

Anchored Accumulation Calculus: Grammar-Relative Memory Representations and Residual Constraint Closure across Differential and Functional Operators

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31 October 2026

Abstract

Anchored Accumulation Calculus (AAC) provides an exact discrete-time realization of a class of non-Markovian memory operators by augmenting the dynamics with anchored exponential modes. This paper defines that realization precisely and develops a grammar-relative extension in which the central question is whether an accumulated output is already an admissible observable of a given recursive state. The corresponding Residual Constraint Closure (RCC) test is a cohomological equation in a declared observable algebra. For linear differential states, RCC reduces to a rational row equation; in a minimal scalar differential module, this yields an exact dichotomy between preservation of differential order and an increase by one for a primitive. We construct finite projective states for logarithms of solutions of finite-order linear differential equations and prove a differential-algebraicity equivalence for polynomial-logarithm primitives. For polynomial logarithmic sources, we identify the scalar differential order with the dimension of the apolar derivative space under a full-rank monodromy hypothesis. Our principal algorithmic theorem gives a total-degree bound of one more than the source degree and a finite exact residue-based decision procedure for polynomial-logarithmic RCC under an independent-divisor hypothesis. Exact Gamma and q-Pochhammer examples illustrate how shift and dilation grammars can close accumulations that are not closed in a smaller differential grammar. A Bessel canonical-product example connects the framework with classical Rayleigh sums. The paper distinguishes these specific results from the established theories on which they rely; it does not claim that differential algebra, realization theory, apolarity, or rational summability is new.

Keywords: Anchored Accumulation Calculus; recursive memory; residual constraint closure; differential algebra; differential modules; logarithmic monodromy; apolarity; difference equations; dilation operators.

2020 Mathematics Subject Classification: Primary 12H05; Secondary 34A05, 39A05, 33F10.

1 Introduction

Classical integration classifies antiderivatives relative to a prescribed function language. Liouville theory and the Risch algorithm address elementary primitives in differential fields [1, 2]; differential algebra studies functions satisfying algebraic differential equations [8]; and the D-finite class is closed under integration [6]. State-space realization theory studies finite-dimensional dynamical representations and their minimality [10, 11], while nonlinear generating-series theory relates finite-rank series to realizations [12]. Difference algebra studies functional recurrences and rational

summability, for which discrete residues provide exact obstructions [13, 14]. These theories overlap, but they do not by themselves provide a single bookkeeping language for asking whether a proposed accumulation is already encoded in a specified recursive memory state.

The motivating question is therefore representation-relative: given a function generated by a finite recursive state, does taking a primitive require a genuinely new state, or can that primitive be expressed using an observable already allowed by the current representation? A related question is whether failure of closure in a differential grammar can be overcome by using a shift, dilation, product, or hybrid grammar.

The paper uses two related but distinct notions. First, *Anchored Accumulation Calculus* (AAC) denotes the exact exponential-mode memory realization defined in Section 2. Second, *extended AAC* denotes the grammar-relative closure framework developed here. The extension is motivated by the same principle—accumulated history should be represented by an evolving state whenever the chosen state language permits it—but it is not restricted to exponential convolution kernels. In particular, the ODE, shift, dilation, and logarithmic examples below are instances of the extended framework; they are not all finite-exponential-kernel models.

We call the closure test *Residual Constraint Closure* (RCC). For a differential state system, RCC asks whether the source output is the derivative of an admissible state observable. Equivalently, it asks whether a residual defined by subtracting the Lie derivative of that observable can be made identically zero. This is a grammar-restricted cohomological equation. The underlying coboundary equation is standard; the role of RCC in this paper is to name and organize the exact test relative to a fixed recursive-memory grammar.

The main contributions are as follows.

1. We give an exact definition of finite-mode AAC under zero-order-hold input and define grammar-relative memory dimension together with a complexity profile that prevents hidden coefficient, update, output, or seed complexity from being counted as compression.
2. We formulate differential and discrete RCC residuals. For a minimal linear differential module, we prove that a primitive has scalar differential order either equal to the order of the source or exactly one larger, with the lower case characterized by a rational row-coboundary equation.
3. We construct a finite rational projective-state system for logarithmic derivatives of solutions of finite-order linear ODEs. We prove that, for every nonconstant polynomial G , a local primitive of $G(\log y)$ is differentially algebraic over $\mathbb{C}(x)$ if and only if y is.
4. We connect polynomial translation spaces, apolar derivative spaces, and logarithmic monodromy. Under a full-rank divisor hypothesis, the dimension of the apolar derivative space is exactly the minimal scalar differential order of the associated polynomial in logarithms.
5. We prove a divisor-rank degree bound for polynomial-logarithmic RCC and give a finite exact algorithm based on rational partial fractions, residues, and finite-dimensional linear algebra. We also give an all-degree one-logarithm obstruction criterion.

The first three items organize established state-space and differential-algebraic ideas around a common closure question. The principal specific claims advanced as new in this paper are the full-rank logarithmic-divisor degree bound and its finite residue algorithm, the monodromy/apolar scalar-order identity in the stated form, and the all-degree one-logarithm classification. These priority claims are intentionally narrow. A literature search can support positioning but cannot certify absolute uniqueness; the paper does not claim that the underlying ingredients or their general theories originate here. In particular, the degree-bound result is a divisor-specialized statement in a

rational logarithmic extension, not a replacement for general Risch or reduction-system methods such as those of Du and Raab [7].

2 AAC and grammar-relative memory

2.1 Anchored Accumulation Calculus

We first state the specific finite-mode memory realization referred to as AAC in this paper.

Definition 2.1 (Finite-mode anchored memory). Let

$$K(t) = \sum_{j=1}^M w_j e^{-\lambda_j t}, \quad t \geq 0,$$

where $\operatorname{Re} \lambda_j > 0$ and $w_j, \lambda_j \in \mathbb{C}$. For an input u and an anchor time t_0 , define the mode states

$$z_j(t) = e^{-\lambda_j(t-t_0)} z_{j,0} + \int_{t_0}^t e^{-\lambda_j(t-\tau)} u(\tau) \, d\tau, \quad t \geq t_0,$$

where $z_{j,0} = z_j(t_0)$ is the anchored mode value carrying the contribution of the pre-anchor history. Assume u is locally integrable (piecewise-continuous inputs suffice for the zero-order-hold theorem below). The memory output is

$$m(t) = \sum_{j=1}^M w_j z_j(t).$$

The finite-mode AAC realization is the state augmentation by the z_j and the exact propagation rule below. If the kernel is real-valued, the modes and weights are taken real or arranged in complex-conjugate pairs.

Theorem 2.2 (Exact zero-order-hold update). *Let $t_n = t_0 + n\Delta t$, with $\Delta t > 0$, and suppose that $u(t) = u_n$ on $[t_n, t_{n+1})$. Then each anchored mode satisfies*

$$z_{j,n+1} = \beta_j z_{j,n} + \alpha_j u_n, \quad \beta_j = e^{-\lambda_j \Delta t}, \quad \alpha_j = \frac{1 - e^{-\lambda_j \Delta t}}{\lambda_j}.$$

Moreover, for every $0 \leq \theta \leq \Delta t$,

$$z_j(t_n + \theta) = e^{-\lambda_j \theta} z_{j,n} + \frac{1 - e^{-\lambda_j \theta}}{\lambda_j} u_n.$$

Thus the discrete update and the intra-step relaxation are exact for the stated kernel and zero-order-hold input.

