

A HOOK-EXCESS INEQUALITY FOR THE SUPPORT OF KRONECKER PRODUCTS OF SYMMETRIC GROUP CHARACTERS

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ABSTRACT. We prove that if the Kronecker coefficient $g(\lambda, \mu, \nu)$ of three partitions of n is positive, then the number of cells of ν outside its first two rows and first two columns is at most the sum of the numbers of cells of λ and of μ outside their first hooks. For two hooks this is the classical theorem of Remmel and Rosas, which our induction uses as its base case; we also give a computer-assisted proof of that base case (an exact rational certificate checked by an independent program) and a Lean 4 formalisation for the Specht modules of every symmetric group, whose only non-standard ingredient is that one certificate check is done by Lean’s compiled evaluation (`native_decide`). The same induction, with a containment theorem of Berele and Regev as base case, gives a family of inequalities indexed by pairs of positive integers (Theorem 6.3); this part is a short written argument that has not been formalised and has not been reviewed. As consequences we bound the Durfee size of the constituents and obtain a necessary condition for a partition to have a universal tensor square. We report exhaustive computations for $n \leq 21$ and compare the inequality with Dvir’s bounds, the depth bound and the vanishing criterion of Young and Vallejo: it certifies about 2–4% of the classes of zeros for $n \leq 15$, but only about 0.2–0.3% beyond the Young–Vallejo criterion, almost all of these lying inside the Kronecker cone, so that no homogeneous linear inequality valid on the support can detect them. For $n = 23$ we exhibit twelve classes of triples with $g = 0$, computed exactly, that satisfy the hypothesis of the inequality and are certified by none of the criteria we tested; we have no general description of such triples. We do not claim novelty beyond the searches we list, and the proofs have not undergone formal peer review.

1. INTRODUCTION

Background. The irreducible complex characters χ^λ of the symmetric group S_n are indexed by partitions λ of n . The *Kronecker coefficient* $g(\lambda, \mu, \nu)$ is the multiplicity of χ^ν in the product $\chi^\lambda \chi^\mu$, that is, in the tensor product of the corresponding representations. Finding a combinatorial rule for these numbers is a well-known open problem, and even deciding when $g(\lambda, \mu, \nu) > 0$ is hard in general. The question has attracted attention for two reasons. First, it is the missing piece in the classical theory of the representations of S_n and GL_n ; second, it enters geometric complexity theory, where positivity and growth of Kronecker coefficients are used to study lower bounds. A concrete instance is the *Saxl conjecture*: the tensor square $\chi^\rho \chi^\rho$ of the staircase $\rho = (k, k-1, \dots, 1)$ contains every irreducible character of S_n , $n = \binom{k+1}{2}$ (see Pak, Panova and Vallejo [3] for an account and for partial results).

Although no general rule is known, the *support* $\{\nu : g(\lambda, \mu, \nu) > 0\}$ is understood in several special cases. Dvir [11] determined the largest first row and the largest number of parts of a constituent. Bessenrodt and Kleshchev [9] bounded the size of the first hook of a constituent, and Bessenrodt and Behns [8] studied the Durfee size. A necessary condition in terms of the first rows (the *depth bound* $|d_\lambda - d_\mu| \leq d_\nu \leq d_\lambda + d_\mu$ with $d_\lambda = n - \lambda_1$) goes back to James and Kerber [19] (see also [20]). Vallejo [17] observed that $g(\lambda, \mu, \nu)$ is at most the multiplicity of the permutation character of the Young subgroup S_ν in $\chi^\lambda \chi^\mu$, which yields a vanishing criterion, and he proved reduction and vanishing theorems when the length of one partition is the product of the lengths of the other two. When one of the three partitions is a near-hook $(a, b, 1^c)$, a formula for g is given by Campbell [18]. For products of two hooks, Remmel [13] and Rosas [15] gave explicit formulas; one consequence is that every constituent lies in a double hook, that is, $\nu_3 \leq 2$. Berele and Regev [7] proved a general containment result for characters

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in hook-type regions (and Vallejo [16], proving a conjecture of Berele and Imbo [6], showed that for two partitions in $H(k, k)$ some constituent lies again in $H(k, k)$), and Blasiak [2] and Ballantine and Hallahan [1] studied the product of a hook with other shapes.

The main result. For a partition λ and an integer $k \geq 0$ let

$$\varepsilon_k(\lambda) = \#\{(i, j) \in \lambda : i > k, j > k\}$$

be the number of cells of the Young diagram of λ lying below row k and to the right of column k . Thus $\varepsilon_1(\lambda)$ counts the cells outside the first hook (the *hook excess*), and $\varepsilon_2(\lambda)$ counts the cells outside the first two rows and the first two columns (the *double-hook excess*); see Figure 1. In particular, $\varepsilon_1(\lambda) = 0$ means that λ is a hook, and $\varepsilon_2(\nu) = 0$ means that $\nu_3 \leq 2$.

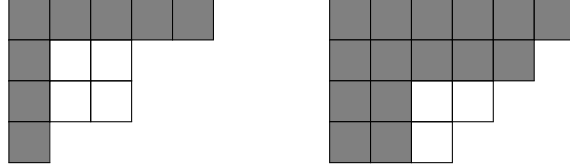


FIGURE 1. Left: $\lambda = (5, 3, 3, 1)$. The shaded cells form the first hook and the four white cells give $\varepsilon_1(\lambda) = 4$. Right: $\nu = (6, 5, 4, 3)$. The shaded cells form the first two rows and columns and the three white cells give $\varepsilon_2(\nu) = 3$.

Theorem 1.1 (Hook-excess inequality). *Let λ, μ, ν be partitions of n . If $g(\lambda, \mu, \nu) > 0$, then*

$$\varepsilon_2(\nu) \leq \varepsilon_1(\lambda) + \varepsilon_1(\mu).$$

When λ and μ are hooks the right-hand side is zero and Theorem 1.1 becomes the Remmel–Rosas statement. Our proof is an induction on n that reduces the general case to this one, using only the branching rule and Frobenius reciprocity. Since g is symmetric in its three arguments, the inequality holds for every permutation of (λ, μ, ν) . Equality occurs with $\varepsilon_2(\nu) > 0$ (Section 3); Theorem 1.1 is not a consequence of the bounds of Dvir or of Bessenrodt and Kleshchev (Theorem 7.1), and it is complementary to the theorem of Berele and Regev (Section 7).

Consequences. Let $d(\nu)$ be the Durfee size of ν , the largest d with $(d, d) \in \nu$.

Corollary 1.2. *If $g(\lambda, \mu, \nu) > 0$, then $d(\nu) \leq 2 + \sqrt{\varepsilon_1(\lambda) + \varepsilon_1(\mu)}$.*

Call a partition $\lambda \vdash n$ *universal* if $g(\lambda, \lambda, \nu) > 0$ for all $\nu \vdash n$. The Saxl conjecture says that the staircase is universal. Put $R_{\max}(n) = \max_{\nu \vdash n} \varepsilon_2(\nu)$; this is of order $n - 4\sqrt{n}$.

Corollary 1.3. *If $\lambda \vdash n$ is universal, then $2\varepsilon_1(\lambda) \geq R_{\max}(n)$.*

Thus a universal partition must have a large part of its diagram outside the first hook, and hooks are never universal for $n \geq 9$. The condition is not the whole story: rectangles may satisfy it but fail for a different reason.

Proposition 1.4. *If $\lambda \vdash n$ is a rectangle and $n \geq 2$, then $g(\lambda, \lambda, (n-1, 1)) = 0$. In particular, no rectangle is universal.*

This is not new. The formula $g(\lambda, \lambda, (n-1, 1)) = c(\lambda) - 1$, with $c(\lambda)$ the number of corners of λ , is Theorem 4.8 of Pak, Panova and Vallejo [3], where it is attributed to Vallejo (see also Saxl and Zisser), and the exclusion of square partitions from the candidates for the tensor square conjecture is their Corollary 4.9. We include the short proof from the branching rule because it uses only the tools of this paper.

Computations suggest that these two necessary conditions together characterise the universal self-conjugate partitions (Theorem 10.1).

Status of the results. We label every claim by the kind of evidence behind it: *paper* (proved by a written argument that we expect a referee to be able to check, possibly using cited published theorems), *computer-assisted* (proved by a finite computation that can be re-checked exactly), *computed* (verified by computation over a finite range; not a proof for all n), *Lean* (formalised in Lean 4 and accepted by it; where the check of the finite certificate uses `native_decide`, this trusts the Lean compiler, as stated in each row) and *conjecture*.

