FIXED-WINDOW GOLDEN-RATIO SUBSET SUMS AND FIBONACCI REPRESENTATION COUNTS

HYDRA RESEARCH PROJECT

ABSTRACT. Let $\varphi = (1+\sqrt{5})/2$ and, for $m \geq 0$ and an integer t, let $\mu_m(t)$ be the number of subsets $S \subseteq \{-2m, \ldots, 2m\}$ with $\sum_{k\in S} \varphi^k = t$. We prove that $\mu_m(t) = R(F_{2m+2}\, t)$ for $0 \leq t \leq L_{2m+1}$, where $R(n)$ is the number of representations of n as a sum of distinct Fibonacci numbers, and that $\mu_m(t) = 0$ for all other integers t; the correspondence is an explicit shift of exponents, and the proof uses only the algebraic conjugation of $\mathbb{Z}[\varphi]$. Combining this transfer with a symmetry and a forced-digit argument, we reduce the extremal problem for μ_m to results of Kocábová, Masáková and Pelantová on the maxima of R over intervals of fixed Zeckendorf length. We obtain $\max_{t\in\mathbb{Z}} \mu_m(t) = F_{2m+1}$ for every $m \geq 0$, and we show that for $m \geq 2$ the maximum is attained exactly at $t \in \{F_{2m}, F_{2m+1}, 2F_{2m}, F_{2m+2}\}$. We compare the transfer with a theorem of Dekking and van Loon on natural base-φ expansions. The results and proofs constitute human-directed, AI-authored mathematical research. They have been checked only by AI-model audits; no independent human mathematical verification has been completed, and novelty has not been established.

1. INTRODUCTION

Every positive integer has a unique finite expansion in base $\varphi = (1+\sqrt{5})/2$ with digits $0, 1$ and no two adjacent ones (Bergman [1]). Without the adjacency restriction an integer has infinitely many finite expansions, because the golden mean flip $100 \to 011$ (from $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$) can be applied indefinitely at the lowest digit. Dekking and van Loon [7] obtained finite counts by two truncation rules (Knott expansions and *natural* expansions) and related the second one to the number $R(n)$ of representations of n as a sum of distinct Fibonacci numbers, a function with a long history [10, 15, 4, 2, 11, 19, 5].

In this note we count expansions inside a *fixed* symmetric window of exponents. For $m \geq 0$ and $t \in \mathbb{Z}$ put

$$\mu_m(t) = \#\Big\{S \subseteq \{-2m, \ldots, 2m\} : \sum_{k\in S} \varphi^k = t\Big\}, \qquad M_m = \max_{t\in\mathbb{Z}} \mu_m(t).$$

Equivalently, $\mu_m(t)$ is the number of subsets of $\{\varphi^{-2m}, \ldots, \varphi^{2m}\}$ whose sum is the integer t. Our first result identifies μ_m with a dilated restriction of R.

Theorem A (Theorem 4.1)**.** *For every $m \geq 0$ and every integer t with $0 \leq t \leq L_{2m+1}$,*

$$\mu_m(t) = R(F_{2m+2}\, t),$$

and $\mu_m(t) = 0$ for all other integers t. The bijection behind the identity is the shift $S \mapsto S+2m+2$ of exponents.

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Human-directed, AI-authored mathematical research. The proofs have not undergone independent human mathematical verification.

The interval in Theorem A is optimal: for every integer $t > L_{2m+1}$ the left-hand side vanishes while $R(F_{2m+2}t) \geq 1$. The essential point of the proof is that a shifted digit string must reproduce *both* coordinates of $t\varphi^{2m+2} = tF_{2m+2}\varphi + tF_{2m+1}$ in the basis $\{\varphi, 1\}$; matching the coefficient of φ alone is not enough (Remark 4.3). We show that inside the stated range the second coordinate is forced by a short conjugation estimate (Lemma 2.2).

Theorem A does not by itself determine M_m: the maximum of R over all arguments below F_{4m+3} is F_{2m+2}, which is larger than the true value. Using a symmetry $t \leftrightarrow L_{2m+1} - t$ and the observation that, after this symmetry, every representation has two digits fixed, we reduce the problem to windows of $4m-1$ consecutive exponents, where the results of Kocábová, Masáková and Pelantová [11] apply sharply.

Theorem B (Theorem 5.4)**.** *For every $m \geq 0$, $M_m = F_{2m+1}$.*

Theorem C (Theorem 6.2)**.** *For $m \geq 2$, $\mu_m(t) = F_{2m+1}$ holds exactly for the four distinct integers $t \in \{F_{2m},\, F_{2m+1},\, 2F_{2m},\, F_{2m+2}\}$. For $m = 0$ the maximizers are $0, 1$, and for $m = 1$ they are $1, 2, 3$.*

Relation to earlier work. For integers t whose Bergman expansion reaches the lowest exponent $-2m$ (these are exactly the integers with $L_{2m-1} < t \leq L_{2m+1}$), the window representations of t are precisely its natural expansions, and Theorem A coincides with [7, Theorem 4.4]. For $0 \leq t \leq L_{2m-1}$ the window contains expansions that are not natural, and the multiplier F_{2m+2} in Theorem A does not depend on t; we did not find this extension in the literature we consulted (Section 8). The extremal input for Theorems B and C is [11, Theorem 4.7, Lemma 3.1 and Corollary 5.1(i)]; our contribution there is the reduction from the window problem to the Fibonacci problem, including the transfer of the equality cases with both coordinates in $\mathbb{Z}[\varphi]$. The lower bound in Theorem B can also be read off from [7, Theorem 4.3]; see Remark 5.5. We make no claim of priority beyond what is stated here.

