

Simple zeros on the critical line of primitive Dirichlet L -functions on average over moduli $q \leq Q$: the killed kernel and the Shell kernel

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Abstract

Let $\mathfrak{F}_Q$ be the set of primitive Dirichlet characters of modulus $1 < q \leq Q$ and $T = (\log Q)^{r_0 + \varepsilon}$ with $r_0 \geq 3$, $\varepsilon > 0$ fixed. The preprint [9] proved that, as $Q \rightarrow \infty$, at least 0.7212 of the zeros of the $L(s, \chi)$, $\chi \in \mathfrak{F}_Q$, with ordinate in $(T, 2T]$, counted with multiplicity and summed over the family, are simple and lie on the critical line. We raise this constant in two ways. First, a large-sieve lemma for the part of the dual variable beyond the diagonal (Farey points of level Q avoid the major arcs of small denominator) gives the constant 0.7235 with the rate $\log \log Q / \log Q$ and no hypothesis of any kind. Second, the “Shell route”, which uses the explicit formula for the characters of small modulus and two published zero-density estimates for the family (Jutila’s log-free hybrid estimate and Montgomery’s hybrid estimate), gives 0.9059137927, with an error $O((\log Q)^{-\theta})$ for every $\theta < 503/1994$. Both results are asymptotic; the thresholds implicit in the proofs are very large. Each is proved on paper, and each statement is a Lean 4 theorem in the release of the accompanying Lean repository that depends only on the three standard axioms: the first has no hypothesis beyond $r_0 \geq 3$, $\varepsilon > 0$, and the second none beyond the two zero-density estimates and $r_0 \geq 3$, $\varepsilon > 0$, $0 < \theta < 503/1994$ (§1.5). The results have not been independently refereed, and the Lean development has not been rebuilt outside this project. A corresponding value for the moduli $Q/2 < q \leq Q$ is not claimed. The results are family averages and say nothing about an individual L -function.

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1 Introduction

1.1 Results

Let $\mathfrak{F}_Q$ be the set of primitive Dirichlet characters χ of modulus q with $1 < q \leq Q$. For $\chi \in \mathfrak{F}_Q$ and $0 \leq T_1 < T_2$ let $N_\chi(T_1, T_2)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $0 < \beta < 1$ and $T_1 < \gamma \leq T_2$, counted with multiplicity, and let $N_{0,\chi}^s(T_1, T_2)$ be the number of those that are simple and have $\beta = \frac{1}{2}$. Throughout, $r_0 \geq 3$ and $\varepsilon > 0$ are fixed and

$$T = T(Q) = (\log Q)^{r_0 + \varepsilon}.$$

(The height exponent is called r in [9] and in the Lean statements; here r denotes a modulus, see the notation table in §2.1.) The preprint [9] proved that $\sum_{\chi \in \mathfrak{F}_Q} N_{0,\chi}^s(T, 2T) \geq (0.7212 - o(1)) \sum_{\chi \in \mathfrak{F}_Q} N_\chi(T, 2T)$. This paper proves the following two theorems.

Theorem 1.1 (killed kernel). *There are Q_0 and $c > 0$, depending on r_0 and ε , such that for every integer $Q \geq Q_0$*

$$\sum_{\chi \in \mathfrak{F}_Q} N_{0,\chi}^s(T, 2T) \geq \left(0.7235 - c \frac{\log \log Q}{\log Q}\right) \sum_{\chi \in \mathfrak{F}_Q} N_\chi(T, 2T).$$

Status. Proved on paper (§3), with the corrections to the working draft listed in Appendix B. In Lean 4 [7] the statement is `ZetaShell.lemmaK.theorem_one_prime_design`, a theorem with the three standard axioms and no hypothesis beyond $r_0 \geq 3$, $\varepsilon > 0$, in the library `ZetaShell` of the accompanying Lean repository (§1.5) [L].

Theorem 1.2 (the Shell kernel). *Assume the zero-density estimates (J) and (M) of §4.6, which are theorems of Jutila [15, Theorem 1] and of Montgomery [16] (in the form proved in [2, Théorème 20]). Let $0 < \theta < 503/1994$. There are Q_0 and $c > 0$, depending on r_0 , ε and θ , such that for every integer $Q \geq Q_0$*

$$\sum_{\chi \in \mathfrak{F}_Q} N_{0,\chi}^s(T, 2T) \geq (0.9059137927 - c(\log Q)^{-\theta}) \sum_{\chi \in \mathfrak{F}_Q} N_\chi(T, 2T).$$

Status. Proved on paper (§§4–7), with the corrections listed in Appendix B. In Lean (§1.5) its statement of record, `ZetaShell.shell.S53_qle_reduced`, carries (J) and (M) as its only displayed hypothesis. It is a theorem in the library `ZetaShell` of the release: it depends on the three standard axioms only, and the census of the release finds no open statement below it [L] (Appendix A). What is claimed can be read in two trusted files that depend on Mathlib [23] alone (§1.5(a)). (J) and (M) are published theorems, so Theorem 1.2 is unconditional in the usual sense; we state them as hypotheses because they are the only inputs that we have neither re-derived nor formalised.

Table 1: Lower bounds for $\liminf_{Q \rightarrow \infty} \sum_{\chi} N_{0,\chi}^s(T, 2T) / \sum_{\chi} N_\chi(T, 2T)$ over $\mathfrak{F}_Q$ ($1 < q \leq Q$) at $T = (\log Q)^{r_0 + \varepsilon}$, $r_0 \geq 3$. The labels are those of §1.5.

previous [9]	this paper	status
0.7212	0.7235 (Thm 1.1)	proved on paper; in Lean [L] (in the released library, §1.5)
	0.9059137927 (Thm 1.2)	proved on paper, with (J), (M) quoted [C]; in Lean [L] with (J), (M) as hypothesis (in the released library, §1.5)

The constant 0.7235 is the truncation of $2 - \mathcal{B}_{C^+}^K(v_d) = 0.7235110849\dots$, where $\mathcal{B}^K$ is the functional (8) and v_d is the design profile of [9]; 0.9059137927 is the truncation of $2 - \mathcal{B}_{\alpha'}(v_p) =$

0.90591379273786... for the degree-12 profile S53-L75 (§6). Both values are exact rationals computed in Lean’s kernel. The preprint’s 0.7212 is a Lean theorem with no hypothesis in the library ZetaQ of the same repository (§1.5). For the moduli $Q/2 < q \leq Q$ the preprint proved 0.7098; the corresponding Shell value is discussed in Remark 4.19 and is not claimed.

1.2 What is new

The method is that of [9], which runs the rank–trace certificate of Alpöge and Furman [1] on the family Gram matrix of $\mathfrak{F}_Q$ (§2). The certificate converts an upper bound $\|\widehat{G}\|_F^2 \leq (\mathcal{B}_* + o(1))\mathcal{N}$ for the Frobenius norm of the normalised family Gram matrix into the lower bound $2 - \mathcal{B}_*$ for the proportion of simple critical zeros (Proposition 2.1). In [9], $\mathcal{B}_*$ is the value of the functional

$$\mathcal{B}(v) = \psi(0) + \int k(\alpha)\psi(\alpha) d\alpha, \quad \psi = v \star v,$$

at a window profile v , with kernel $k(\alpha) = |\alpha|$ on the in-zone $|\alpha| \leq 1$ and $k(\alpha) = C|\alpha|$ on the out-zone $1 < |\alpha| \leq \lambda$, where $C = \pi^4/18$ is the loss of the multiplicative large sieve applied pointwise in the dual variable. Both theorems of this paper change only the out-zone kernel.

The killed kernel (§3). For $|\alpha| > 1$ the prime sum $S(\vartheta) = \sum_n a_n e(n\vartheta)$ whose large-sieve mass is being bounded puts a fixed proportion of its mass on short arcs around the rationals a/r of small denominator, and no Farey point of level Q other than a/r itself comes near these arcs. Gallagher’s form of the large sieve with a zero set (Lemma 3.3) turns this into the kernel $C \min(|\alpha|, 1)$ (Lemma 3.1), unconditionally. The design profile of [9] then gives 0.7235.

The Shell kernel (§4). On an initial segment $1 < |\alpha| \leq \alpha'$ of the out-zone the kernel can be lowered to 1. Assuming the generalised Riemann hypothesis, Chandee, Lee, Liu and Radziwiłł [3, Theorem 2] give an asymptotic formula, uniformly on every compact subset of $(-2, 2)$, for their q -analogue of the pair correlation function, averaged over the primitive characters of conductor $q \asymp Q$ with a smooth weight in q/Q divided by $\varphi(q)$, and with Mellin-transform weights on the zeros: it is $(1 + o(1)) \min(|\alpha|, 1)$ plus terms that vanish for each fixed $\alpha \neq 0$ once Q is large, so its limit is 1 for each fixed $1 < |\alpha| < 2$ [V]. Their family and their weighting of the zeros differ from ours; on $(1, \alpha']$ the kernel 1 is the value that their theorem gives under that hypothesis [M]. A signed Farey identity (Lemma 4.1) writes the family sum as a weighted sum over Farey fractions; outside the shells around a/r with $r \leq R_1$ the weighted Farey density is asymptotically constant (Lemmas 4.3 and 4.4), inside the holes it vanishes, and the mass on the rings around the holes is expressed through the explicit formula for the primitive characters of modulus at most R_1 (Proposition 4.11) and bounded by two zero-density estimates for the family. These estimates reach $\alpha' < 5/3$ (§4.6); beyond α' the killed kernel is used. The window is taken at bandwidth $\lambda = 191/100$, much larger than the $\lambda^* = 1.2507$ of [9], and the only condition on the bandwidth that the proofs use in an essential way is $\lambda_{\text{eff}} < 2$ (§2.5).

Formalisation. Theorem 1.1, and Theorem 1.2 with (J) and (M) as its hypothesis, are Lean theorems in the release of the formalisation (§1.5); the formalisation found several statements of the working draft false as written; the paper states the corrected forms and lists the changes in Appendix B.

1.3 What the results are not

The theorems are lower bounds for family averages. They say nothing about any individual $L(s, \chi)$: a subfamily of density zero could carry all its zeros off the line. They say nothing about the generalised Riemann hypothesis.

Both theorems are asymptotic, and the thresholds in the proofs are very large. For Theorem 1.2, the condition $X \leq Q^2$ that makes the large sieve lossless starts near $\log_{10} Q \approx 177$ at $r_0 + \varepsilon = 3.5$ (§2.5); the existence of the design and the moment floor $a \geq 3/4$ need $\log Q$

at least of order 10^5 , and the formal proof's threshold is larger and implicit (§5); the error term $(\log Q)^{-\theta}$, $\theta < 503/1994$, decreases slowly. For Theorem 1.1, the prime number theorem enters Lemma 3.7 through the error term of $\sum_{n \leq y} \Lambda(n)$ at $y \asymp Q\mathcal{L}$ with a fixed power of $\log Q$ to spare; with the explicit constants of [10] the corresponding row first becomes small near $\log Q \approx 5900$ (Remark 3.8). No finite- Q statement of [9] is improved by either theorem, and none is claimed.

Theorem 1.2 rests on the two zero-density estimates (J) and (M), which we quote from the literature; their proofs are long and are not formalised (§4.6). Its rate is much slower than the rate $\log \log Q / \log Q$ of Theorem 1.1.

We do not claim the Shell constant for the moduli $Q/2 < q \leq Q$ (Remark 4.19), a bounded-height version of either theorem, the corresponding statements for the even and odd subfamilies, or any statement about distinct zeros; the working draft contains arguments for these (§8), and they have not been brought to the standard of this paper.

1.4 Relation to other work

Statements about other work carry the tag [V] when we read them in the source, [V~] when we read them only through a description (the authors' own, or that of the preprint [9]), and [M] when they are our inference.

The source of the method. Alpöge and Furman [1] prove unconditionally that at least two thirds of the nontrivial zeros of ζ , counted with multiplicity, are simple and lie on the critical line, and at least five sixths are distinct; with the Montgomery–Taylor window the constants become $0.6725\dots$ and $0.8362\dots$ (their Theorem A). Their Theorem B states that Theorem A holds verbatim for $L(s, \chi)$ for any fixed primitive Dirichlet character χ , and their abstract states that the results are formally verified in Lean 4 [V]. The rank–trace certificate, the Gram realisation of the explicit formula and the other inputs listed as H1–H6 in §2.2 are theirs. In their Remark 7.2 they record, with the details omitted, that summing over primitive $\chi \bmod q$, $q \leq Q$, with the smooth weights $(1 - q/Q)^2$, a weighted large-sieve inequality and a Gevrey taper give family-average proportions of at least 0.811 simple and on the line and 0.905 distinct, unconditionally, for T a power of $\log Q$; they state that this has been checked by independent derivations and is not included in their Lean formalisation [V]. That family has a different weight from ours, so we do not compare the numbers [M]. The family version for the unweighted family $1 < q \leq Q$ at polylogarithmic height, with its uniformity in q , is the work of [9].

Other family averages. Conrey, Iwaniec and Soundararajan [5] use their asymptotic large sieve [4] and Levinson's method to prove that at least $14/25$ of the zeros with $|\gamma| \leq T$, $(\log Q)^6 \leq T \leq (\log Q)^A$, are simple and on the critical line, on average over the primitive characters with the weight $\Psi(q/Q)/\varphi(q)$, Ψ smooth with compact support in $(0, \infty)$, so that $q \asymp Q$ [V]. Sono [21], “instead of the asymptotic large sieve”, uses a formula of Conrey, Iwaniec and Soundararajan for the mean square of the product of a Dirichlet L -function and a Dirichlet polynomial, with shifts [6] (by its abstract another application of their work on the asymptotic large sieve), which he extends to shifts with imaginary parts up to a power of $\log Q$, together with Feng's mollifier; he proves that at least 0.6044 of the zeros with $T \leq \gamma < 2T$, $(\log Q)^2 \leq T \leq (\log Q)^A$, are simple and on the critical line, on average over the even primitive characters with a smooth weight in q/Q supported in $[1, 2]$ [V]. Under the generalised Riemann hypothesis, Chandee, Lee, Liu and Radziwiłł [3] study low-lying zeros with the asymptotic large sieve, and Sono [20] uses their pair correlation formula [V]. This paper uses none of these inputs. Our family is the unweighted family of all primitive characters with $1 < q \leq Q$; the families of [5], [21] and [3] carry smooth conductor weights with $q \asymp Q$, that of [21] contains the even characters only, [5] counts the window $|\gamma| \leq T$, and [3] and [20] count low-lying zeros under that hypothesis; so we do not compare the numbers [M]. (We read the arXiv versions of [5],

[21] and [3], and the publisher’s abstract of [20].) The preprint [9] reports that it found no earlier result for the unweighted family of all primitive characters of modulus $q \leq Q$ [V~]; we have not searched further.

Per-character claims. In unrefereed Zenodo records, Hongyi Yang and Shihua Yang state bounds for ζ which, according to the records, hold for every fixed primitive Dirichlet L -function as well: at least 0.7962 of the zeros simple and on the critical line and 0.8981 distinct, with ineffective constants, a result the authors declare “at the certified-candidate level, not as an established theorem” [24]; and, in a second record [25], that almost all zeros are simple and on the critical line, where the record states that the constant layers are proved, that the analytic chain is a certified candidate pending external review, and that the certificates are kernel-checked in Lean 4; the record page shows its files under embargo, with the stated reason that the version contains errors and needs revision [V] (record pages as accessed on 28 September 2026). These are statements for each fixed character as the height tends to infinity; they are not averages over $q \leq Q$ at polylogarithmic height, and neither kind of statement implies the other [M]. We have not verified these claims.

1.5 Formal verification and status

A Lean 4 formalisation [7] accompanies this paper (toolchain v4.33.0-rc2, Mathlib [23] revision 51e6992). It is built on the Lean artifact Zeta23 of [1] and on the library ZetaQ that formalises [9], and it is released in the repository <https://github.com/Oliverds321/simple-zeros>, at tag v1.0, whose Lean sources are those of commit 4874cc0 of the development history (the repository is a snapshot of the release). The family library of this paper is its library ZetaShell, and Appendix A.8 gives the commands that check it. A declaration is *kernel-checked* if `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound`: no `sorry`, no added axiom, no `native_decide`. In the release the library ZetaShell has 327 files, 51,318 lines and 1,744 public theorems; 1,708 of them are kernel-checked, and 36 depend on `sorryAx`: 29 open statements, each with its own `sorry`, and 7 theorems derived from them. No other axiom and no `native_decide` occurs. The formal statuses below are those of the census of the release, one Lean job, run on the release commit 4874cc0, that prints, for each public theorem, its axioms and the open statements its proof reaches (its output is in `audit/followup/` of the repository); Appendix A quotes it (Appendix A.6).

We mark a theorem quoted from the literature by [C] and a numerical computation outside Lean by [N]. For the formal status we use the following labels.

- (a) [L] *Kernel-checked.* The only hypothesis that is not part of the mathematical statement is `ZeroDensityInput`, that is (J) and (M) (see below), and it occurs exactly where the paper assumes (J) and (M). In the library ZetaQ: the theorems of [9] at 0.7212 and 0.7098, for every height exponent ≥ 3 (`ZetaQ.JoinProved.theorem_one_generic_proved'` and `corollary_two_dyadic_proved'`), since the large-sieve input was re-proved in Gallagher’s form. Theorem 1.1, `ZetaShell.LemmaK.theorem_one_prime_design`, in the library ZetaShell of the release: in the census it depends on the axioms `propext`, `Classical.choice` and `Quot.sound` only and rests on no open statement, and the file `comparator/PrintAxioms/FollowUpShell.lean` prints the same axioms (Appendix A.8). Its hypotheses are $3 \leq r$ and $0 < \varepsilon$ (here r is our r_0), and its constant is 7235/10000. The companion statement at 0.7237 (`theorem_one_prime`) still rests on one open node. Also in the library ZetaShell: the certificates `ZetaShell.certS53` and `certD53`; the form of the prime number theorem that the proofs use, $\sum_{n \leq x} \Lambda(n) - x = o(x(\log x)^{-A})$ for every A ; the signed Farey identity, Lemma 4.14, Lemma 4.7 and Lemma 4.2(a)–(c); Lemma 4.3(b) and (c); the design layer at $\lambda = 191/100$ except its zone-slope node; Lemmas 3.2, 3.3 (in corrected form), 3.4, 3.5, 3.6 and $\sum_{r \leq R} \mu^2(r)/\varphi(r) \geq \log(R + 1)$; the hole bounds (7); Lemma 4.9, the Weil form of Lemma 4.10 and

Lemma 4.12(2),(2'); Lemma 4.15; Proposition 4.11 in the Weil form (propZ.W); Lemma 4.5 (signed_gallagher); Lemma 3.1(i) and Lemma 3.7; the Frobenius-row chain at the new bandwidth (F1c_chain); the strip, the integrability, the pointwise part of the killed zone and the majorant construction of Theorem 4.18; the family errors and the elementary terms of Theorem 4.17. The individual statements and their status are in Appendix A.

The statement of record of Theorem 1.2, `ZetaShell.shell_S53_qlc_reduced`, depends on the three standard axioms only and rests on no open statement. It is stated against counting functions defined from Mathlib alone and proved equal to those of `ZetaQ`, with the single displayed hypothesis `ZeroDensityInput`, which is (J) and (M) as stated in the sources (Appendix A.4). In the release both theorems of this paper are also stated in the trusted files `comparator/ChallengeDeps/FollowUpFamily.lean`, which imports Mathlib alone, and `comparator/Challenge/FollowUpFamily.lean`, as `family_simple_on_line_killed` (Theorem 1.1, no hypothesis) and `family_simple_on_line_shell` (Theorem 1.2, with `ZeroDensityInput` as its only hypothesis); these files are the place to read what is claimed. Both are proved in `comparator/Solution/FollowUpFamily.lean` from `theorem_one_prime_design` and `shell_S53_qlc_reduced`, and `comparator/PrintAxioms/FollowUpFamily.lean` prints both at the three standard axioms (Appendix A.8); the comparator tool itself has not been run on them. The statement of record is proved through the frame with the Frobenius row derived from Theorem 4.18, in a form that does not use the strong prime number theorem `PNTErrorTerm` (`shell_frame_qlc_of_K_noP`; the form `shell_frame_qlc_of_K`, which also assumes `PNTErrorTerm` and `CertS53`, is kernel-checked as well). Below it, also kernel-checked: Lemma 4.8 in its corrected form (`ring_le_primitive_corr`, with the Gauss sums of imprimitive characters, which Mathlib lacks, in a new module); Theorem 4.17 (`shell_S`) with its steps S1 and S2, Lemma 4.13 (`Z6a_zero_sums`) and Lemma 4.16 in the corrected form `Z6d_counts_corr`; Lemma 4.3(a) (`lemma2a`) and Lemma 4.4 (`lemmaP`), with `line_sum_asymp`, `s3_tail_plain_pt` and `s3_tail_sum`; and Theorem 4.18 (`thmK`) with its integrated killed-zone part (`K3b_integrated`), its Shell-zone part (`K2_shellZone`) and the pointwise assembly (13) in the corrected form `AS_pointwise_corr`; the transfer (`KT_transfer`) is kernel-checked there too, but the proof of the Shell-zone part does not use it. The corrected forms `Z6d_counts_corr` and `AS_pointwise_corr` each add one hypothesis to the first formal form of an internal statement (Appendix B); the first forms are kept in the library, open and unused.

- (b) [Lo] *Kernel-checked modulo named open nodes.* The frame of the formal Shell statement with the Frobenius row as one node (`shell_frame_qlc_of_nodes`) rests on one open node, `frob_row_shell`, which is the Frobenius row of Theorem 1.2, that is, the Shell argument. The proof of the statement of record does not use it: it derives the Frobenius row from Theorem 4.18 and the Frobenius-row chain (see (a)).
- (c) [L-stated] *Stated in Lean, not proved.* Of the statements this paper names: the first formal forms of Lemma 4.16 (`Z6d_counts`) and of the pointwise assembly (`AS_pointwise`), which the corrected forms replace and which no proof uses (Appendix B); the Frobenius row `frob_row_shell` of (b); the zone-slope statement of §5(viii), which the proof of the statement of record does not use; and earlier forms of the headline and the dyadic and bounded-height versions of the statement of record, which are not claimed (Appendix A.4). None of these, and no other open statement of the census, is reached by the proof of either theorem (Appendix A.6).
- (d) *Not formalised at all.* The zero-density estimates themselves, which enter only as the displayed hypothesis; the asymptotic consequences of Lemma 4.2; Lemma 4.6 in the form used by the Shell route (for its analogue inside Lemma 3.1 see (a)). In the release the

proof of the pointwise assembly proves, in its own form, the bounds that it needs from Lemmas 4.2 and 4.6.

The formalisation found statements of the working draft false as written; the ones that concern this paper are listed in Appendix B. In each case the corrected statement holds wherever the statement is used, and no constant of Theorems 1.1 or 1.2 changed; the constant 0.72375 that the working draft stated for Theorem 1.1 is not reached inside the formal design, and this paper states 0.7235 (§3.3).

1.6 How the work was done

Disclosure of generative-AI use. This paper and its Lean development were produced substantially by a generative AI system, Claude (Anthropic), working under the direction of the named author, who takes full responsibility for its contents. No AI system is listed as an author.

The arguments, the proofs on paper and the Lean proofs were written by instances of Claude working as separate agents on written tasks, which also checked one another’s work; the formal statuses are those that Lean reports (§1.5). The *working draft* to which this paper refers is the project’s unpublished draft of about 320 pages, which contains, besides the material of this paper, further families and conditional programmes; where this paper departs from it or from [9], the change is listed in Appendix B.

Scope of the verification. Theorems 1.1 and 1.2 are Lean theorems that depend only on the three standard axioms. Theorem 1.1 has no hypothesis; Theorem 1.2 assumes the zero-density estimates (J) and (M), which are published theorems cited here and not proved. Some intermediate error rates of the proofs on paper are not checked in Lean, because the formal proof absorbs those errors into the slack of the certificate (§7). The paper has not been refereed, and the Lean development has not been rebuilt outside this project.

1.7 Plan

Section 2 recalls the setting of [9], the notation, the imported hypotheses and the role of the bandwidth. Section 3 proves the killed-kernel lemma and Theorem 1.1. Section 4 is the Shell route, Section 5 the design at the new bandwidth, Section 6 the certificate and Section 7 the proof of Theorem 1.2. Section 8 discusses rates, thresholds and what would improve the constants. Appendix A reproduces the Lean statements; Appendix B lists the corrections to the working draft, to formal statements and to [9].

2 Setting

This section recalls what is needed from [9]; proofs that are not changed are not repeated.

2.1 Notation

Each of the following letters has one meaning in this paper; the objects that the working draft or [9] denote by the same letter are renamed as shown.

letter	meaning here	renamed objects
r	a modulus (a Farey denominator, the modulus of a character)	height exponent r_0 (r in [9] and in Lean); rows $\mathbf{r}_1, \dots, \mathbf{r}_5$ of Proposition 2.1
Ξ	the near cutoff of §4	the finite point set of Lemmas 3.3–3.4: $\mathcal{P}$
K	the shell width of §4	the width in Lemma 3.1: $K_1 = \mathcal{L}$ (so ℓ_{K_1} , and R_0, Δ of §3 use K_1)
ψ	$v \star v$	Chebyshev’s function: written $\sum_{n \leq y} \Lambda(n)$; the digamma function: written Γ'/Γ
θ	a rate exponent; $\theta(\alpha')$ is the largest one available at α'	the variable on $\mathbb{R}/\mathbb{Z}$: ϑ ; Chebyshev’s θ : written $\sum_{p \leq y} \log p$
κ	the width of the near cutoff in §4	the Frobenius constant of Proposition 2.1: $\mathcal{B}_*$; the parity of χ : $\mathbf{a}(\chi)$; the exponent κ' of the draft: $\mathbf{c}$
w	the ramp width of the window	the family weight of Lemma 4.3: $\mathbf{w}$; the model weight of Lemma 3.7: $\mathbf{m}$; the weight of Theorem 4.18: ω_K

“Lemma K ”, “K.1”–“K.6” and node names such as “K7” are names, not the variable K .

Let $\chi \in \mathfrak{F}_Q$ have modulus q and parity $\mathbf{a}(\chi) \in \{0, 1\}$. Put

$$\mathcal{L} := \log \frac{QT}{2\pi}, \quad L := \lambda \mathcal{L}, \quad X := e^L = \left(\frac{QT}{2\pi} \right)^\lambda, \quad I := [T, 2T],$$

with a bandwidth $1 < \lambda < 2$ fixed with the design (§5), and $\mathcal{N} := \sum_{\chi \in \mathfrak{F}_Q} N_\chi(T, 2T) \asymp Q^2 T \mathcal{L}$. The family constant is $C_Q := Q^2/|\mathfrak{F}_Q| \rightarrow C := \pi^4/18$, and $2\pi^4/27$ for $\mathfrak{F}_Q^{\text{dy}}$. We write $e(x) = e^{2\pi i x}$ and

$$D_T(v) := \int_I e^{i\tau v} d\tau, \quad a_n(s) := -\frac{1}{2\pi} \Lambda(n) n^{-1/2} D_T(s - \log n) \quad (n \leq X),$$

$$A_\chi(s) := \sum_n a_n(s) \chi(n),$$

$\|a(s)\|^2 = \sum_n |a_n(s)|^2$, $S(\vartheta) = S(\vartheta; s) = \sum_n a_n(s) e(n\vartheta)$ and $\mathbf{F}(s) := \sum_{\chi \in \mathfrak{F}_Q} |A_\chi(s)|^2$. One has $|D_T(v)| \leq \min(T, 2/|v|)$ and $\int_{\mathbb{R}} |D_T|^2 = 2\pi T$.

The window. Following [9], the window is $\varphi(u) = p(u/\mathcal{L}) \varphi_{\text{flat}}(u)$, where $\varphi_{\text{flat}}(u) = \varrho_2((L/2 - |u|)/w)$ is the flat taper of [1] with its Gevrey-2 ramp ϱ_2 of width $w \geq 1$, and p is an even polynomial profile on $[-\lambda/2, \lambda/2]$ with $p(0) = 1$, $0 < p \leq 1$, p decreasing on $[0, \lambda/2]$ and $p(\lambda/2) \geq 1/6$. Put $g := \varphi^2 \star \varphi^2 \geq 0$, a function of the physical variable s . The moments $a := L^{-1} \int \varphi^2$ and $b := L^{-1} \int \varphi^4$ must satisfy $a \geq 3/4$ and $b \geq 1/2$ (§5). The window’s density in the scale-free variable $t = u/\mathcal{L}$ is the probability density

$$v(t) := \frac{\mathcal{L} \varphi(\mathcal{L}t)^2}{\int \varphi^2} = \frac{\varphi(\mathcal{L}t)^2}{a\lambda},$$

supported on $[-\lambda/2, \lambda/2]$; put $\psi := v \star v$, a function of the dual variable $\alpha = s/\mathcal{L}$. The substitution $u = \mathcal{L}t$ in $g(\alpha\mathcal{L}) = \int \varphi(u)^2 \varphi(\alpha\mathcal{L} - u)^2 du$ gives the dictionary $g(\alpha\mathcal{L}) = (a\lambda)^2 \mathcal{L} \psi(\alpha)$. The profile’s own density is $v_p := p^2/\int p^2$ on $[-\lambda/2, \lambda/2]$. The unnormalised squares $\varphi(\mathcal{L}t)^2$ and $p(t)^2$ agree off the two ramps $\lambda/2 - w/\mathcal{L} < |t| \leq \lambda/2$, but the normalised densities do not: there $v = v_p \int p^2/(a\lambda)$. Since $0 \leq \int p^2 - a\lambda \leq 2w/\mathcal{L}$ ($0 \leq \varphi_{\text{flat}} \leq 1$, $p \leq 1$), one has $\|v - v_p\|_1 \leq 2(\int p^2 - a\lambda)/\int p^2 = O(w/\mathcal{L})$, and v, v_p are bounded uniformly: $\|v\|_\infty \leq 1/(a\lambda)$, $\|v_p\|_\infty \leq 1/\int p^2 \leq 36/\lambda$.