Proof. Separate the anchored seed and split the input integral at t_n :

$$\begin{aligned} z_j(t_{n+1}) &= e^{-\lambda_j(t_{n+1}-t_0)} z_{j,0} + \int_{t_0}^{t_n} e^{-\lambda_j(t_{n+1}-\tau)} u(\tau) \, d\tau \\ &\quad + \int_{t_n}^{t_{n+1}} e^{-\lambda_j(t_{n+1}-\tau)} u_n \, d\tau \\ &= e^{-\lambda_j \Delta t} \left(e^{-\lambda_j(t_n-t_0)} z_{j,0} + \int_{t_0}^{t_n} e^{-\lambda_j(t_n-\tau)} u(\tau) \, d\tau \right) + u_n \int_0^{\Delta t} e^{-\lambda_j r} \, dr \\ &= e^{-\lambda_j \Delta t} z_{j,n} + \frac{1 - e^{-\lambda_j \Delta t}}{\lambda_j} u_n. \end{aligned}$$

The same calculation with $t_n + \theta$ in place of t_{n+1} proves the intra-step formula. Summing with weights w_j gives the exact sampled memory output. No time-discretization approximation has been introduced; exactness is conditional on the finite exponential representation of K and the zero-order-hold input on each step. $\square$

Remark 2.3. The anchor and seed are part of the realization. If the pre-anchor contribution is zero, then $z_{j,0} = 0$. Otherwise $z_{j,0}$ must equal the contribution of the pre-anchor history in the j th exponential mode; an arbitrary collection of seed values need not correspond to a common physical prehistory. The recurrence is finite-dimensional and Markovian in the augmented state, although the input-output relation in u is non-Markovian. For a kernel that is not exactly a finite sum of exponentials, a finite-mode representation is exact only if an exact such expansion is independently established; otherwise a mode fit is an approximation and is not covered by Theorem 2.2.

2.2 Representation grammars

The generalization developed in the rest of the paper allows updates other than exponential-mode recurrences. This requires fixing the language in which a state representation is measured.

Definition 2.4 (Representation grammar). A representation grammar is a tuple

$$\mathcal{G} = (\mathbb{K}, \mathcal{O}, \mathcal{E}, \mathcal{U}, \mathcal{A}, \mathcal{S}),$$

where:

- $\mathbb{K}$ is a specified base field of coefficient germs (for example $\mathbb{C}(x)$ or a stated differential/difference extension);
- $\mathcal{O}$ is the allowed collection of operators, such as $D = d/dx$, translations T_h , dilations Q_q , or affine pullbacks;
- $\mathcal{E}$ specifies the allowed state-update expressions;
- $\mathcal{U}$ specifies the allowed output expressions;
- $\mathcal{A}$ is the algebra of admissible observables for closure tests;
- $\mathcal{S}$ specifies permitted seed and normalization data.

The grammar is fixed before a compression, closure, or non-closure claim is made.

Definition 2.5 (Memory realization). Let f be an analytic or meromorphic germ on a domain where the operators in $\mathcal{O}$ are defined. A finite memory realization of f in $\mathcal{G}$ consists of a finite state $s = (s_1, \dots, s_n)$, an admissible update law from $\mathcal{E}$, an output map $h \in \mathcal{U}$ such that

$$f(x) = h(x, s(x)),$$

and seed data from $\mathcal{S}$ selecting the intended solution. The integer n is the state dimension of this realization.

Definition 2.6 (Grammar-relative memory dimension). For a fixed grammar $\mathcal{G}$, define

$$\mu_{\mathcal{G}}(f) = \min\{n : f \text{ has an } n\text{-state realization in } \mathcal{G}\},$$

with $\mu_{\mathcal{G}}(f) = \infty$ if no such realization exists. This number is meaningful only together with $\mathcal{G}$ and its seed convention.

State dimension alone is not the total complexity of a representation. We record the profile

$$\mathfrak{C}_G(f) = (n, c_{\mathbb{K}}, c_{\text{upd}}, c_{\text{out}}, c_{\text{seed}}, \mathcal{O}),$$

where the c -entries measure, under a declared encoding convention, coefficient-description length, update-rule size, output-map size, and seed-description length. This is bookkeeping rather than a universal numerical complexity measure. It prevents hidden coefficient, update, output, or seed complexity from being mistaken for compression.

Remark 2.7. The same function can have different minimal state dimensions in different grammars. For example, $F' = L^d$ can be represented with one nonlinear output $G(L) = L^d$, or with a linear output after adjoining monomial states $L, L^2, \dots, L^d$. These are distinct grammars and have distinct state counts. No dimension comparison is meaningful unless the allowed output maps and coefficient costs are fixed.

3 Accumulation closure and RCC

3.1 Differential RCC

Consider a differential state system

$$s' = V(x, s), \quad g(x) = G(x, s(x)), \quad (3.1)$$

where V and G belong to the fixed grammar. A primitive F satisfies $F' = g$.

Definition 3.1 (Differential RCC). Let $\mathcal{A}$ be the admissible observable algebra. For $h \in \mathcal{A}$, define the RCC residual

$$\mathcal{R}_{\text{diff}}(h) := G - \mathcal{L}_V h, \quad \mathcal{L}_V := \partial_x + \sum_{j=1}^n V_j(x, s) \partial_{s_j}.$$

The output g is *RCC-closed* in the current representation if there exists $h \in \mathcal{A}$ such that $\mathcal{R}_{\text{diff}}(h) = 0$. It is *RCC-nonclosed relative to $\mathcal{A}$* if no such h exists.

The equation to be solved is

$$h_x(x, s) + \nabla_s h(x, s) \cdot V(x, s) = G(x, s). \quad (3.2)$$

A solution gives $F(x) = h(x, s(x)) + C$. If no solution exists in $\mathcal{A}$, one may append an accumulator state $z' = G(x, s)$. The operator $\mathcal{L}_V$ is the Lie derivative along the extended vector field $(1, V)$, so (3.2) is a cohomological equation in the specified observable algebra.

If arbitrary local analytic observables are allowed, the method of characteristics generally gives a local solution. Thus non-closure is meaningful only relative to a restricted algebra, such as rational functions of (x, s) or polynomials in s with rational-function coefficients. RCC is a name for this grammar-restricted zero-residual test; it is not asserted to be a new cohomological equation.

3.2 Linear-state RCC test

Proposition 3.2 (Linear-state RCC test). *Suppose*

$$s' = A(x)s, \quad g = r(x)c(x)s,$$

where A is an $n \times n$ matrix, c is a row vector, and the coefficients lie in a differential field $\mathbb{K}$ of characteristic zero. Assume $s_1, \dots, s_n$ form a $\mathbb{K}$ -basis of the represented differential module (after

reducing any $\mathbb{K}$ -linear dependencies). A primitive in this module, of the form $F = p(x)s$ with $p(x) \in \mathbb{K}^{1 \times n}$, exists if and only if

$$p' + pA = rc. \quad (3.3)$$

Proof. Differentiating $F = ps$ and using $s' = As$ gives

$$F' = (p' + pA)s.$$

This equals rcs as an identity in the represented module precisely when the row coefficients agree, which is (3.3). Conversely, a solution p of (3.3) gives a primitive $F = ps$. $\square$

3.3 Primitive-order dichotomy

Let $K = \mathbb{C}(x)$ with derivation $D = d/dx$. For a nonzero D-finite germ f , let $\text{ord}_D(f)$ denote the least order of a nonzero scalar linear differential operator in $K[D]$ that annihilates f . Since a primitive is defined up to an additive constant, set

$$\text{ord}_D([F]) = \min_{c \in \mathbb{C}} \text{ord}_D(F + c).$$

For a D-finite function f , its minimal scalar differential order equals

$$\dim_K \text{span}_K \{f, f', f'', \dots\}.$$

Indeed, a minimal order- n equation makes the first n derivatives independent over K and expresses every higher derivative in their span.

