

A strictly PF_2 Green kernel with nonreal polynomial roots

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Abstract

We give an explicit counterexample to the polynomial converse posed by Albert and Levandosky (arXiv:2603.23876v1, Remark 3.7). For

$$p(z) = \frac{(z+1)(z+4)((z+8)^2 + 36)}{400},$$

the Fourier symbol $p(\xi^2)$ is positive and elliptic of degree eight, although p has nonreal zeros. Its inverse Fourier kernel is a positive, even C^6 probability density K satisfying $(\log K)'' < 0$ on $\mathbb{R}$. Consequently, all ordered two-by-two minors of the translation kernel $K(x-y)$ are strictly positive. The proof uses an explicit formula in exponentials and trigonometric functions, followed by elementary estimates on a compact interval and on the tail.

1 Introduction

If p is a nonconstant polynomial with positive leading coefficient and only negative real zeros, the inverse Fourier transform of $1/p(\xi^2)$ is a positive multiple of a convolution of two-sided exponentials. It is therefore a Pólya frequency function of every order; see Schoenberg [3]. Albert and Levandosky [1, Remark 3.7] ask whether total positivity of order two forces the same restriction on the zeros of p . We give a counterexample with p of degree four.

We use the Fourier convention

$$\widehat{g}(\xi) = \int_{\mathbb{R}} e^{it\xi} g(t) dt, \quad \mathcal{F}^{-1}h(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} h(\xi) d\xi. \quad (1.1)$$

A nonnegative function K is a Pólya frequency function of order two, written $K \in \text{PF}_2$, if

$$\det \begin{pmatrix} K(x_1 - y_1) & K(x_1 - y_2) \\ K(x_2 - y_1) & K(x_2 - y_2) \end{pmatrix} \geq 0 \quad (x_1 < x_2, y_1 < y_2). \quad (1.2)$$

It is *strictly* PF_2 if $K > 0$ on $\mathbb{R}$ and the determinant in (1.2) is positive for every pair of strictly increasing row and column nodes. This is the strictly $\text{PF}(2)$ convention of [1, Definition 1.5].

The precise implication asked about in [1, Remark 3.7] is the following: for a real polynomial p whose symbol $\alpha(\xi) = p(\xi^2)$ is positive and comparable to $1 + |\xi|^{2s}$ for some $s > 1/2$, does $\mathcal{F}^{-1}(1/\alpha) \in \text{PF}_2$ force every zero of p to be negative and real? Our example satisfies these growth assumptions with $s = 4$ and the strict inequality in (1.2). The positive converse proved there for quartic symbols concerns p of degree two. The reciprocal symbol represents the inverse of a constant-coefficient differential operator; its inverse transform is the corresponding Green kernel, as explained in Section 4.

Earlier rational-transform examples of Belton, Guillot, Khare and Putinar [2, Lemma 8.4] separate finite-order from all-order total positivity, but have nonconstant numerators after cancellation. The example below meets the additional reciprocal-polynomial requirement $1/p(\xi^2)$. We compare these constructions in Section 4.

Theorem 1. *Let*

$$p(z) = \frac{(z+1)(z+4)((z+8)^2+36)}{400}, \quad \alpha(\xi) = p(\xi^2), \quad (1.3)$$

and put $K = \mathcal{F}^{-1}(1/\alpha)$ with convention (1.1). Then p has zeros $-1, -4, -8+6i, -8-6i$, while α is even and strictly positive on $\mathbb{R}$, with

$$\frac{1+|\xi|^8}{400} \leq \alpha(\xi) \leq \frac{117}{40}(1+|\xi|^8). \quad (1.4)$$

The function K is an even C^6 probability density and satisfies

$$K(t) > 0, \quad (\log K)''(t) < 0 \quad (t \in \mathbb{R}). \quad (1.5)$$

In particular, K is strictly PF₂. Explicitly, for $t \geq 0$,

$$K(t) = \frac{1}{663} \left[520e^{-t} - 425e^{-2t} + e^{-3t}(95 \cos t - 45 \sin t) \right], \quad (1.6)$$

and $K(-t) = K(t)$.

2 Fourier representation and regularity

We first establish the formula for K and its regularity. The positivity and strict log-concavity required for Theorem 1 are proved in Section 3.

