness, roughness can be inserted at even Brun order $h = 2\lfloor(\log\log x)^2\rfloor$. Its next divisor support is $z^{h+1} < U^{1/4}$ and its gap density has arbitrary logarithmic decay. For squarefree $\ell$, $\mu(\ell v) = \mu(\ell)\mu(v)\mathbf{1}_{(v,\ell)=1}$. Put $V_\ell = U/\ell$ for the quotient scale. The identity

$$\mu(v)\mathbf{1}_{(v,\ell)=1} = \sum_{\substack{w\mid v\\ p\mid w\Rightarrow p\mid \ell}} \mu(v/w)$$

reduces the small-modulus calculation to the unrough Siegel–Walfisz estimate for $\mu$. Truncation at $w \le V_\ell^{1/4}$ leaves $O(V_\ell^{7/8}\tau(\ell)^2)$; the smallest inner scale is $U^{9/16}$. After summing $\ell \le U^{1/4}$ this is $O(U^{29/32}\log^4 U)$. Large moduli are bounded by the one-bounded interval estimate. Inserting $(n,k) = 1$ deletes only primes $p \mid k$ above $z$ and costs at most $2(U/z+1)\omega_{\mathrm{pf}}(k)$. Partial summation absorbs the allowed variation and twists by choosing a stronger initial logarithmic accuracy. This proves (120) for the actual slots.

The complementary convolution is bounded by a fixed divisor power. If necessary, truncate only this complementary coefficient at a fixed logarithmic threshold, using the same mixed divisor moment bound for nonzero differences and $N^{\theta_*+o(1)}$ at zero. The threshold is chosen after the fixed orders, and the stronger local logarithmic estimate restores its normalization. The medium slot is not truncated. Now apply Proposition 3.2 to the exact tensor representation (111).

If no medium slot occurs, every original group contains a long slot: since $\zeta < a_*/100$, six slots below $X^{3\zeta}$ cannot have product comparable to $R_i$. There are at most two long slots, because three would exceed the complete scale. A truncated Möbius slot is $O(X^{1/3})$ and cannot be long. Thus one or two long unweighted integer variables remain, with their logarithms, smooth twists, individual intervals, and common roughness projector. Their complementary short products are bounded by $x^{20\zeta}$ and $x^{40\zeta}$, respectively. Appendix E estimates both cases with the exact coprime mean.

Finally, the low-rank coefficient pieces have complete support with a fixed gap below $x^{1/2}$. For one original prime use maximal prime Bombieri–Vinogradov. For two or more original primes, apply Lemma D.2 using one original prime slot as explained above. The bounded total coefficient norm permits its absolute form. All fixed atom scalars, logarithmically many charts and the Fourier integral norm are paid by requesting stronger local logarithmic accuracy. The removed prime powers, collisions and trim have fixed power savings. The finite family therefore has one eventual threshold for the asserted estimate. □

## Appendix E. One and two long rough variables

The estimates here supply the branches without a medium slot in the proof of Proposition D.3.

For the one- and two-long-variable cases we use direct progression estimates and the classical complete Kloosterman bound. The argument keeps the exact coprime mean and is uniform in the size of the primitive residue.

E.1. **Brun polynomials with a uniform density gap.** Let $L = \log\log x$, $h = 2\lfloor L^2 \rfloor$, $r = h+1$ and $\log z \le 2\log x/L^3$. Put $P(z) = \prod_{p\le z} p$ and $y = z^r = x^{o(1)}$. Define

$$\lambda^+(d) = \mu(d)\mathbf{1}_{d\mid P(z),\,\omega_{\mathrm{pf}}(d)\le h}, \qquad \lambda^-(d) = \mu(d)\mathbf{1}_{d\mid P(z),\,\omega_{\mathrm{pf}}(d)\le h+1},$$

and

$$U_z(n) = \sum_{d\mid n}\lambda^+(d), \quad L_z(n) = \sum_{d\mid n}\lambda^-(d), \quad G_z = U_z - L_z, \quad r_z(n) = \mathbf{1}_{(n,P(z))=1}.$$

Both divisor sums have coefficients of absolute value at most one and support below $y$. The alternating binomial identity gives

$$L_z \le r_z \le U_z, \qquad U_z \ge 0, \qquad G_z(n) = \binom{\omega_{\mathrm{pf}}((n,P(z)))}{r} \ge 0. \tag{121}$$

This also covers $(n,P(z)) = 1$, when both polynomials are one. The lower polynomial need not be nonnegative.

For every $q$ set

$$A_q = \sum_{(d,q)=1}\frac{\lambda^+(d)}{d}, \quad g_q = \sum_{(d,q)=1}\frac{\lambda^+(d)-\lambda^-(d)}{d}, \quad V_q = \prod_{\substack{p\le z\\ p\nmid q}}\Big(1-\frac{1}{p}\Big).$$

Finite inclusion-exclusion, applied to independent Bernoulli variables of means $1/p$, gives

$$0 \le A_q - V_q \le g_q \le \frac{(\sum_{p\le z}1/p)^r}{r!} =: \epsilon_x. \tag{122}$$

The bound is uniform in the set of primes dividing $q$. The elementary prime harmonic bound $\sum_{p\le z}1/p \ll L$ and $r! \ge (r/e)^r$ yield $\epsilon_x \le \exp(-cL^2\log L)$ eventually. Thus $\epsilon_x$ is smaller than every fixed negative power of $\log x$, whereas $y = x^{o(1)}$ will occur only beside a fixed power saving.

E.2. **Smooth progression cores.** For a compactly supported smooth profile define $N_1(f) = \|f\|_1 + \|f'\|_1$ and $N_2(f) = \|f\|_1 + \|f''\|_1$. Progression summation and Möbius inversion give

$$\sum_{n\equiv b\ (q)} f(n/U) = \frac{U}{q}\int f + O(N_1(f)),$$

$$\sum_{(n,q)=1} f(n/U) = U\frac{\varphi(q)}{q}\int f + O(\tau(q)N_1(f)). \tag{123}$$

These estimates are uniform in $U > 0$ and in the primitive class $b$. For a fixed $m$ coprime to $q$, the discrepancy of $mn \equiv a \pmod q$ from its exact coprime mean is therefore $O((1 + \tau(q)/\varphi(q))N_1(f))$. If $(m, q) > 1$ and $(a, q) = 1$, both terms vanish.

For two variables, use $e(t) = e^{2\pi it}$ and $\widehat{f}(t) = \int f(s)e(-ts)\,ds$. Let $c_q$ be the Ramanujan sum and $S(j, k; q) = \sum^*_{u \bmod q} e((ju + k\bar{u})/q)$. For $b = a\bar{m} \pmod q$, double periodization gives the exact discrepancy

$$\frac{UV}{q^2} \sum_{j,k\in\mathbb{Z}} \widehat{f}(jU/q)\widehat{g}(kV/q) \left( S(bj, k; q) - \frac{c_q(j)c_q(k)}{\varphi(q)} \right). \tag{124}$$

Both axes vanish identically, because $S(0, k; q) = c_q(k)$, $S(bj, 0; q) = c_q(j)$ and $c_q(0) = \varphi(q)$. For $jk \neq 0$, use

$$|S(bj, k; q)| \le \tau(q)\sqrt{q}\,(j, k, q)^{1/2}, \qquad |\widehat{f}(t)| \ll N_2(f)(1 + |t|)^{-2}.$$

For every $H > 0$ and $d \ge 1$, $\sum_{j\neq 0,\, d|j}(1 + |j|/H)^{-2} \le 2H/d$. Expanding common divisors and using $\sum_{d|q} d^{-3/2} \le \zeta(3/2)$ therefore bounds (124) by

$$N_2(f)N_2(g) \left( \sqrt{q}\,\tau(q) + \frac{\tau(q)^2}{\varphi(q)} \right). \tag{125}$$

For the Ramanujan term one uses $|c_q(j)| \le (j, q)$ and $\sum_{d|q} \varphi(d)/d \le \tau(q)$. When $q = 1$ the discrepancy is zero.

Let a short coefficient $\alpha_m$ have $\sum |\alpha_m| \ll M \log^C x$, with $x \asymp MU$ or $MUV$. If $q \le D$ and $|\rho(q)| \le \tau(q)^J \log^B x$ with fixed $J, B$, the sums of the one- and two-variable core errors are

$$O(MD \log^{C'} x), \qquad O(MD^{3/2} \log^{C'} x), \tag{126}$$

respectively. On inserting Brun divisors $d_1, d_2 \le y$, the profiles are at scales $U/d_1, V/d_2$ and their norms in (125) are unchanged. Replace $m$ by $md_1d_2$ only in the primitive residue. The fixed-product bound is uniform in that residue, so summing the divisor pairs costs at most $y^2$.

E.3. **Restoring roughness with the exact mean.** For a nonnegative smooth $F$, expansion of the divisor coefficients in (123) gives

$$\sum_{(n,q)=1} F(n/U)U_z(n) = U\frac{\varphi(q)}{q}A_q \int F + O(y\tau(q)N_1(F)),$$

$$\sum_{(n,q)=1} F(n/U)G_z(n) = U\frac{\varphi(q)}{q}g_q \int F + O(y\tau(q)N_1(F)). \tag{127}$$

Both left sides are nonnegative. They are exact coprime sums; the factor $\varphi(q)/q$ is retained throughout.

For two variables write $U_i = U_z(n_i)$, $r_i = r_z(n_i)$ and $G_i = G_z(n_i)$. The valid product comparison is

$$0 \le U_1U_2 - r_1r_2 \le G_1U_2 + U_1G_2. \tag{128}$$

Indeed $U_1U_2 - r_1r_2 = (U_1 - r_1)U_2 + r_1(U_2 - r_2)$. This argument does not multiply two possibly negative lower polynomials. For complex profiles choose nonnegative smooth majorants $F \ge |f|$, $G \ge |g|$ and use $|\alpha|$ only in estimating the absolute gap. If $C_q$ denotes the congruence count and $M_q$ its exact coprime mean, and $B = G_1U_2 + U_1G_2$, then

$$\begin{aligned} |(C_q - M_q)(\alpha fg(r_1r_2 - U_1U_2))| &\le C_q(|\alpha|FGB) + M_q(|\alpha|FGB) \\ &\le |(C_q - M_q)(|\alpha|FGB)| + 2M_q(|\alpha|FGB). \end{aligned}$$

The first term is covered by expanding the Brun divisors and (126); the second is estimated by (127). Fixed divisor/totient mean bounds give the total two-variable error

$$\ll \log^{C'} x \left\{ MUV\epsilon_x + M(U+V)y + My^2 + My^2D^{3/2} \right\}. \tag{129}$$

The one-variable version is

$$\ll \log^{C'} x\{MU\epsilon_x + My + MyD\}. \tag{130}$$

Only fixed logarithmic factors multiply $\epsilon_x$. For the long alternatives, $M \le x^{20\zeta}$ or $x^{40\zeta}$, $D \le x^{\Theta_{\mathrm{loc}}}$ with $\Theta_{\mathrm{loc}} < 3/5$, and $\min(U,V) \ge x^{\nu}$ for a fixed $\nu > 0$. Choose $\zeta$ so that

$$20\zeta + \Theta_{\mathrm{loc}} < 1, \qquad 40\zeta + \frac{3}{2}\Theta_{\mathrm{loc}} < 1.$$

Every remaining term in (129) and (130) then has a fixed power saving. The errors are consequently $O_A(x \log^{-A} x)$ for every fixed $A$.

E.4. **Individual sharp intervals.** Let $\delta = (\log x)^{-B}$. An interval in a fixed positive scale has a smooth approximation whose error is bounded by nonnegative smooth strips of normalized width $O(\delta)$ at its endpoints. If the interval is shorter than $\delta$, a single strip covers it. Their integrals are $O(\delta)$, their first-derivative norms are $O(1)$, and their second-derivative norms are bounded by a fixed power of $\delta^{-1}$. Multiply by the fixed envelopes for the other profiles.

Apply the proved rough smooth estimates to these positive strips. Their exact means are bounded by

$$\ll \log^{C'} x\{x\delta + M(U+V)y + My^2\}$$

in the two-variable case, and by $\log^{C'} x\{x\delta + My\}$ in the one-variable case. The exponent in the first term uses the strip integral, not its derivatives, and is independent of $B$. The $B$-dependent derivative losses affect only the fixed-power errors. Choose $B$ after the desired logarithmic accuracy. This controls both the count and the exact mean, uniformly in every primitive residue, including a residue equal to an intermediate complete product. It proves the required extension to sharp individual intervals. No assertion about arbitrary sharp coupled domains is used.

## Appendix F. Causal upper sieve policies

This appendix gives the upper-policy density and coefficient realization used in Section 4; its finite savings are evaluated in Appendix I.

We construct the divisor coefficients from finitely many decisions on prime prefixes. Prefixes are ordered by decreasing primes: if $p_i = N^{t_i}$, then $t_1 > \cdots > t_r > 0$. Write $S_j = t_1 + \cdots + t_j$ and $S_0 = 0$. A parent is obtained by deleting the last prime of a prefix. Every decision at that parent is retained when smaller primes are appended.

F.1. **Admissible caps and the parity of first exits.** For a nonempty subset $I \subseteq \{1, \dots, r\}$ write

$$b_I = \sum_{i \in I} t_i, \qquad G_I(t) = \max_{\substack{j \le r \\ j \notin I}} \left( b_I + \sum_{\substack{i<j \\ i \notin I}} t_i + 2t_j \right), \qquad \max \varnothing = 0. \tag{131}$$

The full-square support condition is

$$G_{\mathrm{full}}(t) = \max_{1 \le j \le r} (S_{j-1} + 2t_j). \tag{132}$$

The candidate selected-product endpoint is

$$v(b) = \min\left\{ \frac{1+b}{2}, \frac{57}{64} - \frac{3b}{2} \right\}. \tag{133}$$

A selected branch on a compact prefix region is admitted only when its constant cap $\lambda$ satisfies, with a fixed positive margin,

$$\max\{1/2, G_{\mathrm{full}}(t), G_I(t)\} < \lambda < v(b_I) \tag{134}$$

throughout that region. The separate three-slot branch with endpoint $\Lambda_{\mathrm{cap}} - 3t_3/8$ is subject to the whole-box factorization conditions of Appendix C. A maximum of candidate endpoints is always taken over branches satisfying their support conditions.