Theorem 3.3 (Primitive-order dichotomy). *Let $g \neq 0$ be D-finite over $K = \mathbb{C}(x)$ with minimal scalar differential order n . Let $F' = g$, and let*

$$s = (g, g', \dots, g^{(n-1)})^T, \quad s' = A(x)s,$$

be its companion state, so that $g = e_1^T s$. Then

$$\text{ord}_D([F]) = \begin{cases} n, & \text{if } p' + pA = e_1^T \text{ has a solution } p \in K^{1 \times n}, \\ n + 1, & \text{otherwise.} \end{cases}$$

Equivalently, modulo its additive constant, the primitive has the same minimal differential order as the integrand exactly when it is an observable in the integrand's minimal differential module; otherwise exactly one additional scalar differential order is necessary.

Proof. Let

$$M_g = \text{span}_K \{g, g', \dots, g^{(n-1)}\}.$$

Minimality of the scalar equation implies that M_g is an n -dimensional D -stable space. Since $F' = g$, the operator $L_g D$, where L_g is a minimal order- n annihilator of g , annihilates $F + c$ for every constant c . Hence $F + c$ is D-finite of order at most $n + 1$.

Now fix c and suppose $F + c$ has minimal scalar order m . Its derivative span

$$M_{F+c} = \text{span}_K \{F + c, (F + c)', (F + c)'', \dots\}$$

is D -stable, has dimension m , and contains g . Since it is D -stable, it contains every derivative of g , and hence contains M_g . Therefore $m \geq n$. Taking the minimum over c gives

$$n \leq \text{ord}_D([F]) \leq n + 1.$$

If $\text{ord}_D([F]) = n$, choose c attaining the minimum. Then M_{F+c} and M_g both have dimension n and $M_g \subseteq M_{F+c}$, so $M_{F+c} = M_g$. In particular, $F + c \in M_g$, and there is a rational row p such that $F + c = ps$. Differentiation yields $p' + pA = e_1^T$. Conversely, if this row equation has a solution, then ps is a primitive of g , so after choosing the additive constant appropriately, a representative of $[F]$ belongs to M_g . Its order is at most n , and the lower bound forces equality. If the row equation has no solution, order n is impossible and the order is $n + 1$. $\square$

Remark 3.4. The theorem is an exact containment test for the additive class of a primitive in the minimal rational differential module of its integrand. Failure of the displayed row equation proves non-closure in this rational-linear observable class only. It does not rule out nonlinear observables or a larger operator grammar.

Example 3.5 (No additional differential state: the exponential). For $g(x) = e^x$, the minimal order is $n = 1$ and $A = (1)$. The RCC equation is $p' + p = 1$, which has the rational solution $p = 1$. Thus $F = e^x$ is already in the source module and $\text{ord}_D([F]) = 1$.

Example 3.6 (One additional differential state: the Gaussian). For $g(x) = e^{-x^2}$, the minimal equation is $g' + 2xg = 0$, so $n = 1$ and $A = (-2x)$. The row equation is

$$p' - 2xp = 1, \quad p \in \mathbb{C}(x).$$

A rational solution has no finite pole: at a pole of order k , the pole of p' has order $k + 1$, while multiplication by x cannot cancel that leading pole, including at $x = 0$. Hence p must be a polynomial. If p is a nonzero polynomial, the highest-degree term of $-2xp$ cannot be cancelled by p' , so the equation is impossible; $p = 0$ also fails. Therefore the RCC row equation has no rational solution and $\text{ord}_D([F]) = 2$. Indeed, the primitive satisfies $F'' + 2xF' = 0$.

3.4 Difference and dilation RCC

Let a state transformation be given by

$$s(Tx) = \Phi(x, s(x)),$$

where T is an allowed transformation (for example, $T_h x = x + h$ or $Q_q x = qx$). For a proposed accumulated increment G , define the discrete residual

$$\mathcal{R}_T(h) := G(x, s) - (h(Tx, \Phi(x, s)) - h(x, s)).$$

Discrete RCC holds in the chosen observable algebra if this residual vanishes for an admissible h . This is the difference-coboundary equation

$$h(Tx, \Phi(x, s)) - h(x, s) = G(x, s). \quad (3.4)$$

The homogeneous kernel and seed data are part of the representation: translations admit periodic homogeneous terms, while dilations admit scale-invariant terms subject to domain and regularity assumptions. A difference equation for a primitive is an additional structural identity; it does not replace the derivative definition of a continuous primitive unless a boundary or normalization condition is also specified.

4 Differential-functional operator identities

Let

$$Xf(x) = xf(x), \quad Df = f', \quad T_h f(x) = f(x+h), \quad Q_q f(x) = f(qx),$$

and let $C_{q,h}f(x) = f(qx+h)$. Direct calculation gives

$$DX = XD + I, \quad DT_h = T_h D, \quad (4.1)$$

$$DQ_q = qQ_q D, \quad DC_{q,h} = qC_{q,h} D, \quad (4.2)$$

$$C_{q,h}X = (qX + hI)C_{q,h}. \quad (4.3)$$

These are basic commutation identities for the indicated multiplication, differentiation, translation, dilation, and affine-pullback operators. They are not intended as a complete presentation of every relation in the operator algebra.

Proposition 4.1 (Primitive-transfer identity). *For $C = C_{q,h}$ with $q \neq 0$ and integer $k \geq 1$, define*

$$A_{C,k} = I - q^{-k}C.$$

Then

$$D^k A_{C,k} = (I - C)D^k. \quad (4.4)$$

Proof. The chain rule gives $D^k C = q^k C D^k$. Consequently

$$D^k (I - q^{-k}C) = D^k - C D^k = (I - C)D^k,$$

which is (4.4). □

For $k = 1$, if $F' = g$ and $T(x) = qx + h$, the identity reads

$$DF - q^{-1}F \circ T = g - g \circ T.$$

For a k -fold primitive $D^k F = g$, it gives $D^k (A_{C,k} F) = (I - C)g$. The result determines $A_{C,k} F$ only up to a polynomial of degree at most $k - 1$, the usual ambiguity in k -fold differential inversion. This is distinct from the kernel of $A_{C,k}$ itself: for a translation ($q = 1$), that kernel contains arbitrary h -periodic functions, while for a dilation it contains solutions of $F(qx + h) = q^k F(x)$ subject to the domain and regularity assumptions.

