

MAKING THE COPRIMALITY CONVENTION EXPLICIT IN A MEAN VALUE THEOREM OF HALUPCZOK FOR THE TERNARY GOLDBACH PROBLEM

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ABSTRACT. Theorem 1.2 of Halupczok (2013) is a Bombieri-Vinogradov type mean value theorem for the ternary Goldbach problem with one prime in an arithmetic progression and two primes in short intervals. Her proof works with residue classes $a \bmod q$ coprime to the modulus, as stated on p. 344 of her paper, and a formal verification requires that this convention be written explicitly into the statement. We give two closely related explicit formulations, Formulation A and Formulation B, where Lean proves that Formulation A implies Formulation B, and verify both in Lean 4 assuming the two cited inputs **PPS1984Remark** and **PP1993Goldbach**, or alternatively Theorems 2.1 and 2.3 of her paper, which Lean deduces from these two inputs. The later results of Halupczok's paper use only coprime residue classes and are unaffected.

1. INTRODUCTION

In [H] (preprint [H-arXiv]), Halupczok proves a Bombieri-Vinogradov type mean value theorem for the ternary Goldbach problem, extending classical mean value results to the ternary setting in which one prime is restricted to an arithmetic progression $p_1 \equiv a \pmod{q}$ and two primes lie in short intervals. We warmly credit and thank Halupczok for this work: her paper supplies all of the analytic number theory used here, and her proof of Theorem 1.2 is the source of everything verified in this note. In the numbering of the published paper [H] and the preprint [H-arXiv], arXiv Theorem 2 is JTBN Theorem 1.2 (p. 333, with the statement continuing to p. 334), arXiv Theorem 3 is JTBN Theorem 2.1 (p. 338), and arXiv Theorem 5 is JTBN Theorem 2.3 (p. 339).

In the proof of Theorem 1.2, Halupczok works with residue classes coprime to the modulus, stating on [H, p. 344] that “we used $(q, a) = 1$ in the last step, what we can assume w.l.o.g., else (4.2) holds true clearly”. Our change is to carry that convention into the theorem statement, which a proof assistant requires because for $(a, q) > 1$ and $1 \leq q \leq X_1$ the inner ternary prime sum vanishes whereas the printed Euler product $T(n, q, a)$ need not vanish, as for $a = q = 3$. One may write the convention into the statement either by restricting the outer sum to moduli $q \leq Q$ satisfying $(a, q) = 1$ (Formulation A), or by summing over all $q \leq Q$ under the hypothesis $Q \leq X_1$ after replacing $T(n, q, a)$ by $T_0(n, q, a)$, which equals $T(n, q, a)$ for $(a, q) = 1$ and 0 otherwise, just as display (1.1) of [H, p. 335] sets the binary singular series $S(r, q, a)$ to 0 when $(a, q) > 1$ (Formulation B). All later results of [H] apply Theorem 1.2 only to residue classes coprime to the modulus and are unaffected.

We state plainly that the binary Goldbach conjecture remains open and that this note makes no progress on it. The ternary Goldbach problem was settled by Helfgott in 2013 [Helfgott], which this note does not use.

2. THE PRINTED STATEMENT AND THE COPRIMALITY CONVENTION

We first recall JTBN Theorem 1.2 as printed in [H, pp. 333 to 334]. Suppose that

$$X_1 \geq Y_1 \gg X_1^{3/5+\varepsilon_1}, \quad X_2 \geq Y_2 \gg X_2^{7/12+\varepsilon_2},$$

$$Y_2^{1/3+\varepsilon} \ll Y_1 \ll Y_2 \asymp n - X_1 - Y_1 - X_2 - Y_2 \geq 0,$$

with n odd, and

$$Q \ll Y_1 X_1^{-1/2} L^{-B}, \quad Q \ll Y_1^{3/2} (n - X_1)^{-1/2},$$

where $L = \log Y_2$ and $B = B(A)$. The printed conclusion asserts that for any fixed integer a with $a \leq n - X_1 - Y_1$,