Expanding (1.3) gives

$$p(z) = 1 + \frac{141}{100}z + \frac{23}{50}z^2 + \frac{21}{400}z^3 + \frac{1}{400}z^4. \quad (2.1)$$

Its factorization gives the stated zeros and positivity of α . The lower bound in (1.4) follows from the constant and leading coefficients. For the upper bound use $|\xi|^{2j} \leq 1 + |\xi|^8$ for $0 \leq j \leq 4$ and sum the coefficients in (2.1). Thus the ellipticity assumption holds with $s = 4$. The alternative standard ellipticity form also holds:

$$\frac{(1+\xi^2)^4}{400} \leq \alpha(\xi) \leq (1+\xi^2)^4. \quad (2.2)$$

Indeed, both inequalities follow coefficient by coefficient from (2.1).

The kernel formula follows from partial fractions and the Fourier transform of a two-sided exponential. For $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$, elementary integration gives

$$\mathcal{F}(e^{-a|t|})(\xi) = \frac{2a}{a^2 + \xi^2}. \quad (2.3)$$

Write the last term in the numerator of (1.6), extended evenly, as

$$\frac{95+45i}{2}e^{-(3-i)|t|} + \frac{95-45i}{2}e^{-(3+i)|t|}.$$

Applying (2.3), its transform is

$$\frac{330 + 40i}{\xi^2 + 8 - 6i} + \frac{330 - 40i}{\xi^2 + 8 + 6i} = \frac{660\xi^2 + 4800}{(\xi^2 + 8)^2 + 36}.$$

Together with the two real exponential terms, this gives the partial-fraction identity

$$\frac{1}{663} \left(\frac{1040}{z+1} - \frac{1700}{z+4} + \frac{660z+4800}{(z+8)^2+36} \right) = \frac{400}{(z+1)(z+4)((z+8)^2+36)}. \quad (2.4)$$

This identity follows on multiplying through by the denominator. The even extension of (1.6) is continuous and integrable, and its Fourier transform is therefore $1/\alpha$. Since $1/\alpha$ is integrable, Fourier inversion identifies it with K everywhere.

For $0 \leq j \leq 6$, the function $\xi^j/\alpha(\xi)$ belongs to $L^1(\mathbb{R})$: it is bounded near zero and is $O(|\xi|^{j-8})$ at infinity. Dominated differentiation in (1.1) consequently proves $K \in C^6(\mathbb{R})$. Thus the first two derivatives obtained from the formula on either half-line extend continuously to the origin, where

$$K(0) = \frac{190}{663}, \quad K'(0) = 0, \quad K''(0) = -\frac{50}{221}. \quad (2.5)$$

Finally, $\int_{\mathbb{R}} K(t) dt = \widehat{K}(0) = 1/p(0) = 1$. Positivity, proved next, makes this a probability density.

3 Positivity and strict log-concavity

The two negative real roots produce the slow exponential terms in (1.6); the conjugate roots produce the faster-decaying oscillatory correction. The opposite signs of the slow terms give a negative contribution to $KK'' - (K')^2$. The estimates below show that the correction never reverses this sign. We use trigonometric signs near the origin and amplitude bounds on the tail, splitting at $t = 1$ for convenient rational bounds.

3.1 Positivity

By evenness it suffices to work on $t \geq 0$. If $0 \leq t \leq 1$, then $\cos t \geq 1 - t^2/2 \geq 1/2$ and $0 \leq \sin t \leq 1$. Hence

$$95 \cos t - 45 \sin t \geq \frac{5}{2}, \quad 520e^{-t} - 425e^{-2t} \geq 95e^{-t}.$$

Equation (1.6) is therefore strictly positive on $[0, 1]$. For $t \geq 1$, use $e^{-t} \leq e^{-1} < 1/2$ and

$$|95 \cos t - 45 \sin t| \leq \sqrt{95^2 + 45^2} = \sqrt{11050} < 106.$$

It follows that

$$K(t) \geq \frac{e^{-t}}{663} \left(520 - \frac{425}{2} - \frac{106}{4} \right) = \frac{281}{663} e^{-t} > 0. \quad (3.1)$$

This proves positivity on the whole line.

3.2 Strict log-concavity

Since K is positive, the sign of $(\log K)''$ is the sign of

$$Q(t) = K(t)K''(t) - (K'(t))^2.$$

Differentiating (1.6) and collecting terms gives, for $t \geq 0$,

$$Q(t) = \frac{50}{439569} e^{-3t} H(t), \quad (3.2)$$

where

$$\begin{aligned} H(t) = & -4420 + 52e^{-t}(49 \sin t + 93 \cos t) \\ & - 85e^{-2t}(19 \sin t + 9 \cos t) - 221e^{-3t}. \end{aligned} \quad (3.3)$$

The constant term -4420 comes from the interaction of the two real exponentials; every other term decays as $t \rightarrow \infty$. An amplitude bound therefore controls the tail. On $[0, 1]$, we use $H(0) = -570$ and show that H is decreasing. Direct differentiation yields

$$\begin{aligned} e^t H'(t) = & -2288 \cos t - 7384 \sin t \\ & + 85e^{-t}(47 \sin t - \cos t) + 663e^{-2t}. \end{aligned} \quad (3.4)$$

On $[0, 1]$, $\sin t \geq 0$ and $\cos t \geq 1/2$. Dropping the negative term $-85e^{-t} \cos t$ and using $e^{-t} \leq 1$ gives

$$\begin{aligned} e^t H'(t) & \leq -2288 \cos t - 3389 \sin t + 663 \\ & \leq -1144 + 663 = -481 < 0. \end{aligned}$$

Thus $H(t) \leq H(0) = -570$ throughout this interval.

