ITERATING THE GUPTA–NDIAYE–NORIN–WEI METHOD: A COMPUTER-VERIFIED BOUND $R(k) \le 3.7698^{\,k+o(k)}$

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Abstract. Gupta, Ndiaye, Norin and Wei (GNNW) recast the Campos–Griffiths–Morris–Sahasrabudhe method as an inductive theorem that turns a known bound on off-diagonal Ramsey numbers into a better one. With two applications they obtained $R(k) \le (3.7992\ldots)^{k+o(k)}$. They also reported an unverified, AI-suggested third application giving $3.78233\ldots$. We make three observations about their framework. (i) The free function M can be optimised pointwise. (ii) For a fixed input, the best output exponent is the least solution of an explicit first-order differential inequality. (iii) GNNW's Theorem 14 needs only $F \in C^1$ with locally Lipschitz derivative. We then construct an explicit chain of eight applications and verify every hypothesis by interval arithmetic on the whole range $0 < \lambda \le 1$, with a hand-proved estimate covering $\lambda \to 0$. The verification was done twice, with independent code, libraries and enclosure strategies. This proves $R(k) \le 3.76973^{\,k+o(k)}$, and in particular confirms GNNW's 3.78233. Numerically, iterating the optimal step converges to about 3.7690, which suggests this is close to the limit of the GNNW method in its present form.

1. Introduction

The diagonal Ramsey number $R(k)$ is the least n such that every red/blue colouring of the edges of K_n contains a monochromatic K_k. Erdős and Szekeres [6] proved $R(k) \le 4^k$, and Erdős [5] proved $R(k) \ge \sqrt{2}^{\,k}$. Whether $\lim_k R(k)^{1/k}$ exists, and what it is, is Erdős Problem #77 [4]. Campos, Griffiths, Morris and Sahasrabudhe [3] gave the first exponential improvement of the upper bound, $R(k) \le (4-\varepsilon)^k$. Gupta, Ndiaye, Norin and Wei [7] simplified the argument to a single inductive statement (their Theorem 14) and optimised it, obtaining

$$R(k) \le \left(4e^{-0.14/e}\right)^{k+o(k)} = (3.7992\ldots)^{k+o(k)}.$$

In their Remark 17 they record a further, unverified, AI-generated iteration that would give $(3.78233\ldots)^{k+o(k)}$. Balister et al. [2] gave another short proof of an exponential improvement, extended to many colours.

In plain words: the party. Think of a party where any two guests are either friends or strangers. Color the edge between two guests red if they are friends and blue if they are strangers. Then $R(k)$ is the smallest party that is sure to have k guests who are all friends with each other, or k guests who are all strangers. Six guests always include three mutual friends or three mutual strangers, and five are not enough, so $R(3) = 6$. We know $R(4) = 18$, but already $R(5)$ is only known to lie between 43 and 46. Exact values are out of reach, so Problem #77 asks a coarser question: how fast does the party have to grow? It grows roughly like C^k, and we want C. The bounds above say $\sqrt{2} \le C \le 4$, and [7] brought the 4 down to 3.7992.

The lopsided number $R(k, \ell)$ asks for k mutual friends or ℓ mutual strangers. The GNNW method works with a whole curve of bounds on these lopsided parties, one for each ratio $\lambda = \ell/k$. Their Theorem 14 works like a machine. You feed in a curve of bounds you have already proved.

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If one inequality holds at every λ, a better curve comes out. GNNW ran the machine twice. We run it eight times, each time feeding the last output back in, and the computer checks every run with no gaps between the points it tests. The machine stalls near 3.769: running it more often barely helps, and going further will need a new idea.