Claim	Status
Theorem 1.1 for all n , using the two-hook theorem of Remmel and Rosas as a cited base case	paper; computed for $n \leq 21$
Theorem 1.1 for the Specht modules of every S_n , formalised in Lean without further hypotheses (Section 9)	Lean: conditional only on the standard axioms and on the correctness of the compiled evaluation (<code>native_decide</code>) of the certificate check; it does not use the Remmel–Rosas theorem. The row “paper” above is unchanged: the written proof still uses the cited two-hook theorem or its replacement by the certificate. Definitions audited only by the author (Section 9); not reviewed by anyone else
Two-hook case from the single fact (*) about S_9 (Theorem 4.1)	paper
Fact (*): no product of two hooks of S_9 contains $(3, 3, 3)$	classical (special case of Remmel–Rosas); computed; computer-assisted proof by an exact certificate (Section 4); Lean: derived for the Specht modules of S_9 from the relations of Theorem 4.2 and the certificate, where the certificate check (<code>check_ok</code>) is done by <code>native_decide</code> , i.e. it trusts the Lean compiler (a pure-kernel evaluation ran out of memory)
Theorems 1.2 and 1.3, Theorem 3.3	paper (Theorem 3.3 also computed for $t \leq 11$)
Theorem 1.4	known [3]; paper
Theorem 6.3 (inequalities indexed by pairs, in particular ε_{2ab})	paper (short argument using [7, Thm. 3.26(a)] as base case); not formalised; not reviewed by anyone else; may already be known. Consistency checks: computed for $(a, b) = (1, 2)$, $25 \leq n \leq 32$
Linear shadow of Theorem 1.1 (Theorem 8.1)	paper, using the semigroup property as a cited fact
Comparison with Dvir’s bound (Section 8)	computed (exact for $n \leq 21$, sampled for $22 \leq n \leq 32$)
Comparison with the Young–Vallejo criterion and with the Kronecker cone, $n \leq 15$ (Section 8.4)	computed (exhaustive for the counts of classes certified by the inequality and not by the criterion; samples for the totals when $n \geq 12$)
Twelve classes of triples for $n = 23$ with $g = 0$ not certified by the criteria tested (Section 8.5)	computed (exact g at $n = 23$ and $g(2\cdot)$ at $n = 46$); no general description
The explicit example Q_e (Section 8.6)	$g = 0$: paper (from Theorem 1.1); not certified by Dvir, depth and length bounds for $e \leq 300$: computed; exact $g = 0$ for $e \leq 8$: computed; Young–Vallejo criterion checked for $e \leq 3$
Theorem 10.1	conjecture, supported by computation

We searched the literature listed in Section 7 and zbMATH Open, and we are not aware of a published statement of Theorem 1.1; we did not have access to MathSciNet. This does not establish novelty, and Section 8 shows that part of the content of Theorem 1.1 is already implied by known bounds. The proofs have not undergone formal peer review.

Outline. Section 2 fixes notation and collects the standard facts used. Section 3 proves Theorem 1.1, Section 4 treats the base case through a finite certificate, and Section 5 proves the corollaries and Theorem 1.4. Section 6 proves a family of inequalities that contains Theorem 1.1. Section 7 compares

the result with the literature, Section 8 with Dvir's inequalities, with the criterion of Young and Vallejo and with the Kronecker cone (including the case $n = 23$ and an explicit example), Section 9 describes the computations and the Lean 4 verification, and Section 10 gives a conjecture, open questions and a generalisation that fails.

2. PRELIMINARIES

We write $\lambda \vdash n$ for a partition of n , identify it with its Young diagram $\{(i, j) : 1 \leq j \leq \lambda_i\}$ in English notation, and write $\ell(\lambda)$ for its number of parts. A *corner* of λ is a removable cell. For a corner a we write $\lambda - a$ for the partition obtained by removing it. The *hook size* is $h(\lambda) = \lambda_1 + \ell(\lambda) - 1$, so that $\varepsilon_1(\lambda) = |\lambda| - h(\lambda)$.

The characters χ^λ of S_n are real, so $\langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle$ and g is symmetric in its three arguments. We use the following standard facts.

- **Branching rule.** $\text{Res}_{S_{n-1}}^{S_n} \chi^\mu = \sum_b \chi^{\mu-b}$, the sum over the corners b of μ (multiplicity free); $\text{Ind}_{S_{n-1}}^{S_n} \chi^\kappa = \sum_{\kappa^+} \chi^{\kappa^+}$, the sum over partitions κ^+ obtained by adding one cell.
- **Frobenius reciprocity.** $\langle \text{Ind } \varphi, \psi \rangle = \langle \varphi, \text{Res } \psi \rangle$.
- **Restriction is a ring homomorphism:** $\text{Res}(\psi_1 \psi_2) = \text{Res } \psi_1 \cdot \text{Res } \psi_2$.
- **Two hooks (Remmel, Rosas).** If $\lambda, \mu \vdash n$ are hooks and $g(\lambda, \mu, \nu) > 0$, then $\nu_3 \leq 2$, i.e. $\varepsilon_2(\nu) = 0$ [13, Thm. 2.1(a)], [15, Thm. 13(2)]; see also [10, Prop. 4.1].

Lemma 2.1. *Let λ, μ, ν be partitions of $n \geq 2$.*

- (i) *If $\varepsilon_1(\lambda) \geq 1$, then λ has a corner a outside the first hook, and every such a satisfies $\varepsilon_1(\lambda - a) = \varepsilon_1(\lambda) - 1$.*
- (ii) *For every corner b of μ , $\varepsilon_1(\mu - b) \leq \varepsilon_1(\mu)$.*
- (iii) *For every corner c of ν , $\varepsilon_2(\nu) \leq \varepsilon_2(\nu - c) + 1$.*

Proof. (i) If $\varepsilon_1(\lambda) \geq 1$, some row has length at least 2 beyond the first row; let i be the last row with $\lambda_i \geq 2$. Then $i \geq 2$, and since $\lambda_{i+1} \leq 1$, the cell $a = (i, \lambda_i)$ is a corner with $i \geq 2$, $\lambda_i \geq 2$. Removing a leaves λ_1 and $\ell(\lambda)$ unchanged (row i keeps $\lambda_i - 1 \geq 1$ cells and $i \geq 2$), so $h(\lambda - a) = h(\lambda)$ and $\varepsilon_1(\lambda - a) = \varepsilon_1(\lambda) - 1$. More generally, any corner $a = (i, j)$ with $i, j \geq 2$ has this property for the same reason.

(ii) If b lies in the first row or first column (and $n \geq 2$), removing it lowers h by exactly 1 and n by 1, so ε_1 is unchanged. If b lies outside the hook, h is unchanged and ε_1 drops by 1.

(iii) ε_2 counts cells of ν in the region $\{i \geq 3, j \geq 3\}$; removing one cell lowers this count by at most 1. $\square$

Lemma 2.2. *Let $g(\lambda, \mu, \nu) > 0$ with $\lambda, \mu, \nu \vdash n$, $n \geq 2$. Then for every corner a of λ there are corners b of μ and c of ν with $g(\lambda - a, \mu - b, \nu - c) > 0$.*

Proof. By the branching rule,

$$\sum_{b,c} g(\lambda - a, \mu - b, \nu - c) = \langle \chi^{\lambda-a} \cdot \text{Res } \chi^\mu, \text{Res } \chi^\nu \rangle_{S_{n-1}}.$$

Since all characters are real and restriction is multiplicative, the right-hand side equals

$$\langle \chi^{\lambda-a}, \text{Res}(\chi^\mu \chi^\nu) \rangle_{S_{n-1}},$$

and by Frobenius reciprocity and the branching rule this equals

$$\langle \text{Ind } \chi^{\lambda-a}, \chi^\mu \chi^\nu \rangle_{S_n} = \sum_{\lambda^+} g(\lambda^+, \mu, \nu),$$

the sum running over partitions λ^+ obtained from $\lambda - a$ by adding a cell. One of them is λ itself, and all terms are nonnegative integers, so the sum is at least $g(\lambda, \mu, \nu) > 0$. Hence some summand $g(\lambda - a, \mu - b, \nu - c)$ on the left is positive. $\square$

3. PROOF OF THEOREM 1.1

We argue by induction on n . Let $g(\lambda, \mu, \nu) > 0$ with $\lambda, \mu, \nu \vdash n$.

Case $\varepsilon_1(\lambda) = \varepsilon_1(\mu) = 0$. Then λ and μ are hooks and the claim is the Remmel–Rosas theorem. This case covers $n = 1$.

Case $\varepsilon_1(\lambda) + \varepsilon_1(\mu) \geq 1$. Then $n \geq 2$. Since g is symmetric in its first two arguments we may assume $\varepsilon_1(\lambda) \geq 1$. Let a be a corner of λ as in Theorem 2.1(i). By Theorem 2.2 there are corners b of μ and c of ν with $g(\lambda - a, \mu - b, \nu - c) > 0$ in S_{n-1} . The induction hypothesis, Theorem 2.1(i) and (ii) give

$$\varepsilon_2(\nu - c) \leq \varepsilon_1(\lambda - a) + \varepsilon_1(\mu - b) \leq (\varepsilon_1(\lambda) - 1) + \varepsilon_1(\mu).$$

Finally Theorem 2.1(iii) yields $\varepsilon_2(\nu) \leq \varepsilon_2(\nu - c) + 1 \leq \varepsilon_1(\lambda) + \varepsilon_1(\mu)$. $\square$

Remark 3.1 (The base case). The base case can be seen directly. For $0 \leq k \leq n - 1$ the hook character $\chi^{(n-k, 1^k)} = \wedge^k V$ (with V the standard representation) is a constituent of $\wedge^k P$, P the permutation module, and $\wedge^k P = \text{Ind}_{S_k \times S_{n-k}}^{S_n} (\text{sgn} \otimes 1)$. By Mackey's formula, $\wedge^k P \otimes \wedge^l P$ is a sum of induced modules whose Frobenius characteristics are $h_i e_{k-i} e_{l-i} h_m$, and by iterated Pieri rules the coefficient of s_ν counts semistandard supertableaux of shape ν with two even and two odd letters. By the hook condition of Berele and Regev [7] no such tableau exists if $\nu_3 \geq 3$. This gives an independent argument for the case of two hooks.