Organization. Section 2 collects notation and the tools we use, including the translation of the results of [11] into the indexing of this paper. Section 3 proves the support and symmetry properties of μ_m, Section 4 proves Theorem A, Section 5 proves Theorem B, and Section 6 proves Theorem C. Section 7 lists small cases and boundary examples, Section 8 compares the results with the literature, and Section 9 records limitations and open questions.

2. Notation and tools

2.1. **Fibonacci and Lucas numbers.** Let $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, and $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. We extend F_n to negative indices by $F_{-n} = (-1)^{n+1}F_n$, so that the recurrence holds for all $n \in \mathbb{Z}$. Let $\psi = 1 - \varphi = -\varphi^{-1}$. Binet's formulas are $F_n = (\varphi^n - \psi^n)/\sqrt{5}$ and $L_n = \varphi^n + \psi^n$. For every $k \in \mathbb{Z}$,

$$\varphi^k = F_k\varphi + F_{k-1}, \qquad \psi^k = F_k\psi + F_{k-1}. \tag{2.1}$$

We use the following identities, valid for $m \geq 0$ (and, in (d), for $m \geq 1$):

$$\text{(a)}\ \sum_{j=-m}^{m} \varphi^{2j} = L_{2m+1}; \qquad \text{(b)}\ F_{2m+2}L_{2m+1} = F_{4m+3} - 1; \tag{2.2}$$

$$\text{(c)}\ L_{2m+1} = F_{2m} + F_{2m+2} = F_{2m+1} + 2F_{2m};$$

$$\text{(d)}\ F_{2m}F_{2m+2} = F_{2m+1}^2 - 1 = \sum_{i=1}^{m} F_{4i};$$

$$\text{(e)}\ \varphi^{2m+1} = L_{2m+1} + \varphi^{-2m-1}.$$

For (a), note that $\varphi^{2j+1} - \varphi^{2j-1} = \varphi^{2j}(\varphi - \varphi^{-1}) = \varphi^{2j}$, so the sum telescopes to $\varphi^{2m+1} - \varphi^{-2m-1} = \varphi^{2m+1} + \psi^{2m+1}$. Identity (b) is the case $a = 2m+1$, $b = 2m+2$ of $L_aF_b = F_{a+b} + (-1)^a F_{b-a}$. Identity (c) is $L_n = F_{n-1} + F_{n+1}$. In (d) the first equality is Cassini's identity, and the second follows by induction from $F_{2m+2}F_{2m+4} - F_{2m}F_{2m+2} = F_{2m+2}L_{2m+2} = F_{4m+4}$. Identity (e) is $L_{2m+1} = \varphi^{2m+1} + \psi^{2m+1}$. In particular $L_{2m+1} < \varphi^{2m+1}$, and an integer t satisfies $t \le L_{2m+1}$ if and only if $t < \varphi^{2m+1}$.

Throughout, we write

$$C_m := L_{2m+1} \qquad (m \ge 0).$$

2.2. **The ring $\mathbb{Z}[\varphi]$.** Every element of $\mathbb{Z}[\varphi]$ is uniquely $a\varphi + b$ with $a, b \in \mathbb{Z}$, and it is a rational number if and only if $a = 0$. The conjugation $\sigma(a\varphi + b) = a\psi + b$ is a ring automorphism of $\mathbb{Z}[\varphi]$ that fixes $\mathbb{Z}$. For a finite set $S \subseteq \mathbb{Z}$ we write

$$v(S) = \sum_{k \in S} \varphi^k \in \mathbb{Z}[\varphi].$$

By (2.1), $v(S) \in \mathbb{Z}$ if and only if $\sum_{k \in S} F_k = 0$, and an equality $v(S) = v(S')$ means equality of *both* coordinates in the basis $\{\varphi, 1\}$. Since $\psi^k = (-1)^k \varphi^{-k}$, the term ψ^k is positive for even k and negative for odd k.

2.3. **Fibonacci representations.** For an integer $n \ge 0$ let

$$\mathrm{Rep}(n) = \Big\{ A \subseteq \{2, 3, 4, \dots\} \text{ finite} : \sum_{j \in A} F_j = n \Big\}, \qquad R(n) = \#\,\mathrm{Rep}(n).$$

Thus $R(0) = 1$ (the empty set), and the values $F_2, F_3, \cdots = 1, 2, 3, 5, \dots$ are distinct, so R counts representations of n as sums of distinct Fibonacci numbers; this is sequence A000119 of [14]. By Zeckendorf's theorem [13, 20], every $n \ge 0$ has exactly one representation $Z(n) \in \mathrm{Rep}(n)$ containing no two consecutive integers. In particular $R(n) \ge 1$ for all $n \ge 0$. If $A \in \mathrm{Rep}(n)$ and $n < F_{K+1}$, then $\max A \le K$.

Lemma 2.1 (flip consistency)**.** *If $A, B \subseteq \{2, 3, \dots\}$ are finite and $\sum_{j \in A} F_j = \sum_{j \in B} F_j$, then $v(A) = v(B)$.*

Proof. Let $n = \sum_{j \in A} F_j$. If A contains two consecutive integers, let j be the largest integer with $j, j+1 \in A$. Then $j + 2 \notin A$, since otherwise $j+1, j+2$ would be a higher consecutive pair. Put $A' = (A \setminus \{j, j+1\}) \cup \{j+2\}$. Because $F_{j+2} = F_{j+1} + F_j$ and $\varphi^{j+2} = \varphi^{j+1} + \varphi^j$, the set A' has the same Fibonacci sum and the same value $v(A') = v(A)$, and it has one element fewer. Iterating, we reach after finitely many steps a set without consecutive integers, which is $Z(n)$. Hence $v(A) = v(Z(n))$, and likewise $v(B) = v(Z(n))$. □

