

Existence and Uniqueness of the Minimal Ternary Zero-Sum Structure: The $3 \times 3 \times 3$ Queen-Line Geometry

Jianming Wang

Independent Researcher
Luo Shu Crystal Cell Research Program

October 7, 2026

Abstract

This paper studies the zero-sum display problem on the classical $3 \times 3 \times 3$ queen-line geometry over $\{0, 1, 2\}^3$. The structure contains 27 points and 49 lines: 27 axis-parallel lines, 18 face-diagonals, and 4 space-diagonals. We characterize this structure by a corrected system of combinatorial axioms and prove:

- (1) The value algebra is forced to characteristic 3, and $\mathbb{F}_3$ is the unique minimal ternary alphabet;
- (2) Structure theorem: functions zero-sum on all 49 queen-lines are exactly affine functions, yielding a 4-dimensional solution space of 81 displays;
- (3) Duality theorem: 13 line directions and 13 projective classes of non-constant displays form the projective plane $\text{PG}(2, 3)$ via degeneracy incidence;
- (4) Coordinate reconstruction theorem: any structure satisfying the corrected axioms is isomorphic to the classical queen-line geometry, unique up to the full cube symmetry group of order 48.

This version corrects errors in previous drafts regarding space-diagonal collinearity, degree sequence, automorphism group, and axiom strength, and removes conditions contradicting the classical structure.

Version chain:

v1→RR→v1.1→v1.2→assembly→recon→v3→redteam-11→v4→redteam-12→v5→**v5.3**
(final)

Contents

1	Introduction	2
2	Target Structure	2
3	Corrected Axioms	2
4	Value Algebra: Characteristic 3	3
5	Display Theorem	4
6	Duality Theorem: $\text{PG}(2, 3)$	5
7	Coordinate Reconstruction	6
8	Main Theorem	7
9	Relation to Prior Work	7
	References	8
	Appendix A Display Classes	8
	Acknowledgments	9

1 Introduction

Zero-sum configurations and the cap-set problem are dual over finite fields. The classical $3 \times 3 \times 3$ queen-line geometry is the set of three-point lines along axis, face-diagonal, and space-diagonal directions in the three-dimensional cube. Its order-3 perfect magic cube does not exist over the integers, but over $\mathbb{F}_3$ there is a rich family of affine displays. This paper establishes a rigorous existence and uniqueness theorem at the structural level.

2 Target Structure

Let

$$P = \mathbb{F}_3^3 = \{0, 1, 2\}^3.$$

The classical queen-line set $\mathcal{L}_Q$ contains:

- **Axis-parallel lines:** fix two coordinates, vary the third. Total $3 \cdot 3^2 = 27$.
- **Face-diagonals:** in each coordinate plane, the two diagonals passing through the plane center. Total $3 \cdot 3 \cdot 2 = 18$.
- **Space-diagonals:** the 4 lines connecting opposite vertices of the cube, all passing through the global center $(1, 1, 1)$.

Total:

$$27 + 18 + 4 = 49.$$

The degree sequence of the classical queen-line geometry is

$$13^{\times 1}, \quad 7^{\times 8}, \quad 5^{\times 6}, \quad 4^{\times 12},$$

with degree sum 147. Its automorphism group is the full cube symmetry group

$$S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3,$$

of order $6 \cdot 8 = 48$.

3 Corrected Axioms

Definition 3.1 (Axioms for a 3D complete display). A *three-dimensional complete display* is a pair $(P, \mathcal{L})$, where P is a 27-point set and $\mathcal{L}$ is a family of 3-element subsets (lines), satisfying:

- (S1) **Linearity**: any two points lie on at most one line; $|\mathcal{L}| = 49$.
- (S2) **Nets**: $\mathcal{L}$ contains three pairwise disjoint subclasses A_1, A_2, A_3 , each a partition of P into 9 triples (parallel classes), whose induced equivalence relations pairwise commute.
- (S3) **Face-diagonals**: each 9-point “plane” (section transversal to a net class) determined by the nets contains exactly 2 further lines, and these two intersect at the plane center.
- (S4) **Space-diagonals**: beyond net-lines and face-diagonals there remain exactly 4 lines. These 4 lines all pass through a common point O , and O is simultaneously the center of one transversal plane for A_1 , one for A_2 , and one for A_3 .
- (S5) **Isotropy**: there exists an automorphism of order 3 cyclically permuting $A_1 \rightarrow A_2 \rightarrow A_3$.
- (S6) **Display**: there exists a non-constant map $f : P \rightarrow \mathbb{F}_3$ that is zero-sum on every line.