An ordinary upper residual tree at total exponent $\lambda$, after a prefix of rank $r$, applies the cube tests

$$S_{j-1} + 3t_j < \lambda \qquad (j > r,\ j \text{ odd}). \tag{135}$$

If $r$ is even, these are the ordinary upper residual tests. If $r$ is odd, they are the ordinary lower residual tests. The earlier membership decisions are retained. At an even prefix a change of cap is made after admitting the prefix; it introduces no exclusion at that rank. At an odd prefix a cap may be chosen before its own odd test, provided that the decision depends only on that prefix and the previously fixed policy. Either operation leaves every first exclusion at odd full rank.

**Lemma F.1** (First-exit identity)**.** *Let $\mathcal{T}$ be a prefix-closed collection of squarefree products from a finite prime set $\mathcal{P}$, containing the empty product. Suppose that all first excluded prefixes have odd rank. For every multiplicative $\xi$ with $0 \le \xi(p) \le 1$,*

$$\sum_{d\in\mathcal{T}} \mu(d)\xi(d) = \prod_{p\in\mathcal{P}} (1-\xi(p)) + \sum_{e \text{ first exit}} \xi(e) \prod_{\substack{p\in\mathcal{P}\\ p<p_{\mathrm{last}}(e)}} (1-\xi(p)). \tag{136}$$

*In particular $\sum_{d\mid n,\ d\in\mathcal{T}} \mu(d)$ majorizes $\mathbf{1}_{(n,\prod_{p\in\mathcal{P}} p)=1}$ for every integer $n$.*

*Proof.* Each excluded divisor has a unique first excluded prefix $e$; its remaining prime factors are smaller than $p_{\mathrm{last}}(e)$. Summing the complete continuation gives $\mu(e)\xi(e)\prod_{p<p_{\mathrm{last}}(e)}(1-\xi(p))$. Subtract these disjoint continuations from the complete Euler product. Since $\mu(e) = -1$, the sign in (136) is positive. Take $\xi(p) = \mathbf{1}_{p\mid n}$ for the last assertion. The same finite identity applies at $n = 0$ and at negative integers. □

The square tests persist through descendants. A retained odd cube implies its own square. The following even square follows from ordering:

$$S_{j-1} + 2t_j \le S_{j-2} + 3t_{j-1} < \lambda. \tag{137}$$

After all selected slots have appeared, the expression in (131) for any appended slot equals its full square. Consequently (134), checked at the selected prefix, continues to control the complete descendant vector.

F.2. **The cap changes and their density differences.** Set

$$\rho_0 = \frac{19}{96}, \qquad \mu_0 = \frac{25}{128}, \qquad \Theta = \frac{299}{500}, \qquad \Lambda_{\mathrm{cap}} = \frac{153}{256}.$$

The first prime is less than $N^{\rho_0}$. The three-slot endpoint below $t_1 = \mu_0$ is the envelope of admissible branches of

$$v(t_1),\ v(t_1+t_2),\ v(t_1+t_3),\ v(t_2+t_3),\ v(S_3),\ \Lambda_{\mathrm{cap}} - \frac{3t_3}{8}; \tag{138}$$

above $\mu_0$ the initial three-slot endpoint is $v(t_1)$. Every branch in (138) retains its individual factorization conditions. In particular the displayed list is not an unconditional permission to select any nonleading pair.

At rank four the selected candidates are $M = v(b_I)$ with $4 \in I$, satisfying (134) and exceeding the inherited third cap. Such a change is made only on prefixes that already pass

the third cube. Membership through rank four is unchanged and subsequent odd cubes use the new cap. The limiting density difference on a region $\mathcal{C}$ is

$$4\int_{\mathcal{C}}\frac{\overline{F}((L-S_r)/t_r)-\overline{F}((M-S_r)/t_r)}{t_1\cdots t_{r-1}t_r^2}\,dt_1\cdots dt_r,\qquad \overline{F}=\frac{F}{2e^{\gamma}},\tag{139}$$

where $r=4$, $L$ is the inherited cap and $M$ the selected endpoint. Indeed an even prefix $P_r$ contributes $g_N(P_r)V_N^{+}(N^{\lambda}/P_r,p_r)$ to its complete subtree. Subtracting the two subtrees gives (139); it is an identity of raw densities before taking an asymptotic limit.

At rank five a candidate $v(b_I)$, $5\in I$, is compared with the inherited cap and the fifth cube $S_4+3t_5$. Its cap is fixed before that cube. Thus a fifth prefix rejected at the smaller cap may be retained at the larger one; all earlier decisions remain fixed. The negative fifth subtree has the ordinary lower residual, and its saving is

$$4\int_{\mathcal{C}}\frac{\overline{f}((M-S_5)/t_5)-\overline{f}((L-S_5)/t_5)}{t_1t_2t_3t_4t_5^2}\,dt_1\cdots dt_5,\qquad \overline{f}=\frac{f}{2e^{\gamma}},\tag{140}$$

with the zero extension of $\overline{f}$ below 2. The inherited first-five cap is bounded above by

$$R_5(t)=\max\left\{\Lambda_{\mathrm{cap}}-\frac{3t_3}{8},\ v(b_I):\varnothing\neq I\subseteq\{1,\dots,5\}\right\}.\tag{141}$$

Using an upper bound for the inherited cap in a positive saving is conservative. It does not change the physical cap or its density.

The rank-six changes are even-prefix changes of the form (139). They are applied after preserving the third and fifth admission tests. In particular $L_3-S_6>2t_5-t_6>t_6$ on a region whose fifth cube is controlled by $L_3$. A convex piecewise-affine majorant $U_5\geq R_5$ can be obtained by choosing one defining affine upper plane for each tent in (141). For a new concave endpoint $M$, put $d=M-U_5$ and restrict the region to $M-S_6\leq 3t_6$. Write $L$ for the inherited cap. Where $d>0$, the retained fifth cube gives

$$1<\frac{L-S_6}{t_6}\leq\frac{U_5-S_6}{t_6}<\frac{M-S_6}{t_6}\leq 3.$$

Thus $\overline{F}(s)=1/s$ throughout the interval of comparison, and

$$\overline{F}((L-S_6)/t_6)-\overline{F}((M-S_6)/t_6)\geq\frac{t_6(M-U_5)}{(U_5-S_6)(M-S_6)}.$$

The cap is changed only where $d>0$, giving the saving lower bound

$$\frac{4d_+}{R_oR_n\prod_{i=1}^{6}t_i},\tag{142}$$

where $R_n\geq M-S_6$ and $R_o\geq\min\{U_5-S_6,M-S_6\}>0$ are whole-region upper bounds. The positive part is taken before a lower bound for the positive prime-density measure is extracted. For a rectangular region, concavity gives $\int d_+/\operatorname{Vol}(\mathcal{C})\geq 2^{-6}\sum_{v\text{ vertex}}d(v)$.

Two additional changes occupy disjoint parts of $\mu_0<x<\rho_0$. For the rank-two change, put $a(x)=2v(x)-1=25/32-3x$ and use

$$a(x)<y<x,\qquad L=v(x),\quad M=v(y).$$

The support conditions follow from $2x,x+2y,2x+y<v(x)<v(y)$; the selected slot is 2. The saving is the $r=2$ instance of (139). For the rank-three change, put $t=y+z$ and impose

$$a(x)<t<x,\quad z=tw/2,\quad 5/9\leq w<1,\qquad L=v(x),\quad M=v(t).$$

Here $y<3/16$, whereas the rank-two change has $y\geq 3/16$. Selecting slots $\{2,3\}$ gives the extra support condition $2x+t<M$. The third node is retained and its lower residual is changed, giving the $r=3$ analogue of (140). Its arguments lie in $[2,4]$, where $\overline{f}(s)=\log(s-1)/s$.

At rank eight a selected endpoint $M$ is compared with a valid upper bound $R_6$ for the inherited sixth cap. The retained seventh cube and the additional condition $M - S_8 < 3t_8$ give

$$1 < \frac{\lambda_6 - S_8}{t_8} \le \frac{R_6 - S_8}{t_8} < \frac{M - S_8}{t_8} < 3. \tag{143}$$

Consequently (139) is bounded below by the integral of

$$\frac{4(M - R_6)}{(R_6 - S_8)(M - S_8)\prod_{i=1}^{8} t_i}. \tag{144}$$

Each chosen region must retain every full and selected-complement square, as well as all ancestor tests. Regions acted upon at one stage have disjoint interiors. Changes at different stages are combined by the exact telescope $V_3 - V_6 = (V_3 - V_4) + (V_4 - V_5) + (V_5 - V_6)$. For each density difference we use one lower enclosure.

F.3. **Literal ordinary densities and finite realization.** Let $g_N(p) = \mathbf{1}_{p>2,\ p\nmid N}/(p-1)$ and $V_N(y) = \prod_{p<y}(1 - g_N(p))$. The ordinary cubic Rosser coefficients satisfy, uniformly on compact subsets of the stated ranges,

$$\frac{V_N^+(N^\beta, N^t)}{V_N(N^t)} = F(\beta/t) + o(1) \quad (\beta/t > 1), \qquad \frac{V_N^-(N^\beta, N^t)}{V_N(N^t)} = f(\beta/t) + o(1) \quad (\beta/t \ge 2). \tag{145}$$

Here the densities are those of the literal coefficients, not merely upper and lower bounds for an arbitrary sifted sequence.

To see the uniformity in $N$, compare a normalized fixed-rank exit with the corresponding exit for $g_0(p) = \mathbf{1}_{p>2}/(p-1)$. A surviving prefix gains the multiplier

$$\prod_{\substack{p_{\text{last}} \le p < y \\ p \mid N}} (1 - g_0(p)) \le 1;$$

other prefixes disappear. A rank-$j$ exit has last prime at least $N^{\beta_{\min}/(3\cdot 2^{j-1})}$ on a positive compact parameter range. Deleting prime divisors of $N$ therefore changes any fixed-rank integral by $o(1)$ uniformly. The positive Rosser exit series converges uniformly on compact parameter ranges, and the raw one-sided bounds give the opposite side of a finite-rank sandwich; see [13, Proposition 11.8 and Lemma 14.7]. First choose the rank and then let $N$ increase. This proves (145). For $1 < s \le 3$ the identity at the first cube reduces the upper calculation to cutoff $N^{\beta/3}$, giving $F(s) = 2e^\gamma/s$.

The normalization follows from Mertens' formula. In particular the exact ratio to $2C_N \prod_{p<z}(1 - 1/p)$ is

$$\frac{V_N(z)}{2C_N \prod_{p<z}(1-1/p)} = \prod_{p \ge z} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{\substack{p \mid N \\ p \ge z}} \left(1 - \frac{1}{p-1}\right). \tag{146}$$

Its logarithm is $O((1 + \log N/\log z)/(z-2))$. Thus $V_N(N^t) \sim 2e^{-\gamma} C_N/(t \log N)$ uniformly in $N$. The factor 4 in (139) and (140) is exactly $2e^{-\gamma} \cdot 2e^{\gamma}$.

The finite cell data consist of rational regions, their ranks, selected subsets, endpoints, and parent labels. Every active region has positive margins in its admission and support inequalities; an inherited region keeps its parent's decision. The density is the sum of the exact subtree differences. For a positive saving, any selected subfamily gives a lower bound. For a negative cost, the upper bound must include the whole retained domain.

**Proposition F.2** (Finite-policy realization)**.** *Fix such a finite admissible policy and a positive error $\epsilon$. Assume the coefficient factorization conditions of Appendix C on its selected*

*branches. There is a fixed finite refinement and a bounded squarefree coefficient* $\rho_N^{\mathrm{up}}$*, supported on* $(q,N)=1$*, with the following properties. Its divisor sum is a nonnegative rough-indicator majorant. The signed distribution estimate* (18) *applies to that same coefficient. Its density differs, in the required upper direction, from the ideal policy density by at most* $\epsilon C_N/\log N + o(C_N/\log N)$*.*

*Proof.* First truncate any small-last-prime tail. At the third activation one may use, with $u=\rho_0/2$ and $0<h<u$,

$$T_3(h) \le \frac{18h}{(u-h)^2}\left(\log^2\frac{\rho_0}{h} + 2\log\frac{\rho_0}{h} + 2\right) = o(1). \tag{147}$$

This follows from $\overline{F}(s)-\overline{f}(s)\le 9/s^2$ for $s\ge 3$, the residual lower bound $\lambda - t_1 - t_2 \ge u$, and the ordered reciprocal-prime triangle of mass $\frac{1}{2}\log^2(\rho_0/t_3)$. Higher selected changes have positive last-prime cutoffs. Proper affine walls have arbitrarily small strip volume. Identities and open inherit regions are retained. On the remaining compact pieces choose coarse caps with strict margins and refine them in increasing prefix order. No ancestor decision may depend on an appended prime.

The third endpoint is piecewise affine and Lipschitz. For boxes of width $\chi < e/10$, caps of the form $\lambda_{\mathscr{B}} = \Gamma_{\mathscr{B}} - 2e$ approximate it with error at most $4e+6\chi$. Near a three-slot branch with endpoint $W = \Lambda_{\mathrm{cap}} - 3t_3/8$, an auxiliary factorization cover may exceed the physical cap. If $\inf_{\mathscr{B}} A \ge \lambda_{\mathscr{B}} + e/2$, where $A=(1+t_1)/2$, take $\ell_{\mathrm{aux}} = \lambda_{\mathscr{B}} + e/4$. Otherwise $\sup_{\mathscr{B}} A < \lambda_{\mathscr{B}} + 11e/20$, and $\ell_{\mathrm{aux}} = \lambda_{\mathscr{B}} + e$ lies at least $9e/20$ above the necessary support condition and $e$ below $W$. The other initialization square has margin $3/256$. The auxiliary cover gives a factorization for the coefficient; it never replaces the physical cap in a density or residual level.

Choose a presieve $z_0 = N^{\eta_0}$. At even rough rank use outer endpoint tests and at odd rough rank use inner tests, retaining all ancestor labels. This gives $a_N(d) \ge \lambda_N(d)$ for the raw upper coefficients. If $B^- \le I_0 \le B^+$ are the dimension-one small-prime beta polynomials, attach $B^+$ to positive $a_N(d)$ and $B^-$ to negative $a_N(d)$. Term by term, $a_N(d)B_{\mathrm{sign}} \ge a_N(d)I_0$. Unique small/rough factorization defines the resulting $\rho_N^{\mathrm{up}}$ and preserves squarefreeness and boundedness.