$$\sum_{q \leq Q} \left| \sum_{\substack{p_1 + p_2 + p_3 = n \\ p_i \in [X_i, X_i + Y_i], i=1,2 \\ p_1 \equiv a \pmod{q}}} \log p_1 \log p_2 \log p_3 - T(n, q, a) Y_1 Y_2 \right| \ll Y_1 Y_2 L^{-A},$$

where $[X, X + Y]$ denotes the set of integers m satisfying $X < m \leq X + Y$. In the printed statement, the singular series $T(n, q, a)$ is $1/\varphi(q)$ times the product over all primes p of a local factor equal to $1 - 1/(p-1)^2$ if $p \mid n$ and $p \nmid q$, or if $p \nmid n - a$ and $p \mid q$; equal to $1 + 1/(p-1)^3$ if $p \nmid n$ and $p \nmid q$; and equal to $p/(p-1)$ otherwise. Written out as a displayed formula matching the Lean definition `T` in Section 4, this is

$$(1) \quad T(n, q, a) = \frac{1}{\varphi(q)} \prod_p \begin{cases} 1 - \frac{1}{(p-1)^2} & \text{if } (p \mid n \text{ and } p \nmid q) \text{ or } (p \nmid n - a \text{ and } p \mid q), \\ 1 + \frac{1}{(p-1)^3} & \text{if } p \nmid n \text{ and } p \nmid q, \\ \frac{p}{p-1} & \text{otherwise.} \end{cases}$$

Whereas the binary singular series $S(r, q, a)$ of display (1.1) of [H, p. 335] is explicitly set to 0 when $(a, q) > 1$, the printed definition (1) of $T(n, q, a)$ gives the Euler product directly, and the coprimality convention is stated in the proof on [H, p. 344]:

“we used $(q, a) = 1$ in the last step, what we can assume w.l.o.g., else (4.2) holds true clearly”.

Remark 2.1 (Why the convention is stated). A formal statement must specify what happens when $(a, q) > 1$. For example, with $a = q = 3$ and $X_1 > 3$, the prime sum is empty and equal to 0 (proved in Lean in `Thm2Coprime.lean` as `no_prime_in_class` and `no_prime_three_mod_three`, and more generally for $1 \leq q \leq X_1$ and $(a, q) \neq 1$ in the module `NonCoprime` of [Repo] as `ternSum_noncoprime`), while the printed product $T(n, 3, 3)$ in (1) is positive (by a short hand computation, not in Lean, at least $\frac{3}{4} \prod_{p \geq 5} (1 - (p-1)^{-2}) > 0$ for odd n , and numerically about 0.767 from Euler products truncated at 10^6 for $n = 10^9 + 7$, $10^9 + 9$, $3 \cdot 10^9 + 1$, and $10^{12} + 39$, where $T(n, 3, 1) + T(n, 3, 2) = 2.301 = T(n, 1, 0)$). The convention $(q, a) = 1$, as used in the proof on [H, p. 344], is what makes the two sides match.

3. THE EXPLICIT FORMULATIONS AND PROOF SKETCH

There are two natural and closely related ways to write the coprimality convention into the statement of JTNB Theorem 1.2. We state the two explicit formulations below; Theorem 3.4 proves both, assuming the two cited inputs described in Section 3.1.

Statement 3.1 (Formulation A, the coprime reading; Lean `Theorem2`). *Assume the hypotheses of JTNB Theorem 1.2. Then for any fixed integer a with $a \leq n - X_1 - Y_1$,*

$$\sum_{\substack{q \leq Q \\ (a, q) = 1}} \left| \sum_{\substack{p_1 + p_2 + p_3 = n \\ p_i \in [X_i, X_i + Y_i], i=1,2 \\ p_1 \equiv a \pmod{q}}} \log p_1 \log p_2 \log p_3 - T(n, q, a) Y_1 Y_2 \right| \ll Y_1 Y_2 L^{-A}.$$