Remark 3.2 (Equality). Equality in Theorem 1.1 does occur with $\varepsilon_2(\nu) > 0$. For example, for $n = 9$, $\lambda = (6, 1^3)$, $\mu = (5, 2, 1, 1)$, $\nu = (3, 3, 3)$ one finds $g(\lambda, \mu, \nu) = 1$, $\varepsilon_1(\lambda) = 0$, $\varepsilon_1(\mu) = 1$ and $\varepsilon_2(\nu) = 1$. Equality cases were found at every n tested, so the right-hand side cannot be lowered by a constant. Theorem 3.3 shows this for every excess.

Proposition 3.3 (Sharpness). *For every $t \geq 1$ and every $n \geq 3t + 6$ there are partitions $\lambda, \mu, \nu \vdash n$ with $g(\lambda, \mu, \nu) > 0$ and $\varepsilon_1(\lambda) + \varepsilon_1(\mu) = \varepsilon_2(\nu) = t$.*

Proof. Put $r = t + 2$, $n_0 = 3r$, $\lambda = (2r, 1^r)$ (a hook), $\mu = (r + 2, 2^{r-2}, 1, 1)$ and $\nu = (3^r)$, all partitions of n_0 . Then $\varepsilon_1(\lambda) = 0$, $\varepsilon_1(\mu) = r - 2 = t$ and $\varepsilon_2(\nu) = r - 2 = t$. Write $\text{hook}_j = (n_0 - j, 1^j)$, so $\lambda = \text{hook}_r$, and let $g_j = g(\text{hook}_j, \mu, \nu)$.

Step 1. Since $h_{n_0-m} e_m = s_{\text{hook}_m} + s_{\text{hook}_{m-1}}$ and $s_\mu * (h_{n_0-m} e_m) = \sum_{\alpha \vdash m, \beta \vdash n_0-m} c_{\alpha\beta}^\mu s_{\alpha'} s_\beta$ (induction from $S_{n_0-m} \times S_m$ of trivial $\otimes$ sign), we get

$$g_m + g_{m-1} = G_m := \sum_{\alpha \vdash m, \beta \vdash n_0-m} c_{\alpha\beta}^\mu c_{\alpha'\beta}^\nu.$$

Because $\nu = (3^r)$ is a rectangle, $c_{\alpha'\beta}^\nu \in \{0, 1\}$, and it equals 1 exactly when $\alpha' \subseteq \nu$ and β is the complement of α' in ν , rotated by 180° ; call it $\beta(\alpha)$. In particular $\alpha_1 \leq r$ and $\beta_1 \leq 3$.

Step 2: $G_m = 0$ for $m < r$. If $c_{\alpha\beta}^\mu \neq 0$ then $\alpha + \beta \supseteq \mu$, so $\alpha_1 + \beta_1 \geq \mu_1 = r + 2$, hence $\alpha_1 \geq r - 1$. Thus $|\alpha| < r$ forces $\alpha = (r - 1)$. Then $\alpha' = (1^{r-1})$ and $\beta(\alpha) = (3, 2^{r-1})$, which has r parts and $\beta_r = 2 > 1 = \mu_r$, so $\beta \not\supseteq \mu$ and $c_{\alpha\beta}^\mu = 0$. Hence $g_m = 0$ for $m < r$ by induction on m , and $g_r = G_r$.

Step 3: $G_r \geq 1$. Take $\alpha = (r - 1, 1)$. Then $\alpha' = (2, 1^{r-2})$ and $\beta(\alpha) = (3, 2^{r-2}, 1) \subseteq \mu$. The skew diagram μ/β consists of the $r - 1$ cells of the first row beyond column 3 and the single cell $(r + 1, 1)$. Filling the row with 1's and the cell with 2 gives a Littlewood–Richardson filling of content α , so $c_{\beta\alpha}^\mu \geq 1$. All terms of G_r are non-negative, so $g(\lambda, \mu, \nu) = g_r = G_r \geq 1$.

Step 4. For $n > n_0$ let $a = n - n_0$ and add a to the first row of each of λ, μ, ν . This changes neither ε_1 nor ε_2 , and the semigroup property of Kronecker coefficients [4, 5], applied with $g((a), (a), (a)) = 1$, gives positivity. $\square$

4. THE BASE CASE: TWO HOOKS AND A FINITE CERTIFICATE

The base case of the proof of Theorem 1.1 is the two-hook theorem of Remmel and Rosas. In this section we show that the same descent argument reduces it to a single finite statement about S_9 , and we prove that statement by a computer-assisted argument. This gives a second route to the base case, which does not rely on the published proofs but does rely on the computation described below. (A different, conceptual argument for the base case is sketched in Theorem 3.1.)

Proposition 4.1. *Assume the following finite statement.*

(*) *If $\lambda, \mu \vdash 9$ are hooks, then $g(\lambda, \mu, (3, 3, 3)) = 0$.*

Then for every n and all hooks $\lambda, \mu \vdash n$, $g(\lambda, \mu, \nu) > 0$ implies $\varepsilon_2(\nu) = 0$.

Proof. We argue by induction on n . If $n \leq 8$ then $\varepsilon_2(\nu) = 0$ for every $\nu \vdash n$, because $\varepsilon_2(\nu) > 0$ forces $(3, 3) \in \nu$ and hence $n \geq 9$. Let $n \geq 9$, let λ, μ be hooks and suppose $g(\lambda, \mu, \nu) > 0$ with $\varepsilon_2(\nu) > 0$. If $n = 9$, then $\nu \supseteq (3, 3, 3)$ forces $\nu = (3, 3, 3)$, which contradicts (*). Let $n > 9$. Then $\nu \neq (3, 3, 3)$, and we claim that ν has a corner c with $\varepsilon_2(\nu - c) > 0$. If $\varepsilon_2(\nu) \geq 2$ any corner works, by Theorem 2.1(iii) in the form $\varepsilon_2(\nu - c) \geq \varepsilon_2(\nu) - 1$. If $\varepsilon_2(\nu) = 1$, then $(3, 3)$ is the only cell of ν with both coordinates at least 3; it is a corner of ν , and if it were the only corner then ν would be a rectangle with corner $(3, 3)$, i.e. $\nu = (3, 3, 3)$. So there is a corner $c \neq (3, 3)$, and removing it keeps $(3, 3)$. Now apply Theorem 2.2 with the roles of the three partitions permuted (this is allowed since g is symmetric): there are corners a of λ and b of μ with $g(\lambda - a, \mu - b, \nu - c) > 0$. Removing a corner from a hook of size at least 2 gives a hook, so $\lambda - a$ and $\mu - b$ are hooks of $n - 1$, while $\varepsilon_2(\nu - c) > 0$. This contradicts the induction hypothesis. $\square$

The statement (*) as a linear-algebra lemma. Let Λ be the set of the 30 partitions of 9. For $\alpha, \beta \in \Lambda$ let $P(\alpha, \beta)$ be the number of ways to remove a corner of α and then add a cell so as to obtain β . By the branching rule and Frobenius reciprocity, $P(\alpha, \beta) = \langle \text{Ind Res } \chi^\alpha, \chi^\beta \rangle = \langle \pi \chi^\alpha, \chi^\beta \rangle$, where $\pi = \text{Ind}_{S_8}^{S_9} 1$ is the permutation character; in particular P is symmetric.

Lemma 4.2 (Relations). *The array $m(\alpha, \beta, \gamma) = g(\alpha, \beta, \gamma)$, $\alpha, \beta, \gamma \in \Lambda$, satisfies:*

- (i) **(SYM)** *m is invariant under all permutations of its three arguments, and $m \geq 0$ is integer valued;*
- (ii) **(TRIV)** *$m(\alpha, (9), \gamma) = \delta_{\alpha\gamma}$;*
- (iii) **(PI)** *$\sum_{\kappa} m(\alpha, \beta, \kappa) P(\kappa, \gamma) = \sum_{\kappa} P(\alpha, \kappa) m(\kappa, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in \Lambda$.*

Proof. (i) holds because the characters are real, so $g(\alpha, \beta, \gamma) = \frac{1}{9!} \sum_{x \in S_9} \chi^\alpha(x) \chi^\beta(x) \chi^\gamma(x)$, and g is a multiplicity. (ii) holds because $\chi^{(9)} = 1$. For (iii), since $\pi \chi^\gamma = \sum_{\kappa} P(\gamma, \kappa) \chi^\kappa$ and P is symmetric, the left-hand side equals $\langle \chi^\alpha \chi^\beta, \pi \chi^\gamma \rangle = \frac{1}{9!} \sum_x \chi^\alpha \chi^\beta \pi \chi^\gamma(x)$; the right-hand side equals $\langle \pi \chi^\alpha \cdot \chi^\beta, \chi^\gamma \rangle$, which is the same sum. $\square$

Here (PI) is the statement that multiplication by π commutes with the Kronecker product, and it is the only place where the branching rule enters. We use (PI) only for β a hook different from (9) .

