

# An exponential bound for generalized numerical semigroups of fixed dimension

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## Abstract

Let  $n_{g,d}$  be the number of cofinite additive submonoids of  $\mathbb{N}_0^d$  with complement of cardinality  $g$ . For each fixed  $d$ , we prove  $n_{g,d} = 2^{O_d(g)}$ , answering Question 49 of Li. The main estimate bounds the entire coordinatewise downclosure of the complement by  $4^{d+1}g$ . We apply classical maximal-average estimates to the local density supplied by reflection, and give an elementary finite proof using Abel summation. A closed-word encoding of downsets gives an explicit exponential bound. The core argument is formalized in Lean. A separate, not yet formalized section gives a smaller upper growth constant using elementary downset estimates weaker than known results in multidimensional partition theory.

## 1 The counting problem

A generalized numerical semigroup of dimension  $d \geq 1$  is a submonoid  $S \subseteq \mathbb{N}_0^d$  with finite complement  $G(S) = \mathbb{N}_0^d \setminus S$ . Thus  $0 \in S$  and  $S + S \subseteq S$ . Its genus is  $g(S) = |G(S)|$ . We count submonoids as subsets of the fixed coordinate lattice, and write

$$n_{g,d} = |\{S \subseteq \mathbb{N}_0^d : S \text{ is a generalized numerical semigroup and } g(S) = g\}|.$$

In particular, coordinates are labelled; this is not a count of isomorphism classes. Dimension is fixed in assertions about growth as  $g \rightarrow \infty$ . For foundational algorithms and asymptotics, see Failla–Peterson–Utano [3].

Li [7, Theorem 2] proved  $n_{g,d}^{1/g} < (2e + o(1))(\log g)^{d-1}$  and asked in [7, Question 49, p. 23] whether  $n_{g,d}^{1/g}$  is bounded. The following theorem answers this question. It does not assert convergence of these roots or identify their optimal upper bound.

**Theorem 1.** *For integers  $d, g \geq 1$ , put  $C_d = 4^{d+1}$  and  $B_d = 8d^2$ . Then*

$$n_{g,d} \leq \frac{B_d^{C_d g} - 1}{B_d - 1} < B_d^{C_d g}. \quad (1)$$

*In particular,  $n_{g,d} = 2^{O_d(g)}$  for each fixed  $d$ . For  $g = 0$ , one has  $n_{0,d} = 1$ .*

The gap downclosure and the reflection injection used below already appear in Cisto et al. [1, pp. 3–4 of the preprint], and the half-prism estimate is used by Li [7, Proposition 35]. We retain its local density information to obtain a dimension-only linear bound on the entire gap downclosure and then apply it to genus enumeration. The analytic input is classical maximal-average theory; see Hardy–Littlewood [5] and Jessen–Marcinkiewicz–Zygmund [6]. All estimates needed for Theorem 1 are proved below using finite sums.

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**Formalization scope.** Theorem 1, Lemmas 2, 3, and 6, and Proposition 4 have Lean proofs with Lean 4.24.0 and a pinned mathlib dependency. Finiteness of the counted family is established before its cardinality is used. Remark 5 and Section 5 are outside the formalization. The accompanying theorem correspondence specifies the declarations and boundary cases. The repository for the formalization is <https://github.com/0liverds321/semigroup-genus-growth-lean>.

## 2 A finite maximal-average estimate

**Lemma 2.** *Let  $f_1, \dots, f_L$  be nonnegative real numbers, where  $L \geq 1$ . Set*

$$F_k = \sum_{j=1}^k f_j, \quad a_k = \max_{k \leq n \leq L} \frac{F_n}{n} \quad (1 \leq k \leq L).$$

*Then*

$$\sum_{k=1}^L a_k^2 \leq 4 \sum_{k=1}^L f_k^2. \quad (2)$$

*Proof.* Set  $a_{L+1} = 0$ . The sequence  $(a_k)$  is nonnegative and nonincreasing. If  $a_k > a_{k+1}$ , a maximizing index must be  $k$ , since every larger index also occurs in the maximum for  $a_{k+1}$ . Thus  $ka_k = F_k$  at every strict drop. At other indices,  $a_k - a_{k+1} = 0$ . Weighted telescoping and Abel summation give

$$\begin{aligned} A &:= \sum_{k=1}^L a_k^2 = \sum_{k=1}^L k(a_k^2 - a_{k+1}^2) \\ &\leq 2 \sum_{k=1}^L ka_k(a_k - a_{k+1}) = 2 \sum_{k=1}^L F_k(a_k - a_{k+1}) = 2 \sum_{j=1}^L f_j a_j. \end{aligned}$$