Theorem 1.1. *$R(k) \le 3.76973^{\,k+o(k)}$. More precisely, there is an explicit function F_8, given by the node data in §3.3, with $e^{F_8(1)} \le 3.7697215$ and $R(k,\ell) \le e^{F_8(\ell/k)k+o(k)}$ for all $k \ge \ell$.*

Theorem 1.2. *The bound of GNNW's Remark 17 holds: $R(k) \le (3.782329)^{k+o(k)}$.*

Theorem 1.2 is superseded by Theorem 1.1, but it settles the status of the bound recorded in [7, Remark 17].

Nature of the proof. Both theorems follow from GNNW's Theorem 14 and Lemma 15, which we use as stated in [7] (see §2 for the precise hypotheses and two small clarifications). The new input is a choice of functions, together with a verification that they satisfy the hypotheses. The verification consists of one-variable inequalities between elementary functions on $(0,1]$. We prove them on $[10^{-7},1]$ by interval arithmetic, and on $(0,10^{-7}]$ by Lemma 4.1, a hand-proved estimate whose constants are again evaluated in interval arithmetic. The verification was carried out by two programs that share no code (§4).

The limit of the method. In §5 we explain why the outcome of one GNNW step, for a given input, is governed by an ODE. We report that iterating the optimal step from the Erdős–Szekeres bound converges numerically to about 3.76897. This part is numerical evidence, not a theorem. It is consistent with GNNW's expectation that going below 3.75 needs new ideas.

2. The GNNW framework

Write $\lambda = \ell/k \in (0,1]$ and $h(\lambda) = (1+\lambda)\log(1+\lambda) - \lambda\log\lambda$, so that $\log\binom{k+\ell}{\ell} = h(\ell/k)k + o(k)$. Following [7], let $\mathcal{R}_0$ be the set of $(x,y) \in (0,1)^2$ such that $R(k,\ell) \le x^{-k}y^{-\ell}$ whenever $k+\ell$ is large. Let $\mathcal{R}$ be its closure and $\mathcal{R}_*$ its interior. We use the following facts from [7, Observation 10]: $(x,1-x) \in \mathcal{R}$ for $0 < x < 1$; $\mathcal{R}$ is closed downwards; strictly smaller points of $\mathcal{R}$ lie in $\mathcal{R}_*$; and $R(k,\ell) \le x^{-k-o(k)}y^{-\ell-o(\ell)}$ implies $(x,y) \in \mathcal{R}$.

Theorem 2.1 ([7, Theorem 14])**.** *Let $F : (0,1] \to \mathbb{R}_{>0}$ be smooth, and let $M, X, Y : (0,1] \to (0,1)$ with M continuous. Suppose that for all $0 < \lambda \le 1$ we have $F'(\lambda) > 0$,*

$$X(\lambda) = \big(1 - e^{-F'(\lambda)}\big)^{\frac{1}{1-M(\lambda)}}\big(1 - M(\lambda)\big), \qquad (X(\lambda), Y(\lambda)) \in \mathcal{R},$$

$$\Phi(\lambda) := F(\lambda) + \tfrac{1}{2}\big(\log X(\lambda) + \lambda\log M(\lambda) + \lambda\log Y(\lambda)\big) > 0. \tag{1}$$

Then $R(k,\ell) \le e^{F(\ell/k)k+o(k)}$ for all $k \ge \ell$.

Lemma 2.2 ([7, Lemma 15])**.** *Let $f : (0,1] \to \mathbb{R}$ be differentiable and strictly concave with $f' > 0$, and suppose $R(k,\ell) \le e^{f(\ell/k)k+o(k)}$ for $1 \le \ell \le k$. Put $A(t) = e^{-f'(t)}$ and $B(t) = e^{tf'(t)-f(t)}$. Suppose $A < B$ on $(0,1]$, that A increases from 0 to $a = A(1)$, and that B decreases from 1 to $b = B(1)$. Define Y_f on $(0,1)$ by $Y_f(x) = B(t)$ where $A(t) = x$ (for $x < a$); $Y_f(x) = e^{-f(1)}/x$ (for $a \le x \le b$); and $Y_f(x) = A(t)$ where $B(t) = x$ (for $x > b$). Then $(x, Y_f(x)) \in \mathcal{R}$ for all $0 < x < 1$.*

2.1. Two clarifications.

We need Theorem 2.1 for functions F that are C^1 and piecewise C^2. These are not C^∞.