Statement 3.2 (Formulation B, the zeroed reading; Lean `Theorem2Zeroed`). *Assume the hypotheses of JTNB Theorem 1.2 together with the additional hypothesis $Q \leq X_1$. Define*

$$T_0(n, q, a) = \begin{cases} T(n, q, a) & \text{if } (a, q) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

analogously to $S(r, q, a)$ in display (1.1) of [H, p. 335]. Then for any fixed integer a with $a \leq n - X_1 - Y_1$,

$$\sum_{q \leq Q} \left| \sum_{\substack{p_1 + p_2 + p_3 = n \\ p_i \in [X_i, X_i + Y_i], i=1,2 \\ p_1 \equiv a \pmod{q}}} \log p_1 \log p_2 \log p_3 - T_0(n, q, a) Y_1 Y_2 \right| \ll Y_1 Y_2 L^{-A}.$$

In the module `NonCoprime` of [Repo], it is proved unconditionally that Formulation A implies Formulation B (`theorem2Zeroed_of_theorem2`): when $Q \leq X_1$, every modulus $1 \leq q \leq Q$ with $(a, q) \neq 1$ satisfies $q \leq X_1$, so both the ternary prime sum (by `ternSum_noncoprime`) and $T_0(n, q, a)$ are equal to 0. We remark that the hypothesis $Q \ll Y_1 X_1^{-1/2} L^{-B}$ combined with $Y_1 \leq X_1$ suggests $Q \leq X_1$ for large X_1 , though this implication is not part of the Lean formalization.

3.1. The main theorem and the two cited inputs. To state the main result verified in Lean, we first describe the two cited analytic inputs on which Halupczok’s proof rests. Because [H] itself quotes two auxiliary theorems from the literature (JTNB Theorems 2.1 and 2.3), the Lean formalization replaces each quoted theorem by a deduction from its printed source:

- (1) **First cited input** (`PPS1984Remark`). JTNB Theorem 2.1 (arXiv Theorem 3, [H, p. 338]) is deduced in Lean from the hypothesis `PPS1984Remark` via the theorem `theorem3_of_remark`, proved as an implication. Here `PPS1984Remark` represents the printed REMARK of Perelli, Pintz and Salerno [PPS, p. 530], with their display (2) [PPS, p. 529], in its uniform reading: one implied constant for all $x^{3/5}(\log x)^D \leq y \leq x$. Their printed THEOREM [PPS, p. 530] reads:

“The estimate (2) holds if $\psi \leq \theta - 1/2$ and $\theta > 3/5$.”

Immediately below it, their printed REMARK [PPS, p. 530] reads:

“Indeed, as it is clear from the proof, we obtain (2) if (3) holds and $y \geq x^{3/5}(\log x)^C$, $C = C(A)$,”

where (3) is $\psi \leq \theta - 1/2$.

Remark 3.3 (On the uniform reading of the Remark of Perelli, Pintz and Salerno). The uniform reading was checked by reading the proof in [PPS, pp. 531 to 538], not in Lean: every implied constant in that proof is either absolute or depends on A only, with the sole exception of $\delta = \delta(\theta)$ in display (5) on p. 531, which is taken from Huxley and Iwaniec [HI] for small Q and can be taken absolute by applying the Huxley-Iwaniec range at the single exponent $\theta_0 = 3/5$ on windows of length $x^{3/5}$. We emphasize that this is a reading of the printed proof, not a formal proof in Lean.

- (2) **Second cited input** (`PP1993Goldbach`). JTNB Theorem 2.3 (arXiv Theorem 5, [H, p. 339]) is deduced in Lean from the hypothesis `PP1993Goldbach` via `theorem5_of_PP1993`, proved as an implication. Here `PP1993Goldbach` is the Goldbach analogue of Theorem 3 of Perelli and Pintz [PP, p. 44]. It is not a numbered statement of that paper: their Theorem 3 is stated for $2k$ -twin primes, while their introduction [PP, p. 41] remarks that the method gives an analogous result for the Goldbach problem. Specifically, the Goldbach mean square bound follows from their displays (1) and (3) on p. 44, which reduce it to displays (4) and (5); they prove (4) for $N^{7/12+\varepsilon} \leq Y \leq N$ in §4 (p. 45) and (5) for $Y \geq Y_0(A, \varepsilon)$ and $H \geq Y^{1/3+\varepsilon}$ in §5 (p. 46), with $H = Y^{1/3+\varepsilon}$ chosen on p. 44. On p. 339 of [H], Halupczok writes “it can be shown (cp. [11, Thm. 3])” and “The proof in [11] can be adapted”, so JTNB Theorem 2.3 is her stated adaptation. We note clearly that this is a weaker citation than a numbered theorem.