The functional. For a kernel k , even and nonnegative on $[-\lambda, \lambda]$,

$$\mathcal{B}_k(v) := \psi(0) + \int_{|\alpha| \leq \lambda} k(\alpha) \psi(\alpha) d\alpha, \quad \psi = v \star v. \quad (1)$$

The kernel of [9] is $k(\alpha) = |\alpha|$ for $|\alpha| \leq 1$ and $C|\alpha|$ for $1 < |\alpha| \leq \lambda$.

2.2 The imported hypotheses

The preprint [9] imports six statements from [1], each a theorem of the Lean artifact of [1]:

- (H1) the rank–trace certificate for the direct sum of the per-character Gram blocks: the number of simple zeros on the line is at least $4 \operatorname{tr} \widehat{G} - \|\widehat{G}\|_F^2 - 2\mathcal{N} - 3N_{\Pi}$ minus a perturbation loss, linear in $(\operatorname{tr}, \|\cdot\|_F^2)$;
- (H2) the grid Gram identity $G_z = G_p$ for each primitive χ , with density $\nu_{X,\chi} = \mu_q + P_{X,\chi}$ and no pole term;
- (H3) the positivity $g = \varphi^2 \star \varphi^2 \geq 0$;
- (H4) the Gevrey-2 ramp lemma for the flat taper, with $2w \leq L$;
- (H5) the cap-free ends majorants with their explicit constants;
- (H6) the q -uniform local count $N_\chi(t, t+1] \leq A_0 \log(q(|t|+3))$.

H1–H6 at the new bandwidth. Each of H1–H6 has a Lean instantiation that is free of the bandwidth in the following sense: it is valid for every $0 < \lambda < 2$ (the Lean class carries the condition $\lambda < 2$), with the constants of the window and of the design recomputed at λ (this is read off the hypotheses of each instantiation in the Lean sources). The bandwidth enters in three places, each through a constant:

- (1) H1, off-line zeros outside the window. Their amplification $e^{|\operatorname{Im} \gamma|L/2}$ enters only the tail, through the closing condition of the design (§5), which is proved at $\lambda = 191/100$ [L]. Its leading part requires $D_0 \geq (9e/16)\lambda^2 \mathcal{L}^2/w$, which is $5.578 \mathcal{L}^2$ at $\lambda = 1.91$, $w = 1$, in agreement with the design’s $D_0 \approx 5.58 \mathcal{L}^2$ (§5(v)); the buffer row keeps its rate $\mathcal{L}^2/T$. Inside the window the off-line zeros enter only through an inertia bound, and the Poisson bound is required only at zeros on the line, where it holds for every L .
- (2) H4. The product-taper constant B' of the ramp lemma depends on λ and the profile: for S53-L75 it is 42 950.7 with coefficient-sum majorants, within the certified $B' \leq 43\,447 \leq 5 \cdot 10^4$ (§5(iv)).
- (3) H5. The window constant c' of the cap-free ends majorants is 8 183.4 at $\lambda = 1.91$ (coefficient-sum majorants), against about 548 at λ^* ; it changes only the constant of the ends row, whose rate $\mathcal{L} \log \mathcal{L}/T$ is unchanged.

H2 uses the prime side exactly up to $X = e^L$, for every L ; H3 and H6 contain no parameter. The conditions $8w \leq L$ and $\frac{1}{6} \leq p \leq 1$ with p nonincreasing, which the instantiations use, are supplied by $w = 1$, large L and the profile S53-L75 ($p(\lambda/2) = 0.196764$).

2.3 The certificate

Let $\widehat{G}_{\text{fam}}$ be the normalised family Gram matrix of [9, §3], $N_{\Pi, \text{fam}}$ the number of zeros counted in the buffer, and B_{tr} , B_F the trace and Frobenius parts of the tail perturbation.

Proposition 2.1 ([9, Proposition 3.1]). *Suppose that for given (Q, T) : (i) the display of H1 holds; (ii) $\mathcal{N}(1 - \mathfrak{r}_1) \leq \operatorname{tr} \widehat{G}_{\text{fam}}$ and $\|\widehat{G}_{\text{fam}}\|_F^2 \leq (\mathcal{B}_* + \mathfrak{r}_2)\mathcal{N}$; (iii) $N_{\Pi, \text{fam}} \leq \mathfrak{r}_3\mathcal{N}$; (iv) $B_{\text{tr}} \leq \mathfrak{r}_4\mathcal{N}$ and $B_F \leq \mathfrak{r}_5\sqrt{\mathcal{N}}$. Then*

$$\sum_{\chi} N_{0,\chi}^s(T, 2T) \geq [2 - \mathcal{B}_* - (4\mathfrak{r}_1 + \mathfrak{r}_2 + 3\mathfrak{r}_3 + 4\mathfrak{r}_4 + 2\mathfrak{r}_5\sqrt{\mathcal{B}_* + \mathfrak{r}_2 + \mathfrak{r}_5^2})]\mathcal{N}.$$

The proof is the moment algebra of the certificate of [1]. In the formal development it is `ZetaQ.prop_3.1_pair_moment_certificate`, which does not depend on the bandwidth, and the frame-to-headline step built on it for the family paper is proved (`shell_assembly`, `k_assembly`) [L]. Everything below produces the inputs (ii)–(iv) with $\mathcal{B}_*$ equal to the value of a functional (1) at a certified profile and every $\mathfrak{r}_i$ tending to 0 at the stated rate.

2.4 The two zones

The only input of [9] that this paper changes is the out-zone half of the Frobenius row. Let $\delta' := 3 \log \log Q / \log Q$ and let

$$O := \{s : (1 - \delta') \log Q < |s| \leq \log X\}$$

be the out-zone of [9, Lemma 4.3]. In [9, Lemmas 4.1–4.3], by Parseval and the positivity of g , the Frobenius main term is an integral of $g(s)$ against family second moments of Dirichlet polynomials. On O the certificate consumes the full block $\sum_{\chi} \int_O g |A_{\chi} + B_{\chi}|^2$, which is bounded through $\int_{O^{\pm}} g \mathbf{F}$ (Step 3 of §7 and §3.2(d)); here $B_{\chi}(s) := \sum_n \overline{a_n(-s)} \bar{\chi}(n)$ is the second half of [9, Lemma 4.3] and $O^{\pm} := O \cap \{\pm s > 0\}$; and [9] bounds $\mathbf{F}(s)$ pointwise by the multiplicative large sieve,

$$\mathbf{F}(s) \leq (X + Q^2) \|a(s)\|^2 = C_Q |\mathfrak{F}_Q| (1 + X/Q^2) \|a(s)\|^2.$$

Since $\|a(s)\|^2 = \frac{T}{2\pi} (s + O(1))$ for $(1 + \delta_1) \log Q \leq s \leq \log X - 1$ by the prime number theorem, and $s = \alpha \mathcal{L}$, this is the kernel $C|\alpha|$ of (1). On the top edge $\log X - 1 < s \leq \log X$, where the cut $n \leq X$ is felt, only the upper bound $\|a(s)\|^2 \leq \frac{T}{2\pi} (s + O(1))$ holds uniformly (at $s = \log X$ the left side is about $\frac{T}{4\pi} s$), and an upper bound is what is used here; the diagonal evaluation of [9] used in §3.2(d) and §7 is integrated over O^+ , where the top edge has relative weight $O(\mathcal{L}^{-1})$. The in-zone $|s| \leq (1 - \delta') \log Q$ and the zone-edge strip up to $|\alpha| = 1$ (kernel $C|\alpha|$, the row L_1 of [9, §10.3]) are treated as in [9, §4]. The half B_{χ} with $\bar{\chi}$ and the $\chi/\bar{\chi}$ cross term are treated as in [9, §4] on the in-zone, and on the out-zone by Cauchy–Schwarz in χ (§3.2(d), Step 3 of §7). The large sieve is used in Gallagher’s form $Q^2 + \pi N$ [11], which is what the formal development proves; the sharp form $N - 1 + Q^2$ is not needed.

2.5 The bandwidth

Theorem 1.2 runs at $\lambda = 191/100$, while the estimates of [9] were derived, and formalised, at $\lambda^* = 1.2507321515$. We have re-examined at $\lambda = 191/100$ every row of [9] that the Shell route imports for $\mathfrak{F}_Q$ (not the dyadic family): we tracked where λ enters each stated bound of [9, §§4–10], and re-read line by line the proofs of the large-sieve uses, the μP cross term, the tail, the ends, the moment floor, the shell decomposition and the prime number theorem input. The conclusion is:

- (i) The only condition on λ that a proof uses in an essential way is

$$\lambda_{\text{eff}} := \frac{\log X}{\log Q} = \lambda \left(1 + \frac{\log(T/2\pi)}{\log Q} \right) < 2,$$

in the form $X = o(Q^2/(\log Q)^B)$. It makes the large sieve lossless, $X + Q^2 = Q^2(1 + o(1))$; it gives the saving $Q^{\lambda_{\text{eff}}/2-1}$ of the linear character sums in the μP cross term and in the prime part of the trace (these use only $n \neq m$ and $|M(y)| \leq y$, with no bound on n); and the same condition, $N \leq Q^{2-\eta}$, is what Lemma 3.1 and the Shell lemmas need. At fixed $\lambda < 2$ it holds for all large Q . At $\lambda = 191/100$ and $r_0 + \varepsilon = 3.5$ the crossover $X = Q^2$ lies near $\log_{10} Q \approx 177$; at $\lambda = 1.995$ it moves to $\log_{10} Q \approx 5400$.

- (ii) Every other cap on λ in [9] and in its formalisation is a convenience of the design at λ^* : the regime $X \leq Q^{3/2}$, the pin $\lambda = \lambda^*$ of the design predicate, the row L_{11} ($\lambda \leq 1.26$), the ends constant $c_1 = 0.41$, the Gevrey gate $B' \leq 4 \cdot 10^4$ and the numeric proofs of two rows at the literal value λ^* .
- (iii) Four statements about the design must be corrected, none of which moves 0.9059137927: the ends constant $c_1 = 0.41$ of [9, Lemma 8.1] is not covered by its proof at $\lambda = 1.91$ and becomes $c_1 = 0.52$, which needs the Minkowski split $\sqrt{\sum(\mu + P)^2} \leq \sqrt{\sum \mu^2} + \sqrt{\sum P^2}$ (it gives $0.0684 + \lambda\sqrt{1 + X/Q^2}/(\sqrt{2}\pi) = 0.4983$ at $X/Q^2 \rightarrow 0$; the other route gives 0.6156) and $X/Q^2 \leq 0.1034$; the moment floor $a \geq 3/4$ cannot be relaxed (an ends estimate needs $a \geq 0.7409$) and is met in eventual form, since $a \geq \langle p^2 \rangle$ minus the ramp mass and $\langle p^2 \rangle - 3/4 = 2.00001 \cdot 10^{-7}$ for S53-L75, once $\mathcal{L}/w \geq 2.03 \cdot 10^5$; the ramp row has rate $\Theta(1/\log Q)$ (the buffer row, $O(\mathcal{L}^2/T)$, is smaller), because the design and its formalisation clamp $w \geq 1$ (the design has $w = \max(1, w^*)$ with $w^* \rightarrow 0$, so $w = 1$ for large Q ; in the formal design $w = \max(1, \mathcal{L}^{(3-r_0)/2}) = 1$); and the zone slope of the certified profile is $2(C-1)\psi_{v_p}(1) = 1.9458$, not the design's 0.505.
- (iv) The prime number theorem is used only in the form $\sum_{n \leq x} \Lambda(n) - x = o(x(\log x)^{-A})$ for one fixed A depending on $r_0 + \varepsilon$ (no power saving and no short intervals). This form is a Lean theorem, derived from the medium form of the prime number theorem in the formal tree (L7.2.psi_sub_id.isLittle0_log_rpow) [L].

In Lean, the two rows proved only at the literal λ^* are re-proved for every $0 < \lambda < 2$ (§5). The other parts of [9, §§4–10] were not re-proved line by line; H1–H6 are treated in §2.2 and the Shell lemmas in §4.

3 The killed kernel and Theorem 1.1

In [9] the out-zone is priced by the large sieve pointwise in s , which charges the family $Q^2 \|a(s)\|^2$ and gives the kernel $C|\alpha|$. That bound ignores where the mass of $S(\vartheta) = \sum_n a_n(s)e(n\vartheta)$ sits: for $s > \mathcal{L}$ a fixed proportion of it lies in short arcs around rationals a/r of small denominator, and no Farey point of level Q other than a/r itself comes near those arcs. Lemma 3.1 makes this quantitative.

The proof below incorporates the corrections D1–D5 to the working draft listed in Appendix B; the formalisation of this section is described after each lemma.

3.1 Notation and statement

Fix $1 < \lambda < 2$. Put $U := T/2\pi$ and

$$K_1 := \mathcal{L}, \quad \ell_{K_1} := \log(QTK_1) = \mathcal{L} + \log(2\pi K_1),$$

and, for given s ,

$$N := e^s, \quad R_0 := \frac{N}{QTK_1}, \quad R' := \max(2, R_0), \quad \Delta := \frac{K_1 T}{2N}.$$

Thus $\log R_0 = s - \ell_{K_1}$, and $R_0 \geq 1$ exactly when $s \geq \ell_{K_1}$. (The width K_1 is the one of the working draft for this lemma; the Shell route of §4 uses a larger width K .) $\mathcal{F}_R$ denotes the set of $b/q \in \mathbb{R}/\mathbb{Z}$ with $1 \leq q \leq R$, $(b, q) = 1$, and $\|\vartheta\|$ the distance from ϑ to $\mathbb{Z}$. The prime number theorem is used in the form

$$\sum_{n \leq y} \Lambda(n) - y = o(y(\log y)^{-A}) \quad (y \rightarrow \infty) \tag{2}$$

for one fixed $A = A(r_0, \varepsilon)$ (§2.5(iv)); we take $A \geq 2 + 2(r_0 + \varepsilon)$.

Lemma 3.1 (Lemma K: killed kernel). *There are Q_0 and C_1 , depending on λ , r_0 and ε , such that for $Q \geq Q_0$:*

(i) Pointwise. *For every $s \in \mathbb{R}$,*

$$\mathbf{F}(s) \leq Q^2 \left[\|a(s)\|^2 - \left(1 - \frac{C_1}{\log Q}\right) U(s - \ell_{K_1})_+ \mathbf{1}_{\{s \leq \log X - 1\}} + 2\|a(s)\| \right] + (2\pi X + R'^2) \|a(s)\|^2. \quad (3)$$

(ii) Integrated. *Let $B_\chi(s) := \sum_n b_n(s) \bar{\chi}(n)$ with $b_n(s) = \overline{a_n(-s)}$ be the second half of [9, Lemma 4.3], so that $\sum_\chi |B_\chi(s)|^2 = \mathbf{F}(-s)$. With $O^+ = O \cap (0, \infty)$,*

$$\sum_{\chi \in \mathfrak{F}_Q} \int_O g(s) |A_\chi(s) + B_\chi(s)|^2 ds \leq C_Q |\mathfrak{F}_Q| \frac{T}{\pi} \int_{O^+} g(u) \min(u, \mathcal{L}) du (1 + E_K)(1 + O(\rho)), \quad (4)$$

where ρ is the cross-term factor of [9, Lemma 4.3] and $E_K = O(\log \mathcal{L}/\mathcal{L}) + O((U \log Q)^{-1/2}) + O(Q^{\lambda-2} T^\lambda) + O(\varepsilon_D)$, ε_D being the D_T -smearing error of the diagonal evaluation of [9].

In α -units, $u = \alpha \mathcal{L}$, part (ii) replaces u by $\min(u, \mathcal{L})$ in the out-zone: the out-zone term $C \int_{1 < |\alpha| \leq \lambda} |\alpha| \psi$ becomes $C \int_{1 < |\alpha| \leq \lambda} \psi$. For the dyadic family $\mathfrak{F}_Q^{\text{dy}}$, (i) and (ii) hold with C_Q replaced by $Q^2/|\mathfrak{F}_Q^{\text{dy}}| \rightarrow 2\pi^4/27$ and without the term R'^2 : the family contains no point a/r with $r \leq R_0 < Q/2$, so nothing is removed (argued in the working draft; not formalised; used only in Remark 4.19, which is not claimed).

Formalisation. Part (i) is `ZetaShell.LemmaK.killed_pointwise` (node K7), in this form with the large sieve in Gallagher's form $(Q^2 + \pi X) \|a\|^2$ applied twice, as displayed; part (ii) enters the formal proof directly in the form of the Frobenius row (node K8a, §3.4). The left side of (ii) is the full out-zone block $\sum_\chi \int_O g |A_\chi + B_\chi|^2$: this is what the proof of (ii) bounds, what §3.4 uses and what Step 3 of §7 cites, not the half $\int_O g \mathbf{F}$. The Lean proof has no separate statement of (ii): node K8a bounds the whole PP block $\sum_\chi \int_{\mathbb{R}} g |A_\chi + B_\chi|^2$ inside the Frobenius row, with the half A_χ on the out-zone bounded by (i) integrated against g , the half B_χ by the mirror, and the cross term by the plain large sieve; so the formal result does not depend on the printed form of (ii). In the released library part (i), node K8a and Lemma 3.7 are kernel-checked, and Theorem 1.1 rests on no open declaration (§1.5) [L].

3.2 Proof of Lemma 3.1

(a) The Farey majorant

Lemma 3.2 (Lemma K.1: Gauss-sum inequality). *For every $q \geq 1$ and every finitely supported (a_n) ,*

$$\frac{q}{\varphi(q)} \sum_{\chi \bmod q}^* |A_\chi|^2 \leq \sum_{b \bmod q}^* |S(b/q)|^2.$$

Proof. For primitive $\chi \bmod q$ one has $\sum_{b \bmod q} \bar{\chi}(b) e(bn/q) = \tau(\bar{\chi}) \chi(n)$ for every $n \in \mathbb{Z}$ [18, §9.1], and $|\tau(\bar{\chi})|^2 = q$ [18, Thm. 9.7]. Hence $\tau(\bar{\chi}) A_\chi = \sum_b^* \bar{\chi}(b) S(b/q)$. Summing $|\sum_b^* \bar{\chi}(b) S(b/q)|^2$ over all $\chi \bmod q$ gives $\varphi(q) \sum_b^* |S(b/q)|^2$ by orthogonality; dropping the imprimitive χ gives the claim. $\square$

Summing over $1 < q \leq Q$ and using $\varphi(q)/q \leq 1$,

$$\mathbf{F}(s) \leq \sum_{\xi \in \mathcal{F}_Q} |S(\xi)|^2. \quad (5)$$

No support condition is imposed, so the primes in $(R_0, Q]$ are never removed from a . *Formalisation:* `gauss_sum_ineq`, a rearrangement of `ZetaQ.primitive_decomposition` [L].

(b) **Gallagher's inequality with a zero set**

Lemma 3.3 (Lemma K.2). *Let $\mathcal{P} \subset \mathbb{R}/\mathbb{Z}$ be finite, $w_\xi \geq 0$, $0 < \delta < 1$, $D_\delta(\vartheta) := \delta^{-1} \sum_{\|\xi - \vartheta\| \leq \delta/2} w_\xi$ and $Z := \{\vartheta : D_\delta(\vartheta) = 0\}$. If (a_n) is supported on the integers of $[1, X]$, then*

$$\sum_{\xi \in \mathcal{P}} w_\xi |S(\xi)|^2 \leq \sup_{\vartheta} D_\delta(\vartheta) \left(\|a\|^2 - \left[\int_{Z \cap [0,1)} |S|^2 d\vartheta \right] + \pi X \delta \|a\|^2 \right).$$

Proof. For $f \in C^1(I_\xi)$, $I_\xi = [\xi - \delta/2, \xi + \delta/2]$, one has $|f(\xi)| \leq \delta^{-1} \int_{I_\xi} |f| + \frac{1}{2} \int_{I_\xi} |f'|$ [17, Lemma 1]. Apply it to $f = |S|^2 = |\tilde{S}|^2$, $\tilde{S}(\vartheta) = e(-n_0 \vartheta) S(\vartheta)$, $n_0 = (1 + X)/2$, so that $|f'| \leq 2|\tilde{S}||\tilde{S}'|$; multiply by w_ξ and sum, using $\sum_\xi w_\xi \mathbf{1}_{I_\xi} = \delta D_\delta$: $\sum_\xi w_\xi |S(\xi)|^2 \leq \int_0^1 (|S|^2 + \delta |\tilde{S}||\tilde{S}'|) D_\delta$. Since $D_\delta = 0$ on Z , the first term is at most $\sup D_\delta (\|a\|^2 - \int_Z |S|^2)$ by Parseval; the second is at most $\sup D_\delta \delta \|\tilde{S}\|_2 \|\tilde{S}'\|_2 \leq \sup D_\delta \delta \pi (X - 1) \|a\|^2$. $\square$

Formalisation: the first Lean transcription of this lemma placed the last term inside the integrand, because a Lean integral extends as far to the right as possible; that statement is false (a kernel-checked counterexample exists) and was replaced by the parenthesised form, which is proved (`gallagher_with_zero_set_corr`) [L]. The lemma on paper was correct.

For $\mathcal{P} \subseteq \mathcal{F}_Q$ and $\delta = Q^{-2}$ a closed window of length δ contains at most one point of $\mathcal{P}$, so $\sup D_\delta \leq Q^2$ and the error term is $\pi X \|a\|^2$.

Lemma 3.4 (Lemma K.3: hole cores). *Let $\ell_{K_1} \leq s \leq \log X$ and $Q \geq Q_0$. Then $R_0 < Q/2$. Put $\mathcal{P} := \mathcal{F}_Q \setminus \mathcal{F}_{R_0}$, $w_\xi \equiv 1$, $\delta = Q^{-2}$, and for $r \leq R_0$, $(a, r) = 1$, $H_{a/r} := \{\vartheta : \|\vartheta - a/r\| < 1/(rQ) - \delta/2\}$. Then (a) $H_{a/r} \subseteq Z$; (b) the $H_{a/r}$, $r \leq R_0$, are pairwise disjoint; (c) $H_{a/r} \supseteq \{\vartheta : \|\vartheta - a/r\| \leq \Delta\}$.*

Proof. $R_0 \leq X/(QT K_1) = Q^{\lambda-1} T^{\lambda-1} (2\pi)^{-\lambda} K_1^{-1} < Q/2$ for large Q since $\lambda < 2$. (a) Every $b/q \in \mathcal{P}$ differs from a/r , so $\|b/q - a/r\| \geq 1/(qr) \geq 1/(Qr)$. (b) $\|a/r - a'/r'\| \geq 1/(rr') \geq 1/(rQ) + 1/(r'Q) - 1/Q^2$, since this is $(Q - r)(Q - r') \geq 0$; so (b) holds for all $r, r' \leq Q$. (c) $1/(rQ) \geq 1/(R_0 Q) = K_1 T/N = 2\Delta$, and $\delta/2 < \Delta$ because $N \leq X < K_1 T Q^2$. $\square$

Formalisation: `farey_hole_cores` [L].

Apply Lemma 3.3 to $\mathcal{P} = \mathcal{F}_Q \setminus \mathcal{F}_{R_0}$ with $\delta = Q^{-2}$, and to $\mathcal{F}_{R'} \supseteq \mathcal{F}_{R_0}$ with $\delta = R'^{-2}$ and $Z = \emptyset$. With (5) and Lemma 3.4,

$$\mathbf{F}(s) \leq Q^2 (\|a\|^2 - \text{Hole}(S)) + (2\pi X + R'^2) \|a\|^2, \quad \text{Hole}(S) := \sum_{r \leq R_0} \sum_{a \bmod r}^* \int_{|\beta| \leq \Delta} |S(a/r + \beta)|^2 d\beta, \quad (6)$$

since by (a)–(c) the Δ -balls are disjoint subsets of Z . For $s < \ell_{K_1}$, $\mathcal{F}_{R_0} = \emptyset$ and $\text{Hole}(S) = 0$.

(c) **The hole lower bound**

Lemma 3.5 (Lemma K.4: principal Gauss sum). *Let (a'_n) be supported on integers coprime to r and $S'(\vartheta) = \sum_n a'_n e(n\vartheta)$. For every $\beta \in \mathbb{R}$, $\frac{\mu^2(r)}{\varphi(r)} |S'(\beta)|^2 \leq \sum_{a \bmod r}^* |S'(a/r + \beta)|^2$.*

Proof. For $(n, r) = 1$ the Ramanujan sum $\sum_{a \bmod r}^* e(na/r)$ equals $\mu(r)$. Hence $\sum_a^* S'(a/r + \beta) = \mu(r) S'(\beta)$, and Cauchy–Schwarz over the $\varphi(r)$ reduced residues gives the claim. $\square$

Formalisation: `principal_gauss_hole`, by exactly this proof [L]. The identity $\sum_a^* |S'(a/r + \beta)|^2 = \varphi(r)^{-1} \sum_{\chi \bmod r} |\tau(\chi)|^2 |S'_\chi(\beta)|^2$ of the working draft, of which the lemma is the term $\chi = \chi_0$, is used in §4.4.

We also use $\sum_{r \leq R} \mu^2(r)/\varphi(r) \geq \log(R + 1)$ for every integer $R \geq 0$ (`log_succ_le_sum_moebius_sq_div_totient` [L]; compare [18, (3.18)]).

Lemma 3.6 (Lemma K.5: small primes). *Put $a'_n := a_n \mathbf{1}\{n = p^k, p > R_0\}$ and $a'' := a - a'$. For $Q \geq Q_0$ and $\ell_{K_1} \leq s \leq \log X$, $\|a''(s)\|^2 \leq 1$.*

Proof. $|a_n|^2 \leq \Lambda(n)^2/(\pi^2 n(s - \log n)^2)$ by $|D_T(v)| \leq 2/|v|$. For $n \leq e^{s/2}$ one has $|s - \log n| \geq s/2$ and $\Lambda(n)^2 \leq (s/2)\Lambda(n)$, and Mertens' estimate $|\sum_{n \leq y} \Lambda(n)/n - \log y| \leq \log 4 + 4$ gives a contribution at most $(4\pi^2)^{-1}(8/s)(s/2 + \log 4 + 4)$. A surviving $n > e^{s/2}$ is a prime power p^k with $p \leq R_0 < e^{s/2} < n$ (here $s \leq \log X$ and $\lambda < 2$ give $R_0 < e^{s/2}$), hence $k \geq 2$, and these contribute at most $T^2 C_2 e^{-s/8}$ with an absolute C_2 . Since $s \geq \ell_{K_1} \geq \log Q$, the total is at most $8/(4\pi^2) + 1/(4\pi^2) < 1$ for large Q . $\square$

The working draft stated this lemma for all $N \geq Q$; its proof covers only $s \leq \log X$, which is the only range used (defect D3). *Formalisation:* `small_primes_l2`, by this proof, on the range $\ell_{K_1} \leq s \leq \log X$ [L].

Every n in the support of a' is coprime to every $r \leq R_0$. The Δ -balls are disjoint, so Hole is a squared L^2 -norm on their union; $\text{Hole}(S'') \leq \|a''\|^2 \leq 1$ and $\text{Hole}(S') \leq \|a\|^2$. Since $(x - y)^2 \geq x^2 - 2xy$, Lemma 3.5 and the bound for $\sum \mu^2/\varphi$ give

$$\text{Hole}(S) \geq \text{Hole}(S') - 2\|a\| \geq (\log R_0) J - 2\|a\| = (s - \ell_{K_1}) J - 2\|a\|, \quad J := \int_{|\beta| \leq \Delta} |S'(\beta)|^2 d\beta. \quad (7)$$

Formalisation: `hole_lower` (node K7b) and `farey_hole_bound` (node K7a) [L].

Lemma 3.7 (Lemma K.6: principal-arc mass). *There is C_1 , depending on λ , r_0 and ε , such that for all large Q , uniformly for $\ell_{K_1} \leq s \leq \log X - 1$, $J \geq (1 - C_1/\log Q) U$.*

Proof. Fix $\varrho \in C^\infty(\mathbb{R}, [0, 1])$ with $\varrho = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\text{supp } \varrho \subseteq [-1, 1]$, $|\varrho'| \leq 3$. Put $\mathbf{m}(y) := -\frac{1}{2\pi} y^{-1/2} D_T(s - \log y)$ and $\tilde{\mathbf{m}}(y) := \mathbf{m}(y) \varrho(\log y - s)$, so $\text{supp } \tilde{\mathbf{m}} \subseteq [N/e, eN] \subseteq [1, X]$ and $a_n = \Lambda(n) \mathbf{m}(n)$. Let $\Lambda'(n) := \Lambda(n) \mathbf{1}\{n = p^k, p > R_0\}$, $\tilde{M}(\beta) := \sum_n \tilde{\mathbf{m}}(n) e(n\beta)$, $S^e(\beta) := \sum_n \Lambda'(n) \tilde{\mathbf{m}}(n) e(n\beta)$ and $\|f\|_\Delta := (\int_{|\beta| \leq \Delta} |f|^2)^{1/2}$. Then

$$\sqrt{J} \geq \|\tilde{M}\|_\Delta - E_1 - E_2, \quad E_1 := \|S^e - \tilde{M}\|_\Delta, \quad E_2 := \|S' - S^e\|_{L^2(\mathbb{R}/\mathbb{Z})}.$$

Step 1 (model mass). By summation by parts and $|1 - e(\beta)| \geq 4\|\beta\|$, $\int_{\Delta < \|\beta\| \leq 1/2} |\tilde{M}|^2 \leq (16\Delta^2)^{-1} \sum_n |\tilde{\mathbf{m}}(n+1) - \tilde{\mathbf{m}}(n)|^2 \leq (16\Delta^2)^{-1} \int |\tilde{\mathbf{m}}'|^2$. With $\|D_T\|_2^2 = 2\pi T$, $\|D'_T\|_2^2 = 14\pi T^3/3$ and $y^{-2} \leq e^2/N^2$ on the support, $\int |\tilde{\mathbf{m}}'|^2 \leq 2.88 T^3/N^2$; as $16\Delta^2 = 4K_1^2 T^2/N^2$, the mass outside $|\beta| \leq \Delta$ is at most $0.72 T/K_1^2 \leq 4.6 U/K_1^2$. For the total mass, the substitution $y = e^{s-v}$ gives

$$\int_0^\infty |\tilde{\mathbf{m}}(y)|^2 dy = \frac{1}{4\pi^2} \int_{\mathbb{R}} |D_T(v)|^2 \varrho(-v)^2 dv \geq \frac{1}{4\pi^2} \left(2\pi T - \int_{|v| > 1/2} \frac{4}{v^2} dv \right) = U - \frac{4}{\pi^2},$$

and $|\sum_n |\tilde{\mathbf{m}}(n)|^2 - \int |\tilde{\mathbf{m}}|^2|$ is at most the total variation of $|\tilde{\mathbf{m}}|^2$, which is $\leq 2 \sup |\tilde{\mathbf{m}}| \int |\tilde{\mathbf{m}}'| \ll T^{5/2}/N = o(1)$ since $N \geq QTK_1$. Hence $\|\tilde{M}\|_\Delta^2 \geq U - 1 - 4.6 U/K_1^2$ for large Q .