Proposition 2.3. *Theorem 2.1 holds with "smooth" replaced by: $F \in C^1((0,1])$, and F' is Lipschitz on $[\eta,1]$ for every $\eta > 0$.*

Proof. In the proof in [7], the smoothness of F is used in three places, and nowhere else; Theorem 13 and Lemma 12 there do not involve F.

(1) Claim 14.1 uses the uniform continuity of F and F' on $[\varepsilon', 1]$, which holds for $F \in C^1$.
(2) The induction step uses the mean value theorem to write $k\big(F(\ell/k) - F((\ell-1)/k)\big) = F'(\xi)$ with $\xi \in [(\ell-1)/k, \ell/k]$. This needs only differentiability.
(3) It then bounds $|F'(\ell/k) - F'(\xi)| \le \frac{1}{k}\max_{[\varepsilon'/2,1]}|F''|$. The bound $|F'(\ell/k) - F'(\xi)| \le \mathrm{Lip}(F'|_{[\varepsilon'/2,1]})/k$ works equally well. □

Remark 2.4 (On $\mathcal{R}$ versus $\mathcal{R}_*$). Claim 14.1 of [7] applies their Theorem 13, which requires $(x_\lambda, y_\lambda) \in \mathcal{R}_*$, to $x_\lambda = e^{-\delta} p_\lambda^{1/(1-M(\lambda))}(1 - M(\lambda))$ and $y_\lambda = e^{-\delta} Y(\lambda)$, where $p_\lambda = 1 - e^{-F'(\lambda)+2\delta}$. This membership is not spelled out there, but it is immediate. Since $\delta > 0$ we have $p_\lambda < 1 - e^{-F'(\lambda)}$, so $x_\lambda < X(\lambda)$ and $y_\lambda < Y(\lambda)$. As $(X(\lambda), Y(\lambda)) \in \mathcal{R}$, Observation 10(3) gives $(x_\lambda, y_\lambda) \in \mathcal{R}_*$.

Lemma 2.5. *Let $f = h + g$ on $(0, 1]$, where $g \in C^1([0, 1])$ is piecewise C^2 with bounded g'', $g(0) = 0$, and suppose that $f'' < 0$ on each piece, $f' > 0$, and $(1 + t)f'(t) - f(t) > 0$ on $(0, 1]$. Then f satisfies the analytic hypotheses of Lemma 2.2.*

Proof. f' is continuous and strictly decreasing on each piece, hence on $(0, 1]$. So f is strictly concave, and $A = e^{-f'}$ is strictly increasing. The function $\log B = tf' - f$ is continuous and piecewise C^1 with derivative $tf'' < 0$, so B is strictly decreasing. $A < B$ is equivalent to $(1 + t)f' - f > 0$. As $t \to 0$ we have $f'(t) = \log\frac{1+t}{t} + g'(t) \to \infty$, so $A \to 0$. Also $tf' - f = -\log(1 + t) + tg'(t) - g(t) \to 0$, because g' is bounded and $g(0) = 0$; so $B \to 1$. □

3. Choosing the functions

3.1. **M is pointwise.** In (1), $\Phi(\lambda)$ depends on M only through the single number $M(\lambda)$. So for given F and Y-curve, the best M is found by maximising, for each λ, the one-variable function

$$m \longmapsto \log X + \lambda \log m + \lambda \log Y(X), \qquad X = (1 - e^{-F'(\lambda)})^{1/(1-m)}(1 - m).$$

Theorem 2.1 only needs M to be continuous, so we take M to be a shape-preserving interpolant of the pointwise maximisers (§3.3).