The density direction needs an additional identity. Projecting (136) to primes at least $z_0$ gives

$$V_N(z_0)\sum_{d\ \text{rough}} \lambda_N(d)g_N(d) \le \sum_{d\ \text{full}} \lambda_N(d)g_N(d). \tag{148}$$

Only positive exits with last prime at least $z_0$ survive on the left. For a fixed coarse policy the absolute beta and fine-box errors are $O(e^{-c/\eta}\eta_0^{-2} + C_{\mathrm{policy}}\tau\eta_0^{-5})$ relative to the root Euler product. Choose $\eta,\eta_0$ and then $\tau$ so that these errors and the truncated pieces fit within $\epsilon$.

All these choices precede $N$. The fine rank is bounded by $J_0 > \theta_*/\eta_0 + 2$; support inflation can be kept below $1/1000$, giving $\theta_* = 599/1000$. Finally choose the fixed moment order and complete-scale trim so that $\theta_*/(1-\kappa_0) < 3/5$ and all local inequalities have strict margins. Appendix C then gives (18) for this very coefficient. The pointwise bound follows from Lemma F.1; the density bound follows from (148) and the absolute errors. □

## Appendix G. Lower policies and finite feedback

This appendix supplies the lower-policy realizations and positive marked transition inequalities used in Section 6. Appendix I evaluates the corresponding finite maps and integrals.

G.1. **The ordinary lower tree and fourth-prefix restoration.** Put $a_{\mathrm{s}} = 417/5000$, $h_{\mathrm{s}} = 19/125$ and $\nu = 249/448$. For either $w = a_{\mathrm{s}}$ or $w = h_{\mathrm{s}}$, begin with the ordinary lower

support at total exponent $\nu_\delta = \nu - \delta$, where $0 < \delta$ is fixed and $2h_{\mathrm{s}} < \nu_\delta$. Its first square $2t_1 < \nu_\delta$ is automatic below $w$; its subsequent tests are

$$S_{j-1} + 3t_j < \nu_\delta \qquad (j \text{ even}). \tag{149}$$

At an admitted third prefix, freeze $\lambda_3 \geq \nu_\delta$ for future even cubes. An ordinary fourth change is permitted only after $S_3 + 3t_4 < \lambda_3$; it freezes $\lambda_4 \geq \lambda_3$. The fifth prefix is then admitted, and may freeze $\lambda_5 \geq \lambda_4$ for future even tests. Every active selected cap is below its admissible endpoint and above its parent and all support conditions (131)–(132). For the lower starting families the selected endpoint is

$$v_-(b) = \min\left\{\frac{1+b}{2}, \frac{32-34b}{39}, \frac{57}{64} - \frac{3b}{2}\right\}. \tag{150}$$

The selected branch is kept through descendants; the residual size ratio of a marked state, introduced below, is separate from the local residue exponent in the distribution estimate.

There is also a fourth-prefix restoration. On an otherwise first excluded fourth prefix choose $\lambda_4$ before its own cube, subject to

$$\lambda_3 \leq S_3 + 3t_4 < \lambda_4, \qquad \max\{G_{\mathrm{full}}, G_I\} < \lambda_4 < v_-(b_I). \tag{151}$$

Attach the ordinary lower residual below $p_4$ at level $N^{\lambda_4}/P_4$. Since $\lambda_4 - S_4 > 2t_4$, its first square cannot fail. Its later first exits have even residual rank, hence even full rank. The restored tree remains a lower tree. Its density gain is exactly

$$4\int_{\mathcal{B}} \frac{\overline{f}((\lambda_4 - S_4)/t_4)}{t_1 t_2 t_3 t_4^2}\, dt_1 dt_2 dt_3 dt_4. \tag{152}$$

There was no retained residual at such a first exit, so no parent residual term is subtracted in (152).

Define a finite rational family of restoration regions as follows. Let $r_0 = (\nu - h_{\mathrm{s}})/9$, $n \in \{400, 800\}$, and $\mathscr{B} = \prod_{i=1}^4 [\ell_i, \ell_i + 1/n]$ with $r_0 \leq \ell_4 < \ell_3 < \ell_2 < \ell_1 < h_{\mathrm{s}}$ and $u_1 + 3u_2 < \nu$, where $u_i = \ell_i + 1/n$. Let $U_3$ be the maximum of $\nu$, the upper bounds of all three-slot selected endpoints, and $\Lambda_{\mathrm{cap}} - 3\ell_3/8$. It bounds the inherited third cap from above. With $D = \operatorname{lcm}(448, 256, 78n, 64n, 8n)$, retain a mask $I$ containing 4 only if

$$c_I = \inf_{\mathscr{B}} v_-(b_I) - D^{-1} > \max\{U_3, \sup_{\mathscr{B}} G_{\mathrm{full}}, \sup_{\mathscr{B}} G_I\}.$$

Choose the largest retained $c_I$; write it as $c$. For $A = t_1 + t_2 + t_3 + 3t_4$, clip the box to

$$\max(A_-, U_3 + D^{-1}) < A < \min(A_+, c - D^{-1}),$$

and split this interval into finitely many equal rational layers. Every layer satisfies (151) with positive margins and lies outside previously retained fourth prefixes. On the clipped region,

$$2 < \frac{c - S_4}{t_4} < 2 + \frac{9(\Lambda_{\mathrm{cap}} - \nu)}{\nu - h_{\mathrm{s}}} = 2 + \frac{84375}{90452} < 3.$$

Thus $\overline{f}(s) = \log(s-1)/s$ is increasing throughout. If $A < a_j$ on a layer and $u_4$ is the fourth upper endpoint, use $s_j = 2 + (c - a_j)/u_4$ in the lower bound $4\overline{f}(s_j)\operatorname{Vol}(E_j)/(u_1 u_2 u_3 u_4^2)$. The exact volume of $\mathscr{B} \cap \{A < t\}$ is

$$\frac{1}{72} \sum_{e \in \{0,1\}^4} (-1)^{|e|} \left(t - A_- - \frac{e_1 + e_2 + e_3 + 3e_4}{n}\right)_+^4. \tag{153}$$

This is the inclusion–exclusion formula for weighted box widths $(1, 1, 1, 3)/n$. Differences of (153) give the layer volumes exactly.

G.2. **The lower coefficient and its original prime sequence.** For a prefix-closed tree with even first exits the counterpart of (136) is

$$\sum_{d\in\mathcal{T}}\mu(d)\xi(d)=\prod_{p}(1-\xi(p))-\sum_{e\text{ first exit}}\xi(e)\prod_{p<p_{\mathrm{last}}(e)}(1-\xi(p)). \tag{154}$$

Its proof is identical, using $\mu(e)=1$. Taking $\xi(p)=\mathbf{1}_{p|n}$ gives a rough-indicator minorant. With $\xi=g_N$ and presieve $z_0$ it gives

$$V_N(z_0)R_{\mathrm{rough}}-V_{\mathrm{full}}=\sum_{\substack{e\text{ full even exit}\\ p_{\mathrm{last}}(e)<z_0}}g_N(e)V_N(p_{\mathrm{last}}(e))\geq 0. \tag{155}$$

This is the density direction required for lower coefficient realization.

Round positive even coefficients inward and negative odd coefficients outward, preserving the prefix decisions. Attach beta-minus to a positive rounded coefficient and beta-plus to a negative one. Term by term $a_{\mathrm{rnd}}(d)B_{\mathrm{sign}}\leq a_{\mathrm{rnd}}(d)I_0$. Consequently the physical coefficient satisfies

$$\sum_{q|n}\rho_N^{\mathrm{low}}(q)\leq\mathbf{1}_{(n,P_N(N^w))=1},\qquad\sum_{q}\frac{\rho_N^{\mathrm{low}}(q)}{\varphi(q)}\geq V_{\mathrm{full}}-\text{absolute density error}. \tag{156}$$

The second inequality uses (155) and the absolute beta and box estimates. It does not follow merely from coefficientwise order, and beta-minus need not be nonnegative. The sharp-prime consequence in Appendix H, applied to this same $\rho_N^{\mathrm{low}}$, turns (156) into the ordinary lower estimate for $\{N-p:p<N,\ p\text{ prime}\}$.

For an admissible lower prefix, the third-, fourth-, and fifth-stage density changes are, respectively, upper parent minus upper child, lower child minus lower parent, and upper parent minus upper child. Evaluate each by (145). At ranks $r=3,4,5$, the difference of the ordinary upper and lower densities cancels the apparent inverse-square singularity in the last coordinate, leaving a tail $O(\epsilon(1+\log^{r-1}(h_{\mathrm{s}}/\epsilon)))$. The remaining compact regions admit one common finite refinement that preserves every inherited decision.

G.3. **Finite feedback on positive marked families.** The normalized states are estimates averaged over a nonnegative finite marked family $\sigma(B)$ with a common total level $Q=N^{\lambda}$. Put

$$L=\lambda-\log_N B>0,\qquad\Theta_Q=4\,\mathrm{li}(N)\sum_{B}\frac{\sigma(B)C_{BN}}{\varphi(B)\log(Q/B)},\qquad\alpha=\frac{1-\log_N B}{L}\geq 2.$$

For a singleton child of exponent $Lu$, or a pair of exponents $Lu, Lv$, the residual level and parameter are, respectively,

$$L'=L(1-u),\quad\alpha'=1+\frac{\alpha-1}{1-u};\qquad L'=L(1-u-v),\quad\alpha'=1+\frac{\alpha-1}{1-u-v}. \tag{157}$$

The total $Q$, selected slots and prime ceiling are retained. In particular $\alpha$ increases; the child is not reset to $\alpha=2$.

An admitted marked family here means a finite nonnegative combination of labelled prime-box indicators, with fixed positive lower bounds on all marked prime exponents and on $L$. In the classical case its complete modulus level is below $N^{1/2}$ by a fixed power. In the enhanced case $1/2<\lambda<153/256$ with fixed margins; it retains all full and selected-complement square conditions and one factorization branch from Appendix C. It also retains a number $\chi>0$ such that adjoining primes of exponent below $\chi$ preserves that branch and its support conditions. The number $\chi$ is no larger than the least specified exponent. If a branch uses a bound $c$ for a third slot not yet specified, it also satisfies $\chi\leq c$. Every enhanced ordinary node at argument $s$ has physical cutoff $N^{L/s}$ below $N^{\chi}$. All these restrictions are imposed with fixed margins on each box before endpoint limits are taken. The estimates concern the marked average, rather than an individual fixed prime prefix.

The child-to-parent measures are $du/[u(1-u)]$ and $du\,dv/[uv(1-u-v)]$. They are positive. To specify the one-step switched cost, define

$$I(s,s_0,\alpha)=\int_{1/s_0<u_1<u_2<u_3<1/s}\omega\Big(\frac{\alpha-u_1-u_2-u_3}{u_2}\Big)\frac{du_1\,du_2\,du_3}{u_1u_2^2u_3}. \tag{158}$$

The one-step inequality gives the candidate

$$\begin{aligned}\mathcal{W}_1[H,h](s;s_0,\alpha)&=A_+(s)-A_+(s_0)+H(s_0)-\frac{K}{8}I(s,s_0,\alpha)\\&\quad+\frac12\int_{1/s_0}^{1/s}\frac{A_-(s_0(1-u))+h(s_0(1-u))}{u(1-u)}\,du,\qquad 2\le s<s_0\le 5.\end{aligned} \tag{159}$$

Here $A_+(s)=sF(s)/(2e^\gamma)$ and $A_-(s)=sf(s)/(2e^\gamma)$, with the zero extension below 2. The physical singleton child cutoff is $N^{L/s_0}$ and its argument is $s_0(1-u)$, not the argument at a different total level. The positive switched family is $Bq_1q_2q_3n\le N$ with $N^{L/s_0}\le q_1<q_2<q_3<N^{L/s}$ and $P^-(n)\ge q_2$. Its unsifted mass, in the parent normalization, is $I/4$; the factor $1/2$ in the pointwise weighted inequality gives the cost $KI/8$ in (159). All its prime factors have fixed positive exponents. Its unit quotient has a fixed power saving because $\alpha-3/s\ge 1/2$.

A uniform bound usable in (158) is obtained by putting $c_\omega=70893/125000$ and

$$\omega_{\mathrm{env}}(u)=\begin{cases}1, & u<1,\\ \max\{1/u,c_\omega\}, & u\ge 1.\end{cases} \tag{160}$$

For $v\ge u\ge 1$ one has $\omega(v)\le\omega_{\mathrm{env}}(u)$. On $[1,2]$ this follows from $\omega(v)=1/v$; on $[2,3]$ maximize $1+\log(v-1)-c_\omega v$. The maximum is $-\log c_\omega-c_\omega<0$, since the exact rational inequality $c_\omega\sum_{j=0}^{8}c_\omega^j/j!>1$ implies $c_\omega e^{c_\omega}>1$. The delay equation propagates the bound for $v\ge 3$. Thus replacing the integrand by $\omega_{\mathrm{env}}((2-u_1-u_2-u_3)/u_2)$ bounds the cost for every $\alpha\ge 2$. The branch below 1 is a conservative bound for the switched estimates that use it; unit quotient terms are counted separately when their domains have nonnegligible mass.