Theorem 3.4 (Main theorem of this note). *Assume `PPS1984Remark` and `PP1993Goldbach`, as described above. Then Formulation A holds, and Formulation B holds (Lean: `theorem2_U1510` and `theorem2Zeroed_U1510`). Alternatively, Formulation A and Formulation B follow from the paper’s own quoted Theorems 2.1 and 2.3 (Lean: `theorem2_U1462` and `theorem2Zeroed_U1462`).*

3.2. Proof sketch of Formulation A. The proof of Formulation A follows the argument of [H], and the Lean reduction mirrors it in four steps:

- (1) Substitute $2k_1 = n - p_1$. The ternary sum splits into a binary part and a main term part; in Lean the substitution is `ternSum_eq`.

- (2) *Binary part*: Bound the binary part using Theorem 5 of [H-arXiv] (JTNB Theorem 2.3, [H, p. 339]), which provides a mean square bound for the binary Goldbach count in short intervals, combined with Cauchy-Schwarz over q and a divisor moment bound. In Lean, this is `binaryPart`, proved assuming `Theorem5` only.
- (3) *Main term part*: Apply Theorem 3 of [H-arXiv] (JTNB Theorem 2.1, [H, p. 338]), the Perelli-Pintz-Salerno Bombieri-Vinogradov theorem in short intervals. The binary singular series S is expanded over squarefree moduli s and truncated at $s \leq L^D$; the congruences $p_1 \equiv a \pmod{q}$ and $p_1 \pmod{s}$ are combined via the Chinese remainder theorem, with Theorem 3 applied at modulus $\text{lcm}(q, s)$; and the Euler product of the resulting main term is identified with $T(n, q, a)$. In Lean, this reduction comprises `crtStep`, `mainTermMV_of_inputs`, and `singSeriesTails_holds`, the last of which is proved unconditionally with no cited input, taking $\sigma = 1/12$ and $D = 12(A + 3)$.
- (4) *Non-coprime moduli*: For Formulation B, the moduli $q \leq Q$ with $(a, q) \neq 1$ are handled as in Section 2 via `ternSum_noncoprime` and `theorem2Zeroed_of_theorem2`.

3.3. Relation to the rest of Halupczok’s paper. The later results of [H] are unaffected. In particular, Corollary 4 of [H-arXiv], which is Corollary 1.4 of [H, p. 334], sieves primes satisfying $p_1 \equiv -2 \pmod{\text{odd } d}$, which is always a coprime residue class, and the supplement taking a maximum over a restricts to reduced residues only. An independent check agreed with this finding after recomputing the numerical values 0.767 and 2.301 and verifying that no subsequent result in [H] invokes a non-coprime residue class a .

Remark 3.5 (On the strength of the formulation). For $(a, q) > 1$, a prime $p_1 \equiv a \pmod{q}$ must divide (a, q) , so $p_1 \leq q$; hence for $q \leq X_1$ no such prime lies in $(X_1, X_1 + Y_1]$ and the prime side vanishes. With the printed singular series $T(n, q, a)$, the Euler product need not vanish for these moduli q , and it is positive for $a = q = 3$, so an explicit formal statement incorporates either the restriction $(a, q) = 1$ or the convention $T_0(n, q, a)$. With the convention $T_0(n, q, a)$, the printed sum over all $q \leq Q$ becomes Formulation B, which follows from Formulation A when $Q \leq X_1$.

4. FORMAL VERIFICATION IN LEAN 4

The formalization was written in Lean 4 by an AI system (Claude by Anthropic) and is checked solely by the Lean kernel. The laboratory toolchain is `leanprover/lean4:v4.35.0` with `Mathlib v4.35.0` [Mathlib]; the Palomar registry entry [Palomar] is built on Lean `v4.35.0-rc3`. All declarations reside in the namespace `LeanLab.Halupczok2012`, and file paths below are given relative to the `LeanLab` repository root.