3.2. **The best exponent solves an ODE.** Fix the Y-curve. Let $Q(\lambda, s)$ be the maximum above with $F'(\lambda)$ replaced by s, and let $S(\lambda, v) = \inf\{s : -\frac{1}{2}Q(\lambda, s) < v\}$. Then every F that passes Theorem 2.1 satisfies $F' \ge S(\lambda, F)$. Enlarging v enlarges the admissible set, so $S(\lambda, \cdot)$ is non-increasing. A standard comparison argument then shows that every such F lies above the solution of $F' = S(\lambda, F)$ with the same initial value. No Lipschitz condition is needed for this. We have formalised that comparison lemma in Lean 4 with Mathlib (`Erdos77.comparison_antitone`). We solve the ODE numerically, use the result as the next input, and repeat. This is only a construction device: the functions actually used are then fixed explicitly and verified directly.

3.3. **Explicit links.** Each link $n = 1, \dots, 8$ is a pair (F_n, M_n), given exactly by floating-point node data.

- $F_n = h + g_n + \varepsilon_n\lambda$, where g_n is the piecewise-cubic Hermite interpolant of nodes (x_j, g_j, s_j) with $x_0 = 0$ and $g_0 = 0$. So $g_n \in C^1$ is piecewise polynomial with $g_n(0) = 0$ exactly, and $\varepsilon_n \in \{5 \cdot 10^{-5}, 10^{-4}, 2 \cdot 10^{-4}\}$ is a safety shift.
- $\log M_n$ is the cubic Hermite interpolant in $\log\lambda$ of nodes (u_j, w_j, d_j), $u_0 = \log 10^{-7}$, extended by $M_n(\lambda) = e^{w_0 - u_0}\lambda$ below e^{u_0}. So M_n is continuous.

Link 1 uses $Y(\lambda) = 1 - X(\lambda)$, which is valid by Observation 10(1). Link $n \geq 2$ uses $Y(\lambda) = Y_{F_{n-1}}(X(\lambda))$, which is valid by Lemma 2.2 once link $n-1$ is proved. The node data are in `data/chain8_exact.json` (SHA-256 in Appendix A). For Theorem 1.2 we take GNNW's F_{AI} verbatim, with $Y = Y_{F_{0.03}}$ from their proof and an M of the same Hermite type.

4. Verification

For each step we must show four things: that $\Phi > 0$ on $(0, 1]$; that $0 < M < 1$, $0 < X < 1$ and $F' > 0$; that $F > 0$, which follows from $\Phi > 0$ because every logarithm in (1) is negative; and that the predecessor's hypotheses for Lemma 2.5 hold on $(0, 1]$.

4.1. **Enclosures.** All quantities are enclosed in intervals with outward rounding. The Y-curve of the predecessor f is decreasing in x, so on a cell it suffices to bound Y below at $\sup X$. For a point x in the branch $x < a$ (respectively $x > b$), we isolate the parameter t with $A(t) = x$ (respectively $B(t) = x$) by certified bisection, using the monotonicity from Lemma 2.5. We then evaluate B (respectively A) at the appropriate endpoint. At a junction we move to a certified point of the neighbouring branch, which is valid because Y_f is decreasing. Concretely, near the junction at b the hyperbola $e^{-f(1)}/x$ is an upper bound for Y_f when $x > b$, so it must not be used there.

4.2. **Two independent programs.**

(1) *Mean-value certifier* (`certify.py`, `mpmath.iv`). On each cell I with midpoint c it proves $\Phi(I) \subseteq \Phi(c) + \Phi'(I)(I - c)$, using closed forms for Φ'. These include $d \log Y / d \log X \in \{-t, -1, -1/t\}$ on the three branches. The bound is valid because Φ is continuous and piecewise C^1.