Lemma 2.2 (conjugate bound)**.** *For every finite set $A \subseteq \{2, 3, \dots\}$,*

$$-\varphi^{-2} < \sum_{j \in A} \psi^j < \varphi^{-1}.$$

Proof. The positive terms come from even $j \ge 2$; their sum is less than $\sum_{i \ge 1} \varphi^{-2i} = 1/(\varphi^2 - 1) = \varphi^{-1}$. The negative terms come from odd $j \ge 3$; their sum exceeds $-\sum_{i \ge 1} \varphi^{-2i-1} = -\varphi^{-1} \cdot \varphi^{-1} = -\varphi^{-2}$. Both inequalities are strict because A is finite. □

2.4. **Maxima of R on Zeckendorf-length intervals.** Kocábová, Masáková and Pelantová [11] index the Fibonacci numbers by $F_0 = F_1 = 1$; their F_j is our F_{j+1}, and their function R coincides with ours. Their $\mathrm{Max}(\ell)$ is the maximum of R over $[F_\ell^{\mathrm{KMP}}, F_{\ell+1}^{\mathrm{KMP}}) = [F_{\ell+1}, F_{\ell+2})$, the set of integers whose Zeckendorf word has length ℓ. A Zeckendorf word is written $1\,0^{r_1}\,1\,0^{r_2}\cdots 1\,0^{r_l}$ from the most significant digit, so that $r_l \geq 0$ is the number of trailing zeros. In our indexing the following statements are [11, Theorem 4.7], [11, Lemma 3.1(2)] and [11, Corollary 5.1(i)].

Theorem 2.3 ([11])**.** *For $\ell \geq 1$ let $\mathcal{M}(\ell) = \max\{R(n) : F_{\ell+1} \leq n < F_{\ell+2}\}$.*

(a) *$\mathcal{M}(2h+1) = F_{h+2}$ for $h \geq 0$, $\mathcal{M}(2h+2) = 2F_{h+1}$ for $h \geq 1$, and $\mathcal{M}(2) = 1$.*
(b) *For $k \geq 1$, $\mathcal{M}(4k+3)$ is attained at exactly two integers. Their Zeckendorf words have zero-block lengths $(r_1, \ldots, r_l) = (1, 3, \ldots, 3, 4)$ with $k-1$ entries 3, and $(3, \ldots, 3, 2)$ with k entries 3.*

For $m \geq 1$ put

$$x_m = \sum_{i=1}^{m} F_{4i} = F_{2m}F_{2m+2}, \qquad x'_m = \sum_{i=1}^{m-1} F_{4i+2} + F_{4m}.$$

A word of length $4m-1 = 4(m-1)+3$ ends at our index 2 and begins at our index $4m$. The word with zero blocks $(3, \ldots, 3, 2)$ has its ones at our indices $4, 8, \ldots, 4m$, so it is $Z(x_m)$; the word with zero blocks $(1, 3, \ldots, 3, 4)$ has its ones at $6, 10, \ldots, 4m-2$ and $4m$, so it is $Z(x'_m)$. Summing even-indexed Fibonacci numbers gives $x_m + x'_m = F_{4m+2} - 2$.

Corollary 2.4. *Let $n = 2q+1 \geq 1$.*

(a) *$R(u) \leq F_{q+2}$ for every integer $0 \leq u < F_{n+2}$.*
(b) *If $m \geq 2$, $0 \leq u < F_{4m+1}$ and $R(u) = F_{2m+1}$, then $u \in \{x_m, x'_m\}$.*
(c) *$R(x_m) = F_{2m+1}$ for every $m \geq 1$.*

Proof. (a) We have $R(0) = 1$. If $u \geq 1$, then $F_{\ell+1} \leq u < F_{\ell+2}$ for a unique ℓ with $1 \leq \ell \leq n$, and $R(u) \leq \mathcal{M}(\ell)$. For odd $\ell = 2h+1 \leq 2q+1$ this is $F_{h+2} \leq F_{q+2}$. For even $\ell = 2h+2 \leq 2q$ with $h \geq 1$ it is $2F_{h+1} \leq 2F_q \leq F_{q+2}$, and $\mathcal{M}(2) = 1$.

(b) Here $q = 2m-1$ and $F_{q+2} = F_{2m+1}$. If $\ell \leq 4m-2$, then the bounds in (a) give at most F_{2m} for odd ℓ and at most $2F_{2m-1}$ for even ℓ; both are smaller than $F_{2m+1} = F_{2m} + F_{2m-1}$ because $F_{2m-1} < F_{2m}$ for $m \geq 2$. Also $R(0) = \mathcal{M}(2) = 1$. Hence $R(u) = F_{2m+1}$ forces $\ell = 4m-1$, and Theorem 2.3(b) with $k = m-1 \geq 1$ leaves only the two words described above, namely $Z(x_m)$ and $Z(x'_m)$.

(c) For $m = 1$, $x_1 = 3$ and $R(3) = 2 = F_3$. For $m \geq 2$, [11, Lemma 3.1(2)] gives the value F_{2m+1} for the word with zero blocks $(3, \ldots, 3, 2)$, which is $Z(x_m)$; this is also the statement $\mathrm{Max}(4k+3) = \varrho(3, \ldots, 3, 2)$ in [11, Corollary 5.1(i)]. □

Remark 2.5. The value $R(F_{2m+1}^2 - 1) = F_{2m+1}$ is also contained in a result of Bicknell-Johnson [3] on the smallest integer with a prescribed number of representations, as noted in [11, Remark after Proposition 5.4], and it is an equality case of Stockmeyer's bound $R(n) \leq \sqrt{n+1}$ [19].