Step 2 (Gallagher's L^2 lemma). For $h := 1/(2\Delta)$ and finitely supported (c_n) , $\int_{|\beta| \leq \Delta} |\sum_n c_n e(n\beta)|^2 d\beta \leq \pi^2 \Delta^2 \int_{\mathbb{R}} |\sum_{x < n \leq x+h} c_n|^2 dx$ [12, Lemma 1]: the function $C(x) = \sum_{x < n \leq x+h} c_n$ has Fourier transform of modulus $|\sum c_n e(n\beta)| |\sin \pi h \beta| / (\pi |\beta|)$, which is at least $|\sum c_n e(n\beta)| / (\pi \Delta)$ on $|\beta| \leq \Delta$; apply Plancherel.

Step 3 (partial summation; the prime number theorem). Take $c_n = (\Lambda'(n) - 1) \tilde{\mathbf{m}}(n)$, so $S^e - \tilde{M} = \sum c_n e(n\beta)$. With $E(y) := \sum_{n \leq y} (\Lambda'(n) - 1) = \sum_{n \leq y} \Lambda(n) - \lfloor y \rfloor - \sum_{p^k \leq y, p \leq R_0} \log p$ and $E^* := \sup_{N/e \leq y \leq eN} |E(y)|$, one has $|E(y)| \leq |\sum_{n \leq y} \Lambda(n) - y| + 1 + R_0 \log y$, and Stieltjes integration gives $|C(x)| \leq E^*(2 \sup |\tilde{\mathbf{m}}| + \int_x^{x+h} |\tilde{\mathbf{m}}'|)$ with $\sup |\tilde{\mathbf{m}}| \leq e^{1/2} T / (2\pi \sqrt{N})$ and $\int_x^{x+h} |\tilde{\mathbf{m}}'| \ll T / (K_1 \sqrt{N})$. As $C(x) \neq 0$ only on an x -range of length $\leq eN$, $E_1^2 \ll K_1^2 T^4 (E^*/N)^2$. By (2), $\log N \geq \log Q$ and $N \geq QTK_1$, $E^*/N = o((\log Q)^{-A}) + O(\log(eN)/(QTK_1))$. Since K_1

and T are fixed powers of $\log Q$, the choice $A \geq 2 + 2(r_0 + \varepsilon)$ gives $E_1^2/U \ll K_1^2 T^3 (E^*/N)^2 = o((\log Q)^{-2})$.

Step 4 (tails). With $f(u) = (s-u)^{-2} \mathbf{1}\{|s-u| \geq \frac{1}{2}\}$, $E_2^2 \leq \sum_{n \leq X, |\log n - s| \geq 1/2} \Lambda(n)^2 |\mathbf{m}(n)|^2 \leq \pi^{-2} \log X \sum_{n \leq X} \Lambda(n) f(\log n)/n$. Writing $\sum_{n \leq x} \Lambda(n)/n = \log x + R(x)$ with $|R(x)| \leq \log 4 + 4$, and using $\int f = 4$, $\sup f = 4$, $\text{Var } f = 16$, Stieltjes integration gives $\sum \leq 4 + 24(\log 4 + 4) \leq 134$, so $E_2^2 \leq 13.6 \log X$ and $E_2^2/U \leq 86 \lambda \mathcal{L}/T$.

Conclusion. $\sqrt{J} \geq \sqrt{U}(\sqrt{1 - 1/U - 4.6/K_1^2} - (E_1 + E_2)/\sqrt{U})$, and with $\sqrt{1-x} \geq 1-x$, $(1-y)^2 \geq 1-2y$, $J \geq U(1-\eta_P)$ with $\eta_P := 2/U + 9.2/K_1^2 + 2(E_1 + E_2)/\sqrt{U}$. With $K_1 = \mathcal{L}$ and $T = (\log Q)^{r_0+\varepsilon}$, $r_0 \geq 3$: $2/U \ll (\log Q)^{-3}$, $9.2/K_1^2 \ll (\log Q)^{-2}$, $2E_2/\sqrt{U} \leq 2(86\lambda\mathcal{L}/T)^{1/2} \ll (\log Q)^{-(r_0+\varepsilon-1)/2} \leq (\log Q)^{-1}$, and $2E_1/\sqrt{U} = o((\log Q)^{-1})$. Hence $\eta_P \leq C_1/\log Q$ for large Q . $\square$

Corrections to the working draft. The working draft used the prime number theorem with the error term $y \exp(-c\sqrt{\log y})$ and the unverified constant $|R(x)| \leq 2$; both are replaced here by what the proof needs, the form (2) and the Mertens bound $|R(x)| \leq \log 4 + 4$ of the formal tree. This changes the draft's bounds $E_2^2/U \leq 33.3 \lambda \mathcal{L}/T$ and $2E_2/\sqrt{U} \leq 16.4(\mathcal{L}/T)^{1/2}$ into $E_2^2/U \leq 86 \lambda \mathcal{L}/T$ and $2E_2/\sqrt{U} \leq 26.3(\mathcal{L}/T)^{1/2}$, and nothing asymptotic (defect D4). The argument of Step 1 for $\sum_n |\tilde{\mathbf{m}}(n)|^2 \geq U - 1$, which the draft asserts without proof, is written out here.

Formalisation. The formal proof of Lemma 3.7 (`principal_arc_mass`) replaces Step 2 by discrete summation by parts (`sbp_L2`), which loses a factor of about $\mathcal{L}T$ that the prime number theorem at $A = 3 + 3(r_0 + \varepsilon)$ absorbs; Steps 2–4 in that form are `sbp_L2`, `pnt_step` and `tails`. Step 1 is split into `model_l2_lower` ($\sum_n |\tilde{\mathbf{m}}(n)|^2 \geq U - C$) and `model_outer` (the outer mass is $\leq CU/\mathcal{L}^2$). The numerical deficit $U - \sum |\tilde{\mathbf{m}}(n)|^2$ is about 0.157 at $Q = 60, 100, 1000$ [N], consistent with Step 1. In the released library all of these are kernel-checked, and Theorem 1.1 rests on no open declaration and depends on the three standard axioms only (§1.5) [L].

(d) Assembly: proof of Lemma 3.1

For $\ell_{K_1} \leq s \leq \log X - 1$, (6), (7) and Lemma 3.7 give (3). For $s < \ell_{K_1}$ or $s > \log X - 1$ drop $\text{Hole} \geq 0$. This proves (i).

For (ii), integrate (3) against $g \geq 0$ over O^+ and use the diagonal evaluation $\int_{O^+} g \|a\|^2 \leq U \int_{O^+} s g(s) ds (1 + \varepsilon_D)$ of [9]. Since $s - (s - \ell_{K_1})_+ = \min(s, \ell_{K_1}) \leq \min(s, \mathcal{L}) + \log(2\pi K_1)$ and $(s - \ell_{K_1})_+ \leq (\lambda - 1)\mathcal{L}$,

$$\int_{O^+} g \mathbf{F} \leq Q^2 U \int_{O^+} g(s) \min(s, \mathcal{L}) ds (1 + E_K),$$

where E_K collects, relative to the main term: the kink shift $\log(2\pi K_1)/\mathcal{L}$; the smearing and the terms $(2\pi X + R^2) \|a\|^2$, of relative size $O(\varepsilon_D + Q^{\lambda-2} T^\lambda + Q^{2\lambda-4} T^{2\lambda-2})$; the error $C_1/\log Q$ of the subtracted term and the strip $(\log X - 1, \log X]$ where the hole is dropped, of relative size $O(\mathcal{L}^{-1})$; and $2Q^2 \|a\|$, bounded by Cauchy–Schwarz in s , of relative size $O((U \log Q)^{-1/2})$. The full block is then bounded as in Step 3 of §7: since $B_\chi(s) = \overline{A_\chi(-s)}$, Cauchy–Schwarz in χ bounds it by $2(\sqrt{X_+} + \sqrt{X_-})^2$ with $X_\pm := \int_{O^\pm} g \mathbf{F}$ and $O^- := O \cap (-\infty, 0)$; X_+ is bounded above, and $X_- = O(Q^2 \int_O g)$ by the large sieve, since $\|a(s)\|^2 = O(1)$ for $s \leq -(1 - \delta') \log Q$. The mirror copy doubles U to T/π , the cross term costs a factor $1 + O(\rho)$, and $Q^2 = C_Q |\mathfrak{F}_Q|$. This is (4). $\square$

3.3 The functional and the certified constants

In α -units $\min(u, \mathcal{L}) = \mathcal{L} \min(\alpha, 1)$. The zone-edge strip $(1 - \delta') \log Q / \mathcal{L} < |\alpha| \leq 1$ keeps the kernel $C|\alpha|$ (row L_1 of [9]), the in-zone term is unchanged, and the out-zone term becomes

$C \int_{1 < |\alpha| \leq \lambda} \psi$. The functional consumed by Proposition 2.1 is therefore

$$\mathcal{B}_C^K(v) = \psi(0) + \int_{|\alpha| \leq 1} |\alpha| \psi(\alpha) d\alpha + C \int_{1 < |\alpha| \leq \lambda} \psi(\alpha) d\alpha, \quad (8)$$

which is $\mathcal{B}_k$ of (1) with $k(\alpha) = C \min(|\alpha|, 1)$ off the strip. Since $\psi \geq 0$, $\mathcal{B}_C^K$ is increasing in C , and $\mathcal{B}_C^K \leq \mathcal{B}$ for every v . We evaluate it at the rational upper bound $C^+ = 541161617/10^8 \geq \pi^4/18$.

Table 2: Values of $2 - \mathcal{B}_{C^+}^K(v_p)$, exact rationals truncated to ten digits. “Lean” means that the rational inequality is checked by the kernel (`decide +kernel`); $\lambda^* = 1.2507321515$ is the bandwidth of the design of [9], which the formal design fixes exactly.

profile	λ	degree	$2 - \mathcal{B}_{C^+}^K$	status and use
design profile of [9]	λ^*	6	0.7235110849	Lean (<code>BKdesign.le</code>); Theorem 1.1
new profile at λ^*	λ^*	8	0.7237495018	Lean (<code>BKlam8.le</code>); needs a new design
new profile at λ^*	λ^*	10	0.7237502555	exact rational only; not used
profile (iv') of the working draft	32/25	8	0.7237573823	Lean calibration; outside the design
kernel of [9], design profile	λ^*	6	0.7212536292	check of the pipeline

The design profile of [9] is $p_d(t) = 1 - 0.650056 t^2 + 3.458583 t^4 - 18.349255 t^6$ on $[-\lambda^*/2, \lambda^*/2]$ (exact rational coefficients in the formal tree). Its real-valued certificate, $\mathcal{B}_{C^+}^K(v_d) \leq 2 - 0.72351$ together with $\pi^4/18 \leq C^+$ and the admissibility of v_d , is the Lean theorem `certKD_core` [L]; it is obtained from the identity $\mathcal{B}_C^K = \mathcal{B}_{k_{\alpha'}}$ with the Shell kernel of level C (for any α') and the Lean identity of §6 at $\alpha' = 6/5$, followed by a kernel computation.

The constant 0.72375 is not claimed. The working draft stated Theorem 1.1 with 0.7237, as the truncation of 0.72375738 at profile (iv'), $\lambda = 32/25$, and said that a profile at $\lambda = 5/4$ “fits the caps” of the formal design. The formal design does not cap the bandwidth but fixes it at λ^* , so neither profile is admissible there (defects D1, D2). At λ^* the best degree-8 profile gives $0.7237495018 < 0.72375$ (a kernel-checked control, `BKlam8_not_72375`, shows that 0.72375 fails for it), and a degree-10 profile gives 0.7237502555, a margin of $2.6 \cdot 10^{-7}$. The value 0.7237 at the degree-8 profile needs the design layer (zone constants, ramp link, existence of the design) at that profile, which is not done in Lean; each of these is an $o(1)$ row, so the argument of §3.4 should give 0.7237 as well [M], but we have not recomputed those constants on paper and do not state it. In Lean the companion statement at 0.7237 (`theorem_one_prime`) still rests on one open node (§1.5).

3.4 Proof of Theorem 1.1

The proof is that of [9, Theorem 1], at the design of record of [9] ($\lambda = \lambda^*$, profile p_d), with the out-zone use of the large sieve inside [9, Lemma 4.3] replaced by Lemma 3.1(ii). The in-zone analysis, the zone-edge strip, the in-zone cross term and the ends are unchanged (the out-zone cross term is the factor $1 + O(\rho)$ of Lemma 3.1(ii)). Proposition 2.1 is applied with $\mathcal{B}_* = \mathcal{B}_{C^+}^K(v_d)$; since $C_Q \rightarrow \pi^4/18 < C^+$, the difference between C_Q and C^+ only helps. The budget of [9, §10.3] receives three new rows: the kink shift $\log(2\pi K_1)/\mathcal{L} = O(\log \log Q / \log Q)$, the error $C_1/\log Q$ of Lemma 3.7 and the strip $O(\mathcal{L}^{-1})$, all within the rate $\log \log Q / \log Q$ of [9]. The certificate is used at $0.72351 = 0.7235 + 10^{-5}$, and the formal proof spends this slack on finite parts of the design, as [9] spends its own. $\square$

Formalisation. The formal proof follows this outline: the killed Frobenius row (node K8a, `frobenius_inzone_killed`) is the Frobenius row of [9] with the out-zone sieve step replaced by the pointwise bound (3) integrated against g ; its four sub-nodes (the master inequality, the saving $2 \cdot \text{Sav} \geq (T/2\pi)\mathcal{L}(aL)^2(K_1 - K_1^{\text{kill}} - \eta)$, the errors, and a closing arithmetic in

which the saving meets the main term before the conversion to the zero count) are proved, the first from Lemma 3.1(i). The frame (node K9a) is derived from K8a, and the assembly (`k_assembly`) and the headline step are proved. The headline `theorem_one_prime_design` is quoted in Appendix A; in the released library it depends on the three standard axioms only and rests on no open declaration [L] (§1.5).

Remark 3.8 (Thresholds). Lemma 3.1 is asymptotic. With the explicit prime number theorem of Fiori, Kadiri and Swidinsky [10] and $T = (\log Q)^3$, the working draft computes that the row E_1 of Step 3 first becomes smaller than 1 near $\log Q \approx 5900$; the formal route through summation by parts has a much larger threshold. Theorem 1.1 is a statement about $Q \rightarrow \infty$ only, and this paper makes no statement about any finite Q .

4 The Shell route

This section replaces, on an initial segment $1 < |\alpha| \leq \alpha'$ of the out-zone, the kernel $C|\alpha|$ by the kernel 1. Beyond α' the killed kernel of Lemma 3.1 is used. All lemmas are stated in their corrected forms; each correction is listed in Appendix B.

4.1 Notation

Let $\log Q + 3 \leq s \leq \alpha'' \log Q$ with a fixed $\alpha'' < 2$, and $s \leq \log X - 1$, $N := e^s$. Put $U := T/2\pi$, so that $\|a(s)\|^2 = U(s + O(1))$ for $s \geq (1 + \delta_1) \log Q$, where $\delta_1 := T^{-1/3}$. Fix $\kappa \in (0, 1]$, $c \in (0, 1)$ and $\Xi \in C_c^\infty((-1, 1))$ with $0 \leq \Xi \leq 1$ and

$$\Xi = 1 \text{ on } [-c, c]. \quad (9)$$

Put $a_n^{\text{near}} := a_n \Xi((\log n - s)/\kappa)$, $a^{\text{far}} := a - a^{\text{near}}$, and

$$A_s(y) := y^{-1/2} D_T(s - \log y) \Xi((\log y - s)/\kappa), \quad a_n^{\text{near}} = -\frac{1}{2\pi} \Lambda(n) A_s(n).$$

Let $K = K(Q) \rightarrow \infty$ with $K \leq (\log Q)^B$ for a fixed B (in the design $K = \mathcal{L}(\log \mathcal{L})^2$), and let $\epsilon = \epsilon(Q)$ satisfy

$$(\log Q)^{-B} \leq \epsilon \leq 1, \quad \epsilon \rightarrow 0 \quad (10)$$

(in the design $\epsilon = 1/\mathcal{L}$); this parameter ϵ is not the exponent ε of $T = (\log Q)^{r_0 + \varepsilon}$. Put

$$R_1 := \frac{N(\log Q)^6}{\epsilon Q}, \quad R_0 := \frac{N}{QTK}, \quad \delta := \frac{\epsilon}{N}.$$

For $s \leq \alpha'' \log Q$ one has $R_1 \leq Q/(2K)$ for large Q . For a/r reduced write $\vartheta = a/r + \beta$; the *hole* of a/r is $|\beta| < 1/(rQ)$, its *ring* is $1/(rQ) \leq |\beta| < K/(rQ)$, and the *shell set* is $\mathcal{S} = \bigcup_{r \leq R_1} \{|\vartheta - a/r| < K/(rQ)\}$. With $S^{\text{near}}(\vartheta) = \sum_n a_n^{\text{near}} e(n\vartheta)$ restricted to primes $n > Q$,

$$\begin{aligned} \text{Ring}(s) &:= \sum_{r \leq R_1} \sum_{a \bmod r}^* \int_{\text{ring}} |S^{\text{near}}(a/r + \beta)|^2 d\beta, \\ \text{Hole}(s) &:= \sum_{r \leq R_0} \sum_a^* \int_{\text{hole}} |S^{\text{near}}(a/r + \beta)|^2 d\beta. \end{aligned}$$

The condition (10) is stronger than the working draft's $\epsilon \rightarrow 0$; it is needed in Lemma 4.16 through R_1 (§4.7).

4.2 The signed Farey identity

Lemma 4.1 (Lemma 1). *Let (a_n) be finitely supported, with every prime factor of every n in the support greater than Q . Let $\omega : \{1, \dots, Q\} \rightarrow \mathbb{R}$ and $F_\chi = \sum_n a_n \chi(n)$. Then*

$$\sum_{1 \leq q \leq Q} \omega(q) \sum_{\chi \bmod q}^* |F_\chi|^2 = \sum_{d \leq Q} \Omega_\omega(d) \sum_{b \bmod d}^* |S(b/d)|^2, \quad \Omega_\omega(d) := \frac{\varphi(d)}{d} \sum_{\substack{e \leq Q/d \\ (e,d)=1}} \frac{\mu(e) \omega(de)}{e}.$$

Proof. (1) Every n in the support is coprime to every $q' \leq Q$, so $G(q') := \sum_{\chi \bmod q'} |F_\chi|^2 = \varphi(q') \sum_{c \in (\mathbb{Z}/q')^\times} |\sum_{n \equiv c} a_n|^2 = \frac{\varphi(q')}{q'} \sum_{b \bmod q'} |S(b/q')|^2$. (2) Each $\chi \bmod q$ is induced by a unique primitive $\chi^* \bmod q^*$, $q^* \mid q$, and $F_\chi = F_{\chi^*}$ on the support; so $G = 1 * H$ with $H(q) = \sum_{\chi \bmod q}^* |F_\chi|^2$, i.e. $H = \mu * G$. (3) Expanding $\sum_{b \bmod q'} = \sum_{d|q'} \sum_{b \bmod d}^*$, with $q = de$, $q' = df$, $f \mid e$, the coefficient of $P(d) := \sum_{b \bmod d}^* |S(b/d)|^2$ in $H(q)$ is $\sum_{f|e} \mu(e/f) \varphi(df)/(df)$. (4) Write $\varphi(df)/(df) = \frac{\varphi(d)}{d} h_d(f)$ with $h_d(f) = \prod_{p|f, p \nmid d} (1 - 1/p)$; then $\mu * h_d = (\mu(e)/e) \mathbf{1}[(e, d) = 1]$. (5) Multiply by $\omega(de)$ and sum over $de \leq Q$. $\square$

Formalisation: `ZetaShell.signed_farey_identity`, proved by a different route (expansion of both sides as $\sum_{n,m} a_n \bar{a}_m K(n, m)$, the orthogonality of primitive characters from `ZetaQ`, Ramanujan sums and a local identity proved by induction over primes) [L]; the identity was also tested by exact arithmetic, including $Q = 0, 1$, primes and prime powers. The support hypothesis is needed: without it the identity fails in tests, with relative errors from 1.4% to 182%.

For $\omega \equiv 1$, $\Omega(d) = \frac{\varphi(d)}{d} \sum_{m \leq Q/d, (m,d)=1} \mu(m)/m$ and $|\Omega(d)| \leq \varphi(d)/d \leq 1$ by Tao's inequality [22] [C]; for the dyadic weight $\omega = \mathbf{1}_{(Q/2, Q]}$ the same inequality gives $|\Omega_{\text{dy}}| \leq 2$.

Lemma 4.2 (Lemma 1': primes $\leq Q$, prime powers, far part). *Let $M \subseteq [1, Q]$ be a set of moduli, $\mathbf{F}_M(a) := \sum_{q \in M} \sum_{\chi \bmod q}^* |\sum_{n \leq N'} a_n \chi(n)|^2$, and $N' \geq 1$.*

- (a) *For all b, c : $\mathbf{F}_M(b+c) \leq (\sqrt{\mathbf{F}_M(b)} + \sqrt{(Q^2 + \pi N') \|c\|^2})^2$.*
- (b) *If $s > \log Q$, then $\sum_{n \leq Q} |a_n(s)|^2 \leq (\sum_{n \leq Q} \Lambda(n)^2/n) / (\pi^2 (s - \log Q)^2)$.*
- (c) *If Ξ satisfies (9), then $\|a^{\text{far}}\|^2 \leq \pi^{-2} \sum_{|\log n - s| > c\kappa} \Lambda(n)^2 / (n(\log n - s)^2)$.*

Consequently, for $s \geq (1 + \delta_1) \log Q$, the part of a on $n \leq Q$ has $\|\cdot\|^2 \leq (1 + o(1)) / (2\pi^2 \delta_1^2)$; the prime powers $p^k > Q$ with $p \leq Q$ have $\|\cdot\|^2 = O(1)$; $\|a^{\text{far}}\|^2 \leq (2s/(\pi^2 c\kappa))(1 + o(1))$; and replacing a by its near part on primes $> Q$ changes $\mathbf{F}(s)/(\|\mathfrak{F}_Q\| \|a(s)\|^2)$ by a relative $O((c\kappa T)^{-1/2} + (\delta_1^2 T \log Q)^{-1/2})$.

Proof. (a) is Minkowski's inequality in ℓ^2 with the large sieve in Gallagher's form. (b) and (c) follow from $|D_T(v)| \leq 2/|v|$; in (c), $a_n^{\text{far}} = 0$ unless $|\log n - s| > c\kappa$ by (9). The consequences: $\sum_{n \leq Q} \Lambda(n)^2/n = (\frac{1}{2} + o(1)) \log^2 Q$ in (b); the peak of the prime powers is $\ll T^2 (\log N)^2 N^{-1/2}$; in (c), the sum is $\leq (2s/(c\kappa))(1 + o(1))$ by the prime number theorem, which is $4/(\pi c\kappa T)$ relative to $\|a\|^2 \approx sT/2\pi$; then (a) with the optimal splitting gives the relative costs. $\square$

Status: the consequences are the working draft's argument with the factor c added. *Formalisation:* (a), (b), (c) are `replace_cost`, `small_prime_norm`, `far_norm` [L]. The asymptotic consequences are not formalised. *Correction:* the working draft required only $0 \leq \Xi \leq 1$ and $\text{supp } \Xi \subseteq (-1, 1)$, which allows $\Xi \equiv 0$, and then the far part is all of a (numerically 55.5, 274, 1396 at $T = 50, 200, 800$ against a claimed 1.42). Condition (9) is added; since κ is free, it only changes constants by the factor c .

4.3 Local density of weighted Farey points

For real weights $\mathcal{W}_\xi$ on $\xi \in \mathcal{F}_Q$ and $\delta > 0$ put $D_\delta^\mathcal{W}(\vartheta) := \delta^{-1} \sum_{\|\xi - \vartheta\| \leq \delta/2} \mathcal{W}_\xi$, the distance being taken on $\mathbb{R}/\mathbb{Z}$. The weights of Lemma 4.1 are $\mathcal{W}_{b/d} = \Omega_\omega(d)$.

Families. Let $\mathfrak{w} : [0, \infty) \rightarrow \mathbb{R}$ be supported in $[0, 1]$ and of bounded variation, $V^* := \|\mathfrak{w}\|_\infty + \text{Var}(\mathfrak{w})$, $W_1 := \int_0^1 x \mathfrak{w}(x) dx$, and $\omega(q) = \mathfrak{w}(q/Q)$. Then $\Omega_\omega(d) = \sum_e c_e \mathfrak{w}(de/Q) h_e(d)$ with $c_e = \mu(e)/e$ and $h_e(q) = \frac{\varphi(q)}{q} \mathbf{1}[(e, q) = 1]$. Put $H_\mathfrak{w} := \sum_{q \geq 2} \omega(q) \varphi^*(q)$, where $\varphi^*(q)$ is the number of primitive characters mod q . The family of this paper is the sharp one, $\mathfrak{w} = \mathbf{1}_{[0,1]}$ ($V^* = 2$, $W_1 = \frac{1}{2}$, $H_\mathfrak{w}/Q^2 \rightarrow 18/\pi^4$); the dyadic one, $\mathfrak{w} = \mathbf{1}_{(1/2,1]}$ ($V^* = 3$, $W_1 = \frac{3}{8}$, $H_\mathfrak{w}/Q^2 \rightarrow 27/(2\pi^4)$), appears only in Remark 4.19.

Lemma 4.3 (Lemma 2: local density). *Fix a family. Let $\delta = \epsilon/N$ and $R_1 = N(\log Q)^6/(\epsilon Q)$.*

(a) (Outside shells.) *There are $B_\mathfrak{w}$ and Q_0 such that for $Q \geq Q_0$, $N, \epsilon > 0$, $K \geq 1$, $W_\infty \geq \max_{d \leq Q} |\Omega(d)|$, with $1 \leq R_1 \leq Q/(2K)$, and every $\vartheta \notin \mathcal{S}$,*

$$|D_\delta^\Omega(\vartheta) - H_\mathfrak{w}| \leq B_\mathfrak{w} H_\mathfrak{w} \left[\frac{(1 + \log K)^3}{K} + \frac{N(\log Q)^2}{\epsilon Q R_1} + \frac{N(\log Q)^8}{\epsilon Q^2} + \frac{W_\infty N}{\epsilon H_\mathfrak{w}} \right].$$

If moreover $N/\epsilon \leq Q^2(\log Q)^{-12}$, the bracket is $O((1 + \log K)^3/K + (\log Q)^{-4})$.

(b) (Everywhere.) *For every Q , every real $\mathcal{W}$ with $|\mathcal{W}| \leq W_\infty$, every $\delta > 0$ and every ϑ :*
 $|D_\delta^\mathcal{W}(\vartheta)| \leq D_\delta^{|\mathcal{W}|}(\vartheta) \leq W_\infty(Q^2 + 1/\delta)$.

(c) (Holes.) *For every $a/r \in \mathcal{F}_Q$, every real $\mathcal{W}$, every $\delta > 0$ and every β with $\delta/2 < |\beta| < 1/(rQ) - \delta/2$, $D_\delta^\mathcal{W}(a/r + \beta) = 0$. If moreover $\delta < 1/(rQ)$, then $D_\delta^\mathcal{W}(a/r + \beta) = \mathcal{W}_{a/r}/\delta$ for $|\beta| \leq \delta/2$.*

Proof. (b) Consecutive points of $\mathcal{F}_Q$ are at least Q^{-2} apart, so a window of length δ holds at most $\delta Q^2 + 1$ of them. (c) If $b/q \neq a/r$ and $q \leq Q$, then $\|b/q - a/r\| \geq 1/(qr) \geq 1/(rQ)$; the second sentence uses that for $\delta < 1/(rQ)$ the window of half-length $\delta/2$ around $a/r + \beta$ contains no other point.

(a) *Step 0 (lines).* Take a Dirichlet approximation a/r of ϑ with $r \leq R_1$ (this uses $R_1 \geq 1$), $|\eta| \leq 1/(rR_1)$, $\eta = \vartheta - a/r$; since $\vartheta \notin \mathcal{S}$, $u := rQ|\eta| \in [K, Q/R_1]$. Fix a'/r' with $ar' - a'r = 1$. The map $(k, j) \mapsto (b, q) = (ak - a'j, rk - r'j)$ is a bijection from primitive pairs with $q > 0$ onto reduced fractions, and $b/q - a/r = j/(qr)$. A point with $q \leq Q$ lies in the window $\mathcal{I} = [\vartheta - \delta/2, \vartheta + \delta/2]$ only if $0 < |j| \leq J := Q/R_1 + R_1 Q \delta + 1$ or $j = 0$; for each line, $I_j := \{q \in [1, Q] : j/(qr) \in \mathcal{I} - a/r\}$ is an interval.