For a two-step action write $a_0=1/s_0$, $a_i=1/\phi_i$ and $b=1/s$. The parameter conditions are

$$\begin{aligned}&2<s<\phi_3<\phi_2<\phi_1<s_0\le 5,\qquad a_3+2a_1<1,\quad 3a_1<1,\quad a_2+2a_1<1,\\&a_0<(1-a_2)/2<1/3<a_2<1/2,\qquad s_0(1-b)>2,\quad s_0-1\le 4,\quad s_0(2-2a_2)>2.\end{aligned} \tag{161}$$

Its three singleton regions are

$$(a_0,b),\qquad (a_0,a_2),\qquad (a_0,a_3),\quad \rho(u)=s_0(1-u),\quad \text{physical cutoff } N^{La_0}. \tag{162}$$

The regions on which the ordinary pair estimate and its feedback are used are

| | region | child argument |
|---|---|---|
| $D_5^+$ | $a_0<u<v<a_2,\ v+2u<1$ | $s_0(1-u-v)$ |
| $D_6$ | $(a_0,a_1)\times(a_2,a_3)$ | $s_0(1-u-v)$ |
| $D_7$ | $a_0<u<v<a_1$ | $(1-u-v)/u$ |
| $D_8$ | $(a_0,a_1)\times(a_1,a_2)$ | $(1-u-v)/u$, |

For the pointwise weighted inequality the fifth term has the complete mother region

$$D_5=\{a_0<u<v<a_2\}=D_5^+\sqcup D_5^-,\qquad D_5^-=D_5\cap\{v+2u\ge 1\}. \tag{163}$$

Thus $\Gamma_5=\Gamma_5^++\Gamma_5^-$; it is not restricted to $D_5^+$ in the combinatorial inequality. The ordinary estimate and child defect are used only on $D_5^+$. On $D_5^-$ the complete count is retained and bounded by the positive product estimate. The test $2v<1$ holds throughout $D_5$, and the other three pair regions satisfy both $2v<1$ and $v+2u<1$ by (G.3). If $J_2^0$ denotes the

ordinary and positive switched base for the chosen admissible parameter tuple, the feedback added to that base is

(164)
$$\frac{1}{5}\left\{4H(s_0)+H(\phi_1)+\sum_{c\in\{b,a_2,a_3\}}\int_{a_0}^{c}\frac{h(s_0(1-u))}{u(1-u)}\,du+\sum_{i=5^+,6,7,8}\iint_{D_i}\frac{H(\rho_i(u,v))}{uv(1-u-v)}\,du\,dv\right\}.$$

We now define this base and prove the sign of every contribution.

The twenty-one-term inequality. For a marked state let $R=Q/B$ and, in this paragraph, put $\mathcal{A}_{N,B}=\{(N-p)/B : p<N,\ p \text{ prime},\ B\mid N-p\}$. Let $\Phi_s$ be its positively marked average sifted at $R^{1/s}$. Write $[l,r]^k_\uparrow$ for $k$ increasingly ordered exponents in $[l,r]$. For a positive integer $n$, let

$$Q_d(y;n)=\mathbf{1}_{d\mid n}\,\mathbf{1}_{P^-(n/d)\ge y}.$$

All additional primes in the following expressions are distinct; collisions with the marked prefix, and square factors, are removed with their fixed power-saving errors. Every expression is then averaged with the same nonnegative marked weights. The first four terms are

$$\Gamma_1=4\Phi_{s_0}+\Phi_{\phi_1},\qquad \Gamma_2,\Gamma_3,\Gamma_4=\sum_{R^{a_0}\le q<R^c}Q_q(R^{a_0};n),\quad c=b,a_2,a_3,$$

where the last expressions include the average over $n\in\mathcal{A}_{N,B}$. For $\Gamma_5$ use the full region $D_5$ in (163); for $\Gamma_6,\Gamma_7,\Gamma_8$ use the other three pair regions above. The summand is $Q_{q_1q_2}(R^{a_0};n)$ for $i=5,6$, and $Q_{q_1q_2}(q_1;n)$ for $i=7,8$, with $q_1=R^u<q_2=R^v$. For $9\le i\le 21$ the ordered exponent domains are

(165)

| | | | |
|---|---|---|---|
| 9 | $[a_1,a_3]^3_\uparrow$ | 10 | $[a_1,a_2]^2_\uparrow[a_2,b]$ |
| 11 | $[a_1,a_2][a_2,a_3]^2_\uparrow$ | 12 | $[a_0,a_1]^2_\uparrow[a_3,b]$ |
| 13 | $[a_0,a_1][a_1,a_2][a_2,b]$ | 14 | $[a_0,a_1][a_2,b]^2_\uparrow$ |
| 15 | $[a_1,a_2][a_2,a_3][a_3,b]$ | 16 | $[a_2,a_3]^4_\uparrow$ |
| 17 | $[a_2,a_3]^3_\uparrow[a_3,b]$ | 18 | $[a_2,a_3]^2_\uparrow[a_3,b]^2_\uparrow$ |
| 19 | $[a_1,a_2][a_3,b]^3_\uparrow$ | 20 | $[a_2,a_3][a_3,b]^4_\uparrow$ |
| 21 | $[a_3,b]^6_\uparrow$. | | |

For a domain with $r$ primes, the summand is $Q_{q_1\cdots q_r}(q_{r-1};n)$. Juxtaposition in (165) denotes independent successive bands, with ordering only within each indicated ordered block. The inequality to be used is

(166)
$$5\Phi_s\le\Gamma_1-\Gamma_2-\Gamma_3-\Gamma_4+\sum_{i=5}^{21}\Gamma_i+o(\Theta_Q).$$

It is the second weighted inequality of [15, Lemma 4.2]; the following verification fixes its complete ranges and multiplicities.

For a squarefree $R^{a_0}$-rough quotient let $A,B',C,D$ count its prime factors in the four bands $[a_0,a_1),[a_1,a_2),[a_2,a_3),[a_3,b)$, respectively. (The symbol $B'$ in this verification is a band count, not the marked prefix $B$.) Pointwise the first four terms are

$$\Gamma_1=4+\mathbf{1}_{A=0},\qquad(\Gamma_2,\Gamma_3,\Gamma_4)=(A+B'+C+D,A+B',A+B'+C).$$

The remaining terms have the following values, where $x_+ = \max(x, 0)$:

$$
\begin{array}{r|l|r|l}
5 & \binom{A+B'}{2} & 6 & AC \\
7 & (A-1)_+ & 8 & \mathbf{1}_{A\ge 1}B' \\
9 & \mathbf{1}_{A=0}(B'+C-2)_+ & 10 & \mathbf{1}_{A=0,B'\ge 2}(C+D) \\
11 & \mathbf{1}_{A=0,B'=1}(C-1)_+ & 12 & \mathbf{1}_{A\ge 2}D \\
13 & \mathbf{1}_{A=1,B'\ge 1}(C+D) & 14 & \mathbf{1}_{A=1,B'=0}(C+D-1)_+ \\
15 & \mathbf{1}_{A=0,B'=1,C\ge 1}D & 16 & \mathbf{1}_{A=B'=0}(C-3)_+ \\
17 & \mathbf{1}_{A=B'=0,C\ge 3}D & 18 & \mathbf{1}_{A=B'=0,C=2}(D-1)_+ \\
19 & \mathbf{1}_{A=0,B'=1,C=0}(D-2)_+ & 20 & \mathbf{1}_{A=B'=0,C=1}(D-3)_+ \\
21 & \mathbf{1}_{A=B'=C=0}(D-5)_+. & &
\end{array}
\tag{167}
$$

Indeed a quotient cutoff at the next-to-last chosen prime forces all preceding chosen primes to be the smallest factors of $n$. This gives each entry of (167). The signed right-hand side in (166) is, respectively,

$$
\begin{array}{c|l}
A\ge 2 & (A+B'-2)(A+B'-3)/2+(A-2)C \\
A=1,\ B'\ge 1 & (B'-1)(B'-2)/2 \\
A=1,\ B'=0 & (1-C-D)_+ \\
A=0,\ B'\ge 2 & (B'-2)(B'-3)/2 \\
A=0,\ B'=1,\ C\ge 1 & 0 \\
A=0,\ B'=1,\ C=0 & (2-D)_+ \\
A=B'=0,\ C\ge 3 & 0 \\
A=B'=0,\ C=2,1,0 & (1-D)_+,\ (3-D)_+,\ (5-D)_+.
\end{array}
$$

All are nonnegative, and at $A=B'=C=D=0$ the value is five. The left side is five times the indicator of that last case. Outside the $R^{a_0}$-rough support every summand vanishes. This proves the pointwise inequality. On a fixed positive exponent box the omitted square and prefix-intersection terms have $N^{1-\epsilon+o(1)}$ total mass for some fixed $\epsilon>0$, which gives the stated error after the finite marked averaging.

Termwise estimates and the ordinary base. The two components of $\Gamma_1$ have upper multipliers $4A_+(s_0)+A_+(\phi_1)$. The three negative terms have lower multipliers

$$
\mathcal{B}(c)=\int_{a_0}^{c}\frac{A_-(s_0(1-u))}{u(1-u)}\,du, \qquad c=b,a_2,a_3. \tag{168}
$$

The support conditions put $s_0(1-u)$ in $(2,4]$ on these intervals, so $A_-(s_0(1-u))=\log(s_0(1-u)-1)$. The merged singleton and pair coefficients satisfy the complete square and selected-complement tests of the parent state. In particular $D_5^+$, $D_6$, $D_7$, $D_8$ have the ordinary upper kernel $(1-u-v)^{-1}$ against $du\,dv/(uv)$. For a child argument below 2, comparison with a smaller sifting cutoff gives this same kernel, since $A_+=1$ up to 3. On $D_5^-=\{a_0<u<v<a_2 : v+2u\ge 1\}$ the ordinary pair is not used. The positive product estimate gives the valid kernel $Kc_\omega s_0/4$, because its rough argument is at least $s_0(2-2a_2)>2$. Define

$$
\begin{aligned}
\mathcal{U}_5 &= \iint_{D_5^+}\frac{du\,dv}{uv(1-u-v)}+\frac{Kc_\omega s_0}{4}\iint_{D_5^-}\frac{du\,dv}{uv}, \\
\mathcal{U}_i &= \iint_{D_i}\frac{du\,dv}{uv(1-u-v)} \qquad (i=6,7,8).
\end{aligned}
\tag{169}
$$

Thus the unsupported corner is replaced by a positive upper estimate; its ordinary mass or feedback is not retained as well.

For a switched domain $\mathcal{D}_i$ with $r$ primes, let $w=u_{r-1}$ and $U=u_1+\cdots+u_r$. Its normalized unsifted mass at the actual parent ratio $\alpha$ is

$$
\mathcal{M}_i(\alpha)=\int_{\substack{\boldsymbol{u}\in\mathcal{D}_i\\ U\le\alpha-w}}\frac{\omega((\alpha-U)/w)}{w\prod_j u_j}\,d\boldsymbol{u}+\int_{\substack{\boldsymbol{u}\in\mathcal{D}_i\\ U=\alpha}}\frac{du_1\cdots du_{r-1}}{\prod_j u_j}. \tag{170}
$$

In the surface integral one coordinate is eliminated with derivative one. The surface counts quotient 1; it is separate from the ordinary Buchstab term. Prime summation and the rough-number formula on fixed positive boxes give (170). The upper count in parent units is $K\mathcal{M}_i(\alpha)/4$.

The following explicit bounds are uniform in $\alpha \ge 2$. Put $\ell_{ij} = \log(\phi_i/\phi_j)$, $\ell_{01} = \log(s_0/\phi_1)$ and $\ell_{3b} = \log(\phi_3/s)$. For $i = 9, \dots, 15$ take $C_i = \max\{c_\omega, 1/\max(1, m_i)\}$, where

| $i$ | $m_i$ | $i$ | $m_i$ |
|---|---|---|---|
| 9 | $2\phi_3 - 3$ | 10 | $2\phi_2 - 2 - \phi_2/s$ |
| 11 | $2\phi_3 - \phi_3/\phi_2 - 2$ | 12 | $2\phi_1 - 2 - \phi_1/s$ |
| 13 | $2\phi_2 - \phi_2/\phi_1 - 1 - \phi_2/s$ | 14 | $2s - s/\phi_1 - 2$ |
| 15 | $2\phi_3 - \phi_3/\phi_2 - 1 - \phi_3/s.$ | | |

These are lower bounds for the rough arguments on the respective domains. Equation (160) gives

$$\begin{aligned}
\widehat{M}_9 &= C_9\{(\phi_1+\phi_3)\ell_{13} - 2(\phi_1-\phi_3)\},\\
\widehat{M}_{10} &= C_{10}\{\phi_1 - \phi_2(1+\ell_{12})\}\log(\phi_2/s),\\
\widehat{M}_{11} &= C_{11}\ell_{12}\{\phi_2(\ell_{23}-1)+\phi_3\},\\
\widehat{M}_{12} &= C_{12}\{s_0 - \phi_1(1+\ell_{01})\}\ell_{3b},\\
\widehat{M}_{13} &= C_{13}\ell_{01}(\phi_1-\phi_2)\log(\phi_2/s),\\
\widehat{M}_{14} &= C_{14}\ell_{01}\{\phi_2(\log(\phi_2/s)-1)+s\},\\
\widehat{M}_{15} &= C_{15}\ell_{12}(\phi_2-\phi_3)\ell_{3b},\\
\widehat{M}_{16} &= \phi_2\ell_{23}^4/24, \qquad \widehat{M}_{17} = \phi_2\ell_{23}^3\ell_{3b}/6,\\
\widehat{M}_{18} &= \phi_3\ell_{23}^2\ell_{3b}^2/4, \qquad \widehat{M}_{19} = \phi_3\ell_{12}\ell_{3b}^3/6,\\
\widehat{M}_{20} &= \phi_3\ell_{23}\ell_{3b}^4/24 + \phi_2\ell_{3b}^4/24,\\
\widehat{M}_{21} &= \phi_3\ell_{3b}^6/720 + \phi_3\ell_{3b}^5/120.
\end{aligned} \tag{171}$$

For $i \le 19$ there is no unit surface because the sum of specified exponents is at most $4/s < 2 \le \alpha$. For $i = 20$, eliminate the first variable on the surface and use $1/u_1 \le \phi_2$; the other four variables are ordered in $[a_3, b]$. This gives the second summand of $\widehat{M}_{20}$. For $i = 21$, eliminate the last variable, use $1/u_6 \le \phi_3$, and integrate the other five ordered variables. The other summands in (171) are obtained by integrating $1/(w\prod u_j)$ over their displayed bands, with $\omega \le C_i$ for $9 \le i \le 15$ and $\omega \le 1$ otherwise. Hence $\mathcal{M}_i(\alpha) \le \widehat{M}_i$.

The ordinary base is now the explicit functional

$$J_2^0 = \frac{1}{5}\left\{5A_+(s) - 4A_+(s_0) - A_+(\phi_1) + \mathcal{B}(b) + \mathcal{B}(a_2) + \mathcal{B}(a_3) - \sum_{i=5}^{8}\mathcal{U}_i - \frac{K}{4}\sum_{i=9}^{21}\widehat{M}_i\right\}. \tag{172}$$

Insert the upper estimates for positive terms and lower estimates for negative terms in (166). The defects in the two components of $\Gamma_1$ subtract $4H(s_0) + H(\phi_1)$ from its upper estimate. The three singleton lower defects subtract their integrals from the negative terms. The four good-pair upper defects subtract the pair integrals from their positive upper estimates. After division by 5 these are exactly (164). Therefore

$$\Phi_s \le \{A_+(s) - J_2^0 - \mathcal{P}_2[H,h] + o(1)\}\Theta_Q, \tag{173}$$

where $\mathcal{P}_2[H,h]$ is the expression in (164). This proves admissibility of a finite two-step action from its mathematical base and positively weighted preceding-generation defects.