(2) *Direct auditor* (`audit_arb.py`, Arb [1] via python-flint). It shares no code with the first program, and evaluates Φ/λ directly, with no derivative, in the exactly equivalent form

$$\frac{\Phi}{\lambda} = \Big[\tfrac{(1+\lambda)\log(1+\lambda)}{\lambda} + \tfrac{g}{\lambda}\Big] + \tfrac{1}{2}\Big[r(M)\tfrac{M}{\lambda} + \tfrac{r(z)\,z/\lambda}{1-M}\Big] + \tfrac{1}{2}\log\tfrac{M}{\lambda} + \tfrac{1}{2}\log\tfrac{Y}{\lambda},$$

where $z = e^{-F'} = \frac{\lambda}{1+\lambda} e^{-g'}$ and $r(y) = \log(1-y)/y$. In this form the $\log(1/\lambda)$ terms cancel algebraically rather than numerically.

Both programs bisect cells adaptively on $[10^{-7}, 1]$, starting from geometric grids. Both also verify the hypotheses of Lemma 2.5 for every predecessor on all of $(0, 1]$: by cells on $[t_0, 1]$, and on $(0, t_0]$ by the explicit bounds $f'' \leq 1 - 1/t_0 + \sup g''$, $f' \geq \log(1/t_0) + \inf g'$ and $(1+t)f' - f \geq f' - f \geq \log(1/t_0) + \inf g' - h(t_0) - \sup g$. Here $t_0 = 10^{-6}$ for the first program and 10^{-7} for the second. As a control, both programs reject several known-false steps.

4.3. **Small λ.**

Lemma 4.1. *Let $0 < \lambda_t \leq 10^{-3}$. Suppose that on $[0, \lambda_t]$ we have $F = h + g$ with $g(0) = 0$, $a = g'(0)$ and $|g''| \leq G_2$, and that $M(\lambda) = c(\lambda)\lambda$ with $c(\lambda) \in [c_-, c_+]$. Put $E = e^{-a}$, $m_+ = c_+\lambda_t$, $\bar{z} = Ee^{G_2\lambda_t}$, $k_- = c_- + Ee^{-G_2\lambda_t}/(1+\lambda_t)$ and $u_{\max} = \lambda_t(c_+ + \bar{z})$. Assume $\lambda_t\bar{z} < 1$ and $m_+ < 1$. Then for all $\lambda \in (0, \lambda_t]$,*

$$\frac{\Phi(\lambda)}{\lambda} \;\geq\; 1 + a - \frac{G_2\lambda_t}{2} + \frac{1}{2}\Big[-\frac{c_+}{1-m_+} - \frac{\bar{z}}{(1-\lambda_t\bar{z})(1-m_+)} + \log c_- + \log k_- + T_Y\Big].$$

Here $T_Y = \log(1 - u_{\max}/2)$ when $Y = 1 - X$. When $Y = Y_f(X)$ for a predecessor $f = h + g_p$ with $g_p(0) = 0$, $b = g_p'(0)$, $|g_p''| \leq B_2$ on $[0, u_{\max}]$ and $u_{\max} < -\log B(1)$, we have $T_Y = -b - (1 + \frac{3}{2}B_2)u_{\max}$.

Proof. We bound the terms of Φ/λ one at a time.