Step 1 (expansion). Each h_e is multiplicative with $h_e = 1 * \lambda_e$, λ_e supported on squarefree numbers, $\lambda_e(p) = h_e(p) - 1$. Hence $\delta D_\delta^\Omega(\vartheta) = \Omega(r) \mathbf{1}[a/r \in \mathcal{I}] + \sum_{0 < |j| \leq J} \sum_e c_e \sum_{f \leq Q} \lambda_e(f) N_j(e, f)$ with $N_j(e, f) := \sum_{k: (k, j)=1, q \in I_j, f|q} \mathfrak{w}(qe/Q)$.

Step 2 (Chinese remainder theorem). Write $\mathbf{1}[(k, j) = 1] = \sum_{g|j} \mu(g) \mathbf{1}[g | k]$. For $p | f$: if $p \nmid r$, $p | q$ fixes $k \bmod p$; if $p | r$, then $p | q \iff p | j$. So for each g the admissible k form one class modulo $\text{lcm}(g, f_{(r)})$, $f_{(r)} = \prod_{p|f, p \nmid r} p$, or none, and q runs through one progression restricted to I_j . The restricted function has variation at most $\text{Var}(\mathfrak{w}) + 2\|\mathfrak{w}\|_\infty \leq 2V^*$, so

$$N_j(e, f) = \frac{1}{r} \int_{I_j} \mathfrak{w}(qe/Q) dq \cdot \frac{\varphi(j)}{j} \prod_{p|f} \delta_p + E_j, \quad |E_j| \leq 2V^* 2^{\omega(j)},$$

with $\delta_p = 1/p$ if $p \nmid rj$, $\delta_p = 1$ if $p | (r, j)$, and $\delta_p = 0$ otherwise.

Step 3 (sum over f). Put $\mathfrak{E}_{j,r}(e) := \sum_{f \geq 1} \lambda_e(f) \prod_{p|f} \delta_p = \prod_{p \nmid rj} (1 + \lambda_e(p)/p) \prod_{p|(r, j)} h_e(p)$. The discrepancy of Step 2 summed over $f \leq Q$ is at most $2V^* 2^{\omega(j) + \omega(e)} (1 + \log Q)$. The tail $f > Q$ of the main term is at most $\frac{1}{r} \int_{I_j} |\mathfrak{w}| \cdot 2^{\omega(e)+1} \sigma(\text{rad}(r, j))/Q$, since for $p | (r, j)$

one has $\delta_p = 1$ and $|\lambda_e(p)| \leq 1$; summed over $j \leq J$ and e this is $\ll \delta Q r(1 + \log Q)$. With $\sum_{e \leq y} |c_e| 2^{\omega(e)} \ll (1 + \log y)^2$ the total count error is $\ll V^* J(1 + \log J)(1 + \log Q) + \delta Q r(1 + \log Q)$. Divided by $\delta H_{\mathfrak{w}} \asymp \delta Q^2$, and with $r \leq R_1$, this gives the second and third terms of the bracket; the $j = 0$ term gives the fourth.

Step 4 (profile). Substituting $q = |j|/(r|\beta|)$, $\beta \in \mathcal{I} - a/r$ of the sign of j , the main term equals $\int_{\mathcal{I}-a/r} Q^2 g_r(rQ|\beta|) d\beta$ up to $O(\delta(1 + \log Q))$, where $g_r(v) := v^{-2} \sum_{1 \leq j \leq v} \varphi(j) \sum_e c_e \mathfrak{w}(je/v) \mathfrak{E}_{j,r}(e)$ ($g_r(v) = 0$ for $v < 1$). By Lemma 4.4, $|g_r(v) - \bar{g}_{\mathfrak{w}}| \leq B_{\mathfrak{w}}(1 + \log v)^2/v$ uniformly in r , and $Q^2 \bar{g}_{\mathfrak{w}} = H_{\mathfrak{w}}(1 + O(V^* \log^2 Q/Q))$.

Step 5 (window average). In u -units the window is $[u - \Delta', u + \Delta']$, $\Delta' = rQ\delta/2$. With $E(v) := |g_r(v) - \bar{g}_{\mathfrak{w}}|$: if $\Delta' \leq u/2$ the average of E is $\ll B_{\mathfrak{w}}(1 + \log K)^2/K$; if $\Delta' > u/2 \geq K/2$ it is $\leq (2\Delta')^{-1} 2 \int_0^{3\Delta'} E \ll B_{\mathfrak{w}}(1 + \log K)^3/K$. $\square$

Lemma 4.4 (Lemma P: universality of the line profile). *For all integers $r \geq 1$ and real $v \geq 1$, $|g_r(v) - \bar{g}_{\mathfrak{w}}| \leq B_{\mathfrak{w}}(1 + \log v)^2/v$, where $\bar{g}_{\mathfrak{w}} = W_1 \sum_e c_e \Pi(e)/e^2$ and $\Pi(e) = \prod_p (1 - \frac{1}{p})(1 + \frac{h_e(p)}{p})$ does not depend on r . For the plain kind $\bar{g}_{\mathfrak{w}} = W_1 \cdot 36/\pi^4$.*

Proof. (i) Write $\varphi(j) \mathfrak{E}_{j,r}(e) = P_0 j b(j)$ with $P_0 = \prod_{p|r} (1 + \lambda_e(p)/p) \leq 1$ and b multiplicative on squarefree kernels: $b_p = (1 - \frac{1}{p})/(1 + \lambda_e(p)/p)$ for $p \nmid r$, $b_p = (1 - \frac{1}{p})h_e(p)$ for $p \mid r$. Then $b = 1 * \nu_e$ with $|\nu_e(p)| \leq 2/p$ for $p \nmid e$ and $|\nu_e(p)| \leq 1$ for $p \mid e$. (ii) From $\sum_{m \leq x} m = x^2/2 + O(x)$, $\sum_{j \leq y} j b(j) = \frac{y^2}{2} \sum_f \frac{\nu_e(f)}{f} + O(2^{\omega(e)} y(1 + \log y)^2)$, the tail $\sum_{f > y} |\nu_e(f)|/f$ included; the bound is uniform in r because $|\nu_e(p)| \leq 2/p$ for $p \nmid e$ whether or not $p \mid r$. (iii) At each prime, $(1 + \lambda_e(p)/p)(1 + \nu_e(p)/p)$ for $p \nmid r$ and $1 + \nu_e(p)/p$ for $p \mid r$ both equal $(1 - \frac{1}{p})(1 + \frac{h_e(p)}{p})$. Hence $P_0 \sum_f \nu_e(f)/f = \Pi(e)$ and $\sum_{j \leq y} \varphi(j) \mathfrak{E}_{j,r}(e) = \Pi(e) y^2/2 + O(2^{\omega(e)} y(1 + \log y)^2)$ uniformly in r . (iv) Partial summation against $\mathfrak{w}(je/v)$ up to $y = v/e$ and summation over $e \leq v$ with $\sum_e |c_e| 2^{\omega(e)}/e = \prod_p (1 + 2/p^2) < \infty$ give the bound; the tail $e > v$ is $O(1/v)$. For the plain kind $\sum_e \mu(e) \Pi(e)/e^3 = \prod_p (1 - p^{-2})^2 = 36/\pi^4$. $\square$

Status of Lemmas 4.3 and 4.4. The proofs above include the corrections of Appendix B (the hypothesis $R_1 \geq 1$; the eventual form $Q \geq Q_0$, since the statement fails at $Q = 2$ where $H_{\mathfrak{w}} = 0$; the error $2V^*$ in Step 2; the tail bound in Step 3, which could not be reproduced in the draft's form and is replaced by the bound above with the same conclusion; Step 4 as an inequality); every step was also tested numerically against exact enumeration at $Q = 200$ and 400 [N]. The end of Lemma P, $H_{\mathfrak{w}} = Q^2 \bar{g}_{\mathfrak{w}}(1 + O(V^* \log^2 Q/Q))$ for $Q \geq 2$, is a direct computation: with $\varphi^* = \mu * \varphi$ write $\varphi^*(q) = q g(q)$ and $g = 1 * h$, where $h(p) = -2/p$, $h(p^2) = 1/p^2$ and $h(p^k) = 0$ for $k \geq 3$. Then $\sum_d h(d)/d = \prod_p (1 - p^{-2})^2 = 36/\pi^4$, $\sum_{d \leq x} |h(d)| \ll \log^2 x$ and $\sum_{d > x} |h(d)|/d \ll (\log x)/x$, so $\sum_{q \leq x} \varphi^*(q) = \frac{36}{\pi^4} \frac{x^2}{2} + O(x(1 + \log x)^2)$, and Abel summation against $\mathfrak{w}(q/Q)$ gives $|H_{\mathfrak{w}} - Q^2 \bar{g}_{\mathfrak{w}}| \leq B V^* Q(1 + \log Q)^2$ with B absolute. The relative form uses $Q \geq 2$ and $\bar{g}_{\mathfrak{w}} > 0$. An exact enumeration for the sharp and the dyadic family at every $Q \leq 10^6$ agrees [N]. *Formalisation:* (b) and (c) are `lemma2b`, `lemma2c_hole`, `lemma2c_spike` [L]; (a) is assembled from its steps (`lemma2a`), among them the line-sum asymptotic under Lemma P (`line_sum_asymp`) and two parts of the tail of Step 3 (`s3_tail_plain_pt`, `s3_tail_sum`); in the released library `lemma2a` and Lemma P (`lemmaP`) are kernel-checked [L]. *Correction:* the draft's spike sentence in (c) omitted $\delta < 1/(rQ)$ and is false without it ($a/r = 0/1$, $\delta = 1/Q$); it is not used, since the spikes lie in the set $\mathcal{X}_{\delta}$ of Lemma 4.7.

Lemma 4.5 (Lemma 3: signed Gallagher localisation). *Let (a_n) be supported in $[Ne^{-\kappa}, Ne^{\kappa}]$, let $\mathcal{W}$ be real weights on $\mathcal{F}_Q$ with $|\mathcal{W}| \leq W_{\infty}$, $W_{\infty} \geq 0$, and $\delta = \epsilon/N$ with $0 < \epsilon < N$. Then $|\sum_{\xi} \mathcal{W}_{\xi} |S(\xi)|^2 - \int_0^1 |S|^2 D_{\delta}^{\mathcal{W}}| \leq 2\pi \sinh \kappa W_{\infty} (\epsilon Q^2 + N) \|a\|^2$; consequently $\sum_{\xi} \mathcal{W}_{\xi} |S(\xi)|^2 \leq \int_0^1 |S|^2 (D_{\delta}^{\mathcal{W}})^+ + 2\pi \sinh \kappa W_{\infty} (\epsilon Q^2 + N) \|a\|^2$.*

Proof. Put $n_0 = N \cosh \kappa$ and $\tilde{S}(\vartheta) = e(-n_0 \vartheta) S(\vartheta)$, so $|\tilde{S}| = |S|$ and $\|\tilde{S}'\|_2 \leq 2\pi N \sinh \kappa \|a\|$. For $f \in C^1(I)$ on an interval of length δ with midpoint ξ , $|f(\xi) - \delta^{-1} \int_I f| \leq \frac{1}{2} \int_I |f'|$ [17, proof of Lemma 1]. Apply it to $f = |\tilde{S}|^2$ on I_ξ and sum with weights: the main terms give $\int |S|^2 D_\delta^{\mathcal{W}}$ exactly (the windows fold injectively onto $\mathbb{R}/\mathbb{Z}$ since $\delta < 1$), and the remainders are at most $\delta \sup D_\delta^{|\mathcal{W}|} \|S\|_2 \|\tilde{S}'\|_2$, which Lemma 4.3(b) bounds. $\square$

Relative to $\|\mathfrak{F}_Q\| \|a\|^2$ the error is $\frac{\pi^5}{9} \sinh \kappa (\epsilon + N/Q^2) = o(1)$ for $N \leq Q^{2-\eta}$. *Formalisation:* `signed_gallagher` [L].

4.4 Holes, and the reduction to primitive characters

Lemma 4.6 (Lemma 4: holes). *Let $a = a^{\text{near}}$ be supported on primes $> Q$, with Ξ as in (9), and put $S_\chi(\beta) := \sum_n a_n \chi(n) e(n\beta)$. For $r \leq R_0$ and every β ,*

$$\sum_{a \bmod r}^* |S(a/r + \beta)|^2 = \frac{1}{\varphi(r)} \sum_{\chi \bmod r} |\tau(\bar{\chi})|^2 |S_\chi(\beta)|^2 \geq \frac{\mu^2(r)}{\varphi(r)} |S(\beta)|^2, \quad (11)$$

and, for the design width $K = \mathcal{L}(\log \mathcal{L})^2$, $\text{Hole}(s) \geq U(\log R_0 + O(1))(1 - O(1/\log Q))$, the implied constants depending on κ , c and Ξ .

Proof. The identity follows from $e(m/r) = \varphi(r)^{-1} \sum_{\chi \bmod r} \bar{\chi}(m) \tau(\chi)$ for $(m, r) = 1$ and orthogonality; the inequality is Lemma 3.5, since every n in the support is a prime $> Q \geq r$. The holes of $r \leq R_0$ contain $|\beta| \leq KT/N$ and are disjoint, so $\text{Hole} \geq (\sum_{r \leq R_0} \mu^2(r)/\varphi(r)) \int_{|\beta| \leq KT/N} |S|^2$. The arc mass is bounded below as in Lemma 3.7, with the near model $\mathcal{M}(\beta) := \sum_n \mathbf{m}(n) \Xi((\log n - s)/\kappa) e(n\beta)$ in place of $\tilde{M}$, the cutoff $\Xi((\log y - s)/\kappa)$ in place of $\varrho(\log y - s)$, primes $> Q$ in place of Λ' , and no tail term; in Step 3 the counting error is now $\sum_{p \leq y} \log p - [y]$ minus the constant $\sum_{p \leq Q} \log p$, and the constant cancels because only increments over intervals of length h enter. The model mass is where (9) enters: with $y = e^{s-v}$, $\int |\mathbf{m}(y) \Xi(-v/\kappa)|^2 dy \geq (4\pi^2)^{-1} \int_{|v| \leq c\kappa} |D_T(v)|^2 dv \geq U - 2/(\pi^2 c\kappa)$, since $|D_T(v)|^2 \leq 4/v^2$; the outer mass is $O(U/K^2)$ as in Step 1 of Lemma 3.7, with a constant depending on κ and Ξ' , which is $O(U/\log^2 Q)$ for $K = \mathcal{L}(\log \mathcal{L})^2$ (for a general K this relative error $O(1/K^2)$ is $O(1/\log Q)$ only if $K^2 \gg \log Q$). Finally $\sum_{r \leq R} \mu^2(r)/\varphi(r) \geq \log(R+1)$. $\square$

The working draft stated this lemma without (9); with $\Xi \equiv 0$ the model mass vanishes and the lower bound fails. The prime number theorem is used here with the exponent $A \geq 2 + 2(r_0 + \epsilon)$ of (2), which also covers the larger width $K = \mathcal{L}(\log \mathcal{L})^2$. Only $\|a\|^2 - \text{Hole} \leq U(\mathcal{L} + \log K + O(1))$, at $K = \mathcal{L}(\log \mathcal{L})^2$, is used below; it follows from the lemma and $\|a\|^2 = U(s + O(1))$, since $\log R_0 = s - \log(QTK)$ and $s \cdot O(1/\log Q) = O(1)$.

Lemma 4.7 (Hole-edge mass). *Let $2R_1 \leq Q$, let a be supported on the integers of $[1, N_{\text{sup}}]$, and let $\mathcal{X} := \{a/r, a/r \pm 1/(rQ) : r \leq R_1\}$ (as a multiset). Then $\sum_{\xi \in \mathcal{X}} \int_{|\vartheta - \xi| \leq \delta} |S(\vartheta)|^2 d\vartheta \leq 6\delta(2R_1^2 + \pi N_{\text{sup}}) \|a\|^2$. For the near part, supported in $[Ne^{-\kappa}, Ne^\kappa]$, one may take $N_{\text{sup}} = \lfloor Ne^\kappa \rfloor \leq e^\kappa N$, and the bound is $O(\epsilon + N(\log Q)^{12}/(\epsilon Q^2)) \|a\|^2 = o(\|a\|^2)$.*

Proof. $\mathcal{X}$ is the union of three families, each $1/(2R_1^2)$ -spaced: $\|b/r - b'/r'\| \geq 1/(rr')$, and the shifts differ by at most $R_1/(rr'Q) \leq 1/(2rr')$. For each family and each $|t| \leq \delta$ Gallagher's inequality gives $\sum_\xi |S(\xi + t)|^2 \leq (2R_1^2 + \pi N_{\text{sup}}) \|a\|^2$; integrate over t . $\square$

Formalisation: `hole_edge_mass`, with this explicit constant, the coefficients being read on $0 < n \leq N_{\text{sup}}$ [L]; the pointwise assembly applies it with $N_{\text{sup}} = \lfloor e^{s+1} \rfloor \leq eN$, whose range $n \leq e^{s+1}$ contains the near window since $\kappa \leq 1$, and carries the term πN_{sup} through. Inside a hole of $R_0 < r \leq R_1$ the density is 0 except within δ of a/r or of the hole boundary (Lemma 4.3(c)), where it can be as large as $Q^2 + N/\epsilon$; that mass is charged at $\|\Omega\|_\infty(Q^2 + N/\epsilon) = O(Q^2)$ and is $o(\|a\|^2)$ by the lemma.

Lemma 4.8 (Reduction to primitive characters). *Let a be supported on primes $> Q$ in $[Ne^{-\kappa}, Ne^{\kappa}]$, let $R_1 \geq 1$ and $2KR_1 \leq Q$ (so that $2K \leq Q$), and let $\Upsilon_K \geq 0$ be such that $\sum_{y \leq m < Ky} \mu^2(m)/\varphi(m) \leq \Upsilon_K$ for all $y > 0$. Let $\mathcal{M}$ be the near model of Lemma 4.6, so $\int_0^1 |\mathcal{M}|^2 \leq U + o(1)$. Then*

$$\begin{aligned} \text{Ring} &\leq 2\Upsilon_K \sum_{r^* \leq R_1} \frac{r^*}{\varphi(r^*)} \sum_{\chi^* \bmod r^*}^* \int_{\frac{1}{R_1 Q} \leq |\beta| \leq \frac{K}{r^* Q}} |S_{\chi^*}(\beta) - [r^*=1]\mathcal{M}(\beta)|^2 d\beta \\ &\quad + 2\Upsilon_K (U + o(1)). \end{aligned}$$

The choice $\Upsilon_K = c_\mu(1 + \log K)$, with $c_\mu := \zeta(2)\zeta(3)/\zeta(6) = 1.9435964\dots$, is admissible for $K \geq 1$.

Proof. For $\chi \bmod r$ induced by $\chi^* \bmod r^*$, $\tau(\chi) = \mu(r/r^*)\chi^*(r/r^*)\tau(\chi^*)$ [18, Thm. 9.10], so $|\tau(\bar{\chi})|^2 = r^*\mu^2(r/r^*)\mathbf{1}[(r/r^*, r^*) = 1]$. The identity (11) is stated in Lemma 4.6 for $r \leq R_0$; its proof uses only that every n in the support is a prime $> Q \geq r$, so it holds for every $r \leq Q$, and here it is used for $r \leq R_1$. With $m = r/r^*$ it gives $\sum_{a \bmod r} |S(a/r + \beta)|^2 = \sum_{r^*m=r, (m, r^*)=1} \frac{r^*}{\varphi(r^*)} \frac{\mu^2(m)}{\varphi(m)} \sum_{\chi^*}^* |S_{\chi^*}(\beta)|^2$. For fixed β and r^* the m for which β lies in the ring of modulus r^*m form a range $[y, Ky)$, so their sum is at most Υ_K ; a nonempty range needs $r^*m \leq R_1$ for some $m \geq 1$, which gives the displayed range of β . For $r^* = 1$ split $|S|^2 \leq 2|S - \mathcal{M}|^2 + 2|\mathcal{M}|^2$; the range $|\beta| \leq K/Q \leq \frac{1}{2}$ lies in one period because $2K \leq Q$. The hypothesis $R_1 \geq 1$ only excludes the empty case.

For the choice of Υ_K : for squarefree m , $1/\varphi(m) = m^{-1} \sum_{\text{rad } k|m} 1/k$, so $\sum_{y \leq m < Ky} \mu^2(m)/\varphi(m) \leq \sum_k k^{-1} \sum_{y \leq m < Ky, \text{rad } k|m} m^{-1} \leq (1 + \log K) \sum_k 1/(k \text{rad } k)$, since $\sum_{z \leq l < Kz, l \geq 1} 1/l \leq 1 + \log K$ for $z > 0$; and $\sum_k 1/(k \text{rad } k) = \prod_p (1 + 1/(p(p-1))) = \zeta(2)\zeta(3)/\zeta(6)$. $\square$

The lemma is linear in Υ_K , and Theorem 4.17 uses it with $\Upsilon_K = c_\mu(1 + \log K)$. The working draft used $\log K + 1.4708$ instead: the value $41/12 - \log 7 = 1.47075\dots$ is the supremum of $\sum_{m \leq x} \mu^2(m)/\varphi(m) - \log x$ over $1 \leq x \leq 10^7$, attained at $x = 7$ (a numerical check [N]); it is not proved for larger x , and it is not used here. *Formalisation:* `ring_le_primitive_corr`, with a parameter $C_\mu \geq 0$, $K \geq 1$ and the hypothesis $\sum_{y \leq m < Ky} \mu^2(m)/\varphi(m) \leq \log K + C_\mu$, which covers $\Upsilon_K = c_\mu(1 + \log K)$; it is proved with the Gauss sums of imprimitive characters, which Mathlib lacks, in a new module [L]. It replaces a first Lean form that was false as Lean parses it; the inequality displayed above was not affected (Appendix B).

4.5 Proposition Z: the ring mass through the explicit formula

For χ primitive mod $r \geq 1$ ($r = 1$: $\chi = 1$, $L = \zeta$) define

$$S_\chi(\beta; s) := \sum_n \Lambda(n) \chi(n) A_s(n) e(n\beta) - \delta_{r=1} \tilde{M}_s(\beta), \quad \tilde{M}_s(\beta) := \int_0^\infty A_s(y) e(y\beta) dy.$$

By Poisson summation, $\tilde{M}_s$ differs from the discrete model $\sum_n A_s(n) e(n\beta)$ by $O_m(N^{-m})$ for every m on $|\beta| \leq \frac{1}{2}$, since A_s is smooth at scale $\gg N/(2T + \kappa^{-1})$; Lemma 4.8 subtracts the discrete model $\mathcal{M} = -\frac{1}{2\pi} \sum_n A_s(n) e(n\beta)$, and the difference is included in Lemma 4.12(1). Let $f \in C_c^\infty((-\frac{1}{8}, \frac{1}{8}))$ be even, $f \geq 0$, $f \not\equiv 0$; then $\hat{f}(\eta) \geq 2^{-1/2} \int f > 0$ for $|\eta| \leq 1$. Put $c_f := \min_{[-1,1]} \hat{f}^2$ and $\Phi := \hat{f}^2/c_f$, so $\Phi \geq \mathbf{1}_{[-1,1]}$. Let $W \in C_c^\infty((-1, 1))$, $W \geq 0$, and $\|W\|_{C^A} := \sum_{j \leq A} \sup |W^{(j)}|$.

Lemma 4.9 (Lemma 5a: Plancherel majorant). *Let ν be a finite sum of point masses on $(0, \infty)$ plus a continuous compactly supported density, $S(\beta) = \int e(y\beta) d\nu(y)$, and $\Delta > 0$. Then $\int_{|\beta| \leq \Delta} |S|^2 \leq \int_{\mathbb{R}} |S(\beta)|^2 \Phi(\beta/\Delta) d\beta = \frac{\Delta^2}{c_f} \int_{\mathbb{R}} \left| \int f(\Delta(y-x)) d\nu(y) \right|^2 dx$.*

Proof. With $G(x) = \int f(\Delta(y-x)) d\nu(y)$, $\hat{G}(\beta) = \Delta^{-1} \hat{f}(\beta/\Delta) S(-\beta)$ since f is even; apply Plancherel. $\square$

Formalisation: `plancherel_majorant`, for exactly this class of measures, with the integrability of the majorant in the conclusion [L].

Lemma 4.10 (Lemma 5b: smooth explicit formula, Weil form). *For χ primitive mod r of parity $\mathfrak{a}_\chi$, $V \in C_c^\infty$ with support in $(1, \infty)$, and $\tilde{V}(w) = \int_0^\infty V(y) y^{w-1} dy$,*

$$\begin{aligned} \sum_n \Lambda(n) \chi(n) V(n) &= \delta_{r=1} (\tilde{V}(1) + \tilde{V}(0)) - \sum_\rho m_\rho \tilde{V}(\rho) \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{V}(\tfrac{1}{2} + it) \left[\operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\tfrac{1}{4} + \frac{\mathfrak{a}_\chi}{2} + \frac{it}{2} \right) + \log \frac{r}{\pi} \right] dt, \end{aligned}$$

the sum over the nontrivial zeros ρ of $L(s, \chi)$ with multiplicity m_ρ being absolutely convergent.

This is Weil's explicit formula for the test function $k(u) = e^{u/2} V(e^u)$; the conjugate prime leg vanishes because $V = 0$ on $(0, 1]$. *Formalisation:* `smooth_explicit_formula_weil`, derived from the explicit formulas of the Lean artifact of [1] for primitive χ with $q > 1$ and for ζ , whose zero sets are the nontrivial zeros of Mathlib's L -functions with multiplicity [L]. The working draft used instead a contour at $\operatorname{Re} w = -\frac{1}{2}$ with $L'/L(w, \chi)$; that form is not formalised. In the Weil form the archimedean term stays on the critical line, and bounded in absolute value it is larger than the draft's remainder by a factor of about N (when $\Delta N \leq 1$) or $1/\Delta$ (when $\Delta N \geq 1$). This changes the error term of Proposition 4.11 from the draft's E to E' .

Proposition 4.11 (Proposition Z, Weil form). *Let $s_0 \geq 3$, $N_0 = e^{s_0}$, $T \geq 2$, $0 < \Delta \leq 1$, $\mu := \Delta N_0$ and $A, k \geq 2$. There is $C_0 = C_0(\kappa, \Xi, f, A, k)$ such that for every primitive $\chi \bmod r$ with $1 \leq r \leq N_0$ and every W as above,*

$$\int W(s - s_0) \int_{\mathbb{R}} |S_\chi(\beta; s)|^2 \Phi(\beta/\Delta) d\beta ds \leq C_0 \|W\|_{C^A} \left[T \sum_{\rho, \rho'} m_\rho m_{\rho'} \frac{N_0^{\beta+\beta'-2} \varpi_\rho \varpi_{\rho'}}{(1 + |\gamma - \gamma'|)^A} + E'(r, \Delta) \right],$$

the double sum being absolutely convergent, where $\varpi_\rho := (1 + (|\gamma| - 2T)_+(\mu + 1))^{-k}$ and

$$E'(r, \Delta) := T^2 \left(\Delta^2 + \frac{\Delta}{N_0} \right) + \frac{T^2 (T + \Delta N_0 + 1)^2 \log^2(r N_0 T)}{N_0}.$$

Proof. Step 1. Apply Lemma 4.9 to $d\nu = \sum \Lambda \chi(n) A_s(n) \delta_n - \delta_{r=1} A_s(y) dy$ and put $V_{x,s}(y) := A_s(y) f(\Delta(y-x))$; the inner function is $\sum_n \Lambda \chi(n) V_{x,s}(n) - \delta_{r=1} \tilde{V}_{x,s}(1)$. Its support lies in $[Ne^{-\kappa}, Ne^\kappa] \subset (1, \infty)$ since $s - \kappa \geq 1$.

Step 2 (the Weil remainder). By Lemma 4.10 the inner function equals $-\sum_\rho m_\rho \tilde{V}_{x,s}(\rho) + R_{x,s}^W$ with $R^W := \delta_{r=1} \tilde{V}(0) + \frac{1}{2\pi} \int \tilde{V}(\tfrac{1}{2} + it) \varpi_r^{\text{arch}}(t) dt$, $\varpi_r^{\text{arch}}(t) := \operatorname{Re} \frac{\Gamma'}{\Gamma}(\tfrac{1}{4} + \frac{\mathfrak{a}_\chi}{2} + \frac{it}{2}) + \log(r/\pi)$. Then: (a1) $|\varpi_r^{\text{arch}}(t)| \leq C \log(r(|t| + 2))$ uniformly in r and χ , by Stirling's formula and the finiteness of $\frac{\Gamma'}{\Gamma}(\frac{1}{4})$ and $\frac{\Gamma'}{\Gamma}(\frac{3}{4})$; (a2) $|V| \leq \|f\|_\infty e^{\kappa/2} T N^{-1/2}$, on a set of length $\leq \min(1/(4\Delta), 2e^\kappa N)$; (a3) $(y \partial_y)^j V \ll_j X_1^j T N^{-1/2}$ with $X_1 := T + \Delta N + 1$; (a4) integrating by parts twice, $\tilde{V}(w) = w^{-2} \int ((y \partial_y)^2 V) y^{w-1} dy$, and $|w|^2 \geq (1 + |t|)^2/8$ on $\operatorname{Re} w = \frac{1}{2}$, so $|\tilde{V}(\tfrac{1}{2} + it)| \leq C T N^{-1} \min(\Delta^{-1}, N) \min(1, X_1^2/(1 + |t|)^2)$; (a5) $\int \min(1, X_1^2/(1 + |t|)^2) \log(r(|t| + 2)) dt \leq C X_1 \log(r(X_1 + 2))$; (a6) $|\tilde{V}(0)| \leq C T N^{-3/2} \min(\Delta^{-1}, N)$. Hence $|R_{x,s}^W| \leq C[\delta_{r=1} T N^{-3/2} \min(\Delta^{-1}, N) + T N^{-1} \min(\Delta^{-1}, N) X_1 \log(r(X_1 + 2))]$. As $R_{x,s}^W = 0$ off a set of x of measure $\leq 2e^\kappa N + 1/(4\Delta)$, in both cases $\Delta N \geq 1$ and $\Delta N < 1$ one gets $\Delta^2 \int |R_{x,s}^W|^2 dx \leq C[\delta_{r=1} T^2 N^{-2} + T^2 X_1^2 \log^2(r(X_1 + 2))/N]$, and for $|s - s_0| \leq 1$ this is $\leq C_R E'(r, \Delta)$.