3. Support and symmetry

Lemma 3.1 (support)**.** *Let $m \geq 0$ and $S \subseteq \{-2m, \ldots, 2m\}$ with $v(S) = t \in \mathbb{Z}$. Then $0 \leq t \leq C_m$. If moreover $m \geq 1$ and $S \subseteq \{-2m+1, \ldots, 2m-1\}$, then $t \leq C_{m-1}$.*

Proof. All powers of φ are positive, so $t \geq 0$. Since σ fixes integers, $t = \sigma(t) = \sum_{k \in S} \psi^k$. Discarding the negative terms (odd k) and adding the missing positive terms (even k in the window), we obtain $t \leq \sum_{j=-m}^{m} \varphi^{-2j} = C_m$ by (2.2)(a). In the second case the even exponents available are $-2m+2, \ldots, 2m-2$, which gives $t \leq C_{m-1}$. □

Lemma 3.2 (symmetry)**.** *For all $m \geq 0$ and $t \in \mathbb{Z}$, $\mu_m(t) = \mu_m(C_m - t)$.*

Proof. Let E be the set of even integers in $[-2m, 2m]$ and, for $S \subseteq [-2m, 2m]$, put $S^* = \{-k : k \in S \triangle E\}$. The map $S \mapsto S^*$ is an involution of the subsets of the window. Using $\varphi^{-k} = \psi^k$ for even k and $\varphi^{-k} = -\psi^k$ for odd k,

$$v(S^*) = \sum_{k \in E \setminus S} \psi^k - \sum_{k \in S \setminus E} \psi^k = \sum_{k \in E} \psi^k - \sum_{k \in S} \psi^k = \sigma(C_m) - \sigma(v(S)) = C_m - \sigma(v(S)).$$

If $v(S) = t \in \mathbb{Z}$, then $v(S^*) = C_m - t$. □

4. The fixed-window transfer

Theorem 4.1. *Let $m \geq 0$, $s = 2m+2$, and let t be an integer with $0 \leq t \leq C_m$. Then $S \mapsto S+s$ is a bijection from $\{S \subseteq \{-2m, \ldots, 2m\} : v(S) = t\}$ onto $\mathrm{Rep}(F_s t)$. In particular,*

$$\mu_m(t) = R(F_{2m+2}\, t) \qquad (0 \leq t \leq C_m).$$

Proof. The map is well defined and injective. If $v(S) = t$, then $S + s \subseteq \{2, \ldots, 4m+2\}$ and $v(S+s) = t\varphi^s = tF_s\varphi + tF_{s-1}$. Comparing the coefficients of φ by (2.1) gives $\sum_{j \in S+s} F_j = tF_s$, so $S + s \in \mathrm{Rep}(F_s t)$. Injectivity is clear.

The map is surjective. Let $A \in \mathrm{Rep}(F_s t)$. First, $F_{\max A} \leq tF_s \leq C_m F_s = F_{4m+3} - 1$ by (2.2)(b), so $A \subseteq \{2, \ldots, 4m+2\}$. Next, by (2.1), $v(A) = tF_s\varphi + b$ with $b = \sum_{j \in A} F_{j-1} \in \mathbb{Z}$. Hence

$$n := v(A) - t\varphi^s = b - tF_{s-1} \in \mathbb{Z}.$$

Applying σ gives $n = \sum_{j \in A} \psi^j - t\psi^s$. Since s is even, $t\psi^s = t\varphi^{-s}$, and since $t \leq C_m < \varphi^{2m+1}$ by (2.2)(e),

$$0 \leq t\varphi^{-s} < \varphi^{2m+1}\varphi^{-2m-2} = \varphi^{-1}.$$

Together with Lemma 2.2 this yields $-1 = -\varphi^{-2} - \varphi^{-1} < n < \varphi^{-1} < 1$, so $n = 0$ and $v(A) = t\varphi^s$. The set $S = A - s$ is contained in $\{-2m, \ldots, 2m\}$ and satisfies $v(S) = \varphi^{-s} v(A) = t$, and $S + s = A$. □

Corollary 4.2. *Let $m \geq 0$.*

(a) *$\mu_m(t) > 0$ if and only if $t \in \{0, 1, \ldots, C_m\}$.*
(b) *For an integer $t \geq 0$, the identity $\mu_m(t) = R(F_{2m+2}t)$ holds if and only if $t \leq C_m$.*

Proof. By Lemma 3.1, $\mu_m(t) = 0$ unless $0 \leq t \leq C_m$. For $0 \leq t \leq C_m$, Theorem 4.1 gives $\mu_m(t) = R(F_{2m+2}t) \geq 1$. For $t > C_m$ we have $\mu_m(t) = 0 < 1 \leq R(F_{2m+2}t)$. □

Remark 4.3 (one coefficient is not enough). Take $m = 1$, $t = 5 > C_1 = 4$ and $S = \{-1, 1, 2\}$. Then $S+4 = \{3, 5, 6\}$ and $F_3 + F_5 + F_6 = 15 = 5F_4$, but $v(S) = \varphi^{-1} + \varphi + \varphi^2 = 3\varphi$. The constant coordinates differ: $F_2 + F_4 + F_5 = 9 \neq 5F_3 = 10$; in the notation of the proof of Theorem 4.1, $n = -1$. Thus the coefficient of φ alone does not identify the value, and the hypothesis $t \leq C_m$ is exactly what forces the second coordinate.