For $t \geq 1$, the amplitude bounds

$$|49 \sin t + 93 \cos t| \leq \sqrt{11050} < 106, \quad |19 \sin t + 9 \cos t| \leq \sqrt{442} < 22$$

and $e^{-t} < 1/2$ imply, on discarding the last negative term of (3.3),

$$H(t) \leq -4420 + \frac{52 \cdot 106}{2} + \frac{85 \cdot 22}{4} = -\frac{2393}{2} < 0. \quad (3.5)$$

It follows from (3.2) that $Q(t) < 0$ for every $t \geq 0$. The regularity proved in Section 2 and evenness show that Q is continuous and even on $\mathbb{R}$. At the origin, explicitly,

$$Q(0) = -\frac{9500}{146523} < 0.$$

Since K is positive,

$$(\log K)'' = \frac{Q}{K^2} < 0 \quad \text{on } \mathbb{R}.$$

This proves (1.5) globally.

3.3 Strict positivity of translation minors

To pass from strict log-concavity to the determinant inequality, put $\ell = \log K$ and let $x_1 < x_2$, $y_1 < y_2$. The fundamental theorem of calculus gives

$$\begin{aligned} & \ell(x_1 - y_1) + \ell(x_2 - y_2) - \ell(x_1 - y_2) - \ell(x_2 - y_1) \\ & = - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \ell''(s - r) dr ds > 0. \end{aligned}$$

Exponentiating proves

$$K(x_1 - y_1)K(x_2 - y_2) > K(x_1 - y_2)K(x_2 - y_1),$$

which is precisely the strict inequality in (1.2). This completes the proof of Theorem 1. $\square$

4 The polynomial converse and earlier examples

The Fourier convention (1.1) and the ellipticity bound (1.4) place the example within Assumption 1.1 of [1]. Together with the nonreal zeros of p , strict PF₂ positivity therefore gives the negative answer to Remark 3.7 of its arXiv version 1. The normalization $p(0) = 1$ makes K a probability density. Replacing p by the monic polynomial $400p$ replaces K by $K/400$ and preserves all strict sign conclusions.

For the constant-coefficient operator $L = p(-d^2/dt^2)$, the Fourier identity gives $LK = \delta_0$ in the sense of tempered distributions. Thus K is a Green kernel for an eighth-order operator. Applications to nonlinear solitary waves require the additional nonlinear and solution hypotheses considered in [1].

For comparison, [2, Lemma 8.4 and its proof] considers

$$M_a(t) = (a + 1)e^{-a|t|} - ae^{-(a+1)|t|}, \quad a > 0.$$

Positive integer powers of M_a are log-concave and hence PF₂. For powers at least two, they are not Pólya frequency functions of all orders, and the cited proof expresses their bilateral Laplace transforms as reduced rational functions with nonconstant numerators. Their reciprocals are therefore not polynomials. Theorem 1 provides an explicit example within the reciprocal-polynomial class, with strict PF₂ positivity and an elementary proof of global log-concavity.

Verification and AI assistance

The accompanying script `checks/verify_pf2.py` checks the rational Fourier, curvature and derivative identities, the values at zero, and the rational estimate constants using exact symbolic arithmetic. The global inequalities are proved above and do not depend on numerical sampling.

The accompanying Lean 4 source project provides the declarations `PF2.paper_main` and `PF2.distributional_green_identity`, which formalize Theorem 1 and the distributional Green identity $p(-d^2/dt^2)K = \delta_0$, respectively. Run `check.sh` with the pinned dependencies to rebuild and audit the project. The mathematical source has a recorded successful build and axiom audit dated 12 October 2026; no fresh Lean run is claimed for this manuscript edition. Source repository: <https://github.com/Oliverds321/pf2-green-kernel-lean>.

This manuscript and its computational checks were developed with substantial assistance from OpenAI GPT-6 Astra, used through the personal assistant dot. This assistance included analysis of the example, proof development, literature comparison, drafting, and preparation of verification code.

References

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