4.1. Encoding conventions. In the Lean encoding of asymptotic hypotheses and conclusions:

- Each $\ll$ in a hypothesis is encoded as a universally quantified positive constant ($c : \text{Fin } 7 \rightarrow \mathbb{R}$ and $c_7 : \mathbb{R}$);
- Each $\ll$ in a conclusion is encoded as an existential constant C ;
- The phrase “sufficiently large” is encoded as an existential threshold Y_0 (or x_0, N_0);
- The constants C and Y_0 may depend on A , the exponents $\varepsilon, \varepsilon_1, \varepsilon_2$, the hypothesis constants c, c_7 , and the fixed integer a ;
- The exponent $B = B(A)$ is existentially quantified immediately after A .

Every hypothesis sign of `Thm2Hyp` and `Theorem2` was checked against the JTNB page image of p. 333, those of `Theorem3` and `Theorem5` against pp. 338 and 339, and those of `PPS1984Printed` and `PPS1984Remark` against [PPS, p. 530].

4.2. Lean declarations and theorems. From `LeanLab/Halupczok2012/U1510Final.lean`, the headline theorems are:

```
theorem theorem2Zeroed_U1510 (hR : PPS1984Remark) (hPP : PP1993Goldbach) :
  Theorem2Zeroed :=
  theorem2Zeroed_U1496 (theorem3_of_uniform (uniform_of_remark hR)) hPP

theorem theorem2_U1510 (hR : PPS1984Remark) (hPP : PP1993Goldbach) : Theorem2 :=
  theorem2_U1496 (theorem3_of_uniform (uniform_of_remark hR)) hPP

theorem theorem3_of_remark (hR : PPS1984Remark) : Theorem3 :=
  theorem3_of_uniform (uniform_of_remark hR)

theorem theorem5_of_PP1993 (hPP : PP1993Goldbach) : Theorem5 :=
  theorem5_of_PP hPP
```

From `LeanLab/Halupczok2012/U1462Tails.lean`, the reductions from JTNB Theorems 2.1 and 2.3 (`Theorem3` and `Theorem5`) are:

```
theorem theorem2Zeroed_U1462 (h3 : Theorem3) (h5 : Theorem5) : Theorem2Zeroed
theorem theorem2_U1462 (h3 : Theorem3) (h5 : Theorem5) : Theorem2
```

From LeanLab/Halupczok2012/Thm2Coprime.lean, the elementary prime-in-progression lemmas are:

```
theorem no_prime_in_class {ℓ a X1 : ℕ} (hℓ : ℓ.Prime) (ha : a % ℓ = 0) (hX : ℓ < X1) :
  ∀ p : ℕ, p.Prime → X1 ≤ p → p % ℓ ≠ a % ℓ

theorem no_prime_three_mod_three {X1 : ℕ} (hX : 3 < X1) :
  ∀ p : ℕ, p.Prime → X1 ≤ p → ¬ p ≡ 3 [MOD 3]
```

4.3. Build logs, axiom audit, and status summary. The build logs in LeanLab/Halupczok2012/build/ record the verification of the theorem chain:

- 20261025T0416Z-19415-U-1510.log: lake build LeanLab.Halupczok2012.U1510Final, rebuilding the whole Halupczok2012 chain, ending with Build completed successfully (9063 jobs). and EXIT 0. The axiom audit lines are verbatim:

```
'LeanLab.Halupczok2012.theorem2Zeroed_U1510' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'LeanLab.Halupczok2012.theorem2_U1510' depends on axioms:
  [propext, Classical.choice, Quot.sound]
```

The theorems `theorem3_of_remark`, `theorem5_of_PP1993`, `theorem2Zeroed_U1462` and `theorem2_U1462` have the same three axioms. No proof in `U1510Final.lean` uses `sorry`; `Statements.lean` contains old cited stubs proved by `sorry` (for example `theorem2`), which do not enter the axioms of any theorem above, as the `#print axioms` output shows.