Remark 4.4 (an alternative argument). Surjectivity can also be derived from Lemma 2.1. If t has one representation S_0 in the window (for instance, for $1 \leq t \leq C_m$ its Bergman expansion lies in the window by [7, Proposition 5.1(b)] and $\varphi^{2m+1} > C_m$), then every $A \in \mathrm{Rep}(F_s t)$ satisfies $v(A) = v(S_0 + s) = t\varphi^s$ by Lemma 2.1. The proof given above avoids this input and yields the existence of a window representation as a by-product.

Remark 4.5 (natural expansions). For an integer $t \geq 1$ let $-\rho(t)$ be the lowest exponent of its Bergman expansion. Dekking and van Loon call an expansion of t *natural* if it does not end in the digits 011 and its lowest exponent is $-\rho(t)$, and they prove $\mathrm{Tot}_\nu(t) = R(F_{\rho(t)+2}\, t)$ for $t > 3$, where

$\mathrm{Tot}_\nu(t)$ is the number of natural expansions [7, Theorem 4.4]. By [7, Proposition 5.1(b)], $\rho(t) = 2m$ exactly when $C_{m-1} < t \le C_m$. For such t every expansion has lowest exponent at most $-2m$ and highest exponent at most that of the Bergman expansion, because every expansion arises from the Bergman expansion by golden mean flips [7, Theorem 2.1]. Hence a window expansion of t has lowest exponent exactly $-2m$; it cannot end in 011, since undoing that final flip would produce an expansion with a higher lowest exponent. Conversely, every natural expansion lies in the window, since its highest exponent is at most that of the Bergman expansion, which is at most $2m$ because $t < \varphi^{2m+1}$. So for $C_{m-1} < t \le C_m$ the window representations are exactly the natural expansions and Theorem 4.1 coincides with [7, Theorem 4.4]. For $0 \le t \le C_{m-1}$ the two statements differ. For example, for $m = 2$ and $t = 4$ we have $\rho(4) = 2$ and $\mathrm{Tot}_\nu(4) = R(12) = 1$, whereas $\mu_2(4) = R(32) = 4$, the four window representations being

$$\{2, 0, -2\}, \quad \{2, 0, -3, -4\}, \quad \{2, -1, -2, -3, -4\}, \quad \{1, 0, -1, -2, -3, -4\}.$$

5. The exact maximum

For a finite set $W \subseteq \mathbb{Z}$ and $x \in \mathbb{R}$ let $N_W(x) = \#\{S \subseteq W : v(S) = x\}$.

Lemma 5.1 (fibres on odd windows)**.** *Let $a \in \mathbb{Z}$, $n = 2q+1 \ge 1$ and $W = \{a, a+1, \dots, a+n-1\}$. Then $N_W(x) \le F_{q+2}$ for every $x \in \mathbb{R}$.*

Proof. Multiplication by φ^{2-a} maps the subsets of W bijectively onto the subsets of $W' = \{2, \dots, n+1\}$ and multiplies values by φ^{2-a}, so we may assume $W = W'$. If $N_W(x) = 0$ there is nothing to prove. Otherwise $x \in \mathbb{Z}[\varphi]$, and all S with $v(S) = x$ have the same coefficient $U = \sum_{j \in S} F_j$ of φ. The complement map $S \mapsto W \setminus S$ is a bijection from the fibre of x onto the fibre of $v(W) - x$, and it replaces U by $U_W - U$, where $U_W = \sum_{j=2}^{n+1} F_j = F_{n+3} - 2$. Replacing x by $v(W) - x$ if necessary, we may assume $U \le U_W/2 < F_{n+2}$. The identity map sends the fibre of x injectively into $\mathrm{Rep}(U)$. Hence $N_W(x) \le R(U) \le F_{q+2}$ by Corollary 2.4(a). □

Theorem 5.2. *For all $m \ge 0$ and $t \in \mathbb{Z}$, $\mu_m(t) \le F_{2m+1}$.*

Proof. For $m = 0$ the only subsets are $\emptyset$ and $\{0\}$, so $\mu_0 \le 1 = F_1$. Let $m \ge 1$. By Lemma 3.1 we may assume $0 \le t \le C_m$, and by Lemma 3.2 we may replace t by $s = \min(t, C_m - t)$, so that $0 \le s \le C_m/2$. Let S be a window representation of s. Since $C_m < \varphi^{2m+1} < 2\varphi^{2m}$, we have $s < \varphi^{2m}$, so $2m \notin S$.

Case I: $s \le C_{m-1}$. As $C_{m-1} < \varphi^{2m-1}$ by (2.2)(e), also $2m - 1 \notin S$. Thus $\mu_m(s) = N_W(s)$ with $W = \{-2m, \dots, 2m-2\}$, a window of $4m-1$ consecutive integers, and Lemma 5.1 with $q = 2m - 1$ gives $\mu_m(s) \le F_{2m+1}$.

Case II: $s > C_{m-1}$. Then $-2m \in S$, for otherwise $S \subseteq \{-2m+1, \dots, 2m-1\}$ and $s \le C_{m-1}$ by Lemma 3.1. Hence $S \mapsto S \setminus \{-2m\}$ is a bijection from the window representations of s onto the subsets of $W = \{-2m+1, \dots, 2m-1\}$ with value $s - \varphi^{-2m}$, and Lemma 5.1 again gives $\mu_m(s) \le F_{2m+1}$. □

Proposition 5.3. *For $m \ge 1$,*

$$\mu_m(F_{2m}) = \mu_m(F_{2m+1}) = \mu_m(2F_{2m}) = \mu_m(F_{2m+2}) = F_{2m+1}.$$

Proof. Since $F_{2m} \le C_m$, Theorem 4.1, identity (2.2)(d) and Corollary 2.4(c) give

$$\mu_m(F_{2m}) = R(F_{2m}F_{2m+2}) = R(x_m) = F_{2m+1}.$$

As $C_m - F_{2m} = F_{2m+2}$ by (2.2)(c), Lemma 3.2 gives the same value at F_{2m+2}.