Put  $E = \sum_j f_j^2$  and  $H = \sum_j f_j a_j$ . Squared Cauchy–Schwarz gives  $H^2 \leq EA$ , so  $A^2 \leq 4H^2 \leq 4EA$ . If  $A = 0$  the conclusion is immediate; otherwise division by  $A$  proves (2).  $\square$

Fix integers  $d, L \geq 1$  and use the common finite cube

$$Q = \{0, \dots, L-1\}^d, \quad L \geq 1.$$

For nonnegative  $f : Q \rightarrow \mathbb{R}$ , define the coordinate operators

$$(A_i f)(x) = \max_{x_i < n \leq L} \frac{1}{n} \sum_{j=0}^{n-1} f(x_1, \dots, x_{i-1}, j, x_{i+1}, \dots, x_d). \quad (3)$$

Applying Lemma 2 on each line and summing gives

$$\sum_{x \in Q} (A_i f(x))^2 \leq 4 \sum_{x \in Q} f(x)^2, \quad \sum_{x \in Q} (A_1 \cdots A_d f(x))^2 \leq 4^d \sum_{x \in Q} f(x)^2. \quad (4)$$

The rightmost operator acts first. The common cube is sufficient for the application below; the origin boxes inside it may have unequal side lengths.

**Lemma 3.** *For  $h \in Q$ , let  $R_h = \prod_{i=1}^d \{0, \dots, h_i\}$ . For every nonnegative  $f$  on  $Q$  and every  $x \in R_h$ ,*

$$(A_1 \cdots A_d f)(x) \geq \frac{1}{|R_h|} \sum_{y \in R_h} f(y). \quad (5)$$

*Proof.* In  $A_d$ , select the admissible prefix of length  $h_d + 1$ . The resulting lower bound holds at every point whose last coordinate is at most  $h_d$ . Next select the prefix of length  $h_{d-1} + 1$  in  $A_{d-1}$  and average the lower bound already obtained. Continue through coordinate 1. At each step, every point being averaged satisfies the coordinate restrictions needed at the previous steps. The final expression is the average over  $R_h$ . This proves box domination separately from the norm estimate; no commutation of coordinate operators is needed.  $\square$

### 3 The size of the gap downclosure

Use the coordinatewise order on  $\mathbb{N}_0^d$ . For a finite set  $G \subseteq \mathbb{N}_0^d$ , define

$$\downarrow G = \{x \in \mathbb{N}_0^d : x \leq h \text{ for some } h \in G\}.$$

This is a finite downset. In the notation of Cisto et al. [1], it is  $C(S)$  when  $G = G(S)$ , and its cardinality is  $c(S)$ .

**Proposition 4.** *Let  $S \subseteq \mathbb{N}_0^d$  be a generalized numerical semigroup of genus  $g \geq 1$ . Then*

$$|\downarrow G(S)| \leq 4^{d+1}g. \quad (6)$$

*Proof.* Write  $G = G(S)$  and  $D = \downarrow G$ . For each  $h \in G$ , the reflection  $s \mapsto h - s$  maps  $S \cap R_h$  injectively into  $G \cap R_h$ . Here subtraction is restricted to  $s \leq h$ . If both  $s$  and  $h - s$  belonged to  $S$ , closure under addition would give  $h \in S$ . Consequently,

$$|R_h| \leq 2|G \cap R_h| \leq 2g. \quad (7)$$

For every coordinate  $i$ , the coordinate line in  $R_h$  gives  $h_i + 1 \leq |R_h| \leq 2g$ . Hence the common cube with side  $L = 2g$  contains  $G$ ,  $D$ , and every  $R_h$  for  $h \in G$ . This coordinate bound also proves finiteness of the genus- $g$  family: its gap sets lie among the subsets of a fixed finite cube, and each gap set determines its submonoid.