Remark 3.1. The original v4 axiom merely required “the remaining 4 lines,” which was insufficient to uniquely determine the classical structure. The corrected (S4) forces the remaining 4 lines to be the classical space-diagonals by the purely combinatorial requirement that they be concurrent at O and that O be the center of one plane from each net. In particular, the previous conditions “intersects each non-central equivalence class in exactly one point” and “central equivalence class”—both contradicting the classical structure—have been removed.

4 Value Algebra: Characteristic 3

Theorem 4.1 (Characterization of characteristic 3). Let R be a unital ring. The following are equivalent:

- (i) Every three-term arithmetic progression $(a, a + d, a + 2d) \subseteq R$ sums to zero;
- (ii) $\text{char}(R) = 3$.

Proof. Condition (i) means $3(a + d) = 0$ for all $a, d \in R$. Taking $d = 0$, $a = 1$ gives $3 \cdot 1 = 0$, which is (ii). Conversely, if $\text{char}(R) = 3$ then $3x = 0$ identically. $\square$

Corollary 4.1 (Minimal uniqueness of $\mathbb{F}_3$). If $|R| = 3$ and R carries a display, then $R \cong \mathbb{F}_3$, uniquely.

Proof. A ring of prime order is necessarily a field: the additive group is cyclic of prime order, and distributivity forces commutative multiplication modulo p , yielding the unique field structure. Hence $\mathbb{F}_3$ is the unique ternary ring carrying a display. $\square$

5 Display Theorem

Theorem 5.1 (Structure theorem). Let $f : \mathbb{F}_3^3 \rightarrow \mathbb{F}_3$. The following are equivalent:

- (i) f is zero-sum on every queen-line (49 lines);
- (ii) f is zero-sum on every axis-parallel line and every face-diagonal (45 lines);
- (iii) f is an affine function, i.e.

$$f(x, y, z) = c + ax + by + dz.$$

The solution space has dimension 4, giving $3^4 = 81$ displays, of which 3 are constant.

Proof. Axial elimination of squares. Fix coordinate i and freeze the others, obtaining a univariate function

$$g(t) = f(\dots, t, \dots) = at^2 + bt + c.$$

The axis-parallel line condition gives $g(0) + g(1) + g(2) = 0$. Over $\mathbb{F}_3$,

$$g(0) + g(1) + g(2) = c + (a + b + c) + (a + 2b + c) = 2a + 3b + 3c = 2a.$$

Hence $a = 0$. Therefore f contains no x_i^2 term in its polynomial representation. So

$$f(x, y, z) = c + ax + by + dz + pxy + qxz + ryz + sxyz.$$

Face-diagonal elimination of cross terms. Consider the diagonal on the plane $z = 0$:

$$(0, 0, 0), (1, 1, 0), (2, 2, 0).$$

Substituting gives

$$f(0, 0, 0) = c, \quad f(1, 1, 0) = c + a + b + p, \quad f(2, 2, 0) = c + 2a + 2b + p.$$

Their sum is $3c + 3a + 3b + 2p = 2p$. Zero-sum forces $p = 0$. Similarly, using face-diagonals on the planes $y = 0$ and $x = 0$ gives $q = 0$ and $r = 0$. Finally, using the diagonal $(0, 0, 1), (1, 1, 1), (2, 2, 1)$ on $z = 1$ yields $2s = 0$, so $s = 0$.

Thus $f(x, y, z) = c + ax + by + dz$, i.e. f is affine.

Conversely, if $f(p) = c + \alpha^\top p$, then on any three-term arithmetic line $\{p, p+v, p+2v\}$,

$$f(p) + f(p+v) + f(p+2v) = 3c + \alpha^\top(3p + 3v) = 0.$$

Hence the displays are exactly the affine functions. □

Corollary 5.1 (Independent directions). Only the 3 axial directions and the 3 same-sign face-diagonal directions (6 directions, 36 lines) are needed to force affinity; opposite-sign face-diagonals and all space-diagonals are redundant.

Corollary 5.2 (Failure occurs in parallel classes). If f is not affine, its zero-sum failures occur as entire axial or face-diagonal parallel classes, never on isolated lines alone.

6 Duality Theorem: $\text{PG}(2, 3)$

Theorem 6.1. The 13 directions D and the 13 projective classes P of non-constant displays (coefficients $\alpha \in \mathbb{F}_3^3 \setminus \{0\}$ modulo scalars) form the projective plane $\text{PG}(2, 3)$ via the degeneracy incidence

$$[v] \text{ and } [\alpha] \text{ are degenerate} \iff \alpha^\top v = 0.$$

Each direction is incident with exactly 4 display classes, each display class degenerates on exactly 4 directions, any two directions share exactly one degenerate display, and any two displays share exactly one degenerate direction.