Sharper upper bounds may replace the positive costs in (169) and (170). Such a bound must cover the same labelled family for every admissible $\alpha \ge 2$, including its unit surfaces.

Replace that cost once in (172), and bound the resulting explicit functional in the required direction to obtain the corresponding transition constant. This preserves the sign argument for the two-step inequality.

G.4. **A finite seed on the admitted marked families.** Put

$$S=\frac{11}{5},\qquad s_0=\frac{227}{50},\qquad (\phi_1,\phi_2,\phi_3)=\left(\frac{353}{100},\frac{29}{10},\frac{61}{25}\right),\qquad K_{\mathrm{seed}}=\frac{681}{100}. \tag{174}$$

Here $K<K_{\mathrm{seed}}$, so Proposition 4.1 also supplies the positive upper estimate with $K_{\mathrm{seed}}$. Let $\mathcal{J}_{\mathrm{seed}}$ be the right-hand side of (172) at the parameters (174), with $s=S$ and $K$ replaced by $K_{\mathrm{seed}}$. In particular its fifth cost is the full split cost (169), and the two unit-surface terms in (171) are included.

**Lemma G.1** (Finite seed)**.** *On an admitted marked family with $\alpha\ge 2$, assume additionally $L/S<\chi$ in the enhanced case. Then*

$$\Phi_S\le\left(1-\frac{2327}{100000}+o(1)\right)\Theta_Q.$$

*The same upper defect is available at every $1\le w\le S$ whenever the state admits the comparison cutoff $N^{L/S}$. The conclusion is uniform over each fixed finite collection of such families, and follows from four weighted steps after an initial seed of $1/50$.*

*Proof.* All primes appended in the twenty-one-term inequality have exponent at most $L/S$. They therefore lie below the retained ceiling, so the largest specified slots, selected subsets and any third-slot bound remain valid. The singleton residual square is $2u<1$. The pair squares are $2v<1$ and $v+2u<1$; they hold on $D_5^+, D_6, D_7, D_8$. Every new complete square is obtained by multiplying the corresponding residual square by $B$. Once the selected old slots have occurred, the new selected-complement square is the same full square. The ordinary residual atoms are evaluated at their actual level $Q/(Bq)$ or $Q/(Bq_1q_2)$. The remaining part $D_5^-$ is retained with its positive cost, using no signed distribution claim there. In the classical case the complete ordinary modulus is below $Q$, and the absolute below-half remainder gives the same conservative functional. The switched terms require only $\alpha\ge 2$ and fixed positive prime cutoffs.

The directed enclosure of the explicit functional is

$$\mathcal{J}_{\mathrm{seed}}\ge\frac{11672890052052913}{576460752303423488}>\frac{20249}{10^6}>\frac{1}{50}. \tag{175}$$

The finite certificate for this inequality is specified in the ancillary data under `seed_certificate`. Its ordinary integrals are those in (168) and (169); its switched masses are the explicit upper bounds in (171). The ordinary double integral is first integrated in one coordinate. The resulting one-dimensional logarithmic integrals are enclosed by rational Darboux intervals and the logarithm remainder of (186). Upper endpoints are used for positive costs and lower endpoints for negative costs in the definition of $\mathcal{J}_{\mathrm{seed}}$. Thus (175) proves the initial defect $J_0=1/50$ before any later array recurrence is used.

Suppose an upper defect $J>0$ on $[1,S]$ has already been proved on these admitted families. Positivity and cutoff comparison give

$$H_J(w)=J\mathbf{1}_{[1,S]}(w).$$

The lower Buchstab identity at the same total level gives the previously proved option

$$h_J(w)=J\log\frac{S}{w-1}\,\mathbf{1}_{[2,1+S]}(w). \tag{176}$$

The normalization is independent of $w$, and $A_+(w)=1$ for $1\le w\le S$. To check the comparison inside (176), a Buchstab child indexed by $v\in[w-1,S]$ has appended exponent $L/(v+1)$ and residual exponent $Lv/(v+1)$. Its seed comparison exponent satisfies

$$\frac{Lv}{S(v+1)}\le\frac{L}{v+1}.$$

It is therefore below that appended prime and the preceding stable ceiling. Its new square holds because $v \geq 1$. The lower child need not itself admit argument $S$; the seed is applied to its upper Buchstab children.

Insert $H_J, h_J$ into (164). Since $s_0, \phi_1 > S$, the unchanged-prefix defects vanish. Write $u_* = 67/227$, $b = 1/S$, and set

$$E_i = \begin{cases} D_i \cap \{u + v \geq 117/227\}, & i = 5^+, 6, \\ D_i \cap \{v \geq 1 - 16u/5\}, & i = 7, 8. \end{cases}$$

The feedback is exactly $\lambda_{\mathrm{seed}} J$, where

$$\lambda_{\mathrm{seed}} = \frac{1}{5}\Bigg\{ \sum_{c \in \{b, a_2, a_3\}} \int_{u_*}^{c} \frac{\log\{S/[s_0(1-u)-1]\}}{u(1-u)}\, du + \sum_{i=5^+,6,7,8} \iint_{E_i} \frac{du\, dv}{uv(1-u-v)} \Bigg\}. \tag{177}$$

For $i = 5^+, 6$ the support condition is $s_0(1-u-v) \leq S$; for $i = 7, 8$ it is $(1-u-v)/u \leq S$. All retained pair-child arguments exceed 1, and the singleton lower arguments exceed 2. There is no feedback from $D_5^-$.

Every boundary in (177) is rational and affine. For a vertical slice $l(u) < v < r(u)$ the exact pair integral is

$$\frac{1}{u(1-u)} \log \frac{r(u)(1-u-l(u))}{l(u)(1-u-r(u))}.$$

For $E_{5^+}$ the successive $u$ intervals have endpoints $50/227, 117/454, 19/58, 1/3$. The domain $E_6$ is its entire rectangle. The nonempty support in $E_7$ starts at $1265/5648$ and changes its lower boundary at $5/21$; the lower boundary of $E_8$ changes at $1265/5648$. Rational Darboux enclosures of these pieces give

$$\lambda_{\mathrm{seed}} \geq \frac{75057365825327405}{576460752303423488} > \frac{13}{100}. \tag{178}$$

The exact endpoints, subdivisions and source correspondence are specified by the same `seed_certificate` data as (175).

Equations (173), (175) and (178) prove

$$J_{n+1} = \frac{20249}{10^6} + \frac{13}{100} J_n.$$

At each step (176) is derived from $J_n$ before the weighted inequality is applied. Starting at $J_0 = 1/50$ the four exact values are

$$\frac{22849}{10^6}, \quad \frac{2321937}{10^8}, \quad \frac{232675181}{10^{10}}, \quad \frac{23273777353}{10^{12}} > \frac{2327}{100000}.$$

There are finitely many appended-prime operations. Their exponents remain positive after all compact choices have been fixed, so the positive-product estimate and the distribution remainders apply at each of them. The finitely many errors are $o(\Theta_Q)$; the strict final surplus absorbs them in the stated order of limits. □

Use at the roots and descendants. For $U_3, U_4$, the specified prime has exponent $t > a_{\mathrm{s}}$, the residual exponent is $L = v(t) - t$, and the upper argument is $\max(2, L/a_{\mathrm{s}})$. On the enhanced range $t < 57/224$, the largest-prime branch has $2t < v(t) \leq (1+t)/2$, so its stable ceiling can be taken below $t$ and arbitrarily close to it. Its physical cutoff exponent $\min(a_{\mathrm{s}}, L/2) \leq a_{\mathrm{s}} < t$ lies below that ceiling. Whenever the seed is used, its comparison exponent $L/S$ is no larger than this physical cutoff. The classical tail uses the below-half interface. For $U_{10}, U_{11}$ the same argument uses the least specified exponent $w$ and physical cutoff $\min(w, L/2)$; the original selected subsets and first three slots are already fixed. For the lower pair terms $L_5, L_6$, the seed occurs only inside (176). The first appended prime is at

most $N^{a_{\mathrm{s}}}$, and every seed comparison prime is below it by the preceding displayed inequality. These comparisons also preserve a third-slot bound equal to $a_{\mathrm{s}}$.

More generally, every finite action in (159) or (164) retains the total $Q$, every old square and the stable ceiling. New singletons have $2u < 1$; new pairs in the ordinary part have $2v < 1$ and $v + 2u < 1$. Their physical cutoffs are no larger than the least newly appended prime. Equation (157) shows that their ratios remain at least 2. Induction therefore keeps the seed and its Buchstab consequences within the admitted families. No seed defect is inserted directly into an empty-prefix lower start.

The finite induction uses the preceding generation on every right-hand side. At each state take the maximum of alternative estimates; when a child argument crosses a knot, use the minimum over all adjoining closed cells. Each step therefore gives a bound for the same positive marked counts. Only finitely many steps are used.

Here is an explicit common approximation bound. The positive operator masses satisfy

$$\begin{aligned} \|\mathcal{B}\| &\le \log 5 < 2, \\ \|\mathcal{W}_1\| &\le 1 + \tfrac{1}{2}\log 4 < 2, \\ \|\mathcal{W}_2\| &\le 1 + \tfrac{3}{5}\log 4 + \tfrac{4}{5}\,125(3/10)^2 < 12. \end{aligned}$$

For the last estimate $1/5 \le u, v \le 1/2$ and $1 - u - v \ge u \ge 1/5$, so the pair kernel is at most 125 on each of four regions. The unchanged-prefix coefficients sum to one. The switched positive costs belong to the base of the action and are not additional feedback edges. If each local oriented approximation error is at most $\delta$, then

$$e_{j+1} \le \delta + 12 e_j, \qquad e_j \le \delta \frac{12^{j+1} - 1}{11}. \tag{179}$$

The absolute coefficients of the eleven-term functional sum to 14 before its factor $1/4$, giving root norm $7/2$.

At depth at most 440, every residual level is at least $\lambda_* 101^{-440} > 0$, and an appended prime exponent is at least this number divided by 6. Recursively refine appended exponents on each fixed positive parent, using an error within $\delta$ times its normalized mass. Proper cap and knot walls are null; structural clamps and inherit identities are retained. The denominators are bounded away from zero. Equation (179) then gives one finite set of queries with arbitrarily small total error. Only after fixing that set does $N$ increase. A positive-mass vertex has a positive limiting $\Theta_Q/H_N$, so the absolute distribution errors become $o(\Theta_Q)$; zero-mass pieces are handled by absolute errors.

## Appendix H. Endpoint estimates and the order of limits

The sharp-prime lemma (Lemma 6.1) applies to the same coefficient constructed above. We record the absolute bounds that allow it to be used for signed lower weights. If $|\rho_N^{\mathrm{low}}(q)| \le C$, its squarefree support is $q \le N^\theta$, $\theta < 1$, and $\delta = (\log N)^{-B}$, then the endpoint contribution is bounded by

$$\sum_{\substack{p \text{ prime} \\ 0 < |p-N| \le \delta N}} \sum_{q \mid N-p} |\rho_N^{\mathrm{low}}(q)| \le 2C \sum_{1 \le n \le \lceil \delta N \rceil} \tau(n) \ll \delta N \log N + \log N. \tag{180}$$

The difference $p - N = 0$ is impossible for even $N > 2$. The exact-coprime mean boundary has the same bound because $\sum_{q \le N^\theta} 1/\varphi(q) \ll \log N$. The smooth lower trim below $2N^{1-\kappa_0}$ costs $N^{1-\kappa_0+o(1)}$ in the progression sum and $O(N^{1-\kappa_0}\log N)$ in the mean. These have fixed power savings. Choosing $B > A + 3$ makes all boundary terms smaller than the prescribed logarithmic error.

For the exact-coprime prime mean, let $\omega_{\mathrm{pf}}(q)$ be the number of distinct prime divisors of $q$. Every such divisor is below $N$, and squarefreeness gives

$$\sum_{q} \frac{|\rho_N^{\mathrm{low}}(q)|\omega_{\mathrm{pf}}(q)}{\varphi(q)} \ll \sum_{\ell \le N^\theta} \frac{1}{\ell - 1} \sum_{e \le N^\theta/\ell} \frac{1}{\varphi(e)} \ll \log^2 N. \tag{181}$$

The sum over $\ell$ is over primes; summing over all integers already gives the stated crude bound. Removing the $O(\log N)$ prime divisors of $N$ from the original sequence contributes $N^{o(1)}$ by the divisor bound. These estimates justify using the sharp original prime count in (156), without obtaining a prime lower bound from the positive-product upper constant $K$.

The upper and lower count estimates follow the parameter order in Section 6.6. The fixed moment orders precede the choice of $\kappa$ in (113), and the boundary exponent satisfies $B > A + 3$, as above.

## Appendix I. Finite numerical estimates

We specify the finite quantities that determine the two constants in Lemma 7.1. The construction separates integral enclosures, the finite defect recurrence, and the final rational subtraction, so that each contribution has an explicit mathematical definition and direction.

I.1. **Supplementary data.** The finite data are supplied in *Theorem (1+1.75) on the Goldbach Conjecture — Numerical Supplement*, version 20261106 [2]. They comprise 106 affine transitions, 200-cell defect arrays at depths 248, 312, and 430, and 17572 paired rectangles. Rational quantities in `finite-construction-data.json` are represented as quotients of integers. The fields `upper_saving_bounds`, `lower_root_bounds`, and `lower_integral_bounds` give directed rational endpoints. The map `certificate-map.json` identifies their source certificates and the corresponding domains, subdivisions, and interval witnesses. Table 2 gives the correspondence between the mathematical estimates and their certificate fields.