Now let $s = F_{2m+1}$. We have $C_{m-1} = F_{2m-2} + F_{2m} < F_{2m+1}$ and $F_{2m+1} \le 2F_{2m}$, so $C_{m-1} < s \le C_m/2$. By Case II of the proof of Theorem 5.2, $\mu_m(s) = N_W(s - \varphi^{-2m})$ with

$W = \{-2m+1, \dots, 2m-1\}$. Multiplying by φ^{2m+1} maps W onto $\{2, \dots, 4m\}$ and the value $s - \varphi^{-2m}$ onto

$$y = F_{2m+1}\varphi^{2m+1} - \varphi = (F_{2m+1}^2 - 1)\varphi + F_{2m+1}F_{2m}.$$

By (2.2)(d) this equals $F_{2m}F_{2m+2}\varphi + F_{2m}F_{2m+1} = F_{2m}\varphi^{2m+2}$, and

$$v(Z(x_m)) = \sum_{i=1}^{m} \varphi^{4i} = \varphi^4 \frac{\varphi^{4m} - 1}{\varphi^4 - 1} = \varphi^2 \frac{\varphi^{4m} - 1}{\sqrt{5}} = \varphi^{2m+2}F_{2m},$$

using $\varphi^4 - 1 = \sqrt{5}\,\varphi^2$ and Binet's formula. So the fibre of y on $\{2, \dots, 4m\}$ contains $Z(x_m)$. Since $x_m < F_{4m+1}$, every $A \in \mathrm{Rep}(x_m)$ lies in $\{2, \dots, 4m\}$, and by Lemma 2.1 it satisfies $v(A) = v(Z(x_m)) = y$. Therefore the injection of the fibre of y into $\mathrm{Rep}(x_m)$ is a bijection, and $\mu_m(F_{2m+1}) = R(x_m) = F_{2m+1}$. Finally $C_m - F_{2m+1} = 2F_{2m}$, and Lemma 3.2 gives the value at $2F_{2m}$. □

Theorem 5.4. *For every $m \geq 0$, $M_m = F_{2m+1}$.*

Proof. For $m = 0$, $\mu_0(0) = \mu_0(1) = 1 = F_1$. For $m \geq 1$ combine Theorem 5.2 and Proposition 5.3. □

Remark 5.5 (attainment via natural expansions). For $m \geq 1$ the integers F_{2m+1} and F_{2m+2} lie in $(C_{m-1}, C_m]$, so by Remark 4.5 their window representations are their natural expansions, and [7, Theorem 4.3] gives $\mu_m(F_{2m+2}) = \mathrm{Tot}_\nu(F_{2m+2}) = F_{2m+1}$ and $\mu_m(F_{2m+1}) = \mathrm{Tot}_\nu(F_{2m+1}) = F_{2m+1}$. This yields an independent route to the lower bound in Theorem 5.4.

6. Classification of the maximizers

For $m \geq 2$ put

$$Q_m = \sum_{i=0}^{m-1} \varphi^{4i+2}, \qquad T_m = \sum_{j=0}^{4m-2} \varphi^j.$$

As in the proof of Proposition 5.3, $Q_m = \varphi^{2m}F_{2m}$, and $T_m = (\varphi^{4m-1} - 1)/(\varphi - 1) = \varphi^{4m} - \varphi$. Grouping the terms of $T_m - Q_m$ (the exponents in $\{0, \dots, 4m-2\}$ not congruent to 2 modulo 4) and using $\varphi^j + \varphi^{j+1} = \varphi^{j+2}$ twice, we obtain

$$T_m - Q_m = \sum_{i=0}^{m-1} \varphi^{4i+2} + \sum_{i=0}^{m-2} \varphi^{4i+3} = \sum_{i=1}^{m-1} \varphi^{4i} + \varphi^{4m-2}. \tag{6.1}$$

Lemma 6.1 (equality in Lemma 5.1)**.** *Let $m \geq 2$, $W = \{0, 1, \dots, 4m-2\}$ and $x \in \mathbb{R}$ with $N_W(x) = F_{2m+1}$. Then $x \in \{Q_m,\, T_m - Q_m\}$.*

Proof. Follow the proof of Lemma 5.1 with $n = 4m-1$ after multiplying by φ^2, which maps W onto $\{2, \dots, 4m\}$. Let $x' \in \{x, T_m - x\}$ be the value for which $U \leq U_W/2$. Then $F_{2m+1} = N_W(x') \leq R(U) \leq F_{2m+1}$ with $U < F_{4m+1}$, so $R(U) = F_{2m+1}$ and $U \in \{x_m, x'_m\}$ by Corollary 2.4(b). Moreover, the injection of the fibre of $\varphi^2 x'$ into $\mathrm{Rep}(U)$ is a bijection between finite sets of equal size; in particular $Z(U)$ is the image of a subset in the fibre, so $\varphi^2 x' = v(Z(U))$. Now $v(Z(x_m)) = \sum_{i=1}^{m} \varphi^{4i} = \varphi^2 Q_m$, and by (6.1), $v(Z(x'_m)) = \sum_{i=1}^{m-1} \varphi^{4i+2} + \varphi^{4m} = \varphi^2(T_m - Q_m)$. Hence $x' \in \{Q_m, T_m - Q_m\}$, and since the complement interchanges these two values, also $x \in \{Q_m, T_m - Q_m\}$. □

Theorem 6.2. *Let $m \geq 2$. Then $\mu_m(t) = F_{2m+1}$ if and only if $t \in \{F_{2m}, F_{2m+1}, 2F_{2m}, F_{2m+2}\}$, and these four integers are distinct. For $m = 0$ the maximizers are $0, 1$; for $m = 1$ they are $1, 2, 3$.*

Proof. For $m = 0$ see Theorem 5.4. For $m = 1$, a direct enumeration of the 32 subsets of $\{-2, \dots, 2\}$ gives $(\mu_1(0), \dots, \mu_1(4)) = (1, 2, 2, 2, 1)$.