Put  $f = \mathbf{1}_G$  on this cube. Every  $x \in D$  lies in  $R_h$  for some  $h \in G$ . By Lemma 3 and (7),  $(A_1 \cdots A_d f)(x) \geq 1/2$ . Therefore (4) gives

$$\frac{|D|}{4} \leq \sum_{x \in Q} (A_1 \cdots A_d f(x))^2 \leq 4^d \sum_{x \in Q} f(x)^2 = 4^d g.$$

$\square$

The qualitative bound  $|\downarrow G| = O_d(|G|)$  also follows from classical fixed-threshold indicator halo bounds. Extend  $\mathbf{1}_G$  by zero to  $\mathbb{Z}^d$ . Reflection gives

$$\downarrow G \subseteq \{x : \widetilde{M}_S \mathbf{1}_G(x) > 1/3\},$$

where  $\widetilde{M}_S$  is the discrete strong maximal operator. Its Tauberian constant at this fixed threshold is finite; see Hagelstein–Parissis [4, Lemma 1]. The finite argument above gives the explicit constant  $4^{d+1}$  without using measure theory.

*Remark 5.* The constant in (6) is not intended to be sharp. The one-dimensional bound  $|\downarrow G(S)| \leq 2g$  does not persist unchanged in dimension two. For example, the complement of

$$G = \{(1, 0), (2, 0), (2, 1), (5, 0)\}$$

is a submonoid of  $\mathbb{N}_0^2$ , whereas  $|G| = 4$  and  $|\downarrow G| = 9$ . Every decomposition of a point of  $G$  into two nonzero nonnegative lattice points has at least one summand in  $G$ , which verifies closure of the complement.

## 4 An explicit exponential bound

**Lemma 6.** *For  $N \geq 1$ , the number of finite downsets in  $\mathbb{N}_0^d$  of cardinality  $N$  is at most  $(2d)^{2(N-1)}$ .*

*Proof.* Every nonempty downset contains the origin. We show by induction on  $N$  that it is the visited set of a closed word of  $2(N-1)$  unit directions, starting at the origin. The singleton origin has the empty word. Otherwise choose a maximal nonzero point  $v$  of the downset  $D$ . Deleting  $v$  leaves a downset  $D'$  containing the origin. Choose a coordinate  $i$  with  $v_i > 0$ . The predecessor  $u = v - e_i$  belongs to  $D'$ . By induction, a closed word visits exactly  $D'$ . Split it at a visit to  $u$  and insert the two moves  $+e_i, -e_i$ . The resulting word is closed, visits exactly  $D$ , and has the required length.

For each downset choose one such word. Evaluating its directions from the origin recovers the visited set, so this choice is injective. There are  $(2d)^{2(N-1)}$  words of that length. Words that do not describe a valid downset traversal may simply be included in this upper bound.  $\square$

Stronger bounds for lower sets are known. In particular, Dai et al. [2, Theorem 1.4] give  $d^{N-1}$  for  $d \geq 2$ ; their Theorem 1.5 gives subexponential growth for fixed dimension. The elementary word bound suffices here and is the one formalized. The deletion-and-excursion proof above is also the proof used in Lean; a depth-first traversal of a spanning tree gives another word encoding.

*Proof of Theorem 1.* For genus  $g \geq 1$ , Proposition 4 places the gap downclosure  $D$  among the downsets of sizes  $1 \leq N \leq C_d g$ . For each  $D$ , at most  $2^{N-1}$  subsets can be gap sets, since the origin cannot be a gap. Including subsets that fail additive closure only enlarges the upper bound. Lemma 6 gives

$$n_{g,d} \leq \sum_{N=1}^{C_d g} (2d)^{2(N-1)} 2^{N-1} = \sum_{k=0}^{C_d g-1} B_d^k = \frac{B_d^{C_d g} - 1}{B_d - 1} < B_d^{C_d g},$$

where the last inequality uses  $B_d \geq 2$ . Thus  $n_{g,d} < (B_d^{C_d})^g$ . An empty complement forces  $S = \mathbb{N}_0^d$ , giving  $n_{0,d} = 1$ .  $\square$

## 5 A smaller upper growth constant

*This section is proved mathematically below but is not yet included in the Lean formalization.* The preceding proof already settles boundedness of the genus roots. The following separate downset estimate improves the resulting constant. These are established topics in multidimensional partition theory: downsets in  $\mathbb{N}_0^d$  are  $(d-1)$ -dimensional partition diagrams. Oganesyan [8, Lemma 1 and Theorem 1] proves stronger maximal-element and counting estimates, including the fixed- $d$  bound  $\log p_d(N) = O_d(N^{1-1/d})$  for the number  $p_d(N)$  of downsets of size exactly  $N$ . We include a short elementary proof of weaker bounds sufficient here.