Proof. Directions are 1-dimensional subspaces of $\mathbb{F}_3^3$, totaling $(3^3 - 1)/(3 - 1) = 13$; display classes (projective classes of α) are also 1-dimensional subspaces, again 13. The incidence $\alpha^\top v = 0$ is a point–line relation in $\text{PG}(2, 3)$: fixing $[\alpha]$, the solutions $[v]$ form a 2-dimensional subspace, i.e. a line in $\text{PG}(2, 3)$ containing $q + 1 = 4$ points. For $q = 3$, standard finite-geometry counting applies: $q^2 + q + 1 = 13$ points and 13 lines, each line containing 4 points, each point on 4 lines, with unique joins and meets. $\square$

Corollary 6.1 (Constant-line counts by display type). No rainbow display (taking three distinct values on every queen-line) exists. Each non-constant display is constant on all queen-lines in its 4 degenerate directions. The numbers of constant queen-lines are:

- Axial-type display class: $2 \cdot 9 + 2 \cdot 3 = 24$ lines;
- Face-diagonal-type display class: $9 + 3 + 2 \cdot 1 = 14$ lines;
- Space-diagonal-type display class: $3 \cdot 3 + 1 = 10$ lines.

Proof. In the classical queen-line geometry, axial directions contribute 9 parallel lines each, face-diagonal directions 3 each, and space-diagonal directions 1 each. Counting by degenerate direction type yields the claim. $\square$

Remark 6.1. The 3 constant displays belong to no projective class; they are the unique displays that are zero-sum on all 49 lines but nowhere rainbow.

7 Coordinate Reconstruction

Theorem 7.1. Any structure satisfying (S1)–(S6) is isomorphic to the queen-line geometry on $\{0, 1, 2\}^3$; the isomorphism is unique up to the full cube symmetry group of order 48.

Proof. The proof proceeds in five steps.

Step 1: Nets to product coordinates. By (S2), the three nets A_1, A_2, A_3 are pairwise disjoint, each a partition of P into 9 triples, inducing three equivalence relations E_1, E_2, E_3 that pairwise commute. By (S1) and the pairwise disjointness of the three nets in (S2), no two points can lie on the same line in two different nets; if they belonged to different lines in all three nets, they would lie on at least two lines, violating (S1). Hence

$$E_1 \cap E_2 \cap E_3 = \text{the identity relation.}$$

By the standard product theorem for commuting equivalence relations,

$$P \cong X \times Y \times Z, \quad |X| = |Y| = |Z| = 3.$$

We may therefore identify $P = \mathbb{F}_3^3$. The three nets are then the axis-parallel line families, uniquely determining the 27 axis-parallel lines.

Step 2: Face-diagonals are uniquely determined. Consider any coordinate plane, e.g. fixing $x = a$, yielding $\{a\} \times \mathbb{F}_3 \times \mathbb{F}_3$. It contains lines from the other two nets, forming a 3×3 grid. By (S3), this plane contains exactly 2 further lines, intersecting at the plane center. In a 3×3 grid, the only three-point lines through the center that are not net-lines are the two diagonals. Hence these two lines are uniquely determined as the face-diagonals of that plane. Repeating for all 9 coordinate planes yields 18 face-diagonals, exactly the classical ones.

Step 3: The remaining 4 lines must be space-diagonals. We have already determined $27 + 18 = 45$ lines. By (S4), exactly 4 lines remain, and they all pass through a common point O that is the center of one transversal plane for each of A_1, A_2 , and A_3 . In product coordinates, a transversal plane to A_1 is $x = a$ with center $(a, 1, 1)$; to A_2 is $y = b$ with center $(1, b, 1)$; to A_3 is $z = c$ with center $(1, 1, c)$. The intersection of these three sets is uniquely $O = (1, 1, 1)$.

Among the 13 lines through O , the 3 axis-parallel lines are already in the 27 axis-parallel lines, and the 6 face-diagonals are already in the 18 face-diagonals. The remaining 4 can only be the four space-diagonal directions

$$(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)$$

modulo overall sign. Each direction has exactly one full line through O , so the remain-

ing 4 lines are necessarily the 4 space-diagonals. Hence $\mathcal{L} = \mathcal{L}_Q$.