Let $m \geq 2$. The four values are attained by Proposition 5.3, and $F_{2m} < F_{2m+1} < 2F_{2m} < F_{2m+2}$ because $F_{2m-1} < F_{2m}$. Conversely, suppose $\mu_m(t) = F_{2m+1}$ and let $s = \min(t, C_m - t)$, so that $\mu_m(s) = F_{2m+1}$ by Lemma 3.2. We use the two cases of the proof of Theorem 5.2, together with (2.1) and $F_{-n} = (-1)^{n+1}F_n$, which give

$$\varphi^{-2m} = -F_{2m}\varphi + F_{2m+1}, \qquad \varphi^{1-2m} = F_{2m-1}\varphi - F_{2m}, \qquad \varphi^{2-2m} = -F_{2m-2}\varphi + F_{2m-1}.$$

Case I: $s \leq C_{m-1}$. The fibre of s on $\{-2m, \dots, 2m-2\}$ has F_{2m+1} elements; multiplying by φ^{2m} and applying Lemma 6.1 gives $s \in \{\varphi^{-2m}Q_m,\ \varphi^{-2m}(T_m - Q_m)\}$. Here $\varphi^{-2m}Q_m = F_{2m}$, while

$$\varphi^{-2m}(T_m - Q_m) = \varphi^{2m} - \varphi^{1-2m} - F_{2m} = F_{2m-2}\varphi + F_{2m-1}$$

is irrational since $F_{2m-2} \geq 1$. Hence $s = F_{2m}$.

Case II: $s > C_{m-1}$. The fibre of $s - \varphi^{-2m}$ on $\{-2m+1, \dots, 2m-1\}$ has F_{2m+1} elements; multiplying by φ^{2m-1} and applying Lemma 6.1 gives $s \in \{\varphi^{-2m}+\varphi^{1-2m}Q_m,\ \varphi^{-2m}+\varphi^{1-2m}(T_m - Q_m)\}$. The first value is $\varphi^{-2m} + F_{2m}\varphi = F_{2m+1}$. For the second,

$$\varphi^{-2m} + \varphi^{2m+1} - \varphi^{2-2m} - F_{2m}\varphi = (F_{2m+1} + F_{2m-2} - 2F_{2m})\varphi + (F_{2m+1} + F_{2m} - F_{2m-1}) = 2F_{2m},$$

because $F_{2m+1} + F_{2m-2} - 2F_{2m} = F_{2m-1} + F_{2m-2} - F_{2m} = 0$ and $F_{2m+1} + F_{2m} - F_{2m-1} = F_{2m+1} + F_{2m-2} = 2F_{2m}$. Since $C_m = F_{2m+1} + 2F_{2m}$ and $2F_{2m} > F_{2m+1}$, the value $2F_{2m}$ exceeds $C_m/2$, so $s = F_{2m+1}$.

Thus $s \in \{F_{2m}, F_{2m+1}\}$, and $t \in \{s, C_m - s\} = \{F_{2m}, F_{2m+1}, F_{2m+2}, 2F_{2m}\}$. □

Remark 6.3. Theorem 6.2 for $m \geq 2$ uses the equality classification [11, Corollary 5.1(i)], which requires $k = m - 1 \geq 1$. For $m = 1$ the general formula gives the multiset $\{F_2, F_3, 2F_2, F_4\} = \{1, 2, 2, 3\}$, which agrees with the three maximizers $1, 2, 3$.

7. Small cases and boundary examples

Table 1 lists $\mu_m(t)$ for $m \leq 3$, and Table 2 lists M_m and the set of maximizers for $m \leq 7$. Both tables were computed by exact enumeration with integer pairs (a, b) representing $a\varphi + b$; they agree with Theorems 4.1, 5.4 and 6.2, and they are not used in any proof.

Table 1. Values of $\mu_m(t)$ for $0 \leq t \leq C_m$.

m	C_m	$\mu_m(0), \mu_m(1), \dots, \mu_m(C_m)$
0	1	1, 1
1	4	1, 2, 2, 2, 1
2	11	1, 3, 4, 5, 4, 5, 5, 4, 5, 4, 3, 1
3	29	1, 4, 6, 8, 7, 10, 10, 9, 13, 12, 11, 6, 10, 13, 12, 12, 13, 10, 6, 11, 12, 13, 9, 10, 10, 7, 8, 6, 4, 1

Example 7.1. For $m = 2$ we have $x_2 = F_4 + F_8 = 24$ and $x_2' = F_6 + F_8 = 29$, and $R(24) = R(29) = 5$. Theorem 4.1 gives $\mu_2(3) = R(3F_6) = R(24) = 5$; in the proof of Proposition 5.3 the integer $t = 5 = F_5$ falls into Case II, with $U = F_5^2 - 1 = 24$. The values $t = 6$ and $t = 8$ are the reflections $C_2 - 5$ and $C_2 - 3$.