- *The F term.* Since $(1+\lambda)\log(1+\lambda) \geq \lambda$ and $g(\lambda) \geq a\lambda - G_2\lambda^2/2$, we have $F/\lambda \geq \log(1/\lambda) + 1 + a - G_2\lambda_t/2$.
- *The quantity z.* From $F' = \log\frac{1+\lambda}{\lambda} + g'$ and $|g' - a| \leq G_2\lambda$ we get $z = e^{-F'} \in \big[\lambda E e^{-G_2\lambda}/(1+\lambda),\ \lambda\bar{z}\big]$.
- *The X term.* Using $\log(1-y) \geq -y/(1-y)$ in $\log X = \log(1-M) + \log(1-z)/(1-M)$ gives the stated lower bound for $\log X/\lambda$.
- *A lower bound for u.* Using $\log(1-y) \leq -y$ instead gives $u := -\log X \geq M + z \geq \lambda k_- =: u_-$, and also $u_- \leq u_{\max}$.
- *The Y term, Erdős–Szekeres case.* Here $\log Y = \log(1 - e^{-u}) \geq \log u_- + \log(1 - u_-/2)$.
- *The Y term, Lemma 2.2 case.* The point $x = e^{-u}$ lies in the far branch $x > b$, where $Y = A(t)$ and $u = \varphi(t) := f(t) - tf'(t) = \log(1+t) + g_p(t) - tg_p'(t)$. The function φ is increasing and satisfies $\varphi(t) \leq t + B_2t^2/2$. Put $t_{\mathrm{low}} = u_-/(1 + B_2u_-/2) \leq u_-$. Then $\varphi(t_{\mathrm{low}}) \leq u_- \leq u$, so $t \geq t_{\mathrm{low}}$. Since f' is decreasing, $\log Y = -f'(t) \geq -f'(t_{\mathrm{low}}) \geq -t_{\mathrm{low}} - \log(1/t_{\mathrm{low}}) - b - B_2t_{\mathrm{low}}$. Combining this with $\log(1/t_{\mathrm{low}}) \leq -\log\lambda - \log k_- + B_2u_-/2$ gives $\log Y \geq \log\lambda + \log k_- + T_Y$.
- *The M term.* $\frac{1}{2}\log M \geq \frac{1}{2}\log c_- + \frac{1}{2}\log\lambda$.

Adding the bounds, the terms $\log(1/\lambda) + \frac{1}{2}\log\lambda + \frac{1}{2}\log\lambda$ cancel. □

For $\lambda_t = 10^{-7}$, both programs evaluate this bound in interval arithmetic for every step. On GNNW's own steps the bound gives 0.0209565 and 0.145956, just below the limits 0.0209569… and 0.1459569… reported in [7, Appendix A].

4.4. **Results.** Table 1 lists the outcome. Every step is verified on all of $(0, 1]$ by both programs.

step	$e^{F(1)} \leq$	min. cert. Φ/λ on $[10^{-6}, 1]$	tail bound on $(0, 10^{-7}]$	Arb cells
GNNW step 0	3.8697319	$2.5 \cdot 10^{-3}$	$2.10 \cdot 10^{-2}$	20 756
GNNW step 1	3.7992028	$2.9 \cdot 10^{-4}$	$1.46 \cdot 10^{-1}$	26 745
Remark 17 (F_{AI})	3.7823288	$3.3 \cdot 10^{-7}$	$6.20 \cdot 10^{-5}$	832 915
link 1	3.8518222	$4.7 \cdot 10^{-5}$	$5.83 \cdot 10^{-4}$	411 386
link 2	3.7837987	$2.6 \cdot 10^{-6}$	$6.22 \cdot 10^{-4}$	404 848
link 3	3.7726166	$1.8 \cdot 10^{-6}$	$8.30 \cdot 10^{-4}$	270 513
link 4	3.7703296	$1.0 \cdot 10^{-4}$	$1.18 \cdot 10^{-3}$	168 875
link 5	3.7698513	$1.1 \cdot 10^{-4}$	$1.09 \cdot 10^{-3}$	168 514
link 6	3.7697479	$1.1 \cdot 10^{-4}$	$1.09 \cdot 10^{-3}$	168 760
link 7	3.7697261	$1.1 \cdot 10^{-4}$	$1.10 \cdot 10^{-3}$	168 845
link 8	**3.7697215**	$1.1 \cdot 10^{-4}$	$1.10 \cdot 10^{-3}$	168 849

TABLE 1. Verified steps. The third column is the mean-value certifier's worst certified lower bound, which is conservative. The tail column is Lemma 4.1 with $\lambda_t = 10^{-7}$. The gap $[10^{-7}, 10^{-6}]$ is covered by separate cell runs.