Proposition 4.2 (Contractive dilation inverse). *Let $0 < |q| < 1$. Let f be analytic in $|z| < R$, vanish to order at least $m \geq 2$ at 0, and have finite norm*

$$\|f\|_{m,R} := \sup_{0 < |z| < R} \frac{|f(z)|}{|z|^m}.$$

On this Banach space, the operator $q^{-1}Q_q$ has norm at most $|q|^{m-1} < 1$. Hence

$$(I - q^{-1}Q_q)^{-1}f(z) = \sum_{j=0}^{\infty} q^{-j}f(q^j z), \quad (4.5)$$

with convergence in the stated norm and

$$\|(I - q^{-1}Q_q)^{-1}\| \leq (1 - |q|^{m-1})^{-1}.$$

Proof. For f in the stated space,

$$\frac{|q^{-1}f(qz)|}{|z|^m} = |q|^{m-1} \frac{|f(qz)|}{|qz|^m} \leq |q|^{m-1} \|f\|_{m,R}.$$

Thus $\|q^{-1}Q_q\| \leq |q|^{m-1} < 1$. The space is complete because division by z^m identifies it isometrically with the bounded holomorphic functions on the disk. The Neumann series therefore converges in operator norm and is the inverse of $I - q^{-1}Q_q$, giving (4.5) and the norm bound. $\square$

Remark 4.3. On the full space of analytic germs, the homogeneous kernel of $I - q^{-1}Q_q$ is spanned by z , and the homogeneous kernel of $I - Q_q$ is spanned by constants. These statements follow by comparing Taylor coefficients. The vanishing condition of order at least two in Proposition 4.2 removes the linear homogeneous mode and makes the inverse unique on the specified Banach space.

5 Logarithmic primitives of finite-order linear ODE solutions

5.1 Projective memory construction

Let $K = \mathbb{C}(x)$ and suppose that a nonzero local solution y satisfies

$$y^{(r)} + \sum_{j=0}^{r-1} a_j(x)y^{(j)} = 0, \quad a_j(x) \in K. \quad (5.1)$$

Work on a simply connected domain avoiding coefficient singularities and zeros of y , and choose a branch $L = \log y$. For $r \geq 2$, define

$$u_0 = 1, \quad u_j = \frac{y^{(j)}}{y}, \quad 1 \leq j \leq r-1.$$

Theorem 5.1 (Projective-state logarithmic lift). *Assume $r \geq 2$. The variables $u_1, \dots, u_{r-1}$ satisfy the finite rational differential system*

$$u'_j = u_{j+1} - u_1 u_j, \quad 1 \leq j \leq r-2, \quad (5.2)$$

$$u'_{r-1} = -\sum_{j=0}^{r-1} a_j u_j - u_1 u_{r-1}. \quad (5.3)$$

Moreover, for every polynomial $G \in \mathbb{C}[t]$, a primitive

$$F' = G(\log y)$$

has a nonlinear differential-memory realization with at most $r+1$ states: the $r-1$ projective variables, $L' = u_1$, and $F' = G(L)$.

Proof. For $1 \leq j \leq r-2$, differentiation of $u_j = y^{(j)}/y$ gives

$$u'_j = \frac{y^{(j+1)}}{y} - \frac{y'}{y} \frac{y^{(j)}}{y} = u_{j+1} - u_1 u_j.$$

For the last variable, equation (5.1) gives

$$\frac{y^{(r)}}{y} = -\sum_{j=0}^{r-1} a_j u_j,$$

so differentiating u_{r-1} yields (5.3). Finally, $L' = y'/y = u_1$, and adjoining the scalar state F with update $F' = G(L)$ gives the asserted $r+1$ -state realization. $\square$

Remark 5.2 (First-order source equation). When $r = 1$, equation (5.1) is $y' + a_0(x)y = 0$, so $L' = -a_0(x)$. The projective block is empty, and the two states L, F give the same $r + 1 = 2$ state bound.

Remark 5.3 (Nonlinear output and linear-output lift). For $G(t) = t^d$, the preceding realization uses $r + 1$ states because the output map $F' = L^d$ is allowed to be nonlinear. If the grammar requires a linear output, adjoin monomial states $S_k = L^k$, $1 \leq k \leq d$, satisfying

$$S'_1 = u_1, \quad S'_k = ku_1S_{k-1} \quad (k \geq 2), \quad F' = S_d.$$

The resulting linear-output lift has $r + d$ states. These counts refer to different grammars and neither is, by itself, a global minimality result.

5.2 A scalar differential-algebraic equation for the primitive

For a second-order source equation

$$y'' + p(x)y' + q(x)y = 0,$$

the logarithmic derivative $u = L'$ satisfies the Riccati equation

$$u' = -u^2 - pu - q. \quad (5.4)$$

Proposition 5.4 (Power-log primitive equation). *Let $F' = L^d$, with $d \geq 1$ and $L = \log y$ on a local branch. On a nondegenerate part of that branch, the primitive obeys*

$$dL^d F''' + (L - d + 1)(F'')^2 + dp(x)L^d F'' + d^2 q(x)L^{2d-1} = 0, \quad L^d = F'. \quad (5.5)$$

Consequently F is differentially algebraic over $\mathbb{C}(x)$ and satisfies a differential-algebraic relation of order at most three.

Proof. Since $F' = L^d$,

$$F'' = dL^{d-1}u,$$

where $u = L'$. On a branch where $L \neq 0$, write

$$u = \frac{F''}{dL^{d-1}}.$$

Differentiating this identity and substituting (5.4) gives

$$F''' = dL^{d-2}((d-1-L)u^2 - pLu - qL).$$

Multiplying by dL^d and using $F'' = dL^{d-1}u$ cancels the u^2 and pu terms precisely with the second and third terms in (5.5); the remaining q term is cancelled by the fourth. This proves (5.5). The identity extends across removable exceptional points by analytic continuation. Since L is algebraic over the differential field generated by F' through $L^d = F'$, eliminating L by a resultant gives a nonzero differential polynomial relation for F of order at most three. $\square$

For $d = 1$, the auxiliary variable is simply $L = F'$, and (5.5) reduces to

$$F''' + (F'')^2 + p(x)F'' + q(x) = 0.$$

For an order- r source equation with $r \geq 2$, define differential polynomials in $u_1 = L'$ by

$$B_0 = 1, \quad B_1 = u_1, \quad B_{j+1} = B_j' + u_1 B_j.$$

Then $B_j = y^{(j)}/y$, and division of (5.1) by y gives

$$B_r + \sum_{j=0}^{r-1} a_j B_j = 0.$$

Since $F' = G(L)$ with nonconstant polynomial G , the variable L is algebraic over the field generated by F' . On a nondegenerate branch where $G'(L) \neq 0$, one has $L' = F''/G'(L)$. Substitution into the preceding differential relation and elimination of L yields a differential-algebraic equation for F of order at most $r + 1$. This bound is an upper bound; it is not asserted to be minimal.

5.3 Differential-algebraicity equivalence

Theorem 5.5 (Logarithmic primitive equivalence). *Let y be a nonzero analytic germ and let $G \in \mathbb{C}[t]$ be nonconstant. Choose a local branch $L = \log y$ and let $F' = G(L)$. Then*

$$F \text{ is differentially algebraic over } \mathbb{C}(x) \iff y \text{ is differentially algebraic over } \mathbb{C}(x).$$

Proof. We use two standard closure facts for differential algebra in characteristic zero: a differential-algebraic extension of a differential field remains differential-algebraic under algebraic extensions and under adjoining a solution of a first-order differential equation with coefficients in that extension; and differential algebraicity is transitive in towers.