Lemma 4.3 (Finite certificate). *Let V be the set of the 4960 multisets of size 3 in Λ , and let $m: V \rightarrow \mathbb{R}_{\geq 0}$ satisfy (SYM), (TRIV) and the instances of (PI) with β a hook different from (9) . Then $m(a, c, (3, 3, 3)) = 0$ for all hooks $a, c \in \Lambda$.*

Proof. The relations (TRIV) and (PI) are linear equations $\sum_{v \in V} A_{rv} m_v = b_r$ in the unknowns m_v . We use 1652 of them, namely 1542 of type (PI) and 110 of type (TRIV), and we exhibit integers $Y_1, \dots, Y_{1652}$ and a positive integer Q with 157 decimal digits such that

$$\sum_r Y_r A_{rv} \geq Q w_v \quad \text{for all } v \in V, \quad \sum_r Y_r b_r \leq 0,$$

where $w_v = 1$ if $v = \{a, c, (3, 3, 3)\}$ for hooks a, c and $w_v = 0$ otherwise (there are 45 such v). Since $m \geq 0$,

$$Q \sum_v w_v m_v \leq \sum_v \left(\sum_r Y_r A_{rv} \right) m_v = \sum_r Y_r (Am)_r = \sum_r Y_r b_r \leq 0,$$

so $m_v = 0$ whenever $w_v = 1$. The data (Y, Q) are in the file `h9_certificado.json` of the archive. Checking that they satisfy the inequalities is a finite computation with integers of about 160 digits. The way the certificate was found and checked is described in Theorem 4.4. $\square$

By Theorem 4.2, the array of Kronecker coefficients satisfies the hypotheses of Theorem 4.3, hence (*) holds. Together with Theorem 4.1 this gives the two-hook theorem, and therefore the base case of Theorem 1.1, without citing [13, 15].

Remark 4.4 (Provenance and checking of the certificate). The certificate was found by linear programming (a minimal ℓ^1 dual solution by an interior-point method, then a vertex by the simplex method on its support, then an exact rational solution of the resulting square system with `python-flint`). Its validity does not depend on how it was found. It is checked by a separate program, `verify_h9_cert.py`, which uses only the Python standard library and arbitrary-precision integers, rebuilds every row from its definition (the matrix P , the instances of (PI) and (TRIV)) and tests the inequalities above for all 4960 variables. The program reports no violated inequality and $\sum_r Y_r b_r = 0$. A copy of the certificate with one coefficient altered is rejected. The author reproduced the construction and the check in a second environment (Google Colab). As a control, the real Kronecker coefficients of S_9 computed from the character table satisfy all 1652 equations and the 45 target values vanish. In an earlier numerical experiment, a system in which the (PI) relations were dropped did not force the conclusion (the maximum of the corresponding linear program was positive, equal to 42). This is a computer-assisted proof of Theorem 4.3: its correctness rests on the checker and on the translation of the relations into rows, not on a human-readable argument. It is not a new proof of the two-hook theorem in the sense of a conceptual explanation, and it does not replace the published proofs. The Lean formalisation of this section is described in Section 9.

5. COROLLARIES

Proof of Theorem 1.2. If $d = d(\nu) \geq 3$, then $(d, d) \in \nu$, so all cells (i, j) with $3 \leq i, j \leq d$ belong to ν . They number $(d-2)^2$ and lie outside the first two rows and columns, so $(d-2)^2 \leq \varepsilon_2(\nu) \leq \varepsilon_1(\lambda) + \varepsilon_1(\mu)$. If $d \leq 2$ the bound is trivial. $\square$

Proof of Theorem 1.3. Take $\nu \vdash n$ with $\varepsilon_2(\nu) = R_{\max}(n)$. Universality gives $g(\lambda, \lambda, \nu) > 0$, so $R_{\max}(n) \leq 2\varepsilon_1(\lambda)$ by Theorem 1.1. $\square$

The cells of ν in the first two rows and columns number $\nu_1 + \nu_2 + \nu'_1 + \nu'_2 - 4$ when $\nu_2 \geq 2$, so a partition of n close to a square has $\varepsilon_2(\nu) = n - O(\sqrt{n})$; for $n = m^2$ the $m \times m$ square gives $R_{\max}(n) \geq (m-2)^2 = n - 4\sqrt{n} + 4$. Computed values are in Table 1. Hence a universal λ must satisfy $h(\lambda) \leq n/2 + O(\sqrt{n})$: hooks and thin shapes are excluded. For instance, hooks have $\varepsilon_1 = 0$ and so are not universal as soon as $R_{\max}(n) > 0$, i.e. $n \geq 9$.

n	9	10	11	12	13	14
$R_{\max}(n)$	1	1	1	2	2	2
n	15	16	17	18	19	20
$R_{\max}(n)$	3	4	4	4	5	6

TABLE 1. $R_{\max}(n) = \max_{\nu \vdash n} \varepsilon_2(\nu)$, computed by enumerating all partitions ($R_{\max}(n) = 0$ for $n \leq 8$).

Proof of Theorem 1.4. Let $c(\lambda)$ be the number of corners of λ . By the multiplicity-free branching rule, $\langle \text{Res } \chi^\lambda, \text{Res } \chi^\lambda \rangle_{S_{n-1}} = c(\lambda)$. Let $\pi = \text{Ind}_{S_{n-1}}^{S_n} 1 = \chi^{(n)} + \chi^{(n-1,1)}$ be the permutation character. By Frobenius reciprocity and the multiplicativity of restriction,

$$\langle \chi^\lambda \chi^\lambda, \pi \rangle = \langle \text{Res}(\chi^\lambda \chi^\lambda), 1 \rangle = \langle \text{Res } \chi^\lambda, \text{Res } \chi^\lambda \rangle = c(\lambda).$$

Since $\langle \chi^\lambda \chi^\lambda, \chi^{(n)} \rangle = 1$, we get $g(\lambda, \lambda, (n-1, 1)) = c(\lambda) - 1$. A rectangle has exactly one corner. $\square$

6. FAT HOOKS: A FAMILY OF INEQUALITIES

For integers $k, \ell \geq 1$ and a partition p put

$$\varepsilon_{k,\ell}(p) = \#\{(i, j) \in p : i > k, j > \ell\} = \sum_{i > k} \max(p_i - \ell, 0),$$

so that $\varepsilon_k = \varepsilon_{k,k}$. Write $H(k, \ell) = \{p : p_{k+1} \leq \ell\}$ for the hook region of Berele and Regev; then $\varepsilon_{k,\ell}(p) = 0$ if and only if $p \in H(k, \ell)$. We use the following published theorem.

Theorem 6.1 (Berele–Regev [7, Thm. 3.26(a)]). *Let $\lambda \in H(k_1, \ell_1)$ and $\mu \in H(k_2, \ell_2)$ be partitions of n , and put $K = k_1 k_2 + \ell_1 \ell_2$, $L = k_1 \ell_2 + k_2 \ell_1$. If $g(\lambda, \mu, \nu) > 0$ then $\nu \in H(K, L)$.*

Lemma 6.2. *Let p be a partition and $k, \ell \geq 1$.*

- (i) *If $\varepsilon_{k,\ell}(p) \geq 1$, then p has a corner $a = (i, j)$ with $i > k$ and $j > \ell$, and $\varepsilon_{k,\ell}(p - a) = \varepsilon_{k,\ell}(p) - 1$.*
- (ii) *For every corner b of p , $\varepsilon_{k,\ell}(p - b) \leq \varepsilon_{k,\ell}(p) \leq \varepsilon_{k,\ell}(p - b) + 1$.*

Proof. (i) Since $\varepsilon_{k,\ell}(p) \geq 1$ there is a row $r > k$ with $p_r > \ell$. Let i be the last row with $p_i > \ell$; then $i \geq r > k$ and $p_{i+1} \leq \ell < p_i$, so $a = (i, p_i)$ is a corner in the counted region, and removing it lowers the count by exactly one. (ii) Removing a cell can only remove it from the counted region, and removes at most one cell. $\square$

Theorem 6.3. *Let $k_1, \ell_1, k_2, \ell_2 \geq 1$, $K = k_1 k_2 + \ell_1 \ell_2$ and $L = k_1 \ell_2 + k_2 \ell_1$. If $g(\lambda, \mu, \nu) > 0$ for partitions of n , then*

$$\varepsilon_{K,L}(\nu) \leq \varepsilon_{k_1, \ell_1}(\lambda) + \varepsilon_{k_2, \ell_2}(\mu).$$

In particular, for $a, b \geq 1$: $g(\lambda, \mu, \nu) > 0 \Rightarrow \varepsilon_{2ab}(\nu) \leq \varepsilon_a(\lambda) + \varepsilon_b(\mu)$.

For $a = b = 1$ this is Theorem 1.1 (and then Theorem 6.1 gives the two-hook theorem used as the base case there).