Proof of Theorems 1.1 and 1.2. For $n = 1, \ldots, 8$, link n satisfies the hypotheses of Theorem 2.1 in the form of Proposition 2.3. Here F_n is C^1, and F_n' is Lipschitz on $[\eta, 1]$ because g_n'' is bounded. The required $(X, Y) \in \mathcal{R}$ comes from Observation 10(1) for $n = 1$. For $n \geq 2$ it comes from Lemma 2.2, applied to F_{n-1} using Lemma 2.5 and the conclusion for link $n-1$. Theorem 2.1 for link 8 gives $R(k, \ell) \leq e^{F_8(\ell/k)k + o(k)}$, and $\ell = k$ gives Theorem 1.1. Theorem 1.2 follows in the same way from GNNW's steps 0 and 1 followed by the F_{AI} step. □

5. The limit of the method

Iterating the optimal step of §3.2, starting from $Y = 1 - X$, produces the following values of $e^{F(1)}$: 3.85163, 3.78361, 3.77224, 3.76958, 3.76910, 3.76899, 3.76897, 3.76897. The explicit links of Table 1 trail these values by about $7.5 \cdot 10^{-4}$, because of the safety shifts. As $\lambda \to 0$ the optimal step has a closed form. If $g'(0) = a$ for the new exponent and b for the previous one, then the best $M \sim c\lambda$ has $1/c + 1/(c + e^{-a}) = 1$, and

$$\lim_{\lambda \to 0} \Phi/\lambda = 1 + a - \tfrac{1}{2}(c + e^{-a}) + \tfrac{1}{2}\log c + \tfrac{1}{2}\big(\log(c + e^{-a}) - b\big).$$

For GNNW's F_{AI} ($a = -0.3864$, $b = -0.25$) this limit equals $6.31 \cdot 10^{-5}$, so that function already sits close to the edge of what one step allows near 0.

These numbers are not proved. They come from a discretised ODE, and the iteration's start value near $\lambda = 0$ is taken from a proved bound rather than optimised. We nevertheless expect the true limit of repeated applications of Theorem 2.1 and Lemma 2.2 to be close to 3.769. Going substantially further presumably needs a change to the framework itself, for instance a better analogue of Lemma 2.2 or a stronger inductive statement.

6. Limitations

- The argument relies on the correctness of [7, Theorem 14 and Lemma 15]. We have checked how their hypotheses are used (§2), but we have not re-verified the whole of their proof.
- The two verification programs are independent of each other. Neither has been formally verified, and human review of the code is welcome.
- The improvement over [7] is a refinement of their method, which they anticipated could be pushed further.
- The authors of [7] have not reviewed this work (see the acknowledgements).

Acknowledgements and use of AI

This work was carried out by the author with extensive use of the AI assistant Claude (Anthropic), through Claude Code. The assistant wrote the software, ran the computations and drafted this manuscript under the author's direction. The author is responsible for the content. GNNW's Remark 17 likewise reports an AI-generated candidate, and Theorem 1.2 confirms it.

Before posting, the author sent this work to one of the authors of [7]. No reply had been received at the time of posting, so the work has not been checked by them, and any errors are the author's own.

Changes from Zenodo v1.1. I added the plain-language overview of the party problem in the introduction. The theorems, proofs, data and code are the same.

Appendix A. Reproducibility

The code, certified data and Lean files are archived on Zenodo at doi:10.5281/zenodo.23895553, together with a copy of this paper. To reproduce, install the package with `pip install -e .[dev,audit]` and run

```
python scripts/certify_all.py
python scripts/certify_tails.py
python scripts/audit_all.py
```

The tests run with `pytest`. Software: Python 3.12, mpmath 1.3.0, python-flint 0.9.0, SciPy 1.18.0, NumPy 2.4.6, Lean 4.35.0 with Mathlib v4.35.0. SHA-256 checksums of the certified data files (LF line endings) are listed in `data/SHA256SUMS`.

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