Verification commands and inputs. Run the commands below from the supplement root with Python 3.11 or later, with assertions enabled. The sixth-rank computations also use NumPy. The rows distinguish reconstruction from saved certificates, exact checks of the finite construction, and recalculation of specified witness bounds. All output files requested with `--output` or `--output-dir` are written outside the supplement.

| Entry point | Quantities checked or recomputed | Numerical inputs retained |
|---|---|---|
| `verify-package.py` | File set, sizes and recorded SHA-256 identities. | The versioned `MANIFEST.json`. |
| `finite-data/reproduce-data.py --check` | Exact extraction of finite data and field correspondence from source certificates. | The supplied source-certificate values. |
| `finite-data/verify-finite-data.py` | Finite defect recurrence; paired-cell geometry, masses and loss proxy; $\Gamma_5$ decompositions; density sums and strict scalar margins. | Action coefficients and integral endpoints identified by `certificate-map.json`. |
| `replay-records.py` | Ordered-cover keys, stored integer-floor sums and the cost identity. | Stored floors in the ordered witness. |
| `replay-ordered-cover.py` | The selected-method floor on every row of the grid-48 base cover; use `--grid 32` for its predecessor. | The recorded method choices and generation-248 defect table. |

| Entry point | Quantities checked or recomputed | Numerical inputs retained |
|---|---|---|
| `verify-sixth-domains.py` | Affine sieve-argument ranges on the saved polyhedral cells. | Recorded vertices, coordinate maps and caps. |
| `replay-sixth-rank.py` | Grid-400 sixth-rank cube contributions and their sum, together with the preceding polyhedral range checks. | The supplied cube witness and polyhedral records. |

The recurrence and paired-cell checks use the action and integral inputs listed in Table 2. Their primitive enclosure obligations remain those of the corresponding integrals and policies in this appendix and Appendices F–G. The `enclosure_obligations` entries of `certificate-map.json` list the active assertions, complete records and supplied algorithm sources for each primitive family. Witness generation is described in the package README; in particular `witnesses/generator-0041.py` generates the ordered base cover and `witnesses/generator-0045.py` the sixth-rank cube witness. The generation-248 profile and other numerical inputs retained by a replay are checked through their own entries, so that each verification has an explicit set of inputs. The package README gives the command options; the fixed-point arithmetic for the finite data is specified below.

I.2. **Exact scalar endpoints.** The exact values used in the upper density estimate are

$$E_{\mathrm{up}} = \frac{9979702576710911234850619807816847864907359959}{14759135509622012279467806190993532151398400000}, \tag{182}$$

$$u = \frac{14500796122704831221059342753012800078468 1}{11807308407697609823574244952794825721118720000}. \tag{183}$$

For the lower construction, before the paired-fibre increment, put

$$R_{\mathrm{base}} = \frac{13455751295300037395716814323011888573127547 37}{68573812749466115119844459235455795200000000 0}, \tag{184}$$

$$B_{\mathrm{base}} = \frac{25047483180919428082296942193797993075509 87}{54859050199572892095875567388364636160000000 00}. \tag{185}$$

The two copies of $u$ and of $B_{\mathrm{base}}$ pay for distinct finite approximation and coefficient-realization errors. Quadrature errors are already contained in the oriented integral endpoints. The ancillary fields `scalar_bounds` contain the resulting exact sums and strict surpluses.

I.3. **Directed arithmetic.** All rounding used to prove an upper bound is outward, and all rounding used to prove a lower bound is inward. More explicitly, at denominator $\mathsf{Q} = 2^{b_{\mathrm{fp}}}$ a rational interval $[x, y]$ is represented by $[\lfloor \mathsf{Q}x \rfloor, \lceil \mathsf{Q}y \rceil]$. Addition and subtraction are exact at the integer level. For multiplication the four endpoint products are enclosed by their minimum and maximum and rounded in the indicated directions after division by $\mathsf{Q}$. Division is performed only after verifying that the denominator interval is positive.

For logarithms put $z = (x-1)/(x+1)$. If $1 \le x \le 2$, then

$$2\sum_{j=0}^{n-1} \frac{z^{2j+1}}{2j+1} \le \log x \le 2\sum_{j=0}^{n-1} \frac{z^{2j+1}}{2j+1} + \frac{2z^{2n+1}}{(2n+1)(1-z^2)}. \tag{186}$$

This follows by summing the positive power series for $2\operatorname{arctanh} z$ and bounding the remaining denominators below by $2n+1$. For $n = 96$ the remainder is at most $9/(4 \cdot 193 \cdot 3^{193})$. Range reduction and a separate enclosure of $\log 2$ extend the formula to every positive rational argument used here. The paired-cell calculation uses denominator $2^{224}$ for the series and denominator $2^{160}$ for its returned intervals. Thus no floating-point evaluation of a logarithm determines the direction of an inequality.

I.4. **Enclosing delayed functions and integrals.** It is convenient to propagate the numerators

$$A_+(s) = \frac{sF(s)}{2e^\gamma}, \qquad A_-(s) = \frac{sf(s)}{2e^\gamma}.$$

Their initial branches are $A_+(s) = 1$ for $1 \le s \le 3$ and $A_-(s) = 0$ for $s \le 2$, and subsequent branches satisfy

$$A'_+(s) = \frac{A_-(s-1)}{s-1}, \qquad A'_-(s) = \frac{A_+(s-1)}{s-1}.$$

The equation for a branch involves only the preceding unit interval. One therefore obtains rational interval enclosures successively on a rational mesh, splitting at every integer breakpoint. Lower and upper integral bounds follow from the corresponding interval bounds for the integrand. Chord bounds and curvature bounds sharpen these enclosures without changing their direction. An arbitrary query is enclosed on its entire support, including both adjoining cells when an endpoint is a mesh knot. The analogous construction for $s\omega(s)$ uses $(s\omega(s))' = \omega(s-1)$ and its initial branch $\omega(s) = 1/s$ on $[1,2]$.

The chord integration can be implemented without numerical quadrature. On $[j/n, (j+1)/n]$ the integral of the affine endpoint interpolant of a numerator $V$, divided by its argument, is

$$\big((j+1)\log(1+1/j) - 1\big)V(j/n) + \big(1 - j\log(1+1/j)\big)V((j+1)/n).$$

Both weights are positive because they are integrals of nonnegative affine basis functions against $du/u$. The monotonicities of $F$ and $f$ imply that $A_+$ is convex and $A_-$ is concave on their nonconstant branches. Their numerator curvature on a delayed cell is bounded in absolute value by $1/(j/n - 1)$. Thus the integrated chord error is at most

$$\frac{1}{12nj(j-n)} \qquad (j > n).$$

This quantity is subtracted from a chord lower estimate or added to a chord upper estimate as appropriate. The weights themselves have alternating rational series

$$(j+1)\log(1+1/j) - 1 = \sum_{k\ge1} \frac{(-1)^{k+1}}{k(k+1)j^k}, \qquad 1 - j\log(1+1/j) = \sum_{k\ge1} \frac{(-1)^{k+1}}{(k+1)j^k}.$$

Their fourth and fifth partial sums enclose the exact weights. This supplies rational enclosures even where direct subtraction of nearly equal logarithms would be numerically inconvenient.

The finite integrations use rational boxes or rational polytopes. For a polytope given by affine inequalities, every possible active basis is solved exactly, and each resulting vertex is tested against every original inequality. Boundedness and the face incidences are verified before a pulling triangulation is formed. Polynomial moments are then exact: on a $d$-simplex $T$ with barycentric coordinates $\lambda_0, \dots, \lambda_d$,

$$\int_T \prod_{j=0}^{d} \lambda_j^{\alpha_j}\, dx = d!\,\mathrm{vol}(T)\,\frac{\prod_{j=0}^{d}\alpha_j!}{(d+\sum_j \alpha_j)!}. \tag{187}$$

Rational lower or upper bounds for the remaining kernel factors, together with these moments and (186), give the required integral enclosure. A boundary box in a negative-cost upper integral is retained with an upper bound for its whole contribution. In a positive saving, a box may instead be omitted after establishing that its omitted contribution is nonnegative. These two rules have different signs and are used separately.

I.5. **The upper density.** Write

$$U_0 = \frac{4}{\Theta} + I_1 + I_{3,c} + I_{3,t}, \qquad E_{\mathrm{up}} = U_0 - \sum_{j=1}^{22} G_j.$$

Here $I_{3,c}$ is the integral over the region with last coordinate at least $1/100$, and $I_{3,t}$ bounds its entire complementary tail. Put $D = (F - f)/(2e^{\gamma})$. Its delay equation gives

$$D'(s) = -\frac{D(s) + D(s-1)}{s} \le -\frac{2D(s)}{s} \quad (s \ge 3),$$

using positivity and monotonicity of $D$. Therefore $D(s) \le D(10)(10/s)^2$ for $s \ge 10$. If $\delta = 1/100$ and $\rho_0 = 19/96$, the upper-tree cap inequalities give a residual exponent at least $\rho_0$. For $0 < z < \delta$ its delayed argument is at least $(\rho_0 - z)/z \ge 10$. Enlarging the two remaining ordered coordinates to $z \le y \le x \le \rho_0$ yields the complete tail bound

$$I_{3,t} \le \frac{200D(10)\delta}{(\rho_0 - \delta)^2} \left\{ \log^2 \frac{\rho_0}{\delta} + 2 \log \frac{\rho_0}{\delta} + 2 \right\}. \tag{188}$$

Here the ordered $x, y$ integral is $\frac{1}{2}\log^2(\rho_0/z)$, and the remaining integral over $0 < z < \delta$ is elementary. In particular the logarithmic singularity at $z = 0$ is integrated rather than discarded. The first integral is enclosed by a positive series with its geometric remainder. The compact third integral is covered by a disjoint partition, including conservative bounds on its boundary boxes.

Table 1 gives readable lower bounds for the nonnegative savings $G_j$: its integer entries satisfy $G_j \ge m_j 10^{-14}$. The full rational bounds are in the supplementary data. Successive changes on one region are measured from their respective parents. In particular, the last two rows give only the curvature and mixed increments over the preceding bound for that integral.

Table 1: Rational lower bounds $10^{-14} m_j$ for the upper-density savings.

| $j$ | Saving | $m_j$ |
|---|---|---|
| 1 | Fourth-rank cap replacement | 3507783546602 |
| 2 | Fifth-rank parent subdivision | 1005163763573 |
| 3 | Disjoint fifth-rank region | 136378695993 |
| 4 | Sixth-rank cap replacement | 96281257455 |
| 5 | Second-rank fallback above $\mu_0$ | 4209783419 |
| 6 | Third-rank fallback above $\mu_0$ | 3908979560 |
| 7 | Sixth-rank logarithmic moments | 1525398138 |
| 8 | Sixth-rank descending region | 187855687 |
| 9 | Sixth-rank ascending cap difference | 387157212 |
| 10 | Sixth-rank descending cap difference | 236377744 |
| 11 | Sixth-rank clipped lower window | 846201484 |
| 12 | Sixth-rank descending lower window | 500857555 |
| 13 | Sixth-rank local cap corrections | 50721206 |
| 14 | Sixth-rank nonsingleton regions | 11743733 |
| 15 | Eighth-rank radial bound | 94105689 |
| 16 | Eighth-rank ascending regions | 1543640141 |
| 17 | Eighth-rank descending regions | 590010094 |
| 18 | Eighth-rank unequal-parent moments | 1304343681 |
| 19 | Eighth-rank third-coordinate moments | 357686854 |
| 20 | Eighth-rank prefix-dependent moments | 37235787 |
| 21 | Eighth-rank curvature increment | 97705 |

| $j$ | Saving | $m_j$ |
|---|---|---|
| 22 | Eighth-rank mixed increment | 607514 |

The upper enclosures use 3699127 fourth-rank boxes, 292633 fifth-rank parents subdivided into 18728512 children, 1763420 boxes in the disjoint fifth-rank region, and 632104 sixth-rank cubes. The eighth-rank regions have disjoint families of sizes 30, 15, 13, and 29. Across ranks the cap differences telescope. On a parent region take the maximum of its valid saving bounds, comparing its own bound with the complete child sum.

An upper bound for the four base terms, in the same denominator $10^{14}$, is respectively

$$668896321070235, \quad 76020940667, \quad 11960259849166, \quad 12313775.$$

Let $E_{\mathrm{up},14}$ be their sum minus the sum of Table 1, divided by $10^{14}$. Exact integer arithmetic gives

$$E_{\mathrm{up}} \le E_{\mathrm{up},14}, \qquad E_{\mathrm{up},14} + 2u < K,$$

where $u$ is given in (183); indeed the last difference is greater than $9.65\cdot10^{-8}$. The unrounded rational sum gives (182). The two copies of $u$ pay separately for finite approximation and coefficient realization. Quadrature errors are included in the one-sided integral endpoints.

I.6. **The finite defect recurrence.** Put $\mathsf{Q} = 2^{80}$ and

$$C_i = [1 + i/50, 1 + (i+1)/50], \qquad 0 \le i < 200.$$

All table entries in this subsection are integer numerators at denominator $\mathsf{Q}$. A four-step seed is obtained by starting at $J_0 = 1/50$ and using

$$J_{n+1} = \frac{20249}{10^6} + \frac{13}{100} J_n.$$

The initial defect and the implication represented by this recurrence are supplied by Lemma G.1, with its marked-state and physical-cutoff conditions. The ancillary `seed_certificate` contains the separate ordinary and feedback lower endpoints used there. Thus the arithmetic starts from an analytic saving valid on the required family. Then $J_4 = 0.023273777353 > 2327/100000$. Initialize

$$H_i^{(0)} = \begin{cases} \lfloor \mathsf{Q}\, 2327/100000 \rfloor, & 1 + (i+1)/50 \le 11/5, \\ 0, & \text{otherwise}, \end{cases} \qquad h_i^{(0)} = 0. \tag{189}$$

For completeness the exact integer transition is specified here. An action $\alpha$ consists of an integer $b_\alpha$, a divisor $d_\alpha \in \{1, 5\}$, point pairs $(j, m)$, nonnegative integer weights $w_{\alpha j}, v_{\alpha j}$, and a target set $T_\alpha$. Its candidate value is

$$b_\alpha + \left\lfloor \frac{1}{d_\alpha} \left\{ \sum_{(j,m)} m H_j + \sum_j \left\lfloor \frac{w_{\alpha j} h_j}{\mathsf{Q}} \right\rfloor + \sum_j \left\lfloor \frac{v_{\alpha j} H_j}{\mathsf{Q}} \right\rfloor \right\} \right\rfloor. \tag{190}$$

The 70 one-step actions have $d_\alpha = 1$ and the 36 two-step actions have $d_\alpha = 5$. They are evaluated at the upper constant $K_* = 6.761767 > K$. The action parameters, weights, constants and target sets are given individually in the `actions` array. The target condition for action parameter $s_\alpha$ is $1 + (i+1)/50 \le s_\alpha$.