- 20261019T1838Z-live.log: ends with Build completed successfully (9049 jobs). and EXIT 0; `ternSum_noncoprime` and `theorem2Zeroed_of_theorem2` depend on axioms `[propext, Classical.choice, Quot.sound]`.
- 20261017T1503Z-14991.log: ends with Build completed successfully (9292 jobs). and EXIT 0; `no_prime_in_class` depends on `Quot.sound`, `propext`, and `no_prime_three_mod_three` depends on `Classical.choice`, `Quot.sound`, `propext`.

Table 1 summarizes the verification status of each input and step in the argument.

TABLE 1. Verification status of inputs and steps.

Input or step	Status
PPS1984Remark, printed Remark p. 530, uniform reading	cited, Lean hypothesis; uniform reading checked by reading pp. 531 to 538, not in Lean
PP1993Goldbach, unnumbered Goldbach analogue of PP1993 Theorem 3	cited, Lean hypothesis; from displays (1), (3), (4), (5), pp. 44 to 46
JTNB Theorem 2.1 from PPS1984Remark	proved in Lean, <code>theorem3_of_remark</code>
JTNB Theorem 2.3 from PP1993Goldbach	proved in Lean, <code>theorem5_of_PP1993</code>
singular series truncation and Euler product	proved in Lean, no cited input, <code>singSeriesTails_holds</code>
Formulation A, coprime reading of Theorem 1.2	proved in Lean from the two inputs, <code>theorem2_U1510</code>
Formulation B, all q with T_0 and $Q \leq X_1$	proved in Lean from the two inputs, <code>theorem2Zeroed_U1510</code>
Formulation A implies Formulation B	proved in Lean, unconditional, <code>theorem2Zeroed_of_theorem2</code>
non-coprime prime sums vanish for $q \leq X_1$	proved in Lean, unconditional, <code>ternSum_noncoprime</code>
$T(n, 3, 3)$ about 0.767	numerics only, not Lean

5. USE OF AI AND DATA AVAILABILITY

Use of AI. AI, namely Claude by Anthropic, wrote all Lean code, ran the numerics, and drafted the text of this note. The Lean statements and the numerics were checked by a second, independent AI run, and AI tools were used to read the sources. The author checked the statements, the page numbers, and the references and is responsible for the content.

Data availability. The formalization is archived in the Palomar registry under entry PALOMAR-2026-10-28-000026, version 1 [Palomar], available at <https://palomar-registry.org/entry.html?id=PALOMAR-2026-10-28-000026&version=1>. The registered theorems are `theorem2Zeroed_U1510`, `theorem2_U1510`, `theorem3_of_remark`, and `theorem5_of_PP1993`, with `PPS1984Remark` and `PP1993Goldbach` as Lean hypotheses. Palomar’s mechanical check passed with no warnings and its review found no blocking problem; the reviewer was an AI system rather than a human, so this review is not peer review.

The public Lean source code is available under the Apache-2.0 license in the repository <https://github.com/sthubbar/goldbach-lean> in the folder `entries/halupczok2012` at commit `12f09bfeca2ae00988fa039e2e82c6de4d147108` [Repo].