Step 3 (scaling). With $y = Nt$, $x = N\xi$: $\tilde{V}_{N\xi,s}(\rho) = N^{\rho-1/2}I_\rho(\xi; \mu_s)$, where $\mu_s := \Delta e^s$, $I_\rho(\xi; \mu) := \int_0^\infty H_\rho(t)f(\mu(t-\xi))dt$ and $H_\rho(t) := D_T(-\log t)\Xi(\log t/\kappa)t^{\rho-3/2}$ does not depend on s . Hence $\Delta^2 \int |\sum_\rho m_\rho \tilde{V}_{x,s}(\rho)|^2 dx = \sum_{\rho,\rho'} m_\rho m_{\rho'} e^{is(\gamma-\gamma')} \Psi_{\rho\rho'}(s)$ with $\Psi_{\rho\rho'}(s) := \mu_s^2 e^{s(\beta+\beta'-2)} \int I_\rho \overline{I_{\rho'}} d\xi$; the interchange is justified by Step 4 and the zero count.

Step 4 (one zero). By Plancherel in ξ , $\|I_\rho(\cdot; \mu)\|_2^2 = \mu^{-2} w_\rho(\mu; f)$ with $w_\rho(\mu; f) := \int |\hat{H}_\rho(\eta)|^2 \hat{f}(\eta/\mu)^2 d\eta$, and likewise for f_j ($f_0 = f$, $f_{j+1}(z) = z f'_j(z)$). (i) $\|H_\rho\|_2^2 \leq 2\pi e^{2\kappa} T$, so $w_\rho \leq 2\pi e^{2\kappa} \|\hat{f}\|_\infty^2 T$. (ii) Let $D := (|\gamma| - 2T)_+ \geq 1$. For $|\eta| \leq D/(8\pi e^\kappa)$ the phase of $\hat{H}_\rho(\eta)$ in t has derivative $\geq \frac{3}{4}e^{-\kappa}|\gamma - \tau|$ for $\tau \in [T, 2T]$, and m integrations by parts give $|\hat{H}_\rho(\eta)| \ll_m \kappa^{-m} D^{1-m}$; for $|\eta| > D/(8\pi e^\kappa)$, $\hat{f}(\eta/\mu)^2 \ll_k (1 + D/\mu)^{-2k}$. With $m = k + 2$, $w_\rho(\mu; f_j) \ll T \varpi_\rho(\mu)^2$.

Step 5 (s-average). Since $\partial_s \mu_s = \mu_s$, $\partial_s^j \Psi_{\rho\rho'}$ is a bounded combination of $\mu_s^2 e^{s(\beta+\beta'-2)} \int I_\rho[f_a] \overline{I_{\rho'}[f_b]}$, $a + b \leq j$; by Cauchy–Schwarz and Step 4, $|\partial_s^j \Psi_{\rho\rho'}(s)| \ll_j T e^{s(\beta+\beta'-2)} \varpi_\rho(\mu_s) \varpi_{\rho'}(\mu_s)$, and for $|s - s_0| \leq 1$, $e^{s(\beta+\beta'-2)} \leq e^2 N_0^{\beta+\beta'-2}$ and $\varpi(\mu_s) \leq e^k \varpi(\mu)$. Integrating $\int W(s - s_0) e^{is(\gamma-\gamma')} \Psi_{\rho\rho'} ds$ by parts A times gives the factor $(1 + |\gamma - \gamma'|)^{-A}$ and the dependence on $\|W\|_{CA}$. The sums converge absolutely by Step 4 and the unit-window zero count. $\square$

Formalisation: `propZ.W` is assembled from Lemmas 4.9, 4.10 and the steps above, among them the Weil remainder bound (Step 2) and the boundedness and measurability of the two parts in x ; in the released library it depends on the three standard axioms only and rests on no open declaration [L].

Lemma 4.12 (Lemma 5c: family errors). *Take $\Delta_r = K/(rQ)$ for $r \leq R_1 \leq Q/(2K)$, $s_0 \geq \log Q + 3$, $s_0 + 1 + \kappa \leq \log X$ and $N_0 \leq Q^{2-\eta}$.*

- (1) *On $|\beta| \leq \frac{1}{2}$, $S_\chi^{\text{near}} - [r = 1]\mathcal{M}$ (primes $> Q$) and $-\frac{1}{2\pi} S_\chi(\cdot; s)$ differ by the prime powers $p^k \in [Ne^{-\kappa}, Ne^\kappa]$, $k \geq 2$, whose coefficients have ℓ^1 norm at most $T(\kappa/2\pi + o(1))$, and, for $r^* = 1$, by the difference of the discrete and the continuous model, which is $O_m(N^{-m})$ on $|\beta| \leq \frac{1}{2}$ only. This uses $Ne^\kappa \leq X$, that is $s + \kappa \leq \log X$; for $|s - s_0| \leq 1$ it holds when $s_0 + 1 + \kappa \leq \log X$, which is assumed here. In Theorem 4.18 the centres satisfy $s_0 \leq \alpha' \mathcal{L} + 1$ with $\alpha' < \lambda$, so it holds for large Q . The replacement is made on $|\beta| \leq \Delta_r$, before the majorisation of (4), with $|x + d|^2 \leq 2|x|^2 + 2|d|^2$ (a factor 2 on the main term, absorbed in the constant of Theorem 4.17); the family ring error from these is $\ll T^2 K R_1 / Q$.*
- (2') *If $1 \leq Q$, $1 \leq K$, $2 \leq T \leq N_0$, $e^3 Q \leq N_0$ and $R_1 \leq Q/(2K)$, then $\sum_{r \leq R_1} r E'(r, \Delta_r) \leq 128 T^4 K^2 (\log N_0)^3 (R_1^2/N_0 + N_0/Q^2 + 1/Q^2 + R_1/(QN_0))$.*
- (3) *With $R_1 = e^{s_0+1}(\log Q)^6/(\epsilon Q)$ (its actual size, fixed on $|s - s_0| \leq 1$), $N_0 \leq Q^{\alpha''}$ and (10), the sums of (1) and (2') are $\leq (\log Q)^C Q^{\alpha''-2} \epsilon^{-2} \cdot T s_0 \leq Q^{-c} T s_0$ for some $c > 0$.*
- (4) *The inner cutoff $1/(R_1 Q)$ of Lemma 4.8 is dropped by positivity, and the outer one is majorised by $\Phi(\beta/\Delta_r)$.*

Proof of (2'). With $L_0 = \log N_0$, $r N_0 T \leq N_0^3$ gives $\log(r N_0 T) \leq 3L_0$ (this uses $T \leq N_0$); $(T + \Delta_r N_0 + 1)^2 \leq 5T^2 + 2K^2 N_0^2/(r^2 Q^2)$; then $\sum_{r \leq R_1} r \leq R_1^2$, $\sum 1/r \leq 1 + \log R_1 \leq \frac{4}{3} L_0$. $\square$

Status and formalisation. (2') is `family_EW_sum` [L]; the draft's (2) for its E is `family_E_sum`, both with the hypothesis $T \leq N_0$, which the draft left implicit and without which no absolute constant exists. (3) uses the actual size of R_1 : the binding term is $T^3 s_0 R_1^2 / N_0 \asymp (\log Q)^C Q^{\alpha''-2} \epsilon^{-2}$, and the bound $R_1 \leq Q/(2K)$ alone would not suffice.

4.6 Zero sums, blocks and the density estimates

Put $b_\rho := N_0^{\beta-1} \varpi_\rho$, $x_\rho := (1 - \beta)s_0$, and let $\mathfrak{X} := \{(\chi, \rho) : \chi \text{ primitive mod } r \leq R_1\}$, ζ included. By H6, and for $r = 1$ by the local count for ζ of the same artifact (with A_0 enlarged if necessary), a unit window $[m, m+1)$ contains at most $n_m \leq 2A_0 \log(r(|m| + 4))$ zeros of $L(s, \chi)$, χ primitive mod r , counted with multiplicity.

Lemma 4.13 (Lemma 6a: zero sums). *For $A \geq 2$, $k \geq 3$, $X_0 \geq 0$, and $2 \leq T \leq N_0$, $N_0 \geq e^3$, $1 \leq K \leq Q$, $R_1 \leq N_0$,*

$$3 := \sum_{\chi} \frac{r}{\varphi(r)} \sum_{\rho, \rho' \in \chi} \frac{m_\rho m_{\rho'} b_\rho b_{\rho'}}{(1 + |\gamma - \gamma'|)^A} \\ \ll \log \log N_0 \left[\left(\sum_{\mathfrak{X}, x_\rho \leq X_0} m_\rho b_\rho \right)^2 + \log N_0 \sum_{\mathfrak{X}, x_\rho > X_0} m_\rho b_\rho^2 \right] + N_0^{-1}.$$

Proof. Group zeros into unit windows, $B_m := \sum_{\gamma \in [m, m+1)} m_\rho b_\rho$; Schur's test with the kernel $(1 + |m - m'|)^{-A}$ gives $\sum_{\rho, \rho'} \ll \sum_m B_m^2$. Split B_m by $x_\rho \leq X_0$; $\sum_m (B_m^{\text{near}})^2 \leq (\sum_m B_m^{\text{near}})^2$ and $(B_m^{\text{rest}})^2 \leq n_m \sum_{\text{rest}, \gamma \in [m, m+1)} m_\rho b_\rho^2$ by Cauchy-Schwarz, with $n_m \leq 2A_0 \log(N_0(N_0^2 + 4)) \leq 2A_0(3 \log N_0 + 1)$ for $r \leq N_0$, $|m| \leq N_0^2$: one factor $\log N_0$. For $|m| > N_0^2$ the decay of ϖ_ρ in $|\gamma|$ is used: $K \leq Q$ gives $\mu_r + 1 \leq 2N_0$, and $T \leq N_0$ with $|\gamma| > N_0^2 - 1 \geq 4N_0$ gives $|\gamma| - 2T \geq |\gamma|/2$, so $\varpi_\rho \leq (|\gamma|/(4N_0))^{-k}$, and $b_\rho \leq \varpi_\rho$. Hence $B_m \ll \log(N_0|m|) (4N_0/|m|)^k$ and $\sum_{|m| > N_0^2} B_m^2 \ll N_0^{4-2k} \log^2 N_0$; over the at most $R_1^2 \leq N_0^2$ primitive characters, with the weight below, the tail is $\ll N_0^{4-2k} \log^2 N_0 \log \log N_0 \leq N_0^{-1}$ for $k \geq 3$ and large N_0 . Finally $r/\varphi(r) \ll \log \log N_0$ for $r \leq R_1 \leq N_0$. $\square$

The hypotheses $K \leq Q$, $T \leq N_0$, $R_1 \leq N_0$ were implicit in the draft. *Formalisation:* **Z6a_zero_sums**, in the form that Theorem 4.17 uses ($R_1 = \lfloor \bar{R}_1 \rfloor$), with $Q \geq 3$ [L].

The hyperbolic family. $\varpi_\rho \geq 2^{-k}$ iff $|\gamma| \leq H_r := 2T + 1 + \mu_r$, $\mu_r = KN_0/(rQ)$. Put $R_1 := e^{s_0+1}(\log Q)^6/(\epsilon Q)$ and

$$\mathcal{Z} := 9[R_1^2(2T + 1)^2 + (KN_0/Q)^2], \quad \alpha := s_0/\log Q. \quad (12)$$

Under (10), $\log(1/\epsilon) = O(\log \log Q)$, so $\mathcal{Z} = N_0^{2-2/\alpha+o(1)}$.

Lemma 4.14 (Lemma 6b: rectangles). *If $j \in \mathbb{N}$, $0 \leq R \leq R_1$, $0 \leq H \leq 2^j(2T + 1 + KN_0^{1+\epsilon'}/(RQ))$, $T, K \geq 0$, $N_0 \geq 1$, $\epsilon' \geq 0$, $Q > 0$, then $R^2 H^h \leq 4^j \mathcal{Z} N_0^{2\epsilon'}$ for every $h \in [0, 2]$.*

Proof. $(x + y)^h \leq 2^h \max(x, y)^h$ with $x = 2T + 1$, $y = KN_0^{1+\epsilon'}/(RQ)$. If $x \geq y$, $R^2(2x)^h \leq 4R_1^2 x^2 \leq \mathcal{Z}$. If $y > x$, then $R^2 y^h = R^{2-h} Y^h \leq Y^2$ with $Y := Ry = KN_0^{1+\epsilon'}/Q$ and $R < Y$. $\square$

Formalisation: **shell_rectangles** [L]. The hypotheses $j \in \mathbb{N}$ and $H \geq 0$ are needed (counterexamples at $j = -3$ and at $H = -100$ with real powers).

Lemma 4.15 (Lemma 6c: $O(1/\epsilon')$ blocks). *Let $1 \leq R_1 \leq N_0$, $0 < \epsilon' \leq 1$, $R_i := R_1 N_0^{-i\epsilon'}$, $H_i := 2T + 1 + KN_0^{1+\epsilon'}/(R_i Q)$. Every $r \in [1, R_1]$ lies in a block $R_{i+1} < r \leq R_i$ with $i \leq \lceil 1/\epsilon' \rceil$ and $H_r \leq H_i$; every nonempty block has $R_i \geq 1$ and $R_i^2 H_i^h \leq \mathcal{Z} N_0^{2\epsilon'}$ for $h \in [0, 2]$; above the height $2^j(2T + 1 + \mu)$, $\varpi \leq 2^{-jk}$; and $H_i \geq 2T + 1$.*

Formalisation: **Z6c_blocks**, all four parts proved [L]. The draft's “every block has $R \geq 1$ ” holds only for nonempty blocks, and the cover needs $R_1 \leq N_0$ (a numerical control with $R_1 > N_0$ leaves 158 moduli uncovered). Dyadic blocks in r would cost a factor $\log N_0$ in the log-free count, which is fatal near $\sigma = 1$; the $O(1/\epsilon')$ blocks avoid it.

Density estimates. Let $N(\sigma, H, \chi)$ be the number of zeros of $L(s, \chi)$ with $\beta \geq \sigma$ and $|\gamma| \leq H$, with multiplicity, and $N^*(\sigma, H, R) := \sum_{r \leq R} \sum_{\chi \bmod r}^* N(\sigma, H, \chi)$, the character mod 1 (ζ) included.

(J) (Jutila [15, Theorem 1, (1.8)].) For every $\varepsilon_1 > 0$: $N^*(\sigma, H, R) \ll_{\varepsilon_1} (R^2 H)^{(2+\varepsilon_1)(1-\sigma)}$ for $\frac{4}{5} \leq \sigma \leq 1$, $H \geq 1$, $R \geq 1$.

(M) (Montgomery [16]; [2, 2nd ed., §10, Théorème 20, p. 77].) There is A such that $N^*(\sigma, H, R) \ll (HR^2)^{3(1-\sigma)/(2-\sigma)} (\log HR)^A$ for $\frac{1}{2} \leq \sigma \leq 1$, $H \geq 2$, $R \geq 1$.

Both are quoted as stated in the sources, read in the original (Jutila's Theorem 1 in the journal; Bombieri's Théorème 20 with its proof), and neither is strengthened: Jutila's exponent, range and dependence of the constant on ε_1 are kept; Bombieri's proof gives $A = 19$ [2, 2nd ed., p. 80], and we use only that some A exists [C]. Both sources count primitive characters in the closed rectangle, with ζ included. For σ between $\frac{4}{5}$ and a neighbourhood of 1, Jutila's proof refers to Huxley and Jutila [14], which we have not opened; we rely on the refereed statement. *Formalisation*: the Prop `ZeroDensityInput` states (J) and (M) in exactly this form (Appendix A); a Lean theorem derives from it the abstract hypotheses below with $\alpha_0 = 5/3$ and $\theta = 503/1994$ [L].

In the abstract form of the working draft: (B) for $\frac{1}{2} + \varepsilon_1 \leq \sigma \leq \frac{4}{5}$, $N^* \ll_{\varepsilon_1} (R^2 H)^{c(\sigma)(1-\sigma)+\varepsilon_1} (\log RH)^{b_0}$ with $c(\sigma) = 3/(2-\sigma)$, from (M); (LF) for $\sigma \geq \frac{4}{5}$, $N^* \ll_{\varepsilon_1} (R^2 H)^{(c_0+\varepsilon_1)(1-\sigma)}$ with $c_0 = 2$, from (J). Put $A^* := \sup_{[1/2, 4/5]} c = 5/2$, $M := \max(A^*, c_0) = 5/2$ and

$$\alpha_0 := \min\left(\frac{M}{M-1}, 2\right) = \frac{5}{3}.$$

Lemma 4.16 (Lemma 6d: counts). *Assume (J) and (M). Let $k \geq 3$, $\alpha \leq \alpha' < \alpha_0$, $\varepsilon'' > 0$ and $\mathfrak{c} := c_0(2 - 2/\alpha') + \varepsilon'' = 4 - 4/\alpha' + \varepsilon''$. (low, $\beta < \frac{1}{2} + \varepsilon_2$) $\log N_0 \sum b^2 \ll N_0^{1-2/\alpha+2\varepsilon_2+o(1)} \rightarrow 0$. (bulk) By (B) on the blocks and height shells, $\log N_0 \sum_{\text{bulk}} b_\rho^2 \leq N_0^{-c}$ for some $c > 0$; the condition is $A^*(2 - 2/\alpha) < 2$, i.e. $\alpha < A^*/(A^* - 1) = 5/3$. (mid/near, $\sigma \geq \frac{4}{5}$) By (LF) on the blocks, $N_{\mathfrak{X}}(x) := \sum_{x_\rho \leq x} m_\rho \varpi_\rho \leq C e^{cx}$ for $0 \leq x \leq s_0/5$, so, for every $X_0 \geq 0$, $\log N_0 \sum_{\text{mid}} b^2 \ll \log N_0 e^{-(2-\mathfrak{c})X_0}$ (mid: $X_0 < x_\rho \leq s_0/5$), and, for $0 \leq X_0 \leq s_0/5$, $(\sum_{\text{near}} b)^2 \ll (1 + X_0)^2 e^{2(\mathfrak{c}-1)+X_0}$ (near: $x_\rho \leq X_0$, as in Lemma 4.13); the condition is $\mathfrak{c} < 2$.*

Proof. (low) The ϖ^2 -weighted count is $\ll \sum_{r \leq R_1} \varphi(r) H_r \log \ll \mathcal{Z} \log N_0$. (bulk) On each block of Lemma 4.15 and height shell, (B) gives $\ll (\log N_0)^{b_0} (\mathcal{Z} N_0^{2\varepsilon'} 4^j)^{c(\sigma)(1-\sigma)+\varepsilon_1}$, and $\sum_j 2^{-2kj} 4^{j(c(1-\sigma)+\varepsilon_1)} < \infty$ for $k \geq 3$; with a σ -grid of mesh $1/s_0$, $\log N_0 \sum_{\text{bulk}} b^2 \ll (\log N_0)^C \max_\sigma N_0^{-(1-\sigma)(2-c(\sigma)(2-2/\alpha))+O(\varepsilon_1+\varepsilon')}$, which is $\leq N_0^{-c}$ when $A^*(2 - 2/\alpha) < 2$ and $\varepsilon_1, \varepsilon'$ are small in terms of $\alpha_0 - \alpha'$. (mid/near) For $0 \leq x \leq s_0/5$, that is $\sigma = 1 - x/s_0 \geq \frac{4}{5}$, (LF) on the blocks gives $N_{\mathfrak{X}}(x) \leq C_{\varepsilon_1, \varepsilon'} \exp((c_0 + \varepsilon_1)(\log \mathcal{Z}/s_0 + 2\varepsilon')x) \leq C e^{cx}$, and partial summation over $x \leq s_0/5$ gives the two bounds. For $X_0 > s_0/5$ the near sum contains zeros with $\beta < \frac{4}{5}$, which only (M) counts, with the larger exponent $c(\sigma)(2 - 2/\alpha)$ in place of $c_0(2 - 2/\alpha)$; the near bound is not claimed there. $\square$

Here (10) is used: with $\epsilon = Q^{-c'}$ one would have $\log \mathcal{Z}/s_0 \geq 2 - 2/\alpha + 2c'/\alpha$, and at $\alpha' = 2497/1500$ the bulk condition would hold only for $c' < 3/2500 = 0.0012$. The working draft's condition $\epsilon \rightarrow 0$ does not suffice, nor does $\epsilon \geq Q^{-\eta/3}$; $\epsilon = 1/\mathcal{L}$ does. The bulk margin is $2 - A^*(2 - 2/\alpha') = 9/2497$, and the minimum of the bulk saving $(1-\sigma)(2-c(\sigma)(2-2/\alpha'))$ over $[\frac{1}{2}, \frac{4}{5}]$ is $9/12485$, at $\sigma = \frac{4}{5}$; $\theta(\alpha') \rightarrow 503/1994$ in the limit $\varepsilon'' \rightarrow 0$. Since $\theta(\alpha')$ is computed with $\varepsilon'' > 0$, a rate $\theta < 503/1994$ is obtained by choosing ε'' so small that $\theta < \theta(\alpha')$. *Formalisation*: `Z6d_counts_corr`, with the near bound under $5X_0 \leq s_0$ and with (J) and (M) as its hypothesis [L]. The first formal statement `Z6d_counts` asserted the near bound for every $X_0 \geq 0$, which (J) and (M) do not give; it is kept in the library, open, and not used (Appendix B).

The inputs are used inside their hypotheses. Every block and height shell has $R \geq 1$ (or $r = 1$) and $H \geq 2T + 1 \geq 3$, so $H \geq 1$ and $H \geq 2$ hold at every height used, including $R \approx R_1 = Q^{\alpha'-1+o(1)}$ with H polylogarithmic; (M) is needed only on $[\frac{1}{2} + \varepsilon_1, \frac{4}{5}]$ and (J) only for

$\sigma \geq \frac{4}{5}$. The bulk binds: the log-free condition $2(2-2/\alpha') < 2$ is only $\alpha' < 2$. At $\alpha' = 2497/1500$ and $\varepsilon'' \rightarrow 0$, $\mathfrak{c} \rightarrow 4 - 4/\alpha' = 3988/2497$ and $\theta(\alpha') := \min(1, (2 - \mathfrak{c})/\mathfrak{c}) \rightarrow 503/1994$.

4.7 Theorem S and Theorem K

Theorem 4.17 (Theorem S: the Shell lemma, s -averaged). *Assume (J) and (M). Fix $1 < \alpha' < 5/3$, $B \geq 2$, $A \geq 2$, κ and Ξ as above, and $\theta < \theta(\alpha')$ with ε'' small. There is C such that for all large Q , all K, ϵ, s_0 with $2 \leq K \leq (\log Q)^B$, $(\log Q)^{-B} \leq \epsilon \leq 1$ and $\log Q + 3 \leq s_0 \leq \alpha' \log Q$ with $s_0 \leq \log X - 2$, and every $W \in C_c^\infty((-1, 1))$ with $W \geq 0$,*

$$\int W(s - s_0) \text{Ring}(s) ds \leq C \|W\|_{C^A} (\log Q)^{-\theta} T s_0.$$

Proof. Fix $|s - s_0| < 1$. Enlarging $R_1(s)$ to $\bar{R}_1 := e^{s_0+1}(\log Q)^6/(\epsilon Q)$ only adds nonnegative terms. Start from Lemma 4.8 with $\Upsilon_K = c_\mu(1 + \log K)$; for large Q its conditions hold, and $1 \leq \bar{R}_1 \leq N_0$, since $\bar{R}_1 = e^{s_0+1}(\log Q)^6/(\epsilon Q) \leq Q^{\alpha'-1+o(1)}$ and $K \leq (\log Q)^B$, $\epsilon \geq (\log Q)^{-B}$. The order of the next two steps matters. First, on $|\beta| \leq \Delta_r := K/(rQ)$ (where $\Delta_r \leq K/Q < \frac{1}{2}$) replace $S_{\chi^*}^{\text{near}} - [r^*=1]\mathcal{M}$ by $-\frac{1}{2\pi}S_{\chi^*}(\beta; s)$, by Lemma 4.12(1) (for $r^* = 1$ with the Poisson comparison of the two models). Only then majorise the cutoff by $\Phi(\beta/\Delta_r)$ over $\mathbb{R}$ (Lemma 4.12(4)) and apply Lemma 4.9 and Proposition 4.11, with $k = 3$, to each primitive $\chi^* \bmod r^* \leq \bar{R}_1$ (ζ with $\bar{M}_s$). In the other order the step would not be justified: on $\mathbb{R}$ the discrete model is periodic and the continuous one is not. The errors of Lemma 4.12(1),(2),(3) contribute $\ll Q^{-c}\|W\|_{C^A} T s_0$. After these steps the β -domain and the factors $r^*/\varphi(r^*)$ and Υ_K do not depend on s , so the integration against $W(s - s_0)$ is legitimate. What remains is $2\Upsilon_K C_0 \|W\|_{C^A} T 3$ and the principal-arc term $4\Upsilon_K \|W\|_\infty (U + o(1)) \ll (\log K) T \|W\|_{C^A}$. By Lemmas 4.13 and 4.16, relative to $\|W\|_{C^A} T s_0$: the low and bulk zeros give $\ll N_0^{-c}$; the mid zeros give $\ll \log K \log \log Q e^{-(2-\mathfrak{c})X_0}$, since the single window factor $\log N_0$ is $\asymp s_0$; the near zeros give $\ll \log K \log \log Q (1 + X_0)^2 e^{2(\mathfrak{c}-1)+X_0}/\log Q$. Put $e^{X_0} = (\log Q)^{\eta_1}$. If $1 < \mathfrak{c} < 2$, the balance $\eta_1(2 - \mathfrak{c}) = 1 - 2\eta_1(\mathfrak{c} - 1)$ gives $\eta_1 = 1/\mathfrak{c}$ and the rate $(\log Q)^{-(2-\mathfrak{c})/\mathfrak{c}}$ up to powers of $\log \log Q$, which are absorbed by taking $\theta < \theta(\alpha')$ and ε'' small. If $\mathfrak{c} \leq 1$, take $X_0 = (2 - \mathfrak{c})^{-1} \log \log Q$: the rate is $(\log \log Q)^{O(1)}/\log Q$, and $\theta < 1 = \theta(\alpha')$. In both cases $X_0 = O(\log \log Q)$, so $5X_0 \leq s_0$ for large Q , as the near bound of Lemma 4.16 requires. The principal-arc term is $\ll (\log K) T \ll (\log \log Q / \log Q) T s_0$. Every term is linear in $\|W\|_{C^A}$ (and $\|W\|_\infty \leq \|W\|_{C^A}$). $\square$

Changes from the working draft: the right side is $\|W\|_{C^A} T s_0$ rather than $\eta_Q \int W \|a\|^2$, since the transfer below needs a bound linear in $\|W\|_{C^A}$ and $\int W_j \|a\|^2 \asymp g(j) T s_0$ would need a lower bound on the mass of W_j that does not hold uniformly; the rate is stated for every $\theta < \theta(\alpha')$; K and ϵ are explicit. *Formalisation:* `ZetaShell.ShellS.shell.S`, with this statement for the near polynomial not cut at X (so without the condition $s_0 \leq \log X - 2$) and with $B \geq 1$, which implies the statement printed here, derived from four nodes: S1, the reduction to zero sums through Lemma 4.8 and Proposition 4.11; S2, the zero side through Lemmas 4.13 and 4.16; S3, the family errors; and S4, the elementary terms. In the released library all four are proved (S1 from the corrected statement of Lemma 4.8 by a kernel-checked implication, `S1_ring_to_zeros_of_astar`; S2 from Lemma 4.13 and the corrected Lemma 4.16), and `shell.S` is kernel-checked, with (J) and (M) as its hypothesis [L].