Proof (short argument; not formalised). Induction on n , for all four pairs of parameters at once. If $\varepsilon_{k_1, \ell_1}(\lambda) = \varepsilon_{k_2, \ell_2}(\mu) = 0$, then $\lambda \in H(k_1, \ell_1)$ and $\mu \in H(k_2, \ell_2)$, and Theorem 6.1 gives $\nu \in H(K, L)$, that is $\varepsilon_{K,L}(\nu) = 0$. This covers $n = 1$. Otherwise $n \geq 2$, and since g is symmetric in λ, μ and K, L are symmetric under exchanging the indices $1 \leftrightarrow 2$, we may assume $\varepsilon_{k_1, \ell_1}(\lambda) \geq 1$. Take the corner a of Theorem 6.2(i) for λ . By Theorem 2.2 there are corners b of μ and c of ν with $g(\lambda - a, \mu - b, \nu - c) > 0$ in S_{n-1} . The induction hypothesis and Theorem 6.2 give

$$\varepsilon_{K,L}(\nu) \leq \varepsilon_{K,L}(\nu - c) + 1 \leq (\varepsilon_{k_1, \ell_1}(\lambda) - 1) + \varepsilon_{k_2, \ell_2}(\mu - b) + 1 \leq \varepsilon_{k_1, \ell_1}(\lambda) + \varepsilon_{k_2, \ell_2}(\mu). \quad \square$$

Remark 6.4 (Status and scope). This argument is the proof of Theorem 1.1 with the two-hook theorem replaced by Theorem 6.1; it is short, but it has not been formalised (the theorem of Berele and Regev is not available in Mathlib) and it has not been reviewed by anyone other than the author. We have not searched the literature for this particular statement, and it may be a known consequence of the cited results. For $(a, b) = (1, 2)$ the index is 4, and the index $3 = a + b$ fails (Theorem 10.2); since the index can only get weaker when it grows, 4 is the smallest valid integer index for $(1, 2)$. We did not study minimality for other pairs. By [7, Thm. 3.26(b)] the containment in $H(2k^2, 2k^2)$ of Theorem 6.1 cannot be improved for large n , so for $a = b = k$ the index $2k^2$ cannot be lowered. For $a = b = k$ the theorem is an analogue of Theorem 1.1 for k -fat hooks, which is the question (3) of Section 10; whether it is the *right* analogue is not clear. The existence statement of Vallejo [16] (some constituent of $\chi^\lambda \chi^\mu$ lies in $H(k, k)$ when $\lambda, \mu \in H(k, k)$) points in the opposite direction: it asserts that something of a given shape occurs, whereas Theorem 6.3 excludes shapes.

Remark 6.5 (Consistency checks (computed)). For $(a, b) = (1, 2)$ and $25 \leq n \leq 32$ we enumerated all $\nu \vdash n$ with $\varepsilon_4(\nu) \geq 1$ together with all $\lambda, \mu \vdash n$ with $\varepsilon_1(\lambda) + \varepsilon_2(\mu) < \varepsilon_4(\nu)$, and computed g from the character table modulo two primes below 10^6 (a nonzero residue shows $g > 0$); there is no triple with $g > 0$ among them (4,835,056 candidate triples at $n = 32$). As a control, with the index 3 the same program reproduces the violations of Theorem 10.2 (140 at $n = 16$). For $k_i, \ell_i \leq 2$ and $n \leq 19$ no violation of Theorem 6.3 was found either, but many of these parameter sets are vacuous in that range. These computations are consistent with the theorem; they are not part of its proof.

7. RELATED WORK AND INDEPENDENCE FROM KNOWN BOUNDS

We read the following sources in full or in the relevant parts and found no statement of Theorem 1.1: Dvir [11] (maximal first row $|\lambda \cap \mu|$ and maximal length $|\lambda \cap \mu'|$, the recursion of its Section 2 and the Frobenius-type identity of its Prop. 1.5, which is of the same kind as Theorem 2.2), Bessenrodt–Kleshchev [9], Remmel [13, 14], Rosas [15], Berele–Regev [7], Berele–Imbo [6], Regev [12], Bessenrodt–Behns [8], Vallejo [16, 17] and the extended abstract of Panova and Zhao [22]. We viewed all 16 pages

of Dvir’s paper in a scanned copy, at low resolution and without checking every proof line by line. Of Panova and Zhao we read only the extended abstract; the full version (arXiv:2311.10276v2) was not read in detail. A further set of about twenty-five arXiv papers on Kronecker coefficients was searched by keyword (for example “double hook”, “outside the hook”, “Durfee”) without finding the statement. This does not establish novelty. Garsia–Remmel (1985), Clausen–Meier (1993) and Remmel–Whitehead (1994) could not be obtained. A search of zbMATH Open for the terms “double hook”, “outside the hook” and “Durfee” together with “Kronecker” (October 2026) returned no paper stating it; MathSciNet was not available to us.

Remark 7.1 (Independence from the bounds of Dvir and Bessenrodt–Kleshchev). The inequalities of Dvir (first row $\leq |\lambda \cap \mu|$, length $\leq |\lambda \cap \mu'|$, in all three roles and under conjugation) and the strict bound $h_{11}(\nu) < |\lambda \cap \mu| + |\lambda \cap \mu'| - 1$ of [9, Thm. 3.3] do not imply Theorem 1.1: there are triples with $g = 0$ satisfying all of them but violating the conclusion of Theorem 1.1, for instance $\lambda = \mu = (6, 1^4)$, $\nu = (4, 3, 3)$ for $n = 10$ (here $\varepsilon_2(\nu) = 1 > 0$). The converse also fails, since Theorem 1.1 says nothing about ν_1 or $\ell(\nu)$.

Remark 7.2 (Comparison with Berele–Regev). The hook-region theorem [7, Thm. 3.26(a)] says that if $\nu \in H(k_1, \ell_1; n)$ and $\mu \in H(k_2, \ell_2; n)$ then every constituent of $\chi^\nu \chi^\mu$ lies in

$$H(k_1 k_2 + \ell_1 \ell_2, k_1 \ell_2 + k_2 \ell_1; n).$$

A partition with $\varepsilon_1(\lambda) = e$ lies in $H(e + 1, 1; n)$, so for two such partitions with excesses e_1, e_2 the theorem places the constituents in

$$H(e_1 e_2 + e_1 + e_2 + 2, e_1 + e_2 + 2; n),$$

whereas Theorem 1.1 places them in $H(e_1 + e_2 + 2, 2; n)$, which is contained in the former. On the other hand Theorem 1.1 says nothing for pairs with large excess in thick hook regions. The two results are thus complementary.

8. RELATION TO DVIR’S INEQUALITIES AND TO THE KRONECKER CONE

Since Theorem 1.1 is a necessary condition for $g > 0$, it is natural to ask how much it adds to the known necessary conditions. This section gives a partial answer: a proof that its asymptotic content is linear, and a computation that compares it with Dvir’s bound. It also lists what we did not compare.

8.1. The linear shadow. Write $\ell(\lambda)$ for the number of parts. We use the semigroup property of Kronecker coefficients [4, 5]: if $g(\lambda, \mu, \nu) > 0$ and $g(\lambda', \mu', \nu') > 0$, then $g(\lambda + \lambda', \mu + \mu', \nu + \nu') > 0$. In particular $g(p\lambda, p\mu, p\nu) > 0$ for every $p \geq 1$.

Proposition 8.1. *If $g(\lambda, \mu, \nu) > 0$ for partitions of n , then $\lambda_1 + \mu_1 \leq n + \nu_1 + \nu_2$.*

Proof. If $\ell(\nu) = 1$ then $\nu = (n)$, so $\lambda = \mu$ and the inequality reads $2\lambda_1 \leq 2n$. Let $\ell(\nu) \geq 2$ and $p \geq 2$. All parts of $p\lambda, p\mu, p\nu$ are at least 2, so $\varepsilon_1(p\lambda) = p(n - \lambda_1) - \ell(\lambda) + 1$, similarly for μ , and $\varepsilon_2(p\nu) = p(n - \nu_1 - \nu_2) - 2\ell(\nu) + 4$. By the semigroup property $g(p\lambda, p\mu, p\nu) > 0$, and Theorem 1.1 applied to this triple of partitions of pn gives

$$p(n - \nu_1 - \nu_2) - 2\ell(\nu) + 4 \leq p(2n - \lambda_1 - \mu_1) - \ell(\lambda) - \ell(\mu) + 2.$$

Dividing by p and letting $p \rightarrow \infty$ yields $n - \nu_1 - \nu_2 \leq 2n - \lambda_1 - \mu_1$. □

Thus the part of Theorem 1.1 that survives scaling is a linear inequality, and it is implied by the stronger candidate inequality $\lambda_1 + \mu_1 \leq n + \nu_1$. The theorem is linear only asymptotically; what it adds beyond a linear inequality are the corrections coming from the lengths ℓ and the constants, which matter for small and medium n . We did not prove that $\lambda_1 + \mu_1 \leq n + \nu_1$ follows from Dvir’s bound. What we checked is a data statement: for $n \leq 14$ and all 24 variants described below, every triple that violates $\lambda_1 + \mu_1 \leq n + \nu_1$ also violates Dvir’s inequality $\nu_1 \leq |\lambda \cap \mu|$ (for $n = 8, \dots, 14$); no triple with $g > 0$ violates either inequality. So in these data the linear shadow of Theorem 1.1 is redundant with Dvir’s bound.

8.2. Counting certified zeros. For three partitions v_1, v_2, v_3 of n , the 24 *variants* are obtained by permuting the three partitions (6 ways) and conjugating none or two of them (4 ways, $g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu)$). A triple is *certified to be a zero by (T)* if in some variant $\varepsilon_2(v_3) > \varepsilon_1(v_1) + \varepsilon_1(v_2)$, and *by (D)* if in some variant $(v_3)_1 > |v_1 \cap v_2|$ (Dvir’s bound on the first row, together with its conjugate on the length). Both are necessary conditions for $g > 0$. We also use a convention to describe the coverage by known rules: a triple is *special* if at least one of the three partitions is a hook, has at most two rows or has at most two columns. A complete combinatorial rule is available in the literature when one of the shapes is a hook [2] and for products of two shapes with at most two rows [15]; counting every triple with one such shape as covered is therefore generous to the existing rules, and the proportions reported as “outside the special families” are conservative.