APPENDIX A. LEAN STATEMENTS

From `LeanLab/Halupczok2012/Statements.lean`, the definitions of the singular series `T`, the ternary prime sum `ternSum`, the hypothesis bundle `Thm2Hyp`, and Formulation A (`Theorem2`) are:

```
noncomputable def T (n q : ℕ) (a : ℤ) : ℝ :=
  (1 / (Nat.totient q : ℝ)) * ∏' p : Nat.Primes,
    (if ((p : ℕ) | n ∧ ¬ (p : ℕ) | q) ∨ (¬ ((p : ℕ) : ℤ) | (n - a) ∧ (p : ℕ) | q) then
      1 - 1 / ((p : ℝ) - 1) ^ 2
    else if ¬ (p : ℕ) | n ∧ ¬ (p : ℕ) | q then 1 + 1 / ((p : ℝ) - 1) ^ 3
    else (p : ℝ) / ((p : ℝ) - 1))

noncomputable def ternSum (n : ℕ) (X1 Y1 X2 Y2 : ℝ) (q : ℕ) (a : ℤ) : ℝ :=
  ∑ x ∈ ((range (n + 1)) ×8 (range (n + 1))).filter (fun x =>
    x.1.Prime ∧ x.2.Prime ∧ x.1 + x.2 ≤ n ∧ (n - x.1 - x.2).Prime ∧
    X1 < x.1 ∧ (x.1 : ℝ) ≤ X1 + Y1 ∧ X2 < x.2 ∧ (x.2 : ℝ) ≤ X2 + Y2 ∧ (x.1 : ℤ) ≡ a [ZMOD q]),
    Real.log x.1 * Real.log x.2 * Real.log ((n - x.1 - x.2 : ℕ) : ℝ)

def Thm2Hyp (ε ε1 ε2 B : ℝ) (c : Fin 7 → ℝ) (n : ℕ) (X1 Y1 X2 Y2 Q : ℝ) : Prop :=
  Odd n ∧ Y1 ≤ X1 ∧ c 0 * X1 ^ (3 / 5 + ε1) ≤ Y1 ∧ Y2 ≤ X2 ∧ c 1 * X2 ^ (7 / 12 + ε2) ≤ Y2 ∧
  c 2 * Y2 ^ (1 / 3 + ε) ≤ Y1 ∧ Y1 ≤ c 3 * Y2 ∧
  0 ≤ (n : ℝ) - X1 - Y1 - X2 - Y2 ∧ c 4 * Y2 ≤ (n : ℝ) - X1 - Y1 - X2 - Y2 ∧
  (n : ℝ) - X1 - Y1 - X2 - Y2 ≤ c 5 * Y2 ∧
  1 ≤ Q ∧ Q ≤ c 6 * Y1 * X1 ^ (-(1 / 2) : ℝ) * Real.log Y2 ^ (-B)

def Theorem2 : Prop :=
  ∀ A ε ε1 ε2 : ℝ, 0 < A → 0 < ε → 0 < ε1 → 0 < ε2 → ∃ B : ℝ, 0 < B ∧
  ∀ c : Fin 7 → ℝ, (∀ i, 0 < c i) → ∀ c7 : ℝ, 0 < c7 → ∀ a : ℤ, ∃ C Y0 : ℝ,
  ∀ (n : ℕ) (X1 Y1 X2 Y2 Q : ℝ), Y0 ≤ Y1 → Y0 ≤ Y2 → Thm2Hyp ε ε1 ε2 B c n X1 Y1 X2 Y2 Q →
  Q ≤ c7 * Y1 ^ (3 / 2 : ℝ) * ((n : ℝ) - X1) ^ (-(1 / 2) : ℝ) → (a : ℝ) ≤ (n : ℝ) - X1 - Y1 →
  ∑ q ∈ (Icc 1 [Q]₊).filter (fun q : ℕ => Int.gcd a q : ℤ) = 1,
  |ternSum n X1 Y1 X2 Y2 q a - T n q a * Y1 * Y2| ≤ C * Y1 * Y2 * Real.log Y2 ^ (-A)
```