**Lemma 7.** *For each  $d \geq 1$  there is a constant  $a_d$  such that a nonempty finite downset  $D \subseteq \mathbb{N}_0^d$  of size  $N$  has at most  $a_d N^{1-1/d}$  maximal elements. The number  $P_d(M)$  of nonempty downsets of size at most an integer  $M \geq 1$  satisfies*

$$P_d(M) \leq \exp(O_d(M^{1-1/d} \log(M+2))). \quad (8)$$

*Proof.* For  $d = 1$  a nonempty downset is an interval with one maximal element, and  $P_1(M) = M$ . For the induction step let  $d \geq 2$  and  $L = \lceil N^{1/d} \rceil$ . Every point of  $D$  has some coordinate

less than  $L$ ; otherwise its down-box would have at least  $(L+1)^d > N$  points. For coordinate  $i$  and  $0 \leq k < L$ , let  $D_{i,k}$  be the  $(d-1)$ -dimensional downset obtained by taking the slice  $x_i = k$ , and put  $N_{i,k} = |D_{i,k}|$ . A maximal point in that slice projects to a maximal point of  $D_{i,k}$ . Thus the induction hypothesis gives

$$|\text{Max } D| \leq a_{d-1} \sum_{i=1}^d \sum_{\substack{0 \leq k < L \\ N_{i,k} > 0}} N_{i,k}^{(d-2)/(d-1)}.$$

For  $d > 2$ , concavity and  $\sum_{k < L} N_{i,k} \leq N$  bound each inner sum by  $L^{1/(d-1)} N^{(d-2)/(d-1)}$ . For  $d = 2$ , the sum counts nonempty slices and is at most  $L$ , yielding the same bound without a  $0^0$  convention. Since  $L \leq 2N^{1/d}$ ,

$$|\text{Max } D| \leq d a_{d-1} 2^{1/(d-1)} N^{1-1/d}.$$

For example,  $a_d = 2^{d-1} d!$  suffices.

Every downset of size at most  $M$  lies in  $\{0, \dots, M-1\}^d$ : a coordinate of size  $M$  would force at least  $M+1$  points on a line below it. It is determined by its maximal elements, whose number is at most  $K = \lceil a_d M^{1-1/d} \rceil$ . Consequently, for  $d \geq 2$ ,

$$P_d(M) \leq \sum_{k=0}^K \binom{M^d}{k} \leq (K+1)(M^d)^K,$$

with binomial coefficients zero when  $k > M^d$ . Taking logarithms proves (8).  $\square$

**Corollary 8.** *For every fixed  $d \geq 1$ , with  $C_d = 4^{d+1}$ ,*

$$\limsup_{g \rightarrow \infty} n_{g,d}^{1/g} \leq \frac{C_d^{C_d}}{(C_d - 1)^{C_d - 1}} < e C_d. \quad (9)$$

*Proof.* There are at most  $P_d(C_d g)$  possible gap downclosures, and any one admits at most  $\binom{C_d g}{g}$  subsets of size  $g$ . Hence

$$n_{g,d} \leq P_d(C_d g) \binom{C_d g}{g}.$$

Lemma 7 gives  $\log P_d(C_d g) = o(g)$  for fixed  $d$ . For any fixed integer  $C > 1$ , Stirling's formula gives

$$\lim_{g \rightarrow \infty} \left( \frac{Cg}{g} \right)^{1/g} = \frac{C^C}{(C-1)^{C-1}}.$$

Finally,  $C^C / (C-1)^{C-1} = C(1 + 1/(C-1))^{C-1} < eC$ .  $\square$

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