Example 7.2 (boundary)**.** For every m, $\mu_m(C_m) = \mu_m(0) = 1 = R(F_{2m+2}C_m) = R(F_{4m+3} - 1)$, and $\mu_m(C_m + 1) = 0$ while $R(F_{2m+2}(C_m + 1)) \geq 1$. For $m = 1$, Remark 4.3 exhibits a window subset whose shift is a Fibonacci representation of $5F_4 = 15$ although its value is 3φ.

TABLE 2. The maximum M_m and the maximizers of μ_m.

m	M_m	maximizers
0	1	$0, 1$
1	2	$1, 2, 3$
2	5	$3, 5, 6, 8$
3	13	$8, 13, 16, 21$
4	34	$21, 34, 42, 55$
5	89	$55, 89, 110, 144$
6	233	$144, 233, 288, 377$
7	610	$377, 610, 754, 987$

8. Comparison with earlier work

Dekking and van Loon [7]. This is the closest source. Their Theorem 4.4 is a transfer of the same type as Theorem 4.1, obtained by multiplying by a power of φ; it counts natural expansions and uses the multiplier $F_{\rho(t)+2}$, which depends on t. As explained in Remark 4.5, the two statements coincide for $C_{m-1} < t \le C_m$ and differ for $t \le C_{m-1}$, where the window also contains expansions that are not natural. Their Theorem 4.3 gives the values at F_{2m+1} and F_{2m+2} used in Remark 5.5. Their paper does not study maxima of the counting functions.

Kocábová, Masáková and Pelantová [11]. They determine the maximum of R on every interval $[F_{\ell+1}, F_{\ell+2})$ and classify its arguments. Theorems 5.4 and 6.2 use their Theorem 4.7, Lemma 3.1 and Corollary 5.1(i); without them our proofs are incomplete. Their work does not involve powers of φ or windows of exponents; the link is provided by Lemma 5.1, by the forced digits in the proof of Theorem 5.2, and by Lemma 6.1, which keeps track of both coordinates in $\mathbb{Z}[\varphi]$.

Other work on R. Recursions and matrix formulas for R go back to [10, 15, 4, 2]; Stockmeyer [19] proved $R(n) \le \sqrt{n+1}$ with equality exactly at $n = F_k^2 - 1$, which contains Corollary 2.4(c); Chow and Slattery [5] and Sanna [16] studied averages and power sums of R. None of these concerns windows of powers of φ.

Base-φ representations and numeration systems. Frougny and Sakarovitch [8] construct a finite automaton converting Fibonacci representations into φ-expansions, and Frougny and Solomyak [9] study finiteness and normalization of β-expansions. Normalization maps all representations of a number to one normal form and therefore does not by itself give an equality of counts. Further properties of φ-representations were proved with the Walnut theorem prover by Shallit [17] and by Shallit and Vukusic [18], and the structure of the digits of base-φ expansions is studied in [6]; the extension of [11] to m-bonacci systems is [12].

Within the sources listed above we did not find Theorem 4.1 for $0 \le t \le C_{m-1}$, Theorem 5.4, or Theorem 6.2. Our literature search did not include a systematic search of MathSciNet or zbMATH, so this statement should be read as a report on the sources consulted, not as a claim of priority.

9. Limitations and further questions

The proofs of Theorems 5.4 and 6.2 rely on the published results [11, Theorem 4.7, Lemma 3.1, Corollary 5.1(i)], which we use as stated and do not reprove. Theorem 4.1 and Corollary 4.2 are self-contained apart from Zeckendorf's theorem. Exact computer checks confirm the statements of Theorems 4.1, 5.4 and 6.2 for $m \le 7$ by complete enumeration, the conclusion of Theorem 4.1 for the Zeckendorf representations of $F_{2m+2}t$ for all $t \le C_m$ and $m \le 12$, and Theorem 2.3(a)

for all lengths $\ell \le 34$ and Theorem 2.3(b) for all lengths $\ell = 4k+3 \le 31$; such checks do not replace the proofs. The following questions are natural.

(1) A self-contained proof that $\max_{0\le u<F_{4m+1}} R(u) = F_{2m+1}$ with exactly the two maximizers x_m and x'_m for $m \ge 2$ would make Theorems 5.4 and 6.2 independent of [11].
(2) Lemma 5.1 bounds every fibre, integral or not, on a window of odd length. Which values of $x \in \mathbb{Z}[\varphi]$ attain the bound for general windows, and what is the maximum on windows of even length?
(3) Describe the second largest value of μ_m and the distribution of the values of μ_m on $[0, C_m]$.
(4) Asymmetric windows $\{a, \dots, b\}$, and analogous questions for other quadratic Pisot units, are not treated here.

Research provenance and AI contribution

The results and proofs of this work were generated and audited by AI systems under human direction. The Owner's public research-provenance classification is `HUMAN_DIRECTED_AI_AUTHORED`. The first complete proofs of the upper bound in Theorem 5.4, of Theorem 6.2 and of the transfer identity were produced in HGS-E1-002 Request #14 by OpenAI model "gpt-6-astra" through the DeepSeek Harness. This request-level identity is established; identities for other GOLD runs remain unresolved. An independent reconstruction and adversarial check was carried out by Claude (Anthropic). Literature audits were carried out by an AI system recorded as KIMI; an earlier numerical verification was carried out through the DeepSeek API. The present text, including the conjugation proof of Theorem 4.1 and the attainment argument in Proposition 5.3, was drafted by Claude (historical model label `claude-opus-5-5`). These contributor roles do not by themselves determine the publication author list or the designation of individual systems as authors.

Independent human mathematical verification has not been completed. No human peer-review status, Proof Verification certificate, or priority claim is asserted. The simplified proofs in PUB-001 did not receive a separate independent audit. Formal submission metadata are maintained separately from this manuscript.

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