Step 4: Automorphism group. An automorphism must preserve the set of three nets $\{A_1, A_2, A_3\}$, so in product coordinates it is composed of coordinate permutations and coordinate-wise bijections. To preserve face-diagonals, each coordinate bijection must map the central element to itself. In a 3-element set, the permutations fixing the center are exactly the identity and the swap of the two non-central elements, giving 2 choices per coordinate. Thus there are $2^3 = 8$ coordinate-wise choices, times $3! = 6$ coordinate permutations, for a total of

$$6 \cdot 8 = 48.$$

Hence the automorphism group is $S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$, of order 48.

Step 5: Conclusion. Steps 1–4 compress all degrees of freedom into the classical queen-line geometry, establishing uniqueness. $\square$

Remark 7.1. This proof does not use (S6): the coordinate reconstruction is purely combinatorial. The display theory on the queen geometry is automatic (Theorem 5.1); (S6) incorporates the zero-sum requirement into the definition rather than the uniqueness argument.

8 Main Theorem

Theorem 8.1 (Main theorem). A three-dimensional complete display exists with $|P| = 27$; every such structure is isomorphic to the queen-line geometry on $\{0, 1, 2\}^3$ (27 axis, 18 face-diagonal, 4 space-diagonal, 49 total), unique up to the full cube symmetry group of order 48. The value algebra is forced to be $\mathbb{F}_3$; displays are exactly affine functions (4-dimensional solution space, 81 displays); the 13 directions and 13 projective classes of non-constant displays form $\text{PG}(2, 3)$ via degeneracy incidence; of the 13 directions, only 6 carry independent constraints.

Proof. Existence is given by the classical queen-line geometry. From (S2), the three nets are each partitions into 9 triples, so $|P| = 27$ immediately. Uniqueness is Theorem 7.1. The value algebra is Theorem 4.1 and Corollary 4.1. The display theorem is Theorem 5.1. The duality theorem is Theorem 6.1. Independent directions are Corollary 5.1. $\square$

9 Relation to Prior Work

A companion note axiomatizes the integer layer: it re-derives the known uniqueness of the order-3 magic square and registers the three-dimensional extension as heuris-

tic. This paper works at the structural ($\mathbb{F}_3$) level, proving a genuine existence and uniqueness theorem via corrected combinatorial axioms.

References

- [1] J. E. Olson, A combinatorial problem on finite abelian groups, I, *J. Number Theory* 1 (1969), 8–10.
- [2] P. Erdős, A. Ginzburg, A. Ziv, Theorem in the additive number theory, *Bull. Res. Council Israel* 10F (1961), 41–43.
- [3] C. Chevalley, Démonstration d’une hypothèse de M. Artin, *Abh. Math. Semin. Univ. Hambg.* 11 (1935), 73–75.
- [4] J. S. Ellenberg, D. Gijswijt, On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression, *Ann. of Math.* 185 (2017), 339–343.
- [5] The Hesse configuration, arXiv:1107.4387.
- [6] W. Trump, The successful search for the smallest perfect magic cube, trump.de/magic-squares.
- [7] B. Alspach, K. Heinrich, Perfect magic cubes of order $4m$, *Fibonacci Quart.* 19 (1981), 97–106.
- [8] J. Wang, The unique minimal structure under discrete local zero-sum conditions: the necessity of the order-3 magic square, AiXiv:260702.000001 (2026).
- [9] J. Wang, Coupling period formula for multi-cell coupled systems over finite fields, AiXiv:260930.000001 (2026).

Appendix A Display Classes and Constant-Line Counts

The 13 directions (modulo $\pm$) are classified as:

Type	Representative directions	Count	Lines
Axial	$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	3	9
Face-diagonal	$(1, 1, 0), (1, -1, 0), (1, 0, 1), (1, 0, -1), (0, 1, 1), (0, 1, -1)$	6	3
Space-diagonal	$(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)$	4	1

The corresponding constant-line counts for display classes:

- Axial-type display class: degenerate directions contain 2 axial + 2 face-diagonal, constant-line count $2 \cdot 9 + 2 \cdot 3 = 24$.
- Face-diagonal-type display class: degenerate directions contain 1 axial + 1 face-diagonal + 2 space-diagonal, constant-line count $9 + 3 + 2 \cdot 1 = 14$.
- Space-diagonal-type display class: degenerate directions contain 3 face-diagonal + 1 space-diagonal, constant-line count $3 \cdot 3 + 1 = 10$.

Acknowledgments

The research direction and all scientific judgments were independently proposed and made by the author. Large-scale symbolic verification, complete enumeration, independent red-team review of proof obligations, and text drafting were performed by an AI assistant; its contribution is acknowledged and thanked here per arXiv policy, without co-authorship. Complete version chains of intermediate reports are available from the author upon request.

This preprint has not yet undergone peer review.