All counts below refer to ordered triples. The flags (T), (D) and “special” depend only on the partitions, so they were counted exactly over all triples of partitions of n . Whether a triple is a zero was decided from the support of g . For $n \leq 21$ the full support was available (computed from the character table modulo two primes of size about 2^{26} , as in Section 9), so all numbers for $n \leq 21$ are exact up to the possibility, discussed in Section 9, that a positive coefficient is divisible by both primes. For $22 \leq n \leq 32$ the support was not computed; the number of zeros certified by (T) or (D) is exact, and the number of zeros among the remaining “uncertified” triples (about 90% of all triples) was *estimated* by hash sampling of about $3.4 \cdot 10^6$ triples per n (ratio estimator; standard error about 0.5% relative; the method reproduces the exact fraction at $n = 21$ within 1.5 standard errors). Rows for $n \geq 22$ are estimates.

n	zeros	(D) %	(T) %	only (T) %	neither %	#(T only)	of which not special
9	$1.27 \cdot 10^4$	93.12	1.91	0.02	6.86	3	0
12	$1.77 \cdot 10^5$	91.03	3.18	0.14	8.82	252	0
15	$1.64 \cdot 10^6$	91.93	3.67	0.20	7.87	3342	36 (1.1%)
16	$3.48 \cdot 10^6$	91.42	3.56	0.20	8.38	7089	237 (3.3%)
18	$1.38 \cdot 10^7$	91.21	3.45	0.20	8.58	28 263	2463 (8.7%)
20	$5.14 \cdot 10^7$	90.98	3.29	0.20	8.82	102 342	16 986 (16.6%)
21	$9.58 \cdot 10^7$	90.95	3.21	0.20	8.85	189 426	39 648 (20.9%)
24 [†]	$6.08 \cdot 10^8$	90.40	2.90	0.18	9.41	$1.10 \cdot 10^6$	$3.80 \cdot 10^5$ (34.6%)
28 [†]	$6.00 \cdot 10^9$	89.81	2.54	0.17	10.03	$9.92 \cdot 10^6$	$5.10 \cdot 10^6$ (51.5%)
32 [†]	$5.14 \cdot 10^{10}$	89.14	2.25	0.15	10.71	$7.82 \cdot 10^7$	$5.08 \cdot 10^7$ (64.9%)

TABLE 2. Percentage of the triples with $g = 0$ certified by (D), by (T), by (T) but not (D) (“only (T)”), and by neither. The last two columns count the ordered triples certified by (T) but not (D), and among them those that are not special (no hook, at most two rows or columns). Rows marked [†] are estimates (for $n = 24, 28, 32$ the standard error of the (D) and “neither” percentages is 0.04, 0.04, 0.05 percentage points); the last two columns are exact for all n , and the other percentages for $n \leq 21$ are exact.

8.3. Reading of the data. The following statements describe the data of Table 2; they are not proofs for general n .

- (1) Dvir’s bound certifies about 93% of the zeros at $n = 9$, 91% at $n = 17$ and an estimated 89% at $n = 32$. The theorem certifies between 1.9% and 3.7%, and the zeros certified by the theorem but not by Dvir’s bound are about 0.2% of all zeros for $15 \leq n \leq 21$ and an estimated 0.15% at $n = 32$. In proportion the theorem adds little; in absolute numbers it adds from 3 ($n = 9$) to an estimated $7.8 \cdot 10^7$ ($n = 32$) triples. No triple with $g > 0$ violated (T) or (D): for $n \leq 21$ among all $4 \cdot 10^8$ positive triples of $n = 21$ and the smaller ones, and for $22 \leq n \leq 32$ among between $3.3 \cdot 10^5$ and $5.7 \cdot 10^5$ sampled triples per n in the strata certified by (T) or (D).
- (2) The first triples certified by (T) and not by (D) occur at $n = 9$ (three ordered triples); they come from the base case, that is, from the two-hook theorem, which is classical. The first zero that needs the general case of the theorem ($\varepsilon_1(\lambda) + \varepsilon_1(\mu) > 0$) and is not certified by (D) occurs at $n = 12$, for

example $\lambda = (7, 1^5)$, $\mu = (6, 2, 1^4)$, $\nu = (4, 4, 4)$ (here $\varepsilon_1 = 0, 1$ and $\varepsilon_2(\nu) = 2$). There are 24 such ordered triples at $n = 12$.

- (3) Up to $n = 14$ every zero certified by (T) and not by (D) is special, so that its value is already given by a complete rule for a hook or two-row shapes. The first triples certified by (T), not by (D) and not special occur at $n = 15$ (36 ordered triples), for example $\lambda = \mu = (8, 2, 1^5)$ with $\nu = (5, 5, 5)$. For every n from 9 to 32 (exact flag counts), no triple certified by (T) and not by (D) violates the length bound $\ell(\nu) \leq \ell(\lambda)\ell(\mu)$, so the last column does not change if that bound is added to the known rules. The proportion of non-special triples among those certified only by (T) grows from 1.1% at $n = 15$ to 20.9% at $n = 21$ and, by the exact counts, to 64.9% at $n = 32$.
- (4) In an earlier computation for $n = 12, 14, 16$ we also included the bound $h_{11}(\nu) < |\lambda \cap \mu| + |\lambda \cap \mu'| - 1$ of Bessenrodt and Kleshchev [9, Thm. 3.3] and the hook-region bounds of Berele and Regev [7] for (k, ℓ) -hooks with $k \leq 5$. The union of all the criteria excluding (T) certified 91.6%, 92.0% and 91.8% of the zeros, and (T) certified in addition 24, 480 and 3525 zeros, that is 0.16%, 0.74% and 1.23% of the zeros left uncertified.
- (5) The comparison above does not include the criterion of Young and Vallejo. Once it is added (Section 8.4) the part certified only by (T) shrinks to about 0.2–0.3% of the classes of zeros for $9 \leq n \leq 15$.

8.4. The criterion of Young and Vallejo and the Kronecker cone. For $\nu = (\nu_1, \dots, \nu_r)$ let $\varphi_\nu = \text{Ind}_{S_{\nu_1} \times \dots \times S_{\nu_r}}^{S_n} 1$ be the permutation character of the Young subgroup. By Young's rule $\varphi_\nu = \chi^\nu + \sum_{\kappa \triangleright \nu} K_{\kappa\nu} \chi^\kappa$ with $K_{\kappa\nu} \geq 0$, so

$$g(\lambda, \mu, \nu) \leq \text{lr}(\lambda, \mu; \nu) := \langle \chi^\lambda \chi^\mu, \varphi_\nu \rangle,$$

and $\text{lr}(\lambda, \mu; \nu) = 0$ implies $g(\lambda, \mu, \nu) = 0$ [17, Cor. 4.2]. Call this criterion (L) and use it in all 24 variants. We computed lr exactly from the character table. As a soundness check, none of the criteria excluded a positive triple in our tests (all triples for $n \leq 11$ for the criteria computed from the partitions alone, and 300 random positive triples for each $n \leq 16$ for all criteria including (L)).

In this subsection counts are numbers of *classes* of triples under permutations and under conjugation of two entries, not ordered triples. Among the classes of zeros for $8 \leq n \leq 16$, (L) certifies about 96–98% (computed exactly for $n \leq 11$ and on random samples for larger n), Dvir's bounds about 90%, and the union of all the criteria we tried, including (T), leaves about 2–5% uncertified. The classes certified by (T) and not by (L) are few, and none of them is certified by Dvir's bounds:

n	classes of zeros	certified by (T), not by (L)	of these, with $g(2\cdot) > 0$
8	283	0	—
9	619	1	1
10	1542	2	2
11	3205	8	8
12	7821	16	16
13	15722	43	42
14	34896	84	82
15	70561	186	179

TABLE 3. Classes of zeros certified by (T) and not by (L). Both the third column and the fourth are exhaustive. The fourth column counts classes T with $g(2T) > 0$ (computed exactly, doubling all three partitions); a class counted in the third column and not in the fourth might still satisfy $g(kT) > 0$ for some $k \geq 3$ (we tested $k = 3$ only for $n \leq 9$ and for one class at $n = 13$). The smallest class with $g(2T) > 0$ is $\lambda = (3, 3, 3)$, $\mu = \nu = (5, 1^4)$ at $n = 9$. At $n = 13$ the class $((3, 3, 3, 3, 1), (7, 1^6), (7, 1^6))$ has $g(2T) = g(3T) = 0$.

In proportion, then, Theorem 1.1 adds about 0.2–0.3% of the classes of zeros to what (L) already certifies in this range. The point of the last column is different.

Remark 8.2 (Zeros inside the cone). Zeros T with $g(kT) > 0$ for some $k \geq 2$ are failures of saturation of the Kronecker semigroup; they exist already for small shapes ($g((1,1), (1,1), (1,1)) = 0$ while $g((2,2), (2,2), (2,2)) = 1$; see [23, 24]). By the semigroup property, if $g(T) = 0$ but $g(2T) > 0$ for a triple $T = (\lambda, \mu, \nu)$, then $2T$ lies in the support. If H is a linear form that is homogeneous and satisfies $H \leq 0$ on the whole support, then $H(T) = \frac{1}{2}H(2T) \leq 0$, so T satisfies every inequality of this kind. Hence no homogeneous linear inequality valid for the support (in particular no inequality describing the Kronecker cone) can certify $g(T) = 0$. The inequality of Theorem 1.1 is not homogeneous, and the last column of Table 3 shows that it certifies such zeros: in the cases counted, almost all the zeros certified by (T) and not by (L) are of this kind. This is a measurement for $n \leq 15$, not a theorem for all n . It says nothing about whether other published criteria certify these zeros beyond those listed in Section 8.