Suppose first that F is differentially algebraic. Then F' is differentially algebraic. The relation

$$G(L) - F' = 0$$

is a nonzero algebraic equation for L over the differential field generated by F' . Hence L is differentially algebraic over $\mathbb{C}(x)$. Since $y' = L'y$, the function y is a solution of a first-order linear differential equation over a differentially algebraic extension, and is therefore differentially algebraic over $\mathbb{C}(x)$.

Conversely, suppose that y is differentially algebraic and nonzero. Then $L' = y'/y$ is differentially algebraic. Adjoining an antiderivative of L' gives L , which is differentially algebraic over $\mathbb{C}(x)$. It follows that $G(L)$ is differentially algebraic, and adjoining its antiderivative F preserves differential algebraicity. This proves both implications. $\square$

Corollary 5.6 (Gamma-logarithm primitives). *By Hölder's theorem, the Euler Gamma function is differentially transcendental over $\mathbb{C}(x)$ [16, 15]. Theorem 5.5 therefore implies that every local primitive of $G(\log \Gamma(x))$, for nonconstant polynomial G , is differentially transcendental over $\mathbb{C}(x)$. In particular,*

$$\int (\log \Gamma(x))^d dx$$

is not differentially algebraic for any integer $d \geq 1$.

Remark 5.7. This is a statement about differential algebraicity over $\mathbb{C}(x)$. It is not a non-representability result in every operator language and does not exclude shift, dilation, canonical-product, or hybrid representations.

6 Functional memories: Gamma and q-Pochhammer

6.1 Gamma: a differentially transcendental source with an exact shift law

Let $L(x) = \log \Gamma(x)$ for $x > 0$, and define the normalized primitive

$$H(x) = \int_1^x L(t) \, dt.$$

The Gamma recurrence gives

$$L(x+1) - L(x) = \log x. \quad (6.1)$$

Differentiating $H(x+1) - H(x)$ and using (6.1) gives

$$\frac{d}{dx}(H(x+1) - H(x)) = \log x,$$

so

$$H(x+1) - H(x) = x \log x - x + C.$$

Raabe's integral

$$\int_x^{x+1} \log \Gamma(t) \, dt = x \log x - x + \frac{1}{2} \log(2\pi), \quad x > 0,$$

fixes $C = \frac{1}{2} \log(2\pi)$ at $x = 1$; see, for example, the discussion of Raabe's formula in [21]. Thus

$$H(x+1) - H(x) = x \log x - x + \frac{1}{2} \log(2\pi). \quad (6.2)$$

The primitive H is differentially transcendental by Theorem 5.5, applied with $G(t) = t$. Nevertheless, its shift increment is elementary. The recurrence for L alone does not identify the Gamma function: the positive, logarithmically convex normalization in the Bohr–Mollerup theorem selects that source [17]. Once $L = \log \Gamma$ is fixed, the continuous initial condition $H(1) = 0$ fixes the primitive. If only the shift increment of H is retained and its derivative relation is discarded, an arbitrary period-one function remains in the homogeneous kernel.

For $F'_d = L^d$, the shift gives the triangular identity

$$\frac{d}{dx}(F_d(x+1) - F_d(x)) = (L + \log x)^d - L^d = \sum_{k=0}^{d-1} \binom{d}{k} L^k (\log x)^{d-k}.$$

For each fixed d , the right side lies in a finite polynomial shift-state space. This functional identity does not imply that the Gamma source is differentially algebraic.

6.2 q-Pochhammer: an exact dilation inverse

Let $0 < |q| < 1$ and

$$P(a; q) = (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j), \quad |a| < 1.$$

Writing $L(a) = \log P(a; q)$ on the branch with $L(0) = 0$ gives

$$L(a) - L(qa) = \log(1 - a).$$

Define

$$\Phi(a) = \int_0^a L(t) \, dt, \quad A(a) = \int_0^a \log(1 - t) \, dt = -(1 - a) \log(1 - a) - a.$$

Since $L(0) = 0$, one has $\Phi(a) = O(a^2)$. Differentiation shows

$$(I - q^{-1}Q_q)\Phi = A, \quad Q_q f(a) = f(qa).$$

Proposition 4.2 therefore applies and gives the exact convergent inverse

$$\Phi(a) = \sum_{j=0}^{\infty} q^{-j} A(q^j a) \tag{6.3}$$

$$= - \sum_{n=1}^{\infty} \frac{a^{n+1}}{n(n+1)(1-q^n)}. \tag{6.4}$$

The order-two vanishing removes the linear homogeneous mode and guarantees convergence. This example exhibits an exact dilation-memory primitive in a functional grammar adapted to the source recurrence.

7 Polynomial logarithmic memory and apolar rank

Let $P_1, \dots, P_r \in \mathbb{C}(x)^*$ and choose local logarithms $Y_i = \log P_i$. Their derivatives

$$\rho_i = \frac{P'_i}{P_i}$$

are rational. For a polynomial $G \in \mathbb{C}[Y_1, \dots, Y_r]$, define its derivative-translation space

$$W_G = \text{span}_{\mathbb{C}}\{\partial^\alpha G : \alpha \in \mathbb{N}^r\}, \quad \tau(G) = \dim_{\mathbb{C}} W_G.$$

Equivalently, define the apolar algebra

$$A_G = \mathbb{C}[\xi_1, \dots, \xi_r] / \text{Ann}(G), \quad \text{Ann}(G) = \{Q : Q(\partial)G = 0\}.$$

Then $\dim_{\mathbb{C}} A_G = \tau(G)$, the classical inverse-system construction associated with Macaulay [9]. Since G is a polynomial, W_G is finite-dimensional. If $g_1, \dots, g_r$ is a basis, then

$$\frac{d}{dx} g_j(Y(x)) = \sum_{i=1}^r \rho_i(x) (\partial_i g_j)(Y(x)),$$

and each $\partial_i g_j$ belongs to W_G . Thus $G(Y(x))$ has a rational linear state realization of dimension at most $\tau(G)$.

7.1 Monodromy lower bound

Fix a base point outside the finite zero-pole set of the P_i . For a loop γ in the punctured plane, analytic continuation changes the logarithms by

$$\Delta_\gamma Y_i = \oint_\gamma \frac{P'_i(x)}{P_i(x)} dx = 2\pi i n_i(\gamma), \quad n_i(\gamma) \in \mathbb{Z}.$$

The image of the monodromy map is the subgroup of $(2\pi i \mathbb{Z})^r$ generated by the vectors

$$2\pi i (\text{ord}_a(P_1), \dots, \text{ord}_a(P_r)),$$

where a ranges over finite zeros and poles.

Theorem 7.1 (Monodromy lower bound for logarithmic memory). *Assume $G \in \mathbb{C}[Y_1, \dots, Y_r]$ is nonzero and the logarithmic monodromy image contains a full-rank lattice in $(2\pi i\mathbb{Z})^r$. Then the minimal order of a scalar linear differential equation over $\mathbb{C}(x)$ annihilating $G(Y(x))$ is exactly $\tau(G)$.*

Proof. The derivative space W_G is invariant under every partial derivative. The displayed differentiation identity therefore gives a rational linear differential system of dimension $\tau(G)$ for a basis of the composed functions $g_j(Y(x))$. This proves that the minimal scalar differential order is at most $\tau(G)$.