Let $\ell_j$ be the integer lower bound for $\mathsf{Q}\log((1 + (j+1)/50)/(1 + j/50))$ in the data, and set

$$S_H(j) = \sum_{k=j}^{199} \left\lfloor \frac{H_k \ell_k}{\mathsf{Q}} \right\rfloor, \qquad S_h(j) = \sum_{k=j}^{199} \left\lfloor \frac{h_k \ell_k}{\mathsf{Q}} \right\rfloor.$$

For $50 \le i < 200$ the Buchstab candidates are $S_h(\max(50, i - 49))$ for $H_i$ and $S_H(i - 49)$ for $h_i$. Each new entry is the maximum of its preceding value and all applicable candidates. Every right-hand side is evaluated using only the preceding arrays. This is a simultaneous finite recurrence, with no assumption of convergence to a fixed point.

The analytic action inequalities and their closure under descendants imply by induction that the rational arrays are lower bounds for the defect functions on the whole closed cells. Floors in (190) preserve this direction because the feedback weights are nonnegative. Applying the map 430 times to (189) gives the 400 entries in `defect_tables.430`. The arrays at depth 248 are bounded below by the corresponding finite iteration. The depth-430 arrays are used only in the two occurrences $U_3, U_4$; the other designated occurrences use the depth-248 arrays. Thus a change of table is charged only at its actual occurrence in the lower functional.

I.7. **The integrals attached to the eleven counts.** The correspondence with the arithmetic terms $Y_j$ is fixed as follows. Write $A_+(s) = sF(s)/(2e^\gamma)$, $A_-(s) = sf(s)/(2e^\gamma)$, and put $\nu = 249/448$, $w_* = 25/224$. The reference one-prefix terms are

$$L_1 = \frac{4}{a_\mathrm{s}}\overline{f}(\nu/a_\mathrm{s}),$$
$$L_2 = 4\left\{(1/h_\mathrm{s}+1)\overline{f}\left(\frac{1+h_\mathrm{s}}{2h_\mathrm{s}}\right) - \overline{f}(249/50)\right\} + H_\mathrm{rem},$$

where $H_\mathrm{rem}$ is defined in (194). No defect table is inserted at the unmarked root of $L_1$: its ratio $1/\nu$ is less than 2.

For the two-prefix lower terms put

$$\mathcal{P}_5 = \{a_\mathrm{s} < u < t < h_\mathrm{s}\}, \qquad \mathcal{P}_6 = \{a_\mathrm{s} < u < h_\mathrm{s} < t < d_\mathrm{s}\}.$$

On the enhanced part $t < \mu_0$, let $V(t,u)$ be the maximum of the admitted candidates $v(t), v(t+u), \Lambda_\mathrm{cap} - 3a_\mathrm{s}/8$; on $t > \mu_0$ take $V = v(t)$. Here admission retains the full and selected squares, the selected largest slots and the physical prime ceiling, as in Appendix C. Set $L = V - t - u$. The lower integrals are

$$L_j = 4\iint_{\mathcal{P}_j} \frac{A_-(L/a_\mathrm{s}) + h_{248,\mathrm{adm}}(t,u)}{tuL}\,du\,dt, \qquad j = 5, 6. \tag{191}$$

The defect term equals $h_{248}(L/a_\mathrm{s})$ only on a fully admitted state with $2 \le L/a_\mathrm{s} \le 5$, and is zero otherwise. The ordinary lower term is also zero when $L/a_\mathrm{s} < 2$. Thus an uncertain defect lookup can be discarded without changing an ordinary lower estimate.

For $m = 248, 312, 430$, the one-prefix upper integrals are

$$U_j^{(m)} = 4\int_{I_j} \frac{A_+(s_j(t)) - H_m(s_j(t))}{tL_j(t)}\,dt, \quad I_3 = (a_\mathrm{s}, 1/3), \quad I_4 = (a_\mathrm{s}, d_\mathrm{s}), \tag{192}$$

with $L_j, s_j$ given in (201) below. The reference values in the eleven-term sum use $m = 248$. The two subsequent table improvements are inserted only through (202).

The last two costs use decreasing coordinates $x > y > z > w$ on

$$\mathcal{D}_{10} = \{a_\mathrm{s} < w < z < y < x < h_\mathrm{s}\}, \qquad \mathcal{D}_{11} = \{a_\mathrm{s} < w < z < y < h_\mathrm{s} < x < d_\mathrm{s}\}.$$

The integration measure is

$$d_\mathrm{s}\sigma_4 = \frac{dx\,dy\,dz\,dw}{xyzw}.$$

Put $S = x + y + z + w$. The switched option has kernel $K_0\omega((1-S)/z)/z$. Each admitted ordinary branch with residual exponent $L = \lambda - S$ supplies the kernel $4\{1 - H_{248}(\max(2, L/w))\}/L$; all its full and selected support conditions must hold on the whole cell. The chosen cell majorant bounds the minimum of these options from above. Its integral defines the upper enclosure for $U_{10} + U_{11}$. The cell choices, residual intervals, closed-cell defect minima and complete ordered cover are the data of the four-prefix certificate. The switched option remains available when an ordinary branch fails its support conditions. Since $d_\mathrm{s} + 3h_\mathrm{s} < 1$, these two domains have no unit-quotient contribution. The remaining switched costs $U_7, U_8, U_9$ are defined in Section I.9.

I.8. **Definitions of the lower correction terms.** We specify the correction terms before giving their numerical combination. Put $\nu = 249/448$, $w_* = 25/224$ and write $\overline{F} = F/(2e^{\gamma})$, $\overline{f} = f/(2e^{\gamma})$. The difference between the ordinary start at $h_{\mathrm{s}}$ and the ordinary part of the least-prime decomposition is

$$B_{\mathrm{ord}} = \frac{4}{h_{\mathrm{s}}}\overline{f}(\nu/h_{\mathrm{s}}) - 4\left\{(1/h_{\mathrm{s}}+1)\overline{f}\left(\frac{1+h_{\mathrm{s}}}{2h_{\mathrm{s}}}\right) - \overline{f}(249/50)\right\}. \tag{193}$$

The defect part removed from that decomposition is the whole integral

$$H_{\mathrm{rem}} = 4\int_{w_*}^{h_{\mathrm{s}}} \frac{H_{248}((1-t)/(2t))}{t(1-t)/2}\,dt. \tag{194}$$

The finite ordinary estimate at $h_{\mathrm{s}}$ is inserted only after this entire defect term has been removed. On a defect cell $[s_l, s_r]$, the contributing interval in (194) is $(w_*, h_{\mathrm{s}}) \cap [1/(2s_r+1), 1/(2s_l+1)]$; its primitive is $2\log(t/(1-t))$, before multiplication by the factor 4.

For $w = a_{\mathrm{s}}, h_{\mathrm{s}}$, let $C_3, C_4, C_5$ be the successive caps of the same lower policy in Appendix G. A cap is inherited or selected from the finitely specified subproduct branches subject to all full and selected-complement support conditions. In particular $\nu \le C_3 \le C_4 \le C_5 \le \Lambda_{\mathrm{cap}}$. For decreasing $t_1, \dots, t_r$, write $S_r = \sum_{j\le r} t_j$ and

$$d_{\mathrm{s}}\mu_r = \frac{dt_1 \cdots dt_r}{t_1 \cdots t_{r-1} t_r^2}.$$

The three retained domains are

$$\begin{aligned}
\mathcal{E}_3(w) &= \{0 < t_3 < t_2 < t_1 < w : t_1 + 3t_2 < \nu\},\\
\mathcal{E}_4(w) &= \{(t_1, \dots, t_4) : (t_1, t_2, t_3) \in \mathcal{E}_3(w),\ 0 < t_4 < t_3,\ S_3 + 3t_4 < C_3\},\\
\mathcal{E}_5(w) &= \{(t_1, \dots, t_5) : (t_1, \dots, t_4) \in \mathcal{E}_4(w),\ 0 < t_5 < t_4\}.
\end{aligned}$$

Thus the three nonnegative density gains are

$$I_3(w) = 4\int_{\mathcal{E}_3(w)} \left\{\overline{F}\left(\frac{\nu - S_3}{t_3}\right) - \overline{F}\left(\frac{C_3 - S_3}{t_3}\right)\right\} d_{\mathrm{s}}\mu_3, \tag{195}$$

$$I_4(w) = 4\int_{\mathcal{E}_4(w)} \left\{\overline{f}\left(\frac{C_4 - S_4}{t_4}\right) - \overline{f}\left(\frac{C_3 - S_4}{t_4}\right)\right\} d_{\mathrm{s}}\mu_4, \tag{196}$$

$$I_5(w) = 4\int_{\mathcal{E}_5(w)} \left\{\overline{F}\left(\frac{C_4 - S_5}{t_5}\right) - \overline{F}\left(\frac{C_5 - S_5}{t_5}\right)\right\} d_{\mathrm{s}}\mu_5. \tag{197}$$

The fifth square follows from $S_4 + 2t_5 < S_3 + 3t_4 < C_3 \le C_4$. The cap choices, branch labels and parent cubes are fixed jointly; an integral evaluated with a different fourth cap cannot be appended to (197). The rational certificates may integrate only a subset of these nonnegative gains, so their endpoints are lower bounds, rather than assertions that the selected boxes exhaust the ideal domains.

For the fourth-node restoration, let $\mathcal{R}_{h_{\mathrm{s}}}$ be the disjoint union of the rational slanted slabs specified by (151). On each slab a cap $\lambda_4$ is chosen before the fourth cube, and $C_3 \le S_3 + 3t_4 < \lambda_4$, with the full and selected support conditions retained. Its additional gain is

$$T_{h_{\mathrm{s}}} = 4\int_{\mathcal{R}_{h_{\mathrm{s}}}} \overline{f}\left(\frac{\lambda_4 - S_4}{t_4}\right) d_{\mathrm{s}}\mu_4. \tag{198}$$

The exact slab volume is (153). This term enters the Chen functional with coefficient $1/4$, and its domain is disjoint from the other gains already counted.

I.9. **The fixed upper-cost slope and table changes.** The coefficient $A_{\mathrm{cost}}$ used below is a safe slope of a specified affine upper enclosure of five costs. It need not be the derivative of a minimum of kernels. To state its components, put $K_0 = 6.7619$ and $A_+(s) = s\overline{F}(s)$. The three one-dimensional switched costs are

$$\frac{U_7}{K_0} = \int_{d_{\mathrm{s}}}^{1/3} \frac{\log((1-2t)/t) + A_+(1/t-1) - 1}{t(1-t)}\,dt,$$
$$\frac{U_8}{K_0} = \int_{a_{\mathrm{s}}}^{1/3} \frac{\log(2-3t)}{t(1-t)}\,dt,$$
$$\frac{U_9}{K_0} = \int_{h_{\mathrm{s}}}^{d_{\mathrm{s}}} \frac{\log((1-t-d_{\mathrm{s}})/d_{\mathrm{s}})}{t(1-t)}\,dt.$$

The extra ordinary-function contribution in the first integral can be evaluated without interpolation:

$$\int_{d_{\mathrm{s}}}^{1/4} \frac{A_+(1/t-1)-1}{t(1-t)}\,dt = \int_2^{1/d_{\mathrm{s}}-2} \frac{\log(u-1)}{u} \log\frac{1-d_{\mathrm{s}}}{d_{\mathrm{s}}(u+1)}\,du.$$

Let $\underline{U}_j$ denote the rational lower endpoints for these $K_0$-linear costs from the fourth-derivative Simpson enclosures specified in `five_cost_slope_certificate`. The root cost upper endpoints may use the separate midpoint enclosures; an upper endpoint and a lower endpoint have different roles here. Their contribution to a safe slope is $A_{\mathrm{cost},789} = (2\underline{U}_7 + \underline{U}_8 + \underline{U}_9)/(4K_0)$. For a retained four-prefix cell $\mathscr{B}$ outside the 40000 priority parents, write $A_{\mathscr{B}}$ for its assigned upper mass at $K_0$ and $s_{\mathscr{B}}$ for the supplied nonnegative affine slope. The required cellwise conditions are

$$0 \le s_{\mathscr{B}} \le A_{\mathscr{B}}/K_0, \qquad A_{\mathscr{B}} \ge K_0 \mathcal{T}_{\mathscr{B}}, \tag{199}$$

where $\mathcal{T}_{\mathscr{B}}$ is the exact nonnegative switched mass. They give

$$A_{\mathscr{B}} - (K_0 - K)s_{\mathscr{B}} \ge (K/K_0)A_{\mathscr{B}} \ge K\mathcal{T}_{\mathscr{B}}.$$

No inequality $s_{\mathscr{B}} \le \mathcal{T}_{\mathscr{B}}$ is required. At denominator $\mathsf{Q} = 2^{80}$ the retained row slopes are $s_{\mathscr{B}} = 10000 d_{\mathscr{B}}/\mathsf{Q}$. Their sums for $U_{10}, U_{11}$ are 1277985388931956822 and 7976183963523684871, respectively. Consequently

$$A_{\mathrm{cost}} = A_{\mathrm{cost},789} + \frac{10000(1277985388931956822 + 7976183963523684871)}{4\mathsf{Q}}. \tag{200}$$

The complete retained-cell inequalities are indexed by the `feedback_original_slope` entry of the certificate map.

For two valid defect envelopes $H_m \le H_n$, define

$$\mathcal{T}(H_m, H_n) = \int_{a_{\mathrm{s}}}^{1/3} \frac{(H_n - H_m)(s_3(t))}{tL_3(t)}\,dt + \int_{a_{\mathrm{s}}}^{d_{\mathrm{s}}} \frac{(H_n - H_m)(s_4(t))}{tL_4(t)}\,dt, \tag{201}$$

where $s_j(t) = \max(2, L_j(t)/a_{\mathrm{s}})$, with zero contribution outside $[1,5]$, and, for $e = 57/224$,

$$L_3(t) = \begin{cases} (1-t)/2, & a_{\mathrm{s}} < t < \mu_0, \\ 57/64 - 5t/2, & \mu_0 < t < e, \\ 1/2 - t, & e < t < 1/3. \end{cases}$$

The function $L_4$ uses the first two branches restricted to $(a_{\mathrm{s}}, d_{\mathrm{s}})$. The factors 4 in the two upper counts cancel the single Chen factor $1/4$. Thus

$$T_{312} = \mathcal{T}(H_{248}, H_{312}), \qquad T_{430} = \mathcal{T}(H_{312}, H_{430}). \tag{202}$$

At a knot the lookup uses the smaller adjacent value; on the positive-measure branch $s_j = 2$ this means the minimum of cells 49 and 50. On an affine piece $L = A_0 - B_0 t$ the exact inverse-cell primitive is

$$\int_l^r \frac{dt}{t(A_0 - B_0 t)} = \frac{1}{A_0} \log\frac{r(A_0 - B_0 l)}{l(A_0 - B_0 r)}.$$

This determines each table-change interval and its rational logarithm enclosure without numerical interpolation across a knot.