8.5. Triples not covered by the criteria tested: $n = 23$. Call a partition a *near-hook* if its third part is at most 1 (that is, it has the form $(a, b, 1^c)$). Formulas for g are available when one of the three partitions is a hook [2], has at most two rows [15], or is a near-hook [18], and the same holds for conjugates. The flags (T) and (D), the length bound and these shape conditions depend only on the partitions, so we enumerated them exactly for every n . We counted the ordered triples certified by (T) and not by (D) or by the length bound, in which no partition is a hook, has at most two rows or at most two columns, or is a near-hook or the conjugate of one. The count is 0 for every $n \leq 22$ and 192 for $n = 23$ (for $n = 24, 26, 28, 30$ it is 1,761, 21,093, 164,739 and 991,986; we did not apply further criteria for $n \geq 24$).

Proposition 8.3 (Computed). *The 192 ordered triples at $n = 23$ form 12 classes under permutations and conjugation of two entries. They are the classes of (λ, μ, ν) with*

$$\lambda = (a, 3, 2, 1^{18-a}), \quad \mu = (b, 3, 2, 1^{18-b}), \quad (a, b) \in \{(9, 9), (9, 10), (9, 11), (9, 12), (10, 10), (10, 11)\},$$

and $\nu \in \{(4^5, 3), (5^3, 4^2)\}$. For all of them $\varepsilon_1(\lambda) = \varepsilon_1(\mu) = 3 < 7 = \varepsilon_2(\nu)$, so (T) certifies $g = 0$. Computation shows:

- (a) None of them is certified by (D) or the length bound, by the depth bound, by (L) in all 24 variants, by the Berele–Regev containment for $k, \ell \leq 10$, by the bound of Bessenrodt and Kleshchev, or by the vanishing and reduction results of Vallejo [17, Cor. 4.5, Thm. 4.6].
- (b) $g = 0$ for all 12 classes, computed exactly from the character table of S_{23} (1255 conjugacy classes, Murnaghan–Nakayama), without using Theorem 1.1. Controls: $g(\lambda, \lambda, (23)) = 1$, $g(\lambda, \lambda, (22, 1)) = c(\lambda) - 1$ and $g(\lambda, \lambda, (1^{23})) = 1$ on the same program.
- (c) $g(2\lambda, 2\mu, 2\nu) > 0$ for all 12 classes, computed exactly in S_{46} (105,558 conjugacy classes), with values from 41,251 to 2,113,658,409; so all twelve are zeros inside the Kronecker cone.

We stress what this does and does not show. “Not covered” means not covered by the criteria listed above and by the shape conditions; it does not mean that no published criterion certifies these zeros. At $n = 23$ the value of g is computed directly from the character table, so what Theorem 1.1 offers here is a uniform and cheap reason, not a new way to determine the value. We do not have a description of such triples for general n : the classes above have $n = 23$ only, and for $n \geq 24$ we did not analyse the counts quoted. We therefore do not call this a family.

8.6. An explicit example. For $e \geq 1$ let $\lambda = \mu = (e + 7, e + 1, 1^{2e+4})$, $\nu = (4^{e+3})$ and $n = 4e + 12$. Then $\varepsilon_1(\lambda) = \varepsilon_1(\mu) = e$ and $\varepsilon_2(\nu) = 2e + 2 > 2e$, so $g(\lambda, \mu, \nu) = 0$ by Theorem 1.1 for every $e \geq 1$. By computation: Dvir’s bounds in all 24 variants, the length bound and the depth bound certify none of these triples for $e \leq 300$, while (T) certifies all of them; (L) does not certify them for $e \leq 3$; $g = 0$ exactly for $e \leq 8$ (character computation); and $g(2\lambda, 2\mu, 2\nu) = 2$ for $e = 1$ and 967 for $e = 2$, so these are zeros inside the cone. We expect that the failure of Dvir’s bounds can be checked by hand for all e , but we only checked it by computer.

Here λ is a near-hook $(a, b, 1^c)$, so the formula of Campbell [18] applies to this example; we did not implement it. The example shows that Theorem 1.1 gives a uniform reason for these zeros where the other inequalities we tried do not, but it is not an example of a zero outside the reach of the published formulas, and we do not present it as a new family.

8.7. What was not compared. We make no claim that Theorem 1.1 certifies zeros that no other published criterion certifies. The criteria we compared it with are Dvir’s bounds, the length bound, the depth bound, the Berele–Regev containment (with $k, \ell \leq 10$ for the classes at $n = 23$ and $k, \ell \leq 6$ elsewhere), the bound of Bessenrodt and Kleshchev, the Young–Vallejo criterion (L), the reduction theorem of Vallejo, and, through $g(2\cdot)$, the Kronecker cone. We did not compare it with: the published Horn-type inequalities for the Kronecker cone (Klyachko; Ressayre [21]) — when we implemented the statements of [21] as we read them, they did not agree with exact data, which is probably due to a misreading on our part, so we do not use them, and Theorem 8.2 is a property of any valid homogeneous inequality and not a comparison with specific ones; the necessary condition of Panova and Zhao (Thm. 6.1 of the extended abstract [22], Thm. 6.2 of the full version, arXiv:2311.10276v2), which we did not implement; the formula of Campbell [18] for near-hooks, which we did not implement; results of Manivel and Bessenrodt on vanishing and non-vanishing, apart from the ones named above; or any criterion used in geometric complexity theory for rectangular shapes. The proportions for $n \geq 22$ in Table 2 rest on sampling and on the assumption that the hash sampling is uniform; this was validated only at $n = 21$. A positive triple with $g \equiv 0$ modulo both primes would be classified as a zero; we estimate the probability of this as negligible but it is not zero. We therefore do not claim novelty of Theorem 1.1 beyond what the searches of Section 7 support.

9. COMPUTATIONS AND FORMAL VERIFICATION

The statement of Theorem 1.1 was tested exhaustively, and its proof steps were checked separately.

- **Exhaustive check, $n \leq 21$ (computed).** For every $n \leq 21$ and every ordered triple of partitions of n we computed the support of g from the character table (Murnaghan–Nakayama), using residues modulo two primes of size about 2^{26} for positivity, and checked $\varepsilon_2(\nu) \leq \varepsilon_1(\lambda) + \varepsilon_1(\mu)$, in all 24 variants of Section 8, whenever $g > 0$. There were no violations; for example $n = 20$ has 627 partitions and 195,081,715 ordered triples with $g > 0$, and $n = 21$ has 400,985,763. Orthogonality of the character table was verified at each $n \leq 20$ (for $n = 21$ the support table was produced by the same program, and the check of orthogonality was not repeated). A coefficient divisible by both primes would be recorded as zero; this could only hide a potential violation, and it is not detected by this method alone. For $n \leq 13$ the same check was repeated with exact integer arithmetic (566,932 unordered triples with $g > 0$, no violations).
- **Independent implementations (computed).** The character tables were computed in two ways (Murnaghan–Nakayama and a Jacobi–Trudi construction) and the counts of triples with $g > 0$ agree with a third implementation written in Lean 4 for $n \leq 14$.
- **Proof steps (computed).** Theorem 2.1 was checked for all partitions with $n \leq 22$, and Theorem 2.2 for all triples with $g > 0$ and all corners of λ for $n \leq 17$; no failures. The base case was additionally checked for all pairs of hooks with $n \leq 19$.
- **Lean 4 and Mathlib: Specht modules, branching and the descent argument (Lean).** We formalised the argument in Lean 4 on top of Mathlib (Lean 4.35.0; by a line count of the source files, about 5,350 lines of hand-written code, plus a generated data file of 1,693 lines). The development constructs the Specht modules $S^\lambda \subseteq M^\lambda$ of S_n (polytabloids, column stabilisers), proves James’ submodule theorem, deduces that each S^λ is irreducible and that $S^\lambda \not\cong S^\mu$ for $\lambda \neq \mu$ (via the dominance lemma), and that the number of partitions of n equals the number of conjugacy classes of S_n . It proves Frobenius reciprocity for class functions, the descent step of Theorem 2.2, and the combinatorial induction of Section 3. It also proves the *branching rule* $\text{Res}_{S_n}^{S_{n+1}} \chi^\lambda = \sum_a \chi^{\lambda-a}$ for these Specht modules, without standard bases or Garnir relations: a projection $M^\lambda \rightarrow M^{\lambda-e_r}$ sending polytabloids to polytabloids gives $a_{\lambda\kappa} \geq 1$ on corners; this yields $\dim S^\lambda \geq f^\lambda$ (the number of standard tableaux) by induction; the identities $\sum_\lambda (\dim S^\lambda)^2 = n! = \sum_\lambda (f^\lambda)^2$ (the latter from the relation $DU - UD = I$ on Young’s lattice) then force equality. The reduction of Theorem 4.1 is also formalised. The resulting Lean theorem states that Theorem 1.1 holds for every n for these Specht modules provided $(*)$ holds for them.