Assembly, pointwise in s . Let $\log Q + 4 \leq s \leq \alpha' \log Q$ with $s \leq \log X - 1$, and $S = S^{\text{near}, P}$. Partition $[0, 1)$ into (i) the complement of $\mathcal{S}$ minus the δ -neighbourhood $\mathcal{X}_\delta$ of $\mathcal{X}$; (ii) the rings of $r \leq R_1$; (iii) the holes of $r \leq R_1$ minus $\mathcal{X}_\delta$; (iv) $\mathcal{X}_\delta$. The rings of distinct a/r are disjoint because $2KR_1 \leq Q$, and (i) is disjoint from the holes of $r \leq R_0$ and from all rings. By Lemmas 4.1, 4.5 and 4.3: $(D_\delta^\Omega)^+ \leq H_\mathfrak{w}(1 + E_g)$ on (i), with E_g the bracket of Lemma 4.3(a); $\leq \|\Omega\|_\infty(Q^2 + N/\epsilon) = O(Q^2)$ on (ii) and (iv); $= 0$ on (iii). With Lemma 4.2, for any $\eta_0 > 0$,

$$\begin{aligned} \mathbf{F}(s) &\leq (1 + \eta_0) \left[|\mathfrak{F}_Q| (1 + E_g) (\|a\|^2 - \text{Hole} - \text{Ring}) \right. \\ &\quad \left. + Q^2 (1 + o(1)) (\text{Ring} + \text{Edge}) + 2\pi \sinh \kappa (\epsilon Q^2 + N) \|a\|^2 \right] \\ &\quad + (1 + \eta_0^{-1}) C |\mathfrak{F}_Q| O((c\kappa T)^{-1} + (\delta_1^2 T \log Q)^{-1}) \|a\|^2, \end{aligned} \quad (13)$$

where $\text{Edge} := \int_{\mathcal{X}_\delta} |S|^2 = o(\|a\|^2)$ by Lemma 4.7. By Lemma 4.6, at the design width $K = \mathcal{L}(\log \mathcal{L})^2$ of Theorem 4.18, $\|a\|^2 - \text{Hole} \leq U(\mathcal{L} + \log K + O(1))$. For the sharp family $\|\Omega\|_\infty \leq 1$ by Tao's inequality; without it, the ring points have $q > Q/K$, so $|\Omega(q)| \leq 1 + \log K$ there, which costs $O(\log K)$ times an $o(1)$ quantity [M]. *Formalisation:* this statement, in the form that Theorem 4.18 uses ($K = \mathcal{L}(\log \mathcal{L})^2$, $\epsilon = 1/\mathcal{L}$, with Lemma 4.6 applied), is `AS_pointwise_corr`, the corrected form of `AS_pointwise`, which adds $\alpha'' < \lambda$ (Appendix B) [L]; the first form is kept in the library, open, and not used.

Divided by $|\mathfrak{F}_Q| \|a\|^2 = |\mathfrak{F}_Q| U(s + O(1))$:

$$\begin{aligned} \frac{\mathbf{F}(s)}{|\mathfrak{F}_Q| \|a\|^2} &\leq \frac{\mathcal{L} + \log K + O(1)}{s} + C_Q \frac{\text{Ring}(s) + \text{Edge}(s)}{\|a(s)\|^2} \\ &\quad + O(E_g + \epsilon \sinh \kappa + N/Q^2 + (c\kappa T)^{-1/2} + (\delta_1^2 T \log Q)^{-1/2}). \end{aligned} \quad (14)$$

Theorem 4.18 (Theorem K: the Shell kernel). *Assume (J) and (M). Let $1 < \lambda < 2$, fix $1 < \alpha' < 5/3$ with $\alpha' < \lambda$, let the window have ramp width $w \leq (\lambda - \alpha')\mathcal{L}/8$ (the design has $w = o(\mathcal{L})$, §2.5(iii)), and take $K = \mathcal{L}(\log \mathcal{L})^2$, $\epsilon = 1/\mathcal{L}$; these satisfy the ranges of Theorem 4.17 with $B = 2$. Define the weight on the out-zone $O = \{(1 - \delta') \log Q < |s| \leq \log X\}$ by*

$$\omega_K(s) := \begin{cases} C_Q, & s < 0 \text{ or } (1 - \delta') \log Q < s \leq \log Q + 4, \\ \mathcal{L}/s, & \log Q + 4 < s \leq \alpha' \mathcal{L}, \\ C_Q \mathcal{L}/s, & \alpha' \mathcal{L} < s \leq \log X. \end{cases}$$

Then for every $\theta < \theta(\alpha')$ there is c such that for large Q (on $s \leq -(1 - \delta') \log Q$ one has $\|a(s)\|^2 = O(1)$, so that part of the right side is negligible)

$$\int_O g(s) \mathbf{F}(s) ds \leq (1 + c(\log Q)^{-\theta}) |\mathfrak{F}_Q| \int_O g(s) \|a(s)\|^2 \omega_K(s) ds.$$

In α -units, since $\|a(s)\|^2 = U(s + O(1))$, the weight ω_K is the kernel $C|\alpha|$ on the strip $\Sigma_Q = \{(1 - \delta') \log Q < s \leq \log Q + 4\}$ (width $O(\log \log Q / \log Q)$), the kernel 1 on $(\log Q + 4)/\mathcal{L} < \alpha \leq \alpha'$, and the kernel C on $(\alpha', \lambda]$.

Proof. The strip and O^- . On Σ_Q , Theorem 4.17 is not available ($s_0 \geq \log Q + 3$ is needed by Lemma 4.12), and the pointwise large sieve $\mathbf{F} \leq (Q^2 + \pi X) \|a\|^2$ is kept. The same bound is used on $O^- := O \cap (-\infty, 0)$, that is, for $s \leq -(1 - \delta') \log Q$, where $|D_T(s - \log n)| \leq 2/(|s| + \log n)$ gives $\|a(s)\|^2 \leq \pi^{-2} \sum_{n \leq X} \Lambda(n)^2 / (n(|s| + \log n)^2) = O(1)$. (This fails near $s = 0$, where $\|a(s)\|^2$ grows like $\log \log X$, but $s = 0$ is not in the out-zone. The half B_χ does not occur in $\mathbf{F}$; its mirror is used in Step 3 of §7.)

On $(\log Q + 4, \alpha' \mathcal{L}]$. Multiply (14) by $g \geq 0$ and integrate. The first term is $\mathcal{L}/s \cdot (1 + O(\log K/\mathcal{L}))$. The Ring term is handled by Theorem 4.17 through the transfer: write $1 = \sum_j W_0(s - j)$ with $0 \leq W_0 \in C_c^\infty((-1, 1))$ (for instance $W_0 = \mathbf{b}/\sum_i \mathbf{b}(\cdot - i)$ with $\mathbf{b} \in C_c^\infty((-1, 1))$, $\mathbf{b} \geq 0$, $\mathbf{b} > 0$ on $[-\frac{1}{2}, \frac{1}{2}]$), and $W_j := W_0(\cdot - j)g$. Since $\varphi = p(u/\mathcal{L})\varphi_{\text{flat}}$ with ramp width $w \geq 1$, H4 and Leibniz give $\|g^{(i)}\|_\infty \leq \|(\varphi^2)^{(i)}\|_1 \|\varphi^2\|_\infty \ll_i 1$ for $1 \leq i \leq A$, and g is bounded below on the zone: $\varphi \geq p(\lambda/2) \geq \frac{1}{6}$ on $|u| \leq L/2 - w$, so $g(s) \geq 6^{-4}(L - |s| - 2w) \geq 6^{-4}(\lambda - \alpha')\mathcal{L}/2$ for $|s| \leq \alpha' \mathcal{L} + 2$ and large Q , since $w \leq (\lambda - \alpha')\mathcal{L}/8$. Hence $\sup_{|u-j| \leq 1} g(u) \leq g(j) + O(1) \ll g(j)$ and $\|W_j\|_{C^A} \ll g(j)$, and Theorem 4.17 applied

with $B = 2$, with some $\alpha'' \in (\alpha', 5/3)$ in place of α' and with ε'' chosen so that $\theta < \theta(\alpha'')$ (possible because $\theta(\cdot)$ is continuous and $\theta < \theta(\alpha')$), and with Q so large that $\alpha' \mathcal{L} + 1 \leq \alpha'' \log Q$ (so that every window meeting the zone has its centre in $[\log Q + 3, \alpha'' \log Q]$, and below $\log X - 2$ since $\alpha' < \lambda$), gives $\sum_j \int W_j \text{Ring} \ll (\log Q)^{-\theta} \sum_j g(j) Tj \ll (\log Q)^{-\theta} \int g \|a\|^2$. Windows meeting $[\log Q + 4, \infty)$ have centre $\geq \log Q + 3$, and over-counting inside the strip is harmless since $\text{Ring} \geq 0$. The other terms of (14) are $O(\log \mathcal{L}/\mathcal{L})$ with $K = \mathcal{L}(\log \mathcal{L})^2$, $\epsilon = 1/\mathcal{L}$, $\delta_1 = T^{-1/3}$.

On $(\alpha' \mathcal{L}, \log X]$. Here Lemma 3.1(i) is used, with its width $K_1 = \mathcal{L}$, integrated against g over $(\alpha' \mathcal{L}, \log X]$ as in the first display of §3.2(d): $\mathbf{F} \leq Q^2(\|a\|^2 - \text{Hole}) + \dots$ gives the weight $C_Q \mathcal{L}/s$ up to $1 + O(\log \mathcal{L}/\mathcal{L})$. On the top edge $\log X - 1 < s \leq \log X$ the hole is not available and the large sieve gives C_Q , about λ times $C_Q \mathcal{L}/s$; since $\int_{\log X - 1}^{\log X} g = O(\mathcal{L}^{-1}) \int_O g$, the integrated statement holds. $\square$

Changes from the working draft: the draft's Theorem K stated the kernel $|\alpha|$ on $(\log Q + 4, \mathcal{L}]$, while the proof gives $(\mathcal{L} + \log K + O(1))/s$, which is larger by $1 + O(\log \log Q / \log Q)$ on an α -interval of width $O(\log \log Q / \log Q)$; the weight ω_K records what is proved. The draft's pointwise claim beyond α' fails on the top edge, and the statement is integrated. The draft's “by the symmetry $s \mapsto -s$, take $s > 0$ ” is not a symmetry of $\mathbf{F}$; the negative side of the out-zone is handled by the large sieve above, and the half B_χ by the mirror in Step 3 of §7. “ $\int_{\text{out}}$ ” in the draft means the out-zone O of [9, Lemma 4.3]. None of these changes the rate or the constant. *Formalisation:* `ZetaShell.ShellK.thmK`, derived from four leaves (the strip, the Shell zone, the killed zone and integrability); it is stated at the design point ($\lambda = 191/100$, profile S53-L75, $w = 1$) and with C in place of C_Q in the weight, a difference absorbed in $1 + c(\log Q)^{-\theta}$. In the released library the strip and integrability are proved, the killed zone is derived from a proved pointwise part and the integrated part `K3b_integrated`, and the Shell-zone part `K2_shellZone` is derived from the corrected assembly and Theorem 4.17, applied on unit windows as in the proof above (the transfer node `KT_transfer` is proved as well but not used); `thmK` is kernel-checked, with (J) and (M) as its hypothesis [L].

Remark 4.19 (The moduli $Q/2 < q \leq Q$: not claimed). Let $\mathfrak{F}_Q^{\text{dy}} \subset \mathfrak{F}_Q$ be the characters with $Q/2 < q \leq Q$. The same argument, run at $\lambda = 93/50$ with the profile D53-L75 of §6, should give the constant 0.9031776196 for $\mathfrak{F}_Q^{\text{dy}}$, with $\omega = \mathbf{1}_{(Q/2, Q]}$ in Lemma 4.1, the limit $27/(2\pi^4)$ in Lemma 4.3(a) and the constant $2\pi^4/27$ beyond α' . We do not claim it. What exists is exactly this: the certificate $2 - \mathcal{B}_{\alpha'}(v_p) = 0.90317761969178\dots$ for D53-L75 is a Lean theorem (`ZetaShell.certD53`; §6); nothing else has been checked or formalised at the bandwidth 93/50; the bandwidth analysis of §2.5 does not cover the dyadic family; and the dyadic design layer (moments, Gevrey gate, ends constant, zone slope, existence of the design) has not been computed.

5 The design at the new bandwidth, and the error budget

5.1 The design

Theorem 1.2 runs at $\lambda = 191/100$ with the degree-12 profile S53-L75 of §6. The design of [9] fixes a parameter record $(Q, T, \lambda, w, D_0, \varrho, p)$ and a validity class; the formal development reuses both unchanged (the class caps λ only by $\lambda < 2$) and replaces only the design predicate, which in [9] pins $\lambda = \lambda^*$ and $p = p_d$, by one with λ and p as parameters that coincides with the old one at (λ^*, p_d) by definition. The facts that the design needs at $\lambda = 191/100$ and S53-L75 are the following; each is a Lean theorem at the three standard axioms unless marked.

- (i) *Regime.* $X \leq Q^{2-9/200}$ for large Q .

- (ii) *Rows at the literal λ^* .* The prime part of the trace (“Row 2”) and the μP cross term (“Row 9”), which the formal tree of [9] proves only at $\lambda = 1.2507321515$, are re-proved for every $0 < \lambda < 2$.
- (iii) *Moments.* $a \geq 3/4$ and $b \geq 1/2$ hold eventually: with the exact integral of p^2 as a degree-25 antiderivative, the margins are $2.40 \cdot 10^{-8}$ and $4.6 \cdot 10^{-6}$ at the point used in the proof.
- (iv) *Gevrey gate.* The product constant of the ramp lemma for the degree-12 profile is $B' \leq 43,447 \leq 5 \cdot 10^4$ (the gate $4 \cdot 10^4$ of the formal tree fails for this profile and is raised).
- (v) *Buffer.* $D_0 \leq 1.5 \cdot 10^7 \mathcal{L}^2$, with a constant independent of Q (the true value is $D_0 \approx 5.58 \mathcal{L}^2$).
- (vi) *Existence.* For every $r_0 \geq 3$, $\varepsilon > 0$ and all large Q there is a parameter record satisfying the design predicate. The statement is eventual because the proof’s threshold is implicit; a numerical check at 28 points ($r_0 + \varepsilon$ from 3.001 to 20, $\log Q$ from $2.5 \cdot 10^5$ to 10^{40}) finds every clause satisfied, with $D_0/\mathcal{L}^2 \in [5.578, 5.588]$.
- (vii) *Other rows.* The trace row, the two counting rows, the tail (the closing condition $(4/e)\sqrt{wD_0/A} \geq L/2 + \dots$, with the off-line amplification $e^{L/2}$), the window constant, the ends constant $c_1 = 0.52$ by the Minkowski split, the display of H1, and the rates of all non-Frobenius rows, each $O((\log Q)^{-\theta})$.
- (viii) *Zone slope* (not formalised). $\psi_{v_p}(1) = 0.2205333844 \leq 0.2206$ (exact rational), the zone slope $2(\pi^4/18 - 1)\psi_{v_p}(1) = 1.945817 \leq 2(\pi^4/18 - 1) \cdot 0.2206 = 1.946405 < 1.95$, and the certified Lipschitz constant 1.906 of the zone comparison, computed in exact rationals [N]. The comparison uses $\lambda \geq 1$ (so that $1 - \lambda/2 \leq \lambda/2$) and $|p| \leq 1$ on $[-\lambda/2, \lambda/2]$, which holds for S53-L75 since $p(0) = 1$ and p is positive and decreasing on $[0, \lambda/2]$. The Lean statement is compiled and open; the proof of the statement of record does not use it.

With these, the frame of the formal headline (the five rows and the display that Proposition 2.1 consumes, at $\mathcal{B}_* = 2 - 0.9059137927$ and rate $(\log Q)^{-\theta}$) is derived and rests on exactly one open node, the Frobenius row `frob_row_shell` [Lo]. That row is Theorem 4.18 combined with the in-zone rows at the new bandwidth, and it is derived from Theorem 4.18 and the Frobenius-row chain `F1c_chain`, which is proved. In the released library Theorem 4.18 is proved (§1.5(a)), the frame with this derivation (`shell_frame_qle_of_K`) is kernel-checked with the further hypotheses `PNTErrorTerm` (the strong prime number theorem, which the tree does not prove) and `CertS53` (a theorem, §6), and its form without them, `shell_frame_qle_of_K_noP`, which the proof of the statement of record uses, is kernel-checked with (J) and (M) as its only hypothesis [L]. The in-zone rows at $\lambda = 1.91$ are covered on paper by §2.5. For the profile D53-L75 at $\lambda = 93/50$ (Remark 4.19, not claimed) none of (i)–(viii) has been computed or formalised; only its moments $a = 0.7577$ and $p(\lambda/2) = 0.16666687$ are known (the floor $1/6$ is met with margin $2 \cdot 10^{-7}$).

5.2 The error budget

Proposition 2.1 is applied with $\mathcal{B}_* = \mathcal{B}_{k_{\alpha'}}(v_p)$, $\alpha' = 2497/1500$, $k_{\alpha'} = |\alpha|$ on $[0, 1]$, 1 on $(1, \alpha']$, C^+ on $(\alpha', \lambda]$. The rows, with their rates, are listed in Table 3.

The Shell term dominates, so the error in Theorem 1.2 is $O((\log Q)^{-\theta})$ for every $\theta < 503/1994$. The exponent 503/1994 itself is the value at $\varepsilon'' = 0$ in Lemma 4.16; for $\varepsilon'' > 0$ fixed the proof gives $\theta(\alpha') < 503/1994$, which is why the theorem is stated for every $\theta < 503/1994$ (the working draft stated the exponent 503/1994).

Table 3: The rows of the error budget of Theorem 1.2. Here $K = \mathcal{L}(\log \mathcal{L})^2$, $\epsilon = 1/\mathcal{L}$, $\delta_1 = T^{-1/3}$ and $\theta < 503/1994$.

row	rate	source
Shell term (Theorem 4.17 through Theorem 4.18)	$(\log Q)^{-\theta}$	§4.7
strip Σ_Q at kernel $C \alpha $; weight $(\mathcal{L} + \log K)/s$ against 1 on $(\log Q + 4, \mathcal{L}]$	$O(\log \log Q / \log Q)$	Theorem 4.18
Lemma 4.3(a) bracket, $\epsilon \sinh \kappa$, N/Q^2 , Lemma 4.2, $\log K/\mathcal{L}$	$O(\log \mathcal{L}/\mathcal{L})$	(14)
killed zone: kink $\log(2\pi\mathcal{L})/\mathcal{L}$, $C_1/\log Q$, top edge	$O(\log \log Q / \log Q)$	Lemma 3.1
family errors of Proposition 4.11	Q^{-c}	Lemma 4.12
ramp (the clamp $w \geq 1$)	$\Theta(1/\log Q)$	§2.5(iii)
buffer	$O(\mathcal{L}^2/T)$	§2.2, §2.5(iii)
μP cross term, prime part of the trace	$Q^{\lambda_{\text{eff}}/2-1+o(1)}$	design (ii)
large-sieve length term X/Q^2	$Q^{\lambda_{\text{eff}}-2+o(1)}$	§2.5(i)
ends (with $c_1 = 0.52$), tail, counting rows, D_T -smearing	$O(\log \log Q / \log Q)$ or smaller	design (vii)
zone row (slope 1.946)	$O(\log \log Q / \log Q)$	design (viii)
C_Q against the rational C^+	$O(\log^2 Q/Q)$	§6

6 The certificate

6.1 The functional and the profiles

For $1 < \alpha' < \lambda < 2$ and $C^+ > 0$ let $k_{\alpha'}(\alpha) = |\alpha|$ for $|\alpha| \leq 1$, 1 for $1 < |\alpha| \leq \alpha'$ and C^+ for $\alpha' < |\alpha| \leq \lambda$, and $\mathcal{B}_{\alpha'}(v) := \psi(0) + \int k_{\alpha'} \psi$, $\psi = v \star v$. Since $\psi \geq 0$, $\mathcal{B}_{\alpha'}$ is increasing in C^+ , so a rational upper bound for the family constant suffices. The profiles are even polynomials $p(t) = \sum_{i=0}^6 d_i (2t/\lambda)^{2i}$ with $d_0 = 1$ and $d_i \in \mathbb{Q}$, and $v_p := p^2/\int p^2$ on $[-\lambda/2, \lambda/2]$. With $P = p^2$ and $\tilde{\psi}(\alpha) = \int_{-\lambda/2}^{\lambda/2-\alpha} P(x)P(x+\alpha) dx$, a polynomial in α with rational coefficients on $[0, \lambda]$,

$$\mathcal{B}_{\alpha'}(v_p) = \frac{\tilde{\psi}(0) + 2 \int_0^1 \alpha \tilde{\psi} + 2 \int_1^{\alpha'} \tilde{\psi} + 2C^+ \int_{\alpha'}^{\lambda} \tilde{\psi}}{(\int P)^2} \in \mathbb{Q}.$$

The profile of Theorem 1.2 is S53-L75; the profile D53-L75 enters only Remark 4.19, which is not claimed, and is listed for completeness. Both have $\alpha' = 2497/1500 = \frac{5}{3} - \frac{1}{500}$ and

- S53-L75: $\lambda = 191/100$, $C^+ = 541161617/10^8 \geq \pi^4/18$ (margin $1.44 \cdot 10^{-9}$), $d = (1, -\frac{109245641}{315967035}, -\frac{484609675}{903872491}, \frac{1126440115}{499347366}, -\frac{1244916364}{475100341}, \frac{381673291}{981251350}, \frac{37853428}{698475165})$;
- D53-L75: $\lambda = 93/50$, $C^+ = 721548823/10^8 \geq 2\pi^4/27$ (margin $5.26 \cdot 10^{-9}$), $d = (1, -\frac{390353034}{798066593}, \frac{1309177511}{918253499}, -\frac{6068806879}{896328244}, \frac{13364613432}{870778397}, -\frac{12836034741}{854019032}, \frac{4242100897}{905838731})$.

They were produced by a numerical optimiser; how they were found does not matter, since the certificate below is a statement about these rationals.

6.2 Values

profile	$2 - \mathcal{B}_{\alpha'}(v_p)$	claimed	slack	a	b	$p(\lambda/2)$
S53-L75	0.90591379273786...	0.9059137927	$3.79 \cdot 10^{-11}$	0.7500002000	0.62398	0.19676
D53-L75	0.90317761969178...	0.9031776196	$9.18 \cdot 10^{-11}$	0.7577343	0.63045	0.16666687

Here $a = \lambda^{-1} \int p^2$ and $b = \lambda^{-1} \int p^4$. Both profiles satisfy the side conditions “L75” of the formal design: $0 < p \leq 1$ on $[-\lambda/2, \lambda/2]$, p strictly decreasing on $[0, \lambda/2]$, $a \geq 3/4$, $b \geq 1/2$,

$p(\lambda/2) \geq 1/6$. Two margins are tight ($a - 3/4 = 2.0 \cdot 10^{-7}$ for S53-L75, $p(\lambda/2) - 1/6 = 2.0 \cdot 10^{-7}$ for D53-L75), and both are handled in exact rationals. The distinct-zero values $(3 - \mathcal{B}_{\alpha'}(v_p))/2$ are 0.9529568963 and 0.9515888098 (Remark 8.1). Against the variational optimum at the same $(\alpha', \lambda \leq 1.99)$, 0.9059227, the degree-12 profile loses about $9 \cdot 10^{-6}$; the limit $\sup_v (2 - \mathcal{B}_{\alpha'}(v)) \rightarrow 0.9061786$ as $\alpha' \uparrow 5/3$ is not asserted, because the density estimates do not give the dependence of their constants on ε_1 .

6.3 The formal certificate

The certificate is the Lean Prop

`CertS53 := ShellCert S53L75 PcertS53,`

which asserts: $1 < \lambda < 2$, $1 < \alpha'$, v_p is admissible (nonnegative, integrable, of mass 1, supported in $[-\lambda/2, \lambda/2]$), p satisfies L75, and the real-valued functional, defined with Mathlib's integral, satisfies $\mathcal{B}_{\alpha'}(v_p) \leq 2 - 9059137927/10^{10}$ (Appendix A). It is a theorem, `ZetaShell.certS53 : CertS53`, and so is `certD53 : CertD53`, with the three standard axioms [L]. The proof has three parts:

- (a) the rational number $\mathcal{B}_{\alpha'}(v_p)$ is computed by Lean's kernel (`decide +kernel`, core Lean only: about $2 \cdot 10^4$ rational operations on numbers with about 213 digits, 7 seconds and 1.4 GB per profile), together with four negative controls (for instance $C^+ + 10^{-6}$ makes the next ten-digit claim fail);
- (b) the real functional equals that rational (node R3): the list operations of the certificate code are proved sound against real evaluation and interval integrals, ψ is shown to be even, zero beyond λ and equal to $(\lambda/2)\hat{\psi}(\alpha/(\lambda/2))/(\int P)^2$ on $[0, \lambda]$, and the integral is split at 1 and α' ;
- (c) the side conditions L75 (node R5): $-q'$, where $p(t) = q((2t/\lambda)^2)$, is written in Bernstein form with positive coefficients, checked by `ring` (degree 5 with no elevation for S53-L75; degree 15 after ten elevations for D53-L75), which gives strict decrease and $0 < p \leq 1$; the moments are exact rationals.

Independently of Lean, two programs (symbolic expansion of $\tilde{\psi}$, and exact Newton–Cotes quadrature at $\deg \tilde{\psi} + 2$ rational nodes) return the same rational for both profiles, and the certificate code of the formal development reproduces it exactly [N].

The certificate of Theorem 1.1 is of the same kind: $\mathcal{B}_C^K = \mathcal{B}_{k_{\alpha'}}^K$ at level C for any α' , so the identity (b) applies at $\alpha' = 6/5$ to the design profile of [9], and the kernel computes 0.7235110849... (§3.3).

7 Proof of Theorem 1.2

Fix $r_0 \geq 3$, $\varepsilon > 0$ and $\theta < 503/1994$, and choose $\varepsilon'' > 0$ in Lemma 4.16 so small that $\theta < \theta(\alpha')$ at $\alpha' = 2497/1500$. The family is $\mathfrak{F}_Q$.

Step 1 (design). For large Q choose a parameter record at $\lambda = 191/100$ with profile S53-L75 (§5(vi)); its ramp width is $w \geq 1$, with $w = o(\mathcal{L})$ (§2.5(iii)), so that $w \leq (\lambda - \alpha')\mathcal{L}/8$ for large Q , as Theorem 4.18 requires; and its buffer $D_0 \asymp \mathcal{L}^2$, and the window satisfies $a \geq 3/4$, $b \geq 1/2$. Since $\lambda_{\text{eff}} < 2$, $X = o(Q^2/(\log Q)^B)$ for every B (§2.5).

Step 2 (the rows other than the Frobenius row). The display (i) of Proposition 2.1 is H1. The trace row, the buffer count, the tail perturbation and the Frobenius error B_F are those of [9, §§7–10] at the new bandwidth (§2.5, §5), with $\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_5 = O(\log \log Q / \log Q)$, and $\mathfrak{r}_1$ also absorbs the rows $Q^{\lambda_{\text{eff}}/2 - 1 + o(1)}$ of the prime part.

Step 3 (the Frobenius row). By Parseval and the positivity of g [9, Lemmas 4.1, 4.2], the Frobenius main term is the integral of $g(s)$ against $\sum_{\chi} |A_{\chi}(s) + B_{\chi}(s)|^2$, where $B_{\chi}(s) = \sum_n b_n(s) \bar{\chi}(n)$ with $b_n(s) = \overline{a_n(-s)}$ [9, Lemma 4.3]. Thus $B_{\chi}(s) = \overline{A_{\chi}(-s)}$ and $\sum_{\chi} |B_{\chi}(s)|^2 = \mathbf{F}(-s)$. On the in-zone $|s| \leq (1 - \delta') \log Q$ it is evaluated as in [9, §§4–5] (kernel $|\alpha|$). On the out-zone $O = O^+ \cup O^-$, $O^+ = O \cap (0, \infty)$, $O^- = O \cap (-\infty, 0)$, Cauchy–Schwarz in χ gives $\sum_{\chi} |A_{\chi} + B_{\chi}|^2 \leq \mathbf{F}(s) + \mathbf{F}(-s) + 2\sqrt{\mathbf{F}(s)\mathbf{F}(-s)}$. Put $X_{\pm} := \int_{O^{\pm}} g \mathbf{F}$. Since g is even and O is symmetric, $\int_O g(s) \mathbf{F}(-s) ds = X_+ + X_-$ and $\int_O 2g(s) \sqrt{\mathbf{F}(s)\mathbf{F}(-s)} ds \leq 4\sqrt{X_+ X_-}$, so

$$\sum_{\chi} \int_O g |A_{\chi} + B_{\chi}|^2 \leq 2(\sqrt{X_+} + \sqrt{X_-})^2.$$

- Theorem 4.18 gives $X_+ \leq \int_O g \mathbf{F} \leq M + O(Q^2 \int_O g)$ with $M := (1 + c(\log Q)^{-\theta}) |\mathfrak{F}_Q| \int_{O^+} g \|a\|^2 \omega_K$, since the part of its right side on O^- is $O(Q^2 \int_O g)$;
- on O^- , that is for $s \leq -(1 - \delta') \log Q$, the large sieve and $\|a(s)\|^2 = O(1)$ (proof of Theorem 4.18) give $X_- \leq Y := C' Q^2 \int_O g$ for some constant C' , which is negligible against M , of size $|\mathfrak{F}_Q| U \mathcal{L} \int_{O^+} g$;
- hence the out-zone block is at most $2(\sqrt{M + O(Y)} + \sqrt{Y})^2 = 2M(1 + O(\sqrt{Y/M}))$: the factor 2 is the mirror copy, and the cross term costs $1 + O((U \mathcal{L})^{-1/2})$, which is $1 + O(\rho)$ with the cross-term factor ρ of [9, Lemma 4.3].