I.10. **Mathematical objects and certificate fields.** Table 2 links the count estimates, their integral bounds and the ancillary certificates. Paths in the last column are targets in `certificate-map.json`; each target has the source filename, exact JSON pointer, direction and any rational extraction formula. The map also includes the enclosure and coverage assertions for each primitive family and the corresponding full-witness filenames. The map's `domain_constructions` entries identify the actual $C_3, C_4, C_5$ region generators and their selected records, including the two third-rank starts, the fourth-rank chord and oblique regions, the fifth-rank child increments, and the slanted restoration. For $U_{10}, U_{11}$ they identify the generator `witnesses/generator-0041.py` and the complete ordered witness `witnesses/record-0638.json`. The witness records specify the parent policy, method choices, generation-248 profile, and full facet descriptions of these domains.

Table 2: Definitions, directions and certificate targets.

| Object | Mathematical estimate | Certificate target |
|---|---|---|
| $G_j$, $1 \le j \le 22$ | The specified child–parent density differences in the upper tree; nonnegative savings may use a subdomain. | `/upper_saving_bounds/` $(j-1)$; lower endpoint. |
| Finite actions | (159), (172), (164); complete $\Gamma_5$ with separate payment on $D_5^-$. | `/actions/` $k$; lower base and feedback weights; `/gamma5_cost_decompositions`. |
| Seed and tables | Lemma G.1 and the simultaneous map (190); closed-cell estimates. | `/seed_certificate`, `/defect_tables/` $m$; lower endpoints, $m = 248, 312, 430$. |
| $L_i, U_i$ | The integrals in Section I.7; complete retained domain for each negative cost, lower estimate for each positive term. | `/lower_root_bounds`; lower for $L_i$, upper for $U_i$. |
| $B_{\mathrm{ord}}, H_{\mathrm{rem}}$ | (193), (194); remove the entire generation-248 defect. | `/lower_auxiliary_bounds`; lower for $B_{\mathrm{ord}}$, upper for $H_{\mathrm{rem}}$. |
| $I_r(w), T_{h_s}$ | (195)–(198); one common lower policy and disjoint restoration slabs. | `/lower_integral_bounds` and `/lower_auxiliary_bounds/restoration_L2_lower`; lower. |
| $A_{\mathrm{cost}}$, $T_{312}, T_{430}$ | (199)–(202); upper-cost slope and successive table changes. | `/five_cost_slope_certificate`, `/lower_auxiliary_bounds`; safe lower coefficients. |
| Paired deletion | Assigned positive majorant and discarded lower estimator on $\mathscr{E}_0 \cup \mathscr{E}_1$; (204), (207). | `/paired_rectangles`; exact geometry, mass intervals and lower estimator integers. |
| $E_{\mathrm{up}}, R_{\mathrm{low}}$ and reserves | The unique combinations in (203) and the upper density expression. | `/scalar_bounds`; exact sums and strict surpluses. |

For each negative cost, the mapped witnesses specify a complete cover of its retained domain. For a positive saving, they specify the selected nonnegative subfamily on which its lower bound is evaluated.

I.11. **The lower functional.** For the signed eleven-term expression, use a lower endpoint for each positive term and an upper endpoint for each negative term. The following integers, divided by $10^{14}$, give such endpoints:

| Term | Coefficient before division by 4 | Endpoint numerator |
|---|---|---|
| $L_1$ | 3 | 1346406373358053 |
| $L_2$ | 1 | 710262601191162 |
| $L_5$ | 1 | 237450056438739 |
| $L_6$ | 1 | 301984961684469 |
| $U_3$ | −1 | 2031242249325404 |
| $U_4$ | −1 | 1405868301028074 |
| $U_7$ | −2 | 138343028785273 |
| $U_8$ | −1 | 405707594761181 |
| $U_9$ | −1 | 325703365221150 |
| $U_{10}+U_{11}$ | −1 | 67568703438104 |

The exact rational endpoints, rather than these shortened ones, define $C_{11}$. The four-prefix minimum covers all

$$\binom{51}{4}+48\binom{50}{3}=1190700$$

ordered rows, including the tied-coordinate simplices. The priority subdivision replaces 40000 parents by maxima with their complete sums of 516779 children. Its extra coefficient saving is

$$P=\frac{106520738437439293219}{6044629098073145873530 88}.$$

The terms $B_{\mathrm{ord}}, H_{\mathrm{rem}}$, $I_r(a_{\mathrm{s}}), I_r(h_{\mathrm{s}})$, $T_{h_{\mathrm{s}}}$ are defined in (193)–(198). Their exact bounds are specified by `lower_auxiliary_bounds` and `lower_integral_bounds`. At $K_0=6.7619$ put

$$C_{K_0}=C_{11}+P+\frac{1}{4}\left(B_{\mathrm{ord}}-H_{\mathrm{rem}}+\sum_{r=3}^{5}\{3I_r(a_{\mathrm{s}})+I_r(h_{\mathrm{s}})\}+T_{h_{\mathrm{s}}}\right).$$

The third-rank integrals cover both cutoff regions. The fourth-rank integrals retain all specified masks and the disjoint additional bands. The fifth-rank integration uses 158251 floors and 23662 additional slab or tied cells on the $h_{\mathrm{s}}$ side and 679 integrals on the $a_{\mathrm{s}}$ side. The slanted restoration uses 98668 boxes and 789344 layers. Only nonnegative gains may omit boundary strips. An asymmetric kernel is bounded on its ordered simplex; no factorial symmetry is assumed for such a kernel.

With $A_{\mathrm{cost}}$ as in (200) and $T_{312}, T_{430}$ as in (202), the initial lower endpoint is bounded below by

$$C_{K_0}+A_{\mathrm{cost}}(K_0-K)+T_{312}+T_{430}+0.002095161+0.003799514-0.002372094. \tag{203}$$

Direct rational substitution is at least $R_{\mathrm{base}}$ in (184). The paired-cell refinement below adds exactly $\Delta$ from (45). After allocating the two reserves in (48), the resulting lower endpoint exceeds $c$ by

$$\frac{1345726092686363377177567858471124575 3}{2742952509978644604793778369418231808000000000}>0.$$

Together with the upper calculation, this proves the scalar inequalities in Lemma 7.1 from the stated finite enclosures.

## Appendix J. A paired-cell refinement

The paired rectangles are specified without a geometric rounding convention as follows. Put

$$t_i=\mu_0+i\frac{d_{\mathrm{s}}-\mu_0}{512},\qquad u_j=a_{\mathrm{s}}+j\frac{h_{\mathrm{s}}-a_{\mathrm{s}}}{512}\qquad(0\le i,j\le 512).$$

Each row of `paired_rectangles` specifies a family $\mathscr{E}_0$ or $\mathscr{E}_1$ and the rectangle $[t_i, t_{i+1}] \times [u_j, u_{j+1}]$. The families have 6091 and 11481 cells, respectively, with no common cell. Their boundaries have zero integration measure. For a rectangle $\mathscr{B} = [t_l, t_r] \times [u_l, u_r]$, define

$$Z_{\mathscr{B}} = \min\left(u_l, \frac{3(1 - t_r - u_r)}{13}\right).$$

The rational data satisfy $Z_{\mathscr{B}} > a_{\mathrm{s}}$ and $1 - t_r - u_r - 2Z_{\mathscr{B}} \geq (7/3)Z_{\mathscr{B}}$ in every row. Hence

$$\frac{1 - t - u - z - w}{z} \geq \frac{7}{3} \qquad (a_{\mathrm{s}} < w < z < Z_{\mathscr{B}},\ (t, u) \in \mathscr{B}).$$

On each rectangle, Lemma 6.2 gives $\omega \geq 1/2$ throughout the fibre and $\omega \geq 11/20$ on its specified inner part. The second estimate improves a subset of the first contribution and is used as an increment.

The positive parent majorant can therefore be bounded below by the contribution from $1/2$ on its whole fibre plus $1/20$ on this inner fibre. These are simultaneous nonnegative contributions. After integration and the single Chen factor $1/4$, the additional contribution is

$$G_{\mathscr{B}} = \frac{K}{80} \log\frac{t_r}{t_l} \log\frac{u_r}{u_l} \left\{\frac{1}{a_{\mathrm{s}}} - \frac{1 + \log(Z_{\mathscr{B}}/a_{\mathrm{s}})}{Z_{\mathscr{B}}}\right\}. \tag{204}$$

Indeed the $w$ integral is $\log(z/a_{\mathrm{s}})$ and

$$\int_{a_{\mathrm{s}}}^{Z_{\mathscr{B}}} \frac{\log(z/a_{\mathrm{s}})}{z^2}\, dz = \frac{1}{a_{\mathrm{s}}} - \frac{1 + \log(Z_{\mathscr{B}}/a_{\mathrm{s}})}{Z_{\mathscr{B}}}.$$

The normalization agrees with (40).

Every row of the data contains rational lower and upper endpoints for (204), represented by integer numerators at denominator $2^{160}$. Applying (186) to its three logarithms and summing the lower endpoints gives

$$\sum_{\mathscr{B} \in \mathscr{E}_0 \cup \mathscr{E}_1} G_{\mathscr{B}} > \frac{558583}{10^9}. \tag{205}$$

The sums on the two families are approximately $1.9318091250 \cdot 10^{-4}$ and $3.6540308174 \cdot 10^{-4}$, respectively. Only the rational lower bound (205) is used.

## Appendix K. Discarded lower estimators and the final reserves

The paired deletion removes a positive term together with its assigned lower estimator. To bound the resulting loss, we give both the exact finite sum of that estimator and a direct rational upper enclosure.

On a cell write

$$R(t, u) = \frac{57}{64} - \frac{5}{2}t - u, \qquad r = R(t_r, u_r), \quad R_0 = R(t_l, u_l), \quad M = \log(t_r/t_l)\log(u_r/u_l).$$

Cells of $\mathscr{E}_0$ have $R_0/a_{\mathrm{s}} \leq 2$ and zero assigned lower estimator. There are 227 cells of $\mathscr{E}_1$ with $r/a_{\mathrm{s}} < 2$; their assigned lower estimator is also zero. On the remaining 11254 cells, $2 \leq r/a_{\mathrm{s}} \leq R_0/a_{\mathrm{s}} \leq 9/4$. Put

$$f_l = \frac{\log(r/a_{\mathrm{s}} - 1)}{r/a_{\mathrm{s}}}, \qquad f_h = \frac{\log(R_0/a_{\mathrm{s}} - 1)}{R_0/a_{\mathrm{s}}},$$

and

$$M_R = \frac{57}{64}M - \frac{5}{2}(t_r - t_l)\log(u_r/u_l) - (u_r - u_l)\log(t_r/t_l).$$

Since $r \leq R(t, u) \leq R_0$, the two assigned ordinary estimators, in Chen-quarter units, are bounded above by

$$\frac{f_l M}{a_{\mathrm{s}}}, \qquad \frac{f_l(R_0 M - M_R) + f_h(M_R - rM)}{a_{\mathrm{s}}(R_0 - r)}. \tag{206}$$

The coefficients of $f_l, f_h$ are nonnegative. Replacing lower endpoints for these functions by their exact values therefore increases the estimator. The maximum of the two expressions in (206) remains an upper bound for the maximum of the two assigned alternatives.

Let $h_{\mathscr{B}}^{\min}$ be the minimum of the depth-248 lower table entries over all closed cells meeting $[r/a_{\mathrm{s}}, R_0/a_{\mathrm{s}}]$. Both adjoining cells are included at a knot. The exact support is contained in the outward-rounded lookup support of the assigned estimator, so $h_{\mathscr{B}}^{\min}$ is at least the minimum used there. Thus the additional term $h_{\mathscr{B}}^{\min} M/R_0$ bounds its assigned defect estimator from above. This comparison uses minima of finite lower estimators, not an upper estimate for the mathematical function $h$. Adding it to the maximum in (206) gives the required cell proxy.

The data record the assigned lower estimator on every cell as an integer at denominator $\mathsf{Q} = 2^{80}$. Their sum is $112908068451756520202 70$. After division by $4\mathsf{Q}$,

$$L_{\mathscr{E}} = \frac{5645403422587826010135}{24178516392292583494123 52} < \lambda_{\mathrm{loss}} := \frac{466977}{200000000}. \tag{207}$$

An outward evaluation of the proxies above gives an upper endpoint below the same $\lambda_{\mathrm{loss}}$. For this evaluation one may reduce logarithm arguments to $[1, 3/2]$, use the 320-term alternating series for $\log(1+x)$ and bound its remainder by the next term. All its divisors have positive interval lower endpoints. The resulting proxy sum is enclosed near $0.00233488413635981310147442734879 04$.

The retained whole $L_6$ endpoint before the deletion is $365077417315690091410168 5/\mathsf{Q}$. Its nonenhanced half has the cell decomposition just used. Consequently subtracting $4\lambda_{\mathrm{loss}}$ from that endpoint bounds the retained $L_6$ from below; the enhanced half is unaffected. The decrease of the paid loss relative to the value in (203) is exactly

$$0.002372094 - \lambda_{\mathrm{loss}} = \frac{37209}{10^9}.$$

Combining this with (205) gives the raw increment

$$\Delta = \frac{558583 + 37209}{10^9} = \frac{37237}{62500000}.$$

Allocate $\Delta/8$ to each of the finite-approximation and coefficient-realization reserves. The net retained increment is therefore $3\Delta/4 = 0.000446844$. All estimates refer to this same fixed set of rectangles, and each density contribution appears once. They give $R_{\mathrm{low}} = R_{\mathrm{base}} + \Delta$ and $B_{\mathrm{low}} = B_{\mathrm{base}} + \Delta/8$, as in (47) and (48).

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