- Lean 4: closing (*) for the Specht modules, and the final theorem (Lean).** The certificate lemma Theorem 4.3 is formalised as `H9_of_check` (no `sorry`): if the Boolean function `checkAll`, which recomputes from the certificate data (stored in the generated file) the verification of Theorem 4.3 in integers, returns `true`, and an integer-valued array M on Λ^3 is symmetric, nonnegative and satisfies (TRIV) and the instances of (PI) with β a hook different from (9), then $M(a, c, (3, 3, 3)) = 0$ for all hooks a, c . The link with the Specht modules of S_9 is then proved as follows. (PI) is derived abstractly (files `H9Bridge1`, `H9Bridge5`) from Frobenius reciprocity, the branching rule and the symmetry of the trilinear form $\sum_x a(x)b(x)c(x)$ on class functions, for a subgroup, a family of characters with the branching property and the matrix of common corners. (TRIV) follows because the Specht module of the one-row shape $[n]$ has a single tabloid, so $S^{[n]}$ is one-dimensional and $\chi^{(9)} = 1$ (`H9Bridge4`). The 30 partitions of 9 are enumerated by an explicit equivalence `Fin30` $\simeq$ `Parts9` (`H9Bridge2`), and the identification of the matrix of common corners with the table P stored in the certificate is done *by computation* (`H9Bridge3`). The assembly (`H9Bridge5`) gives `H9_final`, namely (*) for the Specht modules of S_9 , and the theorem `teoremaA_Sn_final`: for the Specht modules of every S_n , $\langle \chi^\lambda \chi^\mu, \chi^\nu \rangle \neq 0$ implies $\varepsilon_2(\nu) \leq \varepsilon_1(\lambda) + \varepsilon_1(\mu)$, with no hypothesis left. The whole project builds with `lake build`. `#print axioms teoremaA_Sn_final` lists `propext`, `Classical.choice`, `Quot.sound` and one further axiom produced by `native_decide` for the statement `check_ok` : `checkAll = true`; there is no `sorry`. *Caveat.* The proof of `check_ok` uses `native_decide`, so it trusts Lean’s compiler and runtime and not only the kernel (about 7 seconds). We first tried `decide +kernel`, which evaluates inside the kernel, but on this formulation it consumed about 48 GB of memory and was stopped; so the kernel-only check was not obtained. The independent Python verifier checks the same certificate with exact integers. Two further caveats concern the scope. First, the Lean statement is about the Specht modules as we constructed them in Lean, and its content as a statement about Kronecker coefficients rests on the standard identification of these with the characters χ^λ , which we formalised ourselves and no one else has reviewed; the claim “for all n ” in Lean depends on that classification. Second, the Lean theorem replaces the cited theorem of Remmel and Rosas by the certificate, but the written proof of Theorem 1.1 in this paper is unchanged. An earlier independent Lean check (Kronecker coefficients defined by the Murnaghan–Nakayama rule, using `native_decide`) verified the statement for all partitions of n with $4 \leq n \leq 12$.
- Audit of the Lean statement (by the author; not independent).** We compared the Lean definitions with those of this paper. The functions `e` and `R` are ε_1 and ε_2 ; `Rem` is the removal of a corner; `e_exists_remove`, `e_Rem_le`, `R_le_succ` and `gAll_P4` correspond to Theorems 2.1 and 2.2; and `main` is the induction of Section 3. The branching rule (`hbranch_Sn`) and the base case (`H9_final`) are proved, not assumed. The Specht modules constructed in Lean are irreducible, pairwise non-isomorphic and as many as the conjugacy classes, so they give all the irreducible characters of S_n . We compiled `#print axioms teoremaA_Sn_final` with Lean 4.35.0 and obtained exactly `propext`, `Classical.choice`, `Quot.sound` and the `native_decide` axiom of `check_ok`; no source file contains `sorry`, `axiom` or `unsafe`. The statement is not vacuous: in Lean, $\varepsilon_1((3, 2, 2)) = 2$, $\varepsilon_2((3, 3, 3)) = 1$ and $\varepsilon_2((4, 4, 4, 4)) = 4$, and $\langle \chi^{(9)} \chi^{(9)}, \chi^{(9)} \rangle = 1$ in S_9 . In summary: the formalisation proves Theorem 1.1 for every n for the characters of the Specht modules as we construct them in Lean (polytabloids in the permutation module on tabloids); the two-hook base case is established by the exact certificate of Section 4; the statement involves only the inner product $\langle \chi^\lambda \chi^\mu, \chi^\nu \rangle$ and the numbers $\varepsilon_1, \varepsilon_2$ defined as in this paper. The one non-standard axiom is the compiled evaluation of the certificate check, so the proof trusts the Lean compiler; the definitions of the Specht modules have not been reviewed by anyone outside this work; and the identity “character of a tensor product equals the pointwise product of characters” is used directly on class functions and is not formalised over `TensorProduct`. The corollaries, the sharpness family, Theorem 6.3 and the comparisons with other criteria are not formalised.

These computations are evidence consistent with the proof; except for the certificate of Theorem 4.3, which is a proof by exact finite verification, they are not part of it.

10. REMARKS AND OPEN PROBLEMS

Conjecture 10.1. *Let $\lambda \vdash n$ be self-conjugate. Then λ is universal if and only if λ is not a rectangle and $2\varepsilon_1(\lambda) \geq R_{\max}(n)$.*

Necessity is Theorem 1.3 together with Theorem 1.4. Sufficiency is supported by computations: the predicted and observed sets of non-universal self-conjugate partitions agree in all 589 cases tested (complete computation for $n \leq 27$ and certified failures for $28 \leq n \leq 42$). Sufficiency is open and is not proved here. For the Saxl conjecture, which concerns the staircase, Theorem 1.3 is consistent but gives no new information.

Remark 10.2 (A natural generalisation fails). It is tempting to conjecture $\varepsilon_{a+b}(\nu) \leq \varepsilon_a(\lambda) + \varepsilon_b(\mu)$ for all $a, b \geq 1$, of which Theorem 1.1 is the case $a = b = 1$ (with $a + b = 2$). This is false: for $(a, b) = (1, 2)$ the triple $\lambda = (12, 1^4)$, $\mu = (6, 6, 2, 2)$, $\nu = (4, 4, 4, 4)$ with $n = 16$ has $g(\lambda, \mu, \nu) = 1$, while $\varepsilon_3(\nu) = 1$, $\varepsilon_1(\lambda) = 0$ and $\varepsilon_2(\mu) = 0$. Violations were found for $(1, 2)$ from $n = 16$ on.

Remark 10.3 (Open questions). (1) Prove or disprove Theorem 10.1. (2) Describe when equality holds in Theorem 1.1; the naive guess that g is then a Littlewood–Richardson coefficient is false (for $\lambda = \mu = (3, 1, 1)$ and $\nu = (3, 2)$ one has $g = 2$). (3) Decide whether Theorem 6.3 with $a = b = k$ is the correct analogue of Theorem 1.1 for k -fat hooks, and whether the index $2ab$ is the smallest valid one for every pair (a, b) (it is for $(1, 2)$). (4) Find a description, valid for all n , of the triples of Theorem 8.3, or show that they do not form a family; decide whether such zeros are certified by criteria that we did not test. (5) Decide whether the finite corrections of Theorem 1.1 follow from known facets of the Kronecker cone or from the necessary condition of Panova and Zhao (Section 8).

Data and code availability. The version 3 archive that accompanies this paper (CC BY 4.0) contains all the items of version 2 and the following additions. Items of version 2: the certificate `h9_certificado.json` and the two programs `h9_certificate.py` (construction) and `verify_h9_cert.py` (independent checker); the programs that produce the tables of Section 8 (`asymptotic_check.py`, `residual_check.py`, `enum_flags.c`, `gtriples.c`, `run_sampling.py`, `report.py` and helper scripts) together with their output files (`tabla.md`, the exact counts for $n \leq 21$ and the sampling summaries for $22 \leq n \leq 32$); the scripts for the exhaustive check, the sharpness family and the character tables; and the Lean 4 sources (without the Mathlib build directory). Additions of version 3: the programs for the criteria of Section 8.4 (`lr_criterion.py`, `other_criteria.py`, `vallejo_reduction.py`, `coverage_lr.py`, `run_lr.py`, `run_big.py`) and for the enumeration of Section 8.5 (`enum_flags_nh.c`); the scripts for the cone comparison, for the example of Section 8.6 and for the consistency checks of Section 6 (`cono_contraste.py`, `cono_nada.py`, `verifica_Qe.py`, `eps2ab_test.py`, `general_BR_test.py`); the scripts and outputs for the 12 classes at $n = 23$ (directory L-032); and a small Python package of vanishing criteria (L-025-kronfilter) used to check that Dvir’s, the depth and the length bounds do not certify the example of Section 8.6. The large intermediate arrays (support tensors and character tables for $n \geq 17$) and the compiled binaries are not included; they can be regenerated with the included programs. The scripts contain absolute paths of the author’s machine, which have to be adapted. A file list is in the `README.md` of the archive.

AI-assistance statement. An AI assistant (a large language model, Claude, Anthropic) was used as a tool for literature organisation, drafting of proofs, writing the verification code, the Lean 4 scripts, the certificate construction, the computations of Section 8, and the LaTeX source. This includes the argument of Section 6, the comparison with the Young–Vallejo criterion and the Kronecker cone, and the computations at $n = 23$ and $n = 46$. The statement and proof of Theorem 1.1 were checked by the author against the sources listed; the computations were cross-checked by independent implementations, and the certificate of Theorem 4.3 was checked by a separate program and reproduced in a second environment. The author is not a specialist in the field. The proofs have not undergone formal peer review, and nobody has reviewed the Lean formalisation, which, although complete for the Specht modules, trusts Lean’s compiled evaluation for the certificate check (Section 9); the audit of its definitions was made by the author. We do not claim novelty beyond the searches described in Section 7. The AI system is not an author, and the author takes responsibility for the content.

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