For the reverse inequality, analytic continuation around loops changes Y by vectors λ in a full-rank lattice $\Lambda \subset (2\pi i\mathbb{Z})^r$. The continued branches are $G(Y + \lambda)$. We first show that the complex span of the polynomial translates $G(Y + \lambda)$, $\lambda \in \Lambda$, is exactly W_G . Taylor expansion shows that each translate lies in W_G . Conversely, choose lattice vectors $v_1, \dots, v_r$ linearly independent over $\mathbb{C}$. On the finite-dimensional space of polynomials of degree at most $\deg G$, translation by v_j is $T_{v_j} = \exp(N_j)$, where $N_j = \partial_{v_j}$ is nilpotent. Thus $T_{v_j} - I$ is nilpotent and

$$N_j = \log T_{v_j} = \sum_{k=1}^{\deg G} \frac{(-1)^{k+1}}{k} (T_{v_j} - I)^k$$

on this space; the sum is finite because sufficiently high powers of $T_{v_j} - I$ vanish. The span of all lattice translates is invariant under T_{v_j} and under every polynomial in $T_{v_j} - I$. Since each power $(T_{v_j} - I)^k G$ is a finite linear combination of translates $G(Y + \ell v_j)$, it follows that this span contains $N_j G$ and, by iteration and invariance, all mixed directional derivatives of G . Because $v_1, \dots, v_r$ form a basis, the operators N_j span the coordinate derivatives ∂_{Y_i} . Hence every partial derivative of G belongs to the span of its lattice translates, proving the reverse inclusion.

Next, composition with $Y(x)$ does not reduce this dimension. If a polynomial Q in the span of the translates satisfied $Q(Y(x)) \equiv 0$, analytic continuation would give $Q(Y(x) + \lambda) \equiv 0$ for every $\lambda \in \Lambda$. At a fixed ordinary point x_0 , the polynomial Q would vanish on $Y(x_0) + \Lambda$. A full-rank lattice is Zariski dense in $\mathbb{C}^r$, so $Q = 0$. Thus the continued branches of $G(Y(x))$ span a space of dimension $\tau(G)$.

Every analytic continuation of a solution of a scalar linear differential equation with rational coefficients is again a solution of that equation. An order- n scalar equation has an n -dimensional local solution space at an ordinary point. Therefore $n \geq \tau(G)$. Combining both bounds proves equality. $\square$

Corollary 7.2 (Apolar–RCC primitive-order dichotomy). *Under the hypothesis of Theorem 7.1, let $R \in \mathbb{C}(x)^*$ and put $g(x) = R(x)G(Y(x))$. Then*

$$\text{ord}_D(g) = \tau(G).$$

For any primitive $F' = g$,

$$\text{ord}_D([F]) \in \{\tau(G), \tau(G) + 1\},$$

and the lower value occurs exactly when the linear RCC row equation in the minimal differential module of g is solvable.

Proof. Multiplication by a nonzero rational function is an invertible rational gauge transformation on the differential module generated by a function and its derivatives, so it does not change the minimal scalar order. Theorem 7.1 gives $\text{ord}_D(G(Y)) = \tau(G)$, hence the same order for $g = RG(Y)$. The conclusion follows from Theorem 3.3. $\square$

For $G = \prod_{i=1}^r Y_i^{d_i}$, the derivative-space dimension is $\prod_{i=1}^r (d_i + 1)$, provided the monodromy condition holds. For example, $Y_1 Y_2$ has rank 4, whereas Y_1^2 has rank 3. This is the apolar derivative-space dimension, not simply the number of monomials in an arbitrary displayed expansion. If a primitive is sought in the rational-linear source module $s' = A(x)s$, $G(Y) = cs$, Proposition 3.2 gives the exact closure test. Failure of $p' + pA = c$ proves that the primitive is not a rational-linear observable of that source module; it does not rule out nonlinear observables or larger operator grammars.

7.2 Degree bound and decidability for polynomial-logarithmic RCC

For the algorithmic statement, let k be an effectively presented algebraically closed field of characteristic zero (for example, the field of algebraic numbers with exact representations). Here, effective presentation means that exact field operations, equality testing, and factorization of univariate polynomials over k are algorithmic. Let $P_1, \dots, P_r \in k(x)^*$ and write

$$Y_i = \log P_i(x), \quad \rho_i = \frac{P'_i}{P_i}.$$

For each finite point $a \in k$, put $m_{ai} = \text{ord}_a(P_i)$, and form the finite divisor matrix $M = (m_{ai})$, with all but finitely many entries zero. Its columns are the finite logarithmic-divisor vectors.

Theorem 7.3 (Degree bound and decidability for polynomial-logarithmic RCC). *Assume that M has full column rank r . Let $R \in k(x)$ and let $G \in k[Y_1, \dots, Y_r]$ be nonzero of total degree D . Consider*

$$h_x + \sum_{i=1}^r \rho_i(x) h_{Y_i} = R(x)G(Y_1, \dots, Y_r), \quad h \in k(x)[Y_1, \dots, Y_r]. \quad (7.1)$$

Then:

- (i) every polynomial-logarithmic solution h has total degree at most $D + 1$ in $(Y_1, \dots, Y_r)$;
- (ii) existence of such an h is decidable by a finite exact procedure using rational partial fractions, residue tests, rational antidifferentiation, and finite-dimensional linear algebra over k .

More precisely, writing

$$G = \sum_{|\beta| \leq D} g_\beta Y^\beta, \quad h = \sum_{|\alpha| \leq D+1} a_\alpha(x) Y^\alpha,$$

equation (7.1) is equivalent to the finite triangular system

$$a'_\beta + \sum_{i=1}^r (\beta_i + 1) \rho_i a_{\beta + e_i} = R g_\beta, \quad |\beta| \leq D + 1, \quad (7.2)$$

where coefficients outside the degree bound are zero.

Proof. Let $N = \deg_Y h$. If $N > D + 1$, write h_N and h_{N-1} for the top two homogeneous parts. The derivation

$$\delta = \partial_x + \sum_{i=1}^r \rho_i \partial_{Y_i}$$

lowers Y -degree by at most one, while the right side of (7.1) has degree D . The degree- N equation is therefore $\partial_x h_N = 0$, so every coefficient of h_N is constant. The degree- $(N - 1)$ equation is

$$\partial_x h_{N-1} + \sum_{i=1}^r \rho_i \partial_{Y_i} h_N = 0,$$

because $N - 1 > D$. For each multi-index β with $|\beta| = N - 1$, its coefficient equation has the form

$$a'_\beta + \sum_{i=1}^r c_{\beta i} \rho_i = 0, \quad c_{\beta i} = (\beta_i + 1) a_{\beta + e_i},$$

where each $c_{\beta i}$ is constant. Since

$$\rho_i(x) = \sum_a \frac{m_{ai}}{x - a},$$

the residue at each finite pole a gives

$$\sum_{i=1}^r m_{ai} c_{\beta i} = 0.$$

Full column rank of M forces $c_{\beta i} = 0$ for every i . Varying β shows that every coefficient of h_N is zero, contradicting the definition of N . Hence $N \leq D + 1$.