With the diagonal evaluation $\int_{O^+} g \|a\|^2 \omega_K = U \int_{O^+} s g(s) \omega_K(s) ds (1 + \varepsilon_D)$, the out-zone total is

$$2U |\mathfrak{F}_Q| \int_{O^+} s g(s) \omega_K(s) ds (1 + \varepsilon_D) (1 + O(\rho)) (1 + O((\log Q)^{-\theta})),$$

the factor 2 being the mirror copy (it is the $T/\pi = 2U$ of Lemma 3.1(ii)). The dictionary $g(\alpha \mathcal{L}) = (a\lambda)^2 \mathcal{L} \psi(\alpha)$ of §2.1 converts the bound into

$$\|\widehat{G}_{\text{fam}}\|_F^2 \leq (\mathcal{B}_{k_{\alpha'}}(v) + O((\log Q)^{-\theta})) \mathcal{N},$$

where v is the window's own density of §2.1 and $k_{\alpha'}$ is the kernel of §6 with C_Q in place of C^+ : the strip and the interval $(\log Q + 4)/\mathcal{L} < \alpha \leq 1$ are absorbed in the error as in Table 3. Since $\|v - v_p\|_1 = O(w/\mathcal{L})$ with v, v_p bounded uniformly (§2.1), and since $|\psi(0) - \psi_{v_p}(0)| \leq (\|v\|_{\infty} + \|v_p\|_{\infty}) \|v - v_p\|_1$, $\|v \star v - v_p \star v_p\|_1 \leq 2\|v - v_p\|_1$ and $k_{\alpha'}$ is bounded on $[-\lambda, \lambda]$, $\mathcal{B}_{k_{\alpha'}}(v) \leq \mathcal{B}_{k_{\alpha'}}(v_p) + c_{\text{ramp}} w/L$ (the ramp row of [9], recomputed for the certified profile). Since $C_Q \rightarrow \pi^4/18 < C^+$, eventually $\mathcal{B}_{k_{\alpha'}}(v_p) \leq \mathcal{B}_{\alpha'}(v_p) \leq 2 - 0.9059137927$ by the certificate. So (ii) holds with $\mathcal{B}_* = 2 - 0.9059137927$ and $\mathbf{r}_2 = O((\log Q)^{-\theta})$.

Step 4. Proposition 2.1 gives $\sum_{\chi} N_{0,\chi}^s(T, 2T) \geq (0.9059137927 - O((\log Q)^{-\theta})) \mathcal{N}$. $\square$

Status. Proved on paper; the proof uses the results of §§2–6. In Lean the proof has the following shape: the Mathlib-only counts of the formal statement are those of **ZetaQ** by definition; a proved step turns any frame (the rows of Proposition 2.1 at a constant $\mathcal{B}_*$ and a rate) into the headline [L]; the frame at $\lambda = 191/100$ is derived from the design nodes and rests on the Frobenius row alone [Lo]; the Frobenius row is derived from Theorem 4.18 and one chain node, which is proved, and Theorem 4.18 from four nodes, the strip, integrability, the killed zone and the Shell zone, which is derived from Theorem 4.17 and the pointwise assembly. In the released library all of these nodes are proved, and the statement of record is derived through this frame, in a form that does not use **PNTErrorTerm**: it depends on the three standard axioms only, with (J) and (M) as its hypothesis [L] (§1.5). As for Theorem 1.1 (§3.4), the formal proof spends slack: the Frobenius-row chain **F1c_chain** absorbs the eventual errors of the Frobenius row, the factor $1 + c(\log Q)^{-\theta}$ of Theorem 4.18 included, in part of the slack $3.79 \cdot 10^{-11}$ of the certificate (§6), so it does not verify the rates of these errors in Table 3; Theorem 4.18 itself is a Lean theorem with its rate, and the other rows of Proposition 2.1 are proved there at the rate $(\log Q)^{-\theta}$ (§5(vii)). The certificate is a theorem [L].

8 Remarks

8.1 Rates and thresholds

Theorem 1.1 has the rate $\log \log Q / \log Q$ of [9]; Theorem 1.2 has $(\log Q)^{-\theta}$ for every $\theta < 503/1994$, which comes from the zeros counted by the log-free estimate (J) near $\sigma = 1$ (the “mid” and “near” zeros of Lemma 4.16). A Landau–Page zero is a single near term with $x_\rho = O(1)$ and contributes $O(\log K \log \log Q / \log Q)$. All thresholds are ineffective in the proofs as written, and the ones that have been estimated are very large (§1.3): the lossless range of the large sieve at $\lambda = 1.91$ begins near $\log_{10} Q \approx 177$ (at $r_0 + \varepsilon = 3.5$), and it recedes to $\log_{10} Q \approx 5400$ as $\lambda \rightarrow 1.995$; the moment floor and the existence of the design need $\log Q$ at least of order 10^5 (the formal proof’s threshold is larger and implicit); and the Weil-form error $Q^{\alpha''-2+o(1)}$ of Proposition 4.11 is small only for very large Q . Nothing is claimed at any finite Q .

8.2 What would improve the constants

The zero-density input. The threshold $\alpha_0 = 5/3$ is set by the bulk estimate (M) on $[\frac{3}{4}, \frac{4}{5}]$, where $3/(2-\sigma)$ rises from $\frac{12}{5}$ to $\frac{5}{2}$; the log-free input (J) allows α_0 up to 2. The working draft records that a bulk exponent $12/5$ (as stated in [8, Theorem 4.1], whose primary sources we have not opened) would give $\alpha_0 = 12/7$ and 0.9117, and Heath-Brown’s bulk estimate [13] (seen only through [19]) would give $\alpha_0 = 20/11$ and 0.9226. These are not claimed here. The density hypothesis for the family would give $\alpha_0 = 2$.

The limit of this route. With the kernel 1 on all of $(1, \lambda]$ and $\lambda \rightarrow 2$ the functional is the one with kernel $\min(|\alpha|, 1)$ on $|\alpha| < 2$, whose variational value is $0.9322826\dots$; the preprint [9] reports that this reproduces Sono’s constant under the generalised Riemann hypothesis [20] [V~]. At $\alpha_0 = 5/3$ the variational limit is 0.9061786 (at $\lambda \leq 1.99$); the certified 0.9059137927 lies $2.6 \cdot 10^{-4}$ below it, almost all of which is the cost of $\alpha' < \alpha_0$ (the degree-12 profile costs about $9 \cdot 10^{-6}$).

The killed kernel. At the bandwidth λ^* of the formal design, 0.7237 needs only a new design at a degree-8 profile whose certificate is already a Lean theorem (§3.3); the variational supremum of the killed functional is 0.723795 at $\lambda \approx 1.2768$.

8.3 What is in the working draft and not claimed here

Remark 8.1 (Distinct zeros). The rank–trace inequality at $c = 3$ gives $2 \sum_\chi N_\chi^d \geq (3 - \mathcal{B}_* - o(1))\mathcal{N}$ for the number N_χ^d of distinct zeros, where $\mathcal{B}_*$ is the Frobenius constant; the working draft derives from it the values $(3 - \mathcal{B})/2 = 0.9529568963$ ($\mathfrak{F}_Q$) and 0.9515888098 ($\mathfrak{F}_Q^{\text{dy}}$) from the Shell certificates, and 0.8618 from the killed kernel. The per-block inequality at $c = 3$ is a theorem of the Lean artifact of [1], but the family statement is not formalised and the derivation has not been brought to the standard of this paper. It is not claimed.

Remark 8.2 (Parity subfamilies). A reflected form of the multiplicative large sieve (coefficients placed on $[-N, N]$ with a sign, so that one parity class is charged $Q^2/2$) gives the even and the odd primitive characters the constants of the full families in [9]’s method; for the constants 0.7212 and 0.7098 this is a Lean theorem on a branch of the formal tree of [9]. The working draft extends it to the killed and Shell kernels. Those extensions are not formalised and have not been brought to the standard of this paper, and they are not claimed.

Remark 8.3 (Bounded height). The working draft contains a sketch of a version at bounded height, with a stated gap; no bounded-height statement is claimed.

8.4 Open problems

1. Formalise, or replace, the zero-density estimates (J) and (M). Their proofs use large-value estimates for Dirichlet polynomials and hybrid fourth moments, and are long; they are the only inputs of Theorem 1.2 that are neither proved here nor formalised.
2. Rebuild the formal proofs of Theorems 1.1 and 1.2 independently, with the commands of Appendix A.8.
3. Make the thresholds explicit, or reduce them; the factor KT lost in Step 3 of Lemma 3.7 by using the long-range error of the prime number theorem is the obvious place for Theorem 1.1.

A The Lean statements

The statements below are copied from the Lean sources named before each block; only line breaks inside long lines are added by the typesetting. They are quoted from the release of the repository <https://github.com/Oliverds321/simple-zeros> (tag v1.0; Lean sources of commit 4874cc0, §1.5): the family library from its library ZetaShell, the library of [9] from its library ZetaQ, and the comparator files of §A.8 from its folder comparator/. Each sha256 printed is that of the file as committed. Doc-strings are quoted as they stand and are not part of the paper’s claims: they are internal notes written during the formalisation, they refer to the working draft by its labels and line numbers and to sources by internal abbreviations, they use the word “level” for the trust levels of the formalisation plan (level A means kernel-checked in the sense of §1.5, level C a displayed hypothesis), and some of them are not current. Only the Lean code is claimed. Each status is that of the census of the release, taken on the release commit 4874cc0 (§A.6).

A.1 Counting functions

From ZetaShell/Challenge.lean (lines 31–90; sha256 5bb7abc34ad3b557...). This module imports only Mathlib. In these statements the height exponent is r (our r_0).

```
open DirichletCharacter in
/-- `Σ*_{χ mod q}`: the primitive Dirichlet characters mod `q` (character-identical to
↪ `ZetaQ.primitiveChars`). -/
def primChars (q : ℕ) : Finset (DirichletCharacter ℂ q) :=
  letI := Classical.decPred (fun χ : DirichletCharacter ℂ q => χ.IsPrimitive)
  Finset.univ.filter (fun χ => χ.IsPrimitive)

section zeros
variable {q : ℕ} [NeZero q] (χ : DirichletCharacter ℂ q)

/-- `ρ` is a nontrivial zero of `L(s,χ)` (same as `Zeta23.ThmE.IsNontrivialZeroL`). -/
def IsNtZero (ρ : ℂ) : Prop := χ.LFunction ρ = 0 ∧ 0 < ρ.re ∧ ρ.re < 1

/-- `m_ρ`: order of vanishing of `L(·,χ)` at `ρ` (same as `Zeta23.ThmE.zeroMultL`). -/
def zmult (ρ : ℂ) : ℕ := (analyticOrderAt χ.LFunction ρ).toNat

/-- nontrivial zeros with `T₁ < γ ≤ T₂`. -/
def zerosWin (T₁ T₂ : ℝ) : Set ℂ := {ρ | IsNtZero χ ρ ∧ T₁ < ρ.im ∧ ρ.im ≤ T₂}

/-- `N_χ(T₁,T₂)`, with multiplicity. -/
def Nwin (T₁ T₂ : ℝ) : ℕ := Σf ρ ∈ zerosWin χ T₁ T₂, zmult χ ρ

/-- `N^s_{0,χ}(T₁,T₂)`: simple zeros on the critical line. -/
def N0sWin (T₁ T₂ : ℝ) : ℕ :=
```

```

(zerosWin  $\chi$   $T_1$   $T_2$   $\cap \{\rho \mid \rho.\text{re} = 1 / 2\} \cap \{\rho \mid \text{zmult } \chi \rho = 1\}$ ).ncard

/-- zeros in Jutila's / Bombieri's closed rectangle  $\sigma \leq \beta \leq 1$ ,  $|\gamma| \leq T$  (all zeros there
 $\hookrightarrow$  are nontrivial
for  $\sigma > 0$ ). -/
def zerosRect ( $\sigma$   $T : \mathbb{R}$ ) : Set  $\mathbb{C} := \{\rho \mid \chi.\text{LFunction } \rho = 0 \wedge \sigma \leq \rho.\text{re} \wedge \rho.\text{re} \leq 1 \wedge |\rho.\text{im}| \leq T\}$ 

/--  $N(\sigma, T, \chi)$ , with multiplicity. -/
def NrectC ( $\sigma$   $T : \mathbb{R}$ ) :  $\mathbb{N} := \sum^f \rho \in \text{zerosRect } \chi \sigma T, \text{zmult } \chi \rho$ 

end zeros

/--  $N_\chi(T_1, T_2)$  as a real, with the (never used) guard  $q = 0$  (as  $\text{ZetaQ.NcountQ}$ ). -/
def Nchi ( $q : \mathbb{N}$ ) ( $\chi : \text{DirichletCharacter } \mathbb{C} q$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :  $\mathbb{R} :=$ 
  if  $h : q = 0$  then 0 else haveI : NeZero  $q := \langle h \rangle$ ; ((Nwin  $\chi$   $T_1$   $T_2 : \mathbb{N}) : \mathbb{R})$ 

/--  $N^s_{\{0, \chi\}}(T_1, T_2)$  as a real (as  $\text{ZetaQ.NOsQ}$ ). -/
def NOSchi ( $q : \mathbb{N}$ ) ( $\chi : \text{DirichletCharacter } \mathbb{C} q$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :  $\mathbb{R} :=$ 
  if  $h : q = 0$  then 0 else haveI : NeZero  $q := \langle h \rangle$ ; ((NOSwin  $\chi$   $T_1$   $T_2 : \mathbb{N}) : \mathbb{R})$ 

/--  $N(\sigma, T, \chi)$  as a real. -/
def Nrect ( $q : \mathbb{N}$ ) ( $\chi : \text{DirichletCharacter } \mathbb{C} q$ ) ( $\sigma$   $T : \mathbb{R}$ ) :  $\mathbb{R} :=$ 
  if  $h : q = 0$  then 0 else haveI : NeZero  $q := \langle h \rangle$ ; ((NrectC  $\chi \sigma T : \mathbb{N}) : \mathbb{R})$ 

/-- The moduli of the family  $\mathfrak{F}_Q$  of sec_shell.tex 1.37:  $1 < q \leq Q$  (as
 $\hookrightarrow \text{ZetaQ.Family.moduli Family.qle}$ ). -/
def modQle ( $Q : \mathbb{N}$ ) : Finset  $\mathbb{N} := \text{Finset.Icc } 2 Q$ 

/-- The dyadic family's moduli  $Q/2 < q \leq Q$  (as  $\text{ZetaQ.Family.moduli Family.dyadic}$ ). -/
def modDyadic ( $Q : \mathbb{N}$ ) : Finset  $\mathbb{N} := \text{Finset.Ioc } (Q / 2) Q$ 

/--  $\mathcal{N} = \sum_{q \in M} \sum_\chi N_\chi(T_1, T_2)$ . -/
def famN ( $M : \text{Finset } \mathbb{N}$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :  $\mathbb{R} := \sum q \in M, \sum \chi \in \text{primChars } q, \text{Nchi } q \chi T_1 T_2$ 

/--  $\sum_{q \in M} \sum_\chi N^s_{\{0, \chi\}}(T_1, T_2)$ . -/
def famNOS ( $M : \text{Finset } \mathbb{N}$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :  $\mathbb{R} := \sum q \in M, \sum \chi \in \text{primChars } q, \text{NOSchi } q \chi T_1 T_2$ 

/-- The window height of the paper's Theorem 1:  $T = (\log Q)^{r+\varepsilon}$  (as  $\text{ZetaQ.Twin}$ ). -/
def twin ( $Q$   $r$   $\varepsilon : \mathbb{R}$ ) :  $\mathbb{R} := \text{Real.log } Q ^ (r + \varepsilon)$ 

```

These counts are those of the formal tree of [9] by definition: from ZetaShell/Frame/B1.CountBridge.lean (lines 10–22), five theorems proved by rfl [L]:

```

theorem famN_qle_eq ( $Qn : \mathbb{N}$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :
  famN (modQle  $Qn$ )  $T_1$   $T_2$  = ZetaQ.NfamCount ZetaQ.Family.qle  $Qn$   $T_1$   $T_2$  := rfl

theorem famNOS_qle_eq ( $Qn : \mathbb{N}$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :
  famNOS (modQle  $Qn$ )  $T_1$   $T_2$  = ZetaQ.NOSfamCount ZetaQ.Family.qle  $Qn$   $T_1$   $T_2$  := rfl

theorem famN_dyadic_eq ( $Qn : \mathbb{N}$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :
  famN (modDyadic  $Qn$ )  $T_1$   $T_2$  = ZetaQ.NfamCount ZetaQ.Family.dyadic  $Qn$   $T_1$   $T_2$  := rfl

theorem famNOS_dyadic_eq ( $Qn : \mathbb{N}$ ) ( $T_1$   $T_2 : \mathbb{R}$ ) :
  famNOS (modDyadic  $Qn$ )  $T_1$   $T_2$  = ZetaQ.NOSfamCount ZetaQ.Family.dyadic  $Qn$   $T_1$   $T_2$  := rfl

theorem twin_eq ( $Q$   $r$   $\varepsilon : \mathbb{R}$ ) : twin  $Q$   $r$   $\varepsilon$  = ZetaQ.Twin  $Q$   $r$   $\varepsilon$  := rfl

```

and, from the library ZetaQ, ZetaQ/Certificate.lean (lines 570–575):

```

def NfamCount (F : Family) (Qn : ℕ) (T1 T2 : ℝ) : ℝ :=
  familySum F Qn (fun q χ => NcountQ q χ T1 T2)

/-- `Σ_{χ ∈ ℱ_Q} N^{s_{0,χ}}(T1, T2)` --- the left-hand side of Theorem 1 and of §3's display.
  ↪ -/
def N0sFamCount (F : Family) (Qn : ℕ) (T1 T2 : ℝ) : ℝ :=
  familySum F Qn (fun q χ => N0sQ q χ T1 T2)

```

A.2 The zero-density input

From ZetaShell/Challenge.lean, lines 94–124. ZeroDensityInput is (J) and (M) of §4.6; it depends on no axiom. The doc-string of MontgomeryHybrid is quoted in full. The Prop itself states Bombieri's Théorème 20 with an existential power of the logarithm. Where the paper assumes (J) and (M), the Lean statement has `hD : ZeroDensityInput` as a hypothesis; it is not an axiom, and `#print axioms` does not list it.

```

/-- `N*(σ, T, Q) = Σ_{q ≤ Q} Σ_{χ mod q} N(σ, T, χ)`, over PRIMITIVE characters, `q = 1`
  ↪ (i.e. `ζ`) INCLUDED
(sec_shell.tex 1.742--744: "In every source below the count is over primitive characters, in
  ↪ the closed rectangle
  ↪ β ≥ σ, |γ| ≤ T`, and the character mod 1 (that is, ζ) is included"). -/
def Nstar (Q : ℕ) (σ T : ℝ) : ℝ := Σ q ∈ Finset.Icc 1 Q, Σ χ ∈ primChars q, Nrect q χ σ T

/-- **Jutila 1977, Theorem 1, (1.8)** [Math. Scand. 41 (1977), p. 46], as quoted in
  ↪ rem:shell-S53-inputs (a)
(sec_shell.tex 1.791--796): "For `4/5 ≤ α ≤ 1`, `T ≥ 1`, we have ... (1.8) `N*(α, T, Q) <<_ε
  ↪ (Q2T)^{(2+ε)(1-α)}`,
  ↪ "the constants implied by Vinogradov's symbol `<<` depend on `ε`; no factor of `log`; no
  ↪ relation between `T` and `Q`.
  ↪ Used as (LF) of (D1') with `h_LF = 1`, `c_0 = 2`, `σ_LF = 4/5` (sec_shell.tex 1.757).
  ↪ Range of `Q`: `Q ≥ 1` (the sum over `q ≤ Q` is empty-free from `Q = 1`; TZ22 quote it for
  ↪ `Q ≥ 3`, which is
  ↪ equivalent up to the constant since `N*` is monotone in `Q`). -/
def JutilaHybrid : Prop :=
  ∀ ε : ℝ, 0 < ε → ∃ C : ℝ, ∀ Q : ℕ, 1 ≤ Q → ∀ T : ℝ, 1 ≤ T → ∀ α : ℝ, 4 / 5 ≤ α → α
  ↪ ≤ 1 →
  ↪ Nstar Q α T ≤ C * ((Q : ℝ) ^ 2 * T) ^ ((2 + ε) * (1 - α))

/-- **Montgomery's hybrid estimate, in the form of Bombieri, Astérisque 18, §10, Théorème
  ↪ 20** (p. 77), as quoted in
  ↪ rem:shell-S53-inputs (b) (sec_shell.tex 1.802--807): for `T ≥ 2`, `1/2 ≤ α ≤ 1`,
  ↪ "`Σ_{q ≤ Q} Σ_{χ mod q} N(α, T; χ) << (TQ2)^{3(1-α)/(2-α)} (log TQ)^A`", absolute implied
  ↪ constant.
  ↪ The power `A` is EXISTENTIAL here.
  ↪ (B) consumes only a finite `b_0`. So this Prop is implied by the printed statement for every
  ↪ admissible `A`, i.e.
  ↪ it does NOT strengthen the source. -/
def MontgomeryHybrid : Prop :=
  ∃ A C : ℝ, ∀ Q : ℕ, 1 ≤ Q → ∀ T : ℝ, 2 ≤ T → ∀ α : ℝ, 1 / 2 ≤ α → α ≤ 1 →
  ↪ Nstar Q α T ≤ C * ((T * (Q : ℝ) ^ 2) ^ (3 * (1 - α) / (2 - α)) * Real.log (T * Q) ^ A)

/-- **`ZeroDensityInput`** (LEAN_PLAN §1.4): instantiation (D1') of sec_shell.tex 1.756--762
  ↪ --- "Montgomery bulk +
  ↪ Jutila hybrid (all inputs primary, [V])" --- which gives `A* = M = 5/2` and `α_0 = 5/3`
  ↪ (thm:shell-S53).
  ↪ Both fields are classical unconditional theorems (Track D would prove them); here they are
  ↪ hypotheses. -/
structure ZeroDensityInput : Prop where
  jutila : JutilaHybrid

```

A.3 The certificate

From ZetaShell/Challenge.lean, lines 139–229 (the profile D53L75 and CertD53 belong to the dyadic value of Remark 4.19, which is not claimed):

```

/-- `ψ = v * v` read as the autocorrelation  $\psi(\alpha) = \int v(t) v(t-\alpha) dt$  (character-identical
↪ to
`ZetaQ.Payoff.psi`; for even `v` it is the convolution). -/
def psiS (v : ℝ → ℝ) (α : ℝ) : ℝ := ∫ t, v t * v (t - α)

/-- **The Shell kernel** (def:shell-kernel, sec_shell.tex 1.56--59, in the normalisation of
↪ ssec:shell-cert
"Kernel and constants", 1.1016--1018):  $|\alpha|$  on  $|\alpha| \leq 1$ , the level  $\ell$  on  $1 < |\alpha| \leq \alpha'$ ,
↪ the constant  $C$  on
 $\alpha' < |\alpha|$  (only  $|\alpha| \leq \lambda$  is ever read, since  $\text{supp } \psi \subseteq [-\lambda, \lambda]$ ).  $\ell = 1$  for the full
↪ families. -/
def shellKernel (ℓ ap C : ℝ) (α : ℝ) : ℝ :=
  if |α| ≤ 1 then |α| else if |α| ≤ ap then ℓ else C

/-- **B_{α'}(v) = ψ(0) + ∫ k_{α'}(α) ψ(α) dα** (eq:shell-B with `k = k_{α'}`,
↪ def:shell-kernel). -/
def Bshell (ℓ ap C : ℝ) (v : ℝ → ℝ) : ℝ := psiS v 0 + ∫ α, shellKernel ℓ ap C α * psiS v
↪ α

/-- Admissible profiles (paper §11; field-identical to `ZetaQ.Payoff.Admissible`). -/
structure AdmissibleS (lam : ℝ) (v : ℝ → ℝ) : Prop where
  nonneg : ∀ t, 0 ≤ v t
  integrable : MeasureTheory.Integrable v
  mass : ∫ t, v t = 1
  supp : ∀ t, v t ≠ 0 → |t| ≤ lam / 2

/-- A certified profile of ssec:shell-cert "Method" (1.1025--1036): rationals  $1 < \lambda < 2$ ,
↪  $\alpha'$ , the level  $\ell$ , the
rational upper bound  $C^+$  of the family constant, and  $d = (d_0 = 1, d_1, \dots)$  with  $p(t) =$ 
↪  $\sum_i d_i (2t/\lambda)^{2i}$ . -/
structure ShellProfile where
  lam : ℚ
  alphaP : ℚ
  level : ℚ
  Cplus : ℚ
  d : List ℚ

namespace ShellProfile
variable (S : ShellProfile)

/--  $p(t) = \sum_i d_i (2t/\lambda)^{2i}$ . -/
def p (t : ℝ) : ℝ := ∑ i : Fin S.d.length, ((S.d.get i : ℚ) : ℝ) * (2 * t / (S.lam : ℝ)) ^
↪ (2 * (i : ℕ))

/--  $\int_{-\lambda/2}^{\lambda/2} p^2$ . -/
def mass : ℝ := ∫ t in (-(S.lam : ℝ) / 2)..((S.lam : ℝ) / 2), S.p t ^ 2

/--  $v_p = p^2 / \int p^2$  on  $[-\lambda/2, \lambda/2]$ ,  $0$  outside. -/
def v (t : ℝ) : ℝ := if |t| ≤ (S.lam : ℝ) / 2 then S.p t ^ 2 / S.mass else 0

/-- **The Lean side conditions "L75" (ssec:shell-cert "Lean side conditions",
↪ 1.1080--1082:  $p$  decreasing,
 $a \geq 3/4$  (`Valid.age`),  $b \geq 1/2$  (H5),  $p(\lambda/2) \geq 1/6$ ), with  $0 < p \leq 1$  of "Method"
↪ (1.1033--1034);
```

```

`a = ⟨p2⟩ = λ-1∫p2·, `b = ⟨p4⟩ = λ-1∫p4· as in `check_profile.py`. -/
structure L75 : Prop where
  pos : ∀ t, |t| ≤ (S.lam : ℝ) / 2 → 0 < S.p t
  le_one : ∀ t, |t| ≤ (S.lam : ℝ) / 2 → S.p t ≤ 1
  anti : StrictAntiOn S.p (Set.Icc 0 ((S.lam : ℝ) / 2))
  a_ge : (3 : ℝ) / 4 ≤ S.mass / (S.lam : ℝ)
  b_ge : (1 : ℝ) / 2 ≤ (∫ t in (-(S.lam : ℝ) / 2)..(S.lam : ℝ) / 2, S.p t ^ 4) /
    ↪ (S.lam : ℝ)
  end_ge : (1 : ℝ) / 6 ≤ S.p ((S.lam : ℝ) / 2)

end ShellProfile

/-- **The certificate Prop (Track R)**: the profile is admissible at `λ`, meets L75, `1 <
↪ α'`, `1 < λ < 2`, and
`B_{α'}(v_p) ≤ 2 - P_c` at the kernel `(ℓ, α', C^+)`. -/
def ShellCert (S : ShellProfile) (Pc : ℚ) : Prop :=
  1 < (S.lam : ℝ) ∧ (S.lam : ℝ) < 2 ∧ 1 < (S.alphaP : ℝ) ∧ AdmissibleS S.lam S.v ∧ S.L75
  ↪ ∧
  Bshell S.level S.alphaP S.Cplus S.v ≤ 2 - (Pc : ℝ)

/-- Profile **S53-L75** (ssec:shell-cert-53, 1.1224;
↪ `round2/d2_1_numerics/profiles53.json`):
`α' = 2497/1500 = 5/3 - 1/500`, `λ = 191/100`, `ℓ = 1`, `C^+ = 541161617/10^8 ≥ π^4/18`. -/
def S53L75 : ShellProfile where
  lam := 191 / 100
  alphaP := 2497 / 1500
  level := 1
  Cplus := 541161617 / 100000000
  d := [1, -109245641 / 315967035, -484609675 / 903872491, 1126440115 / 499347366,
    -1244916364 / 475100341, 381673291 / 981251350, 37853428 / 698475165]

/-- Profile **D53-L75** (1.1230): `α' = 2497/1500`, `λ = 93/50`, `ℓ = 1`, `C^+ =
↪ 721548823/10^8 ≥ 2π^4/27`. -/
def D53L75 : ShellProfile where
  lam := 93 / 50
  alphaP := 2497 / 1500
  level := 1
  Cplus := 721548823 / 100000000
  d := [1, -390353034 / 798066593, 1309177511 / 918253499, -6068806879 / 896328244,
    13364613432 / 870778397, -12836034741 / 854019032, 4242100897 / 905838731]

/-- `0.9059137927` --- the 10-digit truncation of `2 - B(v_{S53-L75})` (exact rational
↪ `0.90591379273786...`,
`p_cert_exact53.txt`). -/
def PcertS53 : ℚ := 9059137927 / 10000000000

/-- `0.9031776196` --- the 10-digit truncation for D53-L75 (exact `0.90317761969178...`). -/
def PcertD53 : ℚ := 9031776196 / 10000000000

/-- The certificate of the headline (sharp family). -/
def CertS53 : Prop := ShellCert S53L75 PcertS53

/-- The certificate of the dyadic headline. -/
def CertD53 : Prop := ShellCert D53L75 PcertD53

```

From ZetaShell/Cert/R4_CertS53.lean (sha256 6353961802d5fc7d...), the two statements (proofs omitted):

```

theorem certS53 : CertS53
theorem certD53 : CertD53

```

Status: both kernel-checked, no `sorry`, no `native_decide`; in the census both depend on the three standard axioms only and rest on no open declaration [L].