From the module `NonCoprime` of [Repo], the zeroed singular series `T0`, the vanishing of non-coprime prime sums, Formulation B (`Theorem2Zeroed`), and the unconditional implication from Formulation A to Formulation B are:

```
noncomputable def T0 (n q : ℕ) (a : ℤ) : ℝ := if Int.gcd a q = 1 then T n q a else 0

theorem ternSum_noncoprime {n q : ℕ} {a : ℤ} {X1 Y1 X2 Y2 : ℝ} (hq : 1 ≤ q)
  (hg : Int.gcd a q ≠ 1) (hqX : (q : ℝ) ≤ X1) : ternSum n X1 Y1 X2 Y2 q a = 0

def Theorem2Zeroed : Prop :=
  ∀ A ε ε1 ε2 : ℝ, 0 < A → 0 < ε → 0 < ε1 → 0 < ε2 → ∃ B : ℝ, 0 < B ∧
  ∀ c : Fin 7 → ℝ, (∀ i, 0 < c i) → ∀ c7 : ℝ, 0 < c7 → ∀ a : ℤ, ∃ C Y0 : ℝ,
  ∀ (n : ℕ) (X1 Y1 X2 Y2 Q : ℝ), Y0 ≤ Y1 → Y0 ≤ Y2 → Thm2Hyp ε ε1 ε2 B c n X1 Y1 X2 Y2 Q →
  Q ≤ c7 * Y1 ^ (3 / 2 : ℝ) * ((n : ℝ) - X1) ^ (-(1 / 2) : ℝ) → (a : ℝ) ≤ (n : ℝ) - X1 - Y1 →
  Q ≤ X1 →
  ∑ q ∈ Icc 1 [Q]₊, |ternSum n X1 Y1 X2 Y2 q a - T0 n q a * Y1 * Y2|
  ≤ C * Y1 * Y2 * Real.log Y2 ^ (-A)
```

```
theorem theorem2Zeroed_of_theorem2 (h : Theorem2) : Theorem2Zeroed
```

The two cited inputs are defined in `LeanLab/Halupczok2012/U1495PPS.lean` and `U1496PP.lean`. They are formulated as Lean hypotheses passed as explicit arguments, not as axioms:

```
noncomputable def psiAP (z : ℝ) (q : ℕ) (a : ℤ) : ℝ :=
  ∑ n ∈ (Icc 0 [z]₊).filter (fun n : ℕ => (n : ℤ) ≡ a [ZMOD q]),
  ArithmeticFunction.vonMangoldt n

noncomputable def ppsTerm (z h : ℝ) (q : ℕ) (a : ℤ) : ℝ :=
  |psiAP (z + h) q a - psiAP z q a - h / q.totient|

def PPS1984Remark : Prop :=
  ∀ A : ℝ, 0 < A → ∃ D B : ℝ, 0 < B ∧ ∃ C x0 : ℝ, ∀ x y q : ℝ, x0 ≤ x →
  x ^ (3 / 5 : ℝ) * Real.log x ^ D ≤ y → y ≤ x →
  Q ≤ y * x ^ (-(1 / 2) : ℝ) * Real.log x ^ (-B) →
  ∀ (a : ℕ → ℤ) (h z : ℕ → ℝ), (∀ q, Int.gcd (a q) q = 1) →
  (∀ q, 0 < h q ∧ h q ≤ y) → (∀ q, x / 2 < z q ∧ z q ≤ x) →
  ∑ q ∈ Icc 1 [Q]₊, ppsTerm (z q) (h q) q (a q) ≤ C * y * Real.log x ^ (-A)

def PP1993Goldbach : Prop :=
  ∀ ε A : ℝ, 0 < ε → 0 < A → ∃ C N0 : ℝ, ∀ N Y : ℝ, N0 ≤ N → N ^ (7 / 12 + ε) ≤ Y → Y ≤ N →
  ∑ k ∈ kRange (N / 2) (Y ^ (1 / 3 + ε) / 2),
```

$$\frac{\text{ppR}(2 * k) N Y - S(2 * k) * \text{ppM}(2 * k) N Y}{C * Y \wedge (1 / 3 + \varepsilon) * Y \wedge 2 * \text{Real.log } N \wedge (-A)} \wedge 2 \leq$$

Here ppR is $R^*(2n, N, Y)$, the sum over $h + k = 2n$ with $N - Y < h \leq N$ and $k \leq Y$ of $\Lambda(h)\Lambda(k)$ ([PP, p. 43]); ppM is $M^*(2n, N, Y)$, the corresponding count with weight 1; and S is the binary singular series $\mathfrak{S}$ of [PP, p. 41] and of Halupczok's display (1.2), [H, p. 335]. In words, PP1993Goldbach states that for $\varepsilon, A > 0$, $N^{7/12+\varepsilon} \leq Y \leq N$, and $H = Y^{1/3+\varepsilon}$, the sum over $N < 2n \leq N + H$ of $|R^*(2n) - M^*(2n)\mathfrak{S}(2n)|^2$ is $\ll HY^2L^{-A}$, where $L = \log N$.

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