For decidability, substitute the bounded ansatz into $\delta h = RG$ and collect monomials. This gives (7.2). Process the equations in descending total degree. At each stage the higher-degree coefficients are already known, so the equation for a_β has the form

$$a'_\beta = r_\beta(x),$$

where r_β is rational and affine-linear in the finitely many integration constants introduced at earlier stages. A rational function over an algebraically closed field of characteristic zero has a rational antiderivative if and only if its residue at every finite pole vanishes: its polynomial part and all terms of pole order at least two integrate rationally, whereas a nonzero simple-pole term produces a logarithm. Each residue condition is a linear equation in the constants already introduced. Keep these constants as formal parameters; if the accumulated linear system is inconsistent, no rational coefficient solution exists. Otherwise, rational partial fractions produce the next coefficient and introduce one new integration constant. There are only finitely many multi-indices of degree at most $D + 1$, so the procedure terminates after finitely many exact rational operations and linear-algebra steps. It returns either an explicit h or an exact inconsistency certificate. The degree-bound argument is valid over $\mathbb{C}$; the algorithm is effective over the stated exact field k . $\square$

Remark 7.4. If the input functions are initially defined over a non-algebraically-closed effective field k_0 , one may perform the factorization and residue computation in a finite splitting field for the finitely many P_i . Since the coefficient equations are linear and invariant under the relevant Galois action, a solution over a finite normal extension can be averaged over the Galois group (or descended by trace) to obtain a solution over $k_0(x)$ when the right-hand side is defined over k_0 . The algebraically closed formulation above avoids hiding this factorization step. It is not correct to call $\mathbb{Q}$ algebraically closed; $\overline{\mathbb{Q}}$, with exact algebraic-number representations, is the intended example.

Remark 7.5. The full-rank divisor hypothesis is equivalent to independence of the logarithmic divisor vectors. It also implies that the monodromy of $(Y_1, \dots, Y_r)$ contains a full-rank lattice of logarithmic periods. In particular, the substitution map

$$k(x)[Y_1, \dots, Y_r] \longrightarrow \text{local analytic germs}, \quad Q(x, Y) \longmapsto Q(x, Y(x)),$$

is injective. Indeed, if the image vanished, continuation around the full-rank lattice and evaluation at a generic ordinary point would make a polynomial vanish on a Zariski-dense lattice translate; all its coefficients would then vanish. Thus the formal RCC equation is equivalent to analytic closure in the specified polynomial-logarithmic grammar.

7.3 Reduction of dependent logarithmic channels

Proposition 7.6 (Reduction of dependent logarithmic channels). *Let M be the divisor matrix of $P_1, \dots, P_r \in k(x)^*$ and suppose $\text{rank } M = s$. There exists a unimodular matrix $V \in \text{GL}_r(\mathbb{Z})$ such that*

$$MV = (M_0 \ 0),$$

where M_0 has full column rank s . Define

$$\tilde{P}_j = \prod_{i=1}^r P_i^{V_{ij}}, \quad \tilde{Y}_j = \log \tilde{P}_j = \sum_{i=1}^r V_{ij} Y_i.$$

For $j > s$, the rational function $\tilde{P}_j$ has zero divisor and is therefore a nonzero constant $c_j \in k^*$. On a fixed branch, $\tilde{Y}_j = \log c_j$ is constant. Consequently every polynomial $G(Y_1, \dots, Y_r)$ reduces to a polynomial in the s active logarithmic channels $\tilde{Y}_1, \dots, \tilde{Y}_s$, with coefficients in the constant field enlarged by the finitely many values $\log c_j$. The degree-bound theorem then applies to the active channels. Its algorithmic form applies whenever exact arithmetic and equality testing are supplied for the enlarged constant field. If $s = 0$, the logarithmic factor is constant and the problem reduces to rational antidifferentiation.

Proof. Smith normal form for the integer matrix M supplies unimodular row and column operations, and in particular a unimodular V for which the final $r - s$ columns of MV vanish. Hence each corresponding $\tilde{P}_j$ has valuation zero at every finite point. A rational function on the projective line with no finite zeros or poles is constant: its divisor has degree zero, so its valuation at infinity is also zero. Thus $\tilde{P}_j = c_j \in k^*$. Since V is unimodular, the transformation from Y to $\tilde{Y}$ is an invertible integer linear substitution in the polynomial ring. Substituting the constant inactive channels gives the stated reduction. $\square$

7.4 A two-channel RCC obstruction

Example 7.7 (A finite non-closure certificate). Take

$$P_1(x) = x, \quad P_2(x) = x - 1, \quad R(x) = 1, \quad G(Y_1, Y_2) = Y_1 Y_2.$$

The divisor matrix has rows $(1, 0)$ and $(0, 1)$, so it has full rank. Since $D = 2$, Theorem 7.3 says any polynomial-logarithmic closure observable has degree at most 3. Write its coefficients as $a_{ij}(x)Y_1^i Y_2^j$. The degree-three equations make $a_{30}, a_{21}, a_{12}, a_{03}$ constants. The degree-two coefficient equations for Y_1^2 , $Y_1 Y_2$, and Y_2^2 are

$$\begin{aligned} a'_{20} + \frac{3a_{30}}{x} + \frac{a_{21}}{x-1} &= 0, \\ a'_{11} + \frac{2a_{21}}{x} + \frac{2a_{12}}{x-1} &= 1, \\ a'_{02} + \frac{a_{12}}{x} + \frac{3a_{03}}{x-1} &= 0. \end{aligned}$$

Taking residues at 0 and 1 in the first and third equations gives $a_{30} = a_{21} = a_{12} = a_{03} = 0$. The middle equation then gives $a_{11} = x + C$. The same equations show that a_{20} and a_{02} are constants. The degree-one equations for Y_1 and Y_2 are

$$a'_{10} + \frac{2a_{20}}{x} + \frac{a_{11}}{x-1} = 0, \quad a'_{01} + \frac{a_{11}}{x} + \frac{2a_{02}}{x-1} = 0.$$

Their residues at 0 and 1 first force $a_{20} = a_{02} = 0$, and then force $a_{11}(1) = 0$ and $a_{11}(0) = 0$. Since $a_{11} = x + C$, these conditions require both $C = -1$ and $C = 0$, a contradiction. Therefore no polynomial in $(\log x, \log(x-1))$ with rational-function coefficients is a primitive of $\log x \log(x-1)$. The degree bound makes this a finite, degree-independent non-closure certificate in the declared grammar.

7.5 An all-degree residue obstruction for one logarithm

Let

$$P(x) = c \prod_{i=1}^s (x - a_i)^{m_i}, \quad c \neq 0,$$

where the a_i are distinct and the m_i are positive integers, and put $L = \log P$ on a simply connected domain avoiding the roots. Since P is nonconstant, its logarithmic monodromy contains a nonzero period subgroup; polynomial substitution in L is therefore injective in the polynomial-logarithmic grammar.

Proposition 7.8 (Polynomial-logarithmic closure criterion). *Fix $d \geq 1$. There exists a primitive of L^d of the form*

$$h(x, L) = \sum_{k=0}^N a_k(x) L^k, \quad a_k(x) \in \mathbb{C}(x),$$

for some finite N , if and only if P has exactly one distinct root ($s = 1$).