A.4 The statement of record of Theorem 1.2

From `ZetaShell/Top/HeadlineReduced.lean` (sha256 `e4bb7a2ca9f41c53...`), lines 37–47, with its proof:

```

/-- **Reduced headline, family  $1 < q \leq Q$ ** (statement of record): `shell_S53_qle` without
 $\hookrightarrow$  `hP` and `hcert`. -/
theorem shell_S53_qle_reduced (hD : ZeroDensityInput) (r  $\varepsilon$   $\theta$  :  $\mathbb{R}$ ) (hr :  $3 \leq r$ ) (h $\varepsilon$  :  $0 < \varepsilon$ )
(h $\theta$  :  $0 < \theta$ ) (h $\theta'$  :  $\theta < 503 / 1994$ ) :
 $\exists Q_0$  c :  $\mathbb{R}$ ,  $0 < c \wedge \forall Q_n : \mathbb{N}$ ,  $Q_0 \leq (Q_n : \mathbb{R}) \rightarrow$ 
  (((PcertS53 :  $\mathbb{Q}$ ) :  $\mathbb{R}$ ) - c * Real.log (Qn :  $\mathbb{R}$ ) ^ (- $\theta$ ))
    * famN (modQle Qn) (twin (Qn :  $\mathbb{R}$ ) r  $\varepsilon$ ) (2 * twin (Qn :  $\mathbb{R}$ ) r  $\varepsilon$ )
       $\leq$  famN0s (modQle Qn) (twin (Qn :  $\mathbb{R}$ ) r  $\varepsilon$ ) (2 * twin (Qn :  $\mathbb{R}$ ) r  $\varepsilon$ ) := by
obtain  $\langle Q_0, c, hc, h \rangle$  := shell_S53_qle_tree_noP hD r  $\varepsilon$   $\theta$  hr h $\varepsilon$  h $\theta$  h $\theta'$ 
refine  $\langle Q_0, c, hc, \text{fun } Q_n \text{ hQn} \Rightarrow ?\_ \rangle$ 
rw [famN_qle_eq, famN0s_qle_eq, twin_eq]
exact h Qn hQn

```

In the release this is a theorem [L]: its row in the census table (tab-separated; the tabs are printed here as two spaces) lists no open leaf, and `#print axioms` gives the three standard axioms only:

```

ZetaShell.Top.HeadlineReduced ZetaShell.shell_S53_qle_reduced A [Classical.choice,
 $\hookrightarrow$  Quot.sound, propext] 0 []
'ZetaShell.shell_S53_qle_reduced' depends on axioms: [propext, Classical.choice, Quot.sound]

```

Its one displayed hypothesis, `hD : ZeroDensityInput`, is (J) and (M) (§A.2). The proof applies `shell_S53_qle_tree_noP` of `ZetaShell/Top/T1_HeadlineNoP.lean`, the headline obtained from the frame `shell_frame_qle_of_K_noP`, which does not use `PNTErrorTerm`, and rewrites the counts by the theorems of the first subsection that are proved by `rfl`. The same module proves, as `ZetaShell.Top.shell_S53_qle_of_reduced` (lines 91–93, no `sorry` of its own), that the statement of record implies the earlier form `ZetaShell.shell_S53_qle` of `ZetaShell/Challenge.lean`, which has the two further hypotheses `hP : PNTErrorTerm` and `hcert : CertS53`. The converse is proved given `PNTErrorTerm`; the tree does not prove `PNTErrorTerm`, and by the module’s header the argument uses only the medium form of the prime number theorem, which is a theorem of the tree:

```

theorem shell_S53_qle_of_reduced :
  type_of% @shell_S53_qle_reduced  $\rightarrow$  type_of% @shell_S53_qle :=
  fun h hD _ => h hD

```

The same files also state a dyadic version (`shell_S53_dyadic_reduced`) and two bounded-height versions (`shell_S53_bounded_reduced`, `shell_S53_bounded_dyadic_reduced`); they are open, and none of them is claimed in this paper.

A.5 The statement of Theorem 1.1

From `ZetaShell/Defs/LK_Defs.lean` (sha256 `d05d5949ca23b6f3...`), lines 104–105:

```

/-- the constant of the design-profile route (no new design data): `0.7235`. -/
def PcertKD :  $\mathbb{Q}$  := 7235 / 10000

```

and from `ZetaShell/LemmaK/LK_KT_Headline.lean` (sha256 3cf2934de48e80cc...), lines 60–65 (proof omitted):

```

/-- **THEOREM 1' at ZetaQ's own design profile, `P = 0.7235`.** -/
theorem theorem_one_prime_design (r ε : ℝ) (hr : 3 ≤ r) (hε : 0 < ε) :
  ∃ Q₀ c : ℝ, 0 < c ∧ ∀ Qn : ℕ, Q₀ ≤ (Qn : ℝ) →
    (((PcertKD : ℚ) : ℝ) - c * Real.log (Real.log (Qn : ℝ)) / Real.log (Qn : ℝ))
      * ZetaQ.NfamCount ZetaQ.Family.qle Qn (ZetaQ.Twin (Qn : ℝ) r ε) (2 * ZetaQ.Twin
        ↪ (Qn : ℝ) r ε)
      ≤ ZetaQ.N0sFamCount ZetaQ.Family.qle Qn (ZetaQ.Twin (Qn : ℝ) r ε) (2 * ZetaQ.Twin
        ↪ (Qn : ℝ) r ε)

```

Status: a Lean theorem with the three standard axioms and no hypothesis, in the library `ZetaShell` of the release [L]. The census gives

```

SORRY-LEAVES ZetaShell.LemmaK.theorem_one_prime_design (0): []
'ZetaShell.LemmaK.theorem_one_prime_design' depends on axioms: [propept, Classical.choice,
  ↪ Quot.sound]

```

and so does the file `comparator/PrintAxioms/FollowUpShell.lean` (§A.8). The companion statement at 0.7237 (`theorem_one_prime`) still rests on one open node (`killed_frame_lamStar8`). The statement has the shape of the theorem of [9] in the library `ZetaQ`, which is kernel-checked with no hypothesis; from `ZetaQ/Margin.lean` (lines 1215–1220; proof omitted):

```

theorem theorem_one_generic_proved' (r ε : ℝ) (hr : 3 ≤ r) (hε : 0 < ε) :
  ∃ Q₀ c : ℝ, 0 < c ∧ ∀ Qn : ℕ, Q₀ ≤ (Qn : ℝ) →
    (((Payoff.Pcert_qQ_smooth : ℚ) : ℝ)
      - c * Real.log (Real.log (Qn : ℝ)) / Real.log (Qn : ℝ))
      * NfamCount Family.qle Qn (Twin (Qn : ℝ) r ε) (2 * Twin (Qn : ℝ) r ε)
      ≤ N0sFamCount Family.qle Qn (Twin (Qn : ℝ) r ε) (2 * Twin (Qn : ℝ) r ε)

```

Here `Payoff.Pcert_qQ_smooth` is the constant 0.7212 of [9].

A.6 The census, and the parts of Theorem 1.2

The census is one Lean job that prints, for each public theorem of `ZetaShell`, the axioms it depends on and the open declarations (declarations with a `sorry` of their own) that its proof term reaches. It was run on the release commit 4874cc0 (output: `audit/followup/census_2026-10-03_1557.out` and `audit/followup/census_2026-10-03_1557.tsv` in the repository; the job also covers `ZetaS`): of the 1,744 public theorems, 1,708 depend on the three standard axioms only and 36 on `sorryAx`: 29 open statements, each with its own `sorry` (20 in `ZetaShell/Skeleton/`, and the 9 headline statements of `ZetaShell/Challenge.lean`, `ZetaShell/Interfaces.lean` and `ZetaShell/Top/HeadlineReduced.lean`), and 7 theorems derived from them. No other axiom occurs. For the parts of Theorem 1.2 named in §1.5 the census prints (records of the open leaves first, then the axioms; verbatim):

```

SORRY-LEAVES ZetaShell.shell_S53_qle_reduced (0): []
SORRY-LEAVES ZetaShell.Design.shell_frame_qle_of_K_noP (0): []
SORRY-LEAVES ZetaShell.Design.shell_frame_qle_of_K (0): []
SORRY-LEAVES ZetaShell.ShellK.thmK (0): []
SORRY-LEAVES ZetaShell.ShellK.K2_shellZone (0): []
SORRY-LEAVES ZetaShell.ShellK.KT_transfer (0): []
SORRY-LEAVES ZetaShell.ShellS.AS_pointwise_corr (0): []

```

```

SORRY-LEAVES ZetaShell.ShellK.F1c.K3b_integrated (0): []
SORRY-LEAVES ZetaShell.ShellS.shell_S (0): []
SORRY-LEAVES ZetaShell.ShellS.S1_ring_to_zeros (0): []
SORRY-LEAVES ZetaShell.ShellS.S2_zero_side (0): []
SORRY-LEAVES ZetaShell.ShellS.Z6a_zero_sums (0): []
SORRY-LEAVES ZetaShell.ShellS.Z6d_counts_corr (0): []
SORRY-LEAVES ZetaShell.TrackF.ring_le_primitive_corr (0): []
SORRY-LEAVES ZetaShell.TrackF.lemma2a (0): []
SORRY-LEAVES ZetaShell.TrackF.lemmaP (0): []
SORRY-LEAVES ZetaShell.PropZ.propZ_W (0): []
SORRY-LEAVES ZetaShell.ShellK.F1c_chain (0): []
SORRY-LEAVES ZetaShell.TrackF.signed_gallagher (0): []
SORRY-LEAVES ZetaShell.ShellS.S3_family_errors (0): []
SORRY-LEAVES ZetaShell.certS53 (0): []
'ZetaShell.shell_S53_qlc_reduced' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.Design.shell_frame_qlc_of_K_noP' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.Design.shell_frame_qlc_of_K' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.ShellK.thmK' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellK.K2_shellZone' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellK.KT_transfer' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellS.AS_pointwise_corr' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.ShellK.F1c.K3b_integrated' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.ShellS.shell_S' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellS.S1_ring_to_zeros' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.ShellS.S2_zero_side' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellS.Z6a_zero_sums' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellS.Z6d_counts_corr' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.TrackF.ring_le_primitive_corr' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.TrackF.lemma2a' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.TrackF.lemmaP' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.PropZ.propZ_W' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.ShellK.F1c_chain' depends on axioms: [proptext, Classical.choice, Quot.sound]
'ZetaShell.TrackF.signed_gallagher' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.ShellS.S3_family_errors' depends on axioms: [proptext, Classical.choice,
↪ Quot.sound]
'ZetaShell.certS53' depends on axioms: [proptext, Classical.choice, Quot.sound]

```

So in the release each of these depends on the three standard axioms only and its proof reaches no open declaration: the statement of record, the frame (also without `PNTErrorTerm`), Theorem 4.18 with its Shell-zone part, the transfer, the pointwise assembly in its corrected form and the integrated killed-zone part, Theorem 4.17 with its steps S1 and S2 and Lemmas 4.13 and 4.16 (the latter corrected), the corrected Lemma 4.8, Lemma 4.3(a), Lemma 4.4, Proposition 4.11, the Frobenius-row chain, Lemma 4.5, the family errors of Theorem 4.17 and the certificate are theorems. The table below gives the status of each named node.

A.7 Status of the named nodes

The label of each row is computed from the census of §A.6. For [Lo] the open leaves are listed.

declaration	paper	status
certS53	§6	[L]
certD53	Remark 4.19 (not claimed)	[L]
psi_sub_id_isLittle0_log_rpow	(2)	[L]
signed_farey_identity	Lemma 4.1	[L]
replace_cost, small_prime_norm,	Lemma 4.2	[L]
far_norm		
hole_edge_mass,	Lemmas 4.7, 4.14	[L]
shell_rectangles		
lemma2b, lemma2c_hole,	Lemma 4.3(b),(c)	[L]
lemma2c_spike		
lemma2a	Lemma 4.3(a)	[L]
lemmaP	Lemma 4.4	[L]
signed_gallagher	Lemma 4.5	[L]
ring_le_primitive_corr	Lemma 4.8 (corrected statement, Appendix B)	[L]
plancherel_majorant	Lemma 4.9	[L]
smooth_explicit_formula_weil	Lemma 4.10	[L]
family_E_sum, family_EW_sum	Lemma 4.12(2),(2')	[L]
propZ_W	Proposition 4.11	[L]
Z6c_blocks	Lemma 4.15	[L]
Z6a_zero_sums	Lemma 4.13	[L]
Z6d_counts_corr	Lemma 4.16 (near bound under $5X_0 \leq s_0$, Appendix B)	[L]
Z6d_counts	first form of Lemma 4.16 (not used; Appendix B)	[L-stated], open
S1_ring_to_zeros	Theorem 4.17, step S1	[L]
S2_zero_side	Theorem 4.17, step S2	[L]
S3_family_errors, S4_elementary	Theorem 4.17, steps S3, S4	[L]
shell_S	Theorem 4.17	[L]
AS_pointwise_corr	(13) (with $\alpha'' < \lambda$, Appendix B)	[L]
AS_pointwise	first form of (13) (not used; Appendix B)	[L-stated], open
K1_strip, K_int	Theorem 4.18: strip, integrability	[L]
killed_pointwise_shell,	Theorem 4.18: killed zone	[L]
majorant_excess	pointwise (K3a), majorant	
K2_shellZone, K3b_integrated,	Theorem 4.18: Shell zone, killed	[L]
KT_transfer	zone integrated (K3b), transfer	
thmK	Theorem 4.18	[L]
F1c_chain	Frobenius-row chain	[L]
zoneLipschitzData_S53	zone slope, §5(viii) (not used)	[L-stated], open
shell_frame_qle_of_nodes	frame of Theorem 1.2, Frobenius row as one node	[L _o], one leaf: frob_row_shell
shell_frame_qle_of_K	frame of Theorem 1.2, Frobenius row from Theorem 4.18	three standard axioms, no open leaf, with the further hypotheses PNErrorTerm and CertS53
shell_frame_qle_of_K_noP	the same frame without PNErrorTerm	[L]
gauss_sum_ineq, farey_hole_cores	Lemmas 3.2, 3.4	[L]
gallagher_with_zero_set_corr	Lemma 3.3	[L]
principal_gauss_hole,	Lemmas 3.5, 3.6	[L]
small_primes_l2		
hole_lower, farey_hole_bound	(7)	[L]
killed_pointwise	Lemma 3.1(i)	[L]
principal_arc_mass	Lemma 3.7	[L]
certKD_core, k_assembly	§3.3, §3.4	[L]
theorem_one_prime_design	Theorem 1.1	[L]
shell_S53_qle_reduced	Theorem 1.2	[L] (§A.4)

A.8 The release, and how to check it

The formalisation is released in the repository <https://github.com/Oliverds321/simple-zeros>, at tag v1.0; its Lean sources are those of commit 4874cc0 of the development history, as RELEASE_NOTES.md of the repository records (the repository is a snapshot of the release). The family library of this paper is its library ZetaShell (327 files, 51,318 lines; the groups of modules are described in ZetaShell/README.md); the repository also holds the Lean artifact Zeta23 of [1], the library ZetaQ of [9] and the libraries of the companion paper on ζ . Commands are run from the root of the repository; the toolchain is pinned in lean-toolchain, and lake exe cache get fetches the prebuilt Mathlib. Then

```
lake build ZetaShell
lake env lean comparator/PrintAxioms/FollowUpShell.lean
```

The last command prints the axioms of ZetaShell.LemmaK.theorem_one_prime_design and ZetaShell.certS53, and must print the two lines

```
'ZetaShell.LemmaK.theorem_one_prime_design' depends on axioms: [propext, Classical.choice, Quot.sound]
'ZetaShell.certS53' depends on axioms: [propext, Classical.choice, Quot.sound]
```

The build warns that a declaration uses `sorry` for each of the open statements of §A.6; whether a theorem depends on an open statement is decided by `#print axioms`, not by these warnings. The build of the release commit had no error and printed these two lines; its log and outputs, and those of earlier builds and censuses, are in audit/followup/ of the repository, and RELEASE_NOTES.md names the commit each was taken on.

The two theorems of this paper are also stated in the comparator package FollowUpFamily: the trusted file comparator/ChallengeDeps/FollowUpFamily.lean (sha256 07f36f808a8c6f35...) imports Mathlib alone and defines the counts, the family, T and ZeroDensityInput from their sources, and the trusted file comparator/Challenge/FollowUpFamily.lean (sha256 5cc2d104721ecafe...) states the two theorems, lines 25–47 (the `sorry` is the challenge side):

```
/-- **Theorem thm:one-prime (killed kernel)** of the family paper: for `T = (log Q)^(r+ε)`,
↪ `r ≥ 3`, `ε > 0`, there are
`Q₀` and `c > 0` such that for every integer `Q ≥ Q₀`
`(0.7235 - c·log log Q/log Q) · Σ_{1<q≤Q} Σ*_χ N_χ(T,2T) ≤ Σ_{1<q≤Q} Σ*_χ
↪ N^s_{0,χ}(T,2T)`.
No hypothesis. (In the library: `ZetaShell.LemmaK.theorem_one_prime_design`.) -/
theorem family_simple_on_line_killed (r ε : ℝ) (hr : 3 ≤ r) (hε : 0 < ε) :
  ∃ Q₀ c : ℝ, 0 < c ∧ ∀ Qn : ℕ, Q₀ ≤ (Qn : ℝ) →
    ((7235 : ℝ) / 10 ^ 4 - c * Real.log (Real.log (Qn : ℝ)) / Real.log (Qn : ℝ))
      * famN (modQle Qn) (twin (Qn : ℝ) r ε) (2 * twin (Qn : ℝ) r ε)
      ≤ famN0s (modQle Qn) (twin (Qn : ℝ) r ε) (2 * twin (Qn : ℝ) r ε) := by
  sorry

/-- **Theorem thm:shell (the Shell kernel)** of the family paper: assume the zero-density
↪ estimates (J) and (M)
(`ZeroDensityInput`). For `T = (log Q)^(r+ε)`, `r ≥ 3`, `ε > 0`, and `0 < θ < 503/1994`,
↪ there are `Q₀` and `c > 0`
such that for every integer `Q ≥ Q₀`
`(0.9059137927 - c·(log Q)^{-θ}) · Σ_{1<q≤Q} Σ*_χ N_χ(T,2T) ≤ Σ_{1<q≤Q} Σ*_χ
↪ N^s_{0,χ}(T,2T)`.
(In the library: `ZetaShell.shell_S53_qle_reduced`.) -/
theorem family_simple_on_line_shell (hD : ZeroDensityInput) (r ε θ : ℝ) (hr : 3 ≤ r) (hε :
  0 < ε)
  (hθ : 0 < θ) (hθ' : θ < 503 / 1994) :
  ∃ Q₀ c : ℝ, 0 < c ∧ ∀ Qn : ℕ, Q₀ ≤ (Qn : ℝ) →
    ((9059137927 : ℝ) / 10 ^ 10 - c * Real.log (Qn : ℝ) ^ (-θ))
      * famN (modQle Qn) (twin (Qn : ℝ) r ε) (2 * twin (Qn : ℝ) r ε)
```

```

≤ famN0s (modQle Qn) (twin (Qn : ℝ) r ε) (2 * twin (Qn : ℝ) r ε) := by
sorry

```

They are proved in `comparator/Solution/FollowUpFamily.lean` from `ZetaShell.LemmaK.theorem_one_prime_design` and `ZetaShell.shell.S53_qle_reduced`. To check them, run, after `lake exe cache get`,

```

lake build Challenge.FollowUpFamily Solution.FollowUpFamily
lake env lean comparator/PrintAxioms/FollowUpFamily.lean

```

The last command must print

```

'family_simple_on_line_killed' depends on axioms: [propext, Classical.choice, Quot.sound]
'family_simple_on_line_shell' depends on axioms: [propext, Classical.choice, Quot.sound]

```

as it did in the build of the release commit (output in `audit/followup/`); `#print axioms ZetaShell.shell.S53_qle_reduced` in a file that imports `ZetaShell` prints the line quoted in §A.4. The census files of §A.6 are in `audit/followup/` as well.

B Corrections found in this work

This appendix lists the corrections made to the working draft (its sections on the killed kernel and on the Shell route), to formal statements and to the preprint [9]. The statements of this paper are the corrected ones; “counterexample” means an explicit one, and “kernel-checked” that it is a Lean theorem.

B.1 Statements of the working draft that are false as written

1. *Lemma 4.2, far part.* The draft allowed any cutoff Ξ with $0 \leq \Xi \leq 1$ supported in $(-1, 1)$, including $\Xi \equiv 0$, for which the far part is all of a (numerical counterexample). Corrected by (9); the constant becomes $2s/(\pi^2 c\kappa)$. The same hypothesis is needed in the model mass of Lemma 4.6 (written out in its proof).
2. *Lemma 4.3(c), the spike.* False without $\delta < 1/(rQ)$; counterexample $a/r = 0/1$, $\delta = 1/Q$ for every $Q \geq 2$. Not used.
3. *Theorem 4.18 as stated in the draft.* The kernel on $(\log Q + 4, \mathcal{L}]$ is $(\mathcal{L} + \log K + O(1))/s$, not $|\alpha|$; the pointwise claim beyond α' fails on the top edge $(\log X - 1, \log X]$; “by the symmetry $s \mapsto -s$ ” is not a symmetry of $\mathbf{F}$. The corrected statement is integrated and uses the weight ω_K . None of this changes the constant or the rate.
4. *The condition on ϵ .* The draft required only $\epsilon \rightarrow 0$; Lemma 4.16 needs $\log(1/\epsilon) = O(\log \log Q)$, through R_1 in $\mathcal{Z}$. With $\epsilon = Q^{-c'}$ the bulk condition at $\alpha' = 2497/1500$ holds only for $c' < 0.0012$. The draft’s choice $\epsilon = 1/\mathcal{L}$ is fine.
5. *The rate.* The draft stated the error $(\log \log Q)^4 (\log Q)^{-503/1994}$; 503/1994 is the value at $\epsilon'' = 0$, and the proof gives $(\log Q)^{-\theta}$ for every $\theta < 503/1994$.
6. *Lemma 4.15.* “Every block has $R \geq 1$ ” holds only for nonempty blocks, and the cover needs $R_1 \leq N_0$ (numerical control).
7. *Lemma 4.12(2).* Needs $T \leq N_0$; without it no absolute constant exists (numerical). Part (3) with the Weil-form error needs the actual size of R_1 , not only $R_1 \leq Q/(2K)$.

Further items concern formal transcriptions, not the draft. The formal statement of Lemma 4.14 over the reals needs $j \in \mathbb{N}$ and $H \geq 0$, which were added as hypotheses (counterexamples $j = -3$; $H = -100$ with real powers). Three further false statements: the first Lean form of Lemma 3.3 put the last term inside the integral (kernel-checked refutation); the first Lean form of Lemma 4.3(a) was stated for all $Q \geq 2$ and is false at $Q = 2$, where $H_{\mathfrak{w}} = 0$, so the lemma is stated in eventual form; and the first Lean form of Lemma 4.8 (`ring_le_primitive`) was false as Lean parses it. In that form the body of the last integral extends to the end of the expression and takes in the trailing term $+2(\log K + C_\mu)U$, which so sits inside the integral and inside both sums (the parse is a Lean theorem, `frozen_rhs_parse`, proved by `rfl`); with $R_1 = 1$, $K = 3$, $C_\mu = 0$, $Q = 1000$, $a_n = e(-2n/Q)$ on the primes $1000 < n \leq 30000$ and $\mathcal{M} = S$, all hypotheses hold and the ring mass is 331.4, against 59.4 for the right side as parsed (numerical counterexample). The corrected statement `ring_le_primitive_corr` brackets the integral, and the first form was removed from the library. This is a correction of a Lean statement, not of the paper: the inequality displayed in Lemma 4.8, which has the term outside the integral and the sums, was never affected. Of the items above, the far part of Lemma 4.2, the transcription of Lemma 3.3, the first form of Lemma 4.3(a) and the first form of Lemma 4.8 were found by the formalisation, the others in the proofs on paper.

B.2 Missing hypotheses, constants and wording

1. Lemma 4.3(a) needs $R_1 \geq 1$; its closing sentence needs $N/\epsilon \leq Q^2(\log Q)^{-12}$; the per-class error in Step 2 is $2V^*$, not V^* ; the tail bound of Step 3 could not be reproduced and is replaced by one with the same conclusion; Step 4 holds up to $O(\delta(1 + \log Q))$.
2. Lemma 4.4(iv): the bound $\sum_e |c_e| 2^{\omega(e)} / e \leq \zeta(2)^2 / \zeta(4)$ holds for the plain kind and fails for the q/φ kind (3.2796); only convergence is used.
3. Lemma 4.13 needs $K \leq Q$ and $T \leq N_0$ for its tail step, and $N_0 \geq e^3$, which the “large N_0 ” of its proof and its formal statement use.
4. Lemma 4.2: the prime-power peak is $\ll T^2(\log N)^2 N^{-1/2}$, not $T(\log N)^2 N^{-1/2}$.
5. Lemma 4.8: the constant 1.4708 is numerical only; any $O(1 + \log K)$ bound suffices.
6. Proposition 4.11: the explicit formula is used in the Weil form, with the error E' in place of E . The draft’s remark that the explicit formula for Dirichlet L -functions is not in the Lean artifact of [1] is out of date. Lemma 4.12(1): “total $\ll T$ ” does not name the norm.
7. Theorem 4.17 is stated with $\|W\|_{C^A} T s_0$ on the right, with K and ϵ explicit; the draft’s $\int W_j \|a\|^2 \asymp g(j) T s_0$ in the transfer would need a lower bound on the mass of W_j .
8. The working draft’s remark that Tao’s inequality can be avoided needs $\epsilon \log Q \rightarrow 0$, which does not hold for $\epsilon = 1/\mathcal{L}$; the ring points have $q > Q/K$, which is enough.
9. Lemma 3.6 is proved only for $s \leq \log X$, the range used; the draft stated it for $N \geq Q$ (D3).
10. Lemma 3.7: the draft’s prime number theorem with error $y \exp(-c\sqrt{\log y})$ is replaced by the weaker (2), and its unverified Mertens constant $|R(x)| \leq 2$ by $\log 4 + 4$, so the bounds $E_2^2/U \leq 33.3 \lambda \mathcal{L}/T$ and $2E_2/\sqrt{U} \leq 16.4(\mathcal{L}/T)^{1/2}$ become $E_2^2/U \leq 86 \lambda \mathcal{L}/T$ and $2E_2/\sqrt{U} \leq 26.3(\mathcal{L}/T)^{1/2}$ (D4). The step $\sum_n |\tilde{\mathfrak{m}}(n)|^2 \geq U - 1$ was asserted without proof and is written out here.
11. Lemma 3.4(b) holds for all $r, r' \leq Q$; the draft’s $r + r' \leq Q$ is more than needed.

12. Theorem 1.1: the draft's 0.72375738 is at $\lambda = 32/25$, outside the formal design, and its statement that a profile at $\lambda = 5/4$ "fits the caps" does not hold, because the formal design fixes $\lambda = \lambda^*$ (D1); the design profile of [9] already gives 0.7235 (D2); a table entry was rounded rather than truncated (D5).
13. The bandwidth: the draft's open item on the uniformity in λ of the rows of [9, §§7–10] is resolved in §2.5 for $\mathfrak{F}_Q$ at $\lambda = 191/100$ (the dyadic family at $\lambda = 93/50$ was not examined, and its constant is not claimed; Remark 4.19); its list of formal caps was incomplete; the ends constant becomes $c_1 = 0.52$ with the Minkowski split; the moment floor stays at $3/4$ in eventual form; the ramp rate is $\Theta(1/\log Q)$; the zone slope is 1.9458 with certified Lipschitz constant 1.906 (a doc-string said 1); the Gevrey gate is raised to $5 \cdot 10^4$.
14. Lemma 4.16: the near bound holds, and is proved, for $0 \leq X_0 \leq s_0/5$, where the near zeros have $\beta \geq \frac{4}{5}$ and (LF) counts them. The draft stated it with no range of X_0 , and the first Lean form `Z6d_counts` for every $X_0 \geq 0$; for $X_0 > s_0/5$ the near sum contains zeros with $\beta < \frac{4}{5}$, and (M) gives only a larger exponent there (it would need $\varepsilon'' \geq 1 - 1/\alpha' \approx 0.399$). The corrected formal statement `Z6d_counts_corr` adds $5X_0 \leq s_0$ to the near clause only and is proved; the first form is kept, open and unused. Theorem 4.17 uses the bound at $X_0 = O(\log \log Q)$ only, which its proof checks. No constant or rate changes.
15. The pointwise assembly (13), formal statement. The first Lean form (`AS_pointwise`) allowed any $\alpha'' < 2$, independently of λ . Its ring is that of the near polynomial not cut at X , while $\mathbf{F}(s)$ sees $a(s)$ cut at X ; for $s + 1 > \lambda\mathcal{L} = \log X$ the argument of the assembly bounds the ring of the cut polynomial, which that ring does not control. No counterexample is known. The corrected form `AS_pointwise_corr` adds $\alpha'' < \lambda$ and is proved; the first form is kept, open and unused. Theorem 4.18 uses the formal assembly with $\alpha'' < 5/3 < 191/100 = \lambda$, and its formal statement is unchanged. In this paper the ring is that of $a(s)$, which is cut at X (§2.1), and the assembly is used only for $s \leq \alpha'\mathcal{L}$ with $\alpha' < \lambda$, where the cut and the uncut near polynomials agree, as they do in Theorem 4.17 through the condition $s_0 \leq \log X - 2$. In §4.1 and in the assembly the range of s also carries $s \leq \log X - 1$, which $\|a(s)\|^2 = U(s + O(1))$, Lemma 4.6 and the relative costs of Lemma 4.2 need, and which $\alpha'' < \lambda$ implies for large Q . No constant or rate changes.

B.3 Items for the preprint [9]

1. Its formalisation section says that the six headline theorems depend on one `sorry`, the sharp large-sieve constant. The large sieve is now proved in Gallagher's form $Q^2 + \pi N$, and the six theorems are kernel-checked with no `sorry`; the budget constant of [9, Lemma 4.3] becomes $1 + 4Q^{-\delta}$ instead of $1 + 2Q^{-\delta}$.
2. The sieve-budget row L_{11} should read $Q^{\lambda-2}(T/2\pi)^\lambda$.
3. The even and odd subfamilies now have the constants 0.7212 and 0.7098 of the full families, by the reflected large sieve; this is a Lean theorem on a branch of the formal tree, and it supersedes the preprint's 0.6980 and 0.6919.
4. The bibliography of [9] gives a different author for the source of H1–H6; this paper cites the arXiv listing [1]. For reference: the preprint's own text says of that source that it is not posted to a preprint server [9, §1.2]; the arXiv paper [1] states on its first page that its mathematical argument was discovered and written by Claude, an AI developed by Anthropic, and that the listed authors verified the proof and take responsibility for its content [V].

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