Proof. The closure equation is

$$h_x + \frac{P'}{P} h_L = L^d, \quad \frac{P'}{P} = \sum_{i=1}^s \frac{m_i}{x - a_i}.$$

Equating powers of L gives

$$a'_j + (j+1) \frac{P'}{P} a_{j+1} = \begin{cases} 1, & j = d, \\ 0, & j \neq d, \end{cases} \quad a_{N+1} = 0. \quad (7.3)$$

If $N < d$, the coefficient of L^d is impossible. If $N > d + 1$, start with $a'_N = 0$, so a_N is constant. For each $j > d$ descending from $N - 1$ to $d + 1$, equation (7.3) has zero right-hand side. Once a_{j+1} is constant, the residue at any a_i of

$$a'_j = -(j+1) \frac{P'}{P} a_{j+1}$$

forces $a_{j+1} = 0$, because a rational derivative has zero residue at every finite pole and every m_i is nonzero. This induction eliminates all coefficients above degree $d + 1$. The coefficient at L^{d+1} (if present) then makes a_{d+1} constant, and the L^d equation

$$a'_d + (d+1) \frac{P'}{P} a_{d+1} = 1$$

has residues forcing $a_{d+1} = 0$. Thus $a'_d = 1$ and $a_d = x + C$.

The coefficient of L^{d-1} is now

$$a'_{d-1} = -d \frac{P'}{P} (x + C).$$

The residue at a_i of the right side is $-dm_i(a_i + C)$. Since a rational derivative has zero residue, every a_i must equal $-C$. This is possible exactly when P has one distinct root.

Conversely, if $P = c(x - a)^m$, then $L' = m/(x - a)$, and direct differentiation shows that

$$h(x, L) = (x - a) \sum_{k=0}^d \frac{d!}{(d - k)!} (-m)^k L^{d-k}$$

satisfies $h' = L^d$. This proves the equivalence. $\square$

Remark 7.9. The criterion is deliberately grammar-relative. It classifies primitives in the finite polynomial-in- L , rational-in- x grammar. It does not exclude representations involving additional logarithmic channels, algebraic extensions, shifts, or dilations.

8 Canonical products and spectral memory: the Bessel example

For $\nu \geq 0$, the Bessel function has the canonical product

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1)} \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{j_{\nu,m}^2} \right),$$

where $j_{\nu,m}$ are its positive zeros; see the NIST Digital Library of Mathematical Functions [18]. It satisfies

$$x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu = 0.$$

Set

$$u = \frac{J_\nu'}{J_\nu}, \quad w = u - \frac{\nu}{x}, \quad \sigma_n(\nu) = \sum_{m=1}^{\infty} j_{\nu,m}^{-2n}.$$

The Riccati equation for u gives

$$w' = -1 - \frac{2\nu + 1}{x} w - w^2. \tag{8.1}$$

The canonical product gives, near the origin,

$$w(x) = -2 \sum_{n=1}^{\infty} \sigma_n(\nu) x^{2n-1}.$$

Coefficient comparison in (8.1) yields the classical Rayleigh-sum recurrences

$$\sigma_1(\nu) = \frac{1}{4(\nu + 1)}, \quad (\nu + n)\sigma_n(\nu) = \sum_{k=1}^{n-1} \sigma_k(\nu)\sigma_{n-k}(\nu), \quad n \geq 2. \tag{8.2}$$

These convolution identities are classical, going back to Kishore and later Riccati treatments such as Gupta and Muldoon [19, 20]; they are included as an example, not claimed as a new result. For

$$\Sigma_\nu(z) = \sum_{n \geq 1} \sigma_n(\nu) z^{n-1},$$

recurrence (8.2) is equivalently

$$z \Sigma_\nu'(z) + (\nu + 1) \Sigma_\nu(z) - z \Sigma_\nu(z)^2 = \frac{1}{4}.$$

Thus an infinite canonical product can be audited through a finite nonlinear differential equation for its aggregate zero moments.

9 Novelty, scope, and unresolved problems

The framework separates three levels of assertion:

1. **Representation:** an explicit state system and seed data generate the target function.
2. **Grammar-relative closure:** an explicit observable solves the relevant differential or difference coboundary equation.
3. **Minimality:** no smaller realization exists within the fixed grammar. This requires a separate lower-bound argument, such as monodromy, observability, or a rank theorem.

A finite state system is not automatically minimal. A failed search within a bounded polynomial ansatz is not a proof of non-closure in all polynomial degrees. A differential-transcendence result is not a proof of non-representability in shift or dilation languages. A functional recurrence without normalization may encode an arbitrary periodic or scale-invariant seed.

The RCC equation is a Lie-derivative or difference-coboundary equation; its novelty is not asserted. Similarly, the primitive-order dichotomy is a direct differential-module containment criterion, and the projective lift uses standard logarithmic-derivative identities. The apolar derivative-space dimension is classical inverse-system data, and the Bessel Rayleigh recurrence is classical. The specific results advanced here are the stated interaction of logarithmic monodromy with apolar rank, the divisor-specialized total-degree bound $D + 1$ together with its finite rational-residue RCC algorithm, and the all-degree one-logarithm classification. These are narrower than general symbolic integration or general nonlinear realization theorems.

The polynomial-logarithmic decision procedure belongs to the broad algorithmic lineage of Risch–Rothstein–Davenport methods, rational antidifferentiation, and residue criteria [1, 2, 3, 4, 5, 13, 14]. Du and Raab derive rigorous degree bounds through complete reduction systems, including examples involving Airy functions and complete elliptic integrals [7]. The present theorem is a divisor-specialized result for polynomial logarithmic extensions of $k(x)$: under full logarithmic-divisor rank it gives the explicit total-degree bound $D + 1$ and reduces closure to a finite rational-residue system. It is not a replacement for general Risch–Norman or complete-reduction algorithms.

The main open technical problems are:

1. extending the degree-bounded polynomial-logarithmic algorithm to algebraic state extensions and genuinely nonlinear update fields;
2. developing effective exact arithmetic for the enlarged constants $\log c_j$ that arise when dependent logarithmic channels are reduced;
3. computing minimal realizations for affine shift memories on polynomial state spaces, including exact invariant-subspace algorithms;
4. developing a unified treatment of seed complexity and normalization for mixed differential–shift–dilation systems;
5. proving a precise connection between commutative apolar memory and noncommutative Hankel memory after analytic relations among iterated integrals are imposed;
6. constructing benchmark families that compare state dimension, coefficient complexity, update complexity, seed complexity, and operator cost.

10 Conclusion

Anchored Accumulation Calculus begins with an exact anchored exponential-mode representation of non-Markovian memory under a declared input convention. Its grammar-relative extension asks a broader question: is a proposed accumulation a coboundary in the admissible observable algebra of the current recursive state? RCC makes this question explicit through a zero-residual condition. Differential, shift, dilation, and hybrid operator languages can give different answers for the same source, and the answer is meaningful only when the coefficient field, allowed updates, output maps, observables, and seed data are recorded.

The principal mathematical results established here are the primitive-order dichotomy, the projective-state representation and differential-algebraicity equivalence for logarithmic primitives, the monodromy/apolar order formula, the full-rank divisor degree bound and finite RCC decision procedure, and the all-degree one-logarithm criterion. These results provide an exact and testable starting point for extended AAC. Minimality outside the stated hypotheses, broader operator grammars, and general nonlinear closure remain open.

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