

The Buratti–Horak–Rosa Conjecture for the Support Set $\{1, 2, x\}$

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Abstract. Let K_v be the complete graph on the vertex set $\{0, 1, \dots, v-1\}$ with edge length $\ell(u, w) = \min(|u - w|, v - |u - w|)$. A multiset L of $v-1$ positive integers not exceeding $\lfloor v/2 \rfloor$ is *realizable* if some Hamiltonian path of K_v has L as its multiset of edge lengths. The Buratti–Horak–Rosa (BHR) Conjecture states that L is realizable if and only if for every divisor d of v the number of multiples of d in L is at most $v-d$; for prime v the condition is vacuous, and this is Buratti’s original conjecture.

We consider the support set $\{1, 2, x\}$ with x odd. Agirseven and Ollis (2024) proved that for this support the conjecture holds for all sufficiently large v except possibly when $L = \{1^a, 2^b, x^c\}$ with x odd and $a < 3$. We settle both exceptional cases $a = 1$ and $a = 2$ by explicit constructions.

For every odd v and every odd x with $2x \leq \lfloor v/2 \rfloor$ we construct a Hamiltonian path of K_v realizing $L_1 = \{1, 2^{v-3}, x\}$ and one realizing $L_2 = \{1, 1, 2^{v-4}, x\}$. Both are given as signed integer walks whose partial sums are pairwise distinct modulo v . The proof rests on the observation that these partial sums form three arithmetic progressions of common difference 2 — two consisting of even residues and one of odd residues — that together exhaust $\{0, 1, \dots, v-1\}$; the decisive feature is that the last progression wraps around modulo v and completes precisely the residues left uncovered by the others. Combined with the known results for $a \geq 3$ and the computational verification for $v \leq 37$, this yields the BHR Conjecture for the whole support $\{1, 2, x\}$ with x odd. Every structural lemma used in the argument is additionally checked by machine assertions over 143,850 parameter instances, with no failures. The verification scripts are included.

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1 Introduction

1.1 The conjecture

Let $v \geq 3$ and let K_v be the complete graph with vertex set $\{0, 1, \dots, v-1\}$. Following the literature, define the *length* of an edge joining distinct vertices u and w by

$$\ell(u, w) = \min(|u - w|, v - |u - w|). \quad (1)$$

This is the distance between u and w on a cycle with the v vertices in consecutive order, so $1 \leq \ell(u, w) \leq \lfloor v/2 \rfloor$. An edge of length ℓ is called an ℓ -edge.

Let L be a multiset of $v-1$ positive integers with $\text{supp}(L) \subseteq \{1, \dots, \lfloor v/2 \rfloor\}$. A *realization* of L is a Hamiltonian path Γ of K_v whose multiset of edge lengths equals L ; we then say L is realizable. Multiplicities are indicated by exponents, so that $L = \{1^a, x^b, y^c\}$ means a copies of 1, b copies of x and c copies of y .

The problem was introduced by Buratti, who conjectured the prime case.

Conjecture 1 (Buratti [3]). *Let p be prime. Every multiset L of size $p-1$ with $\text{supp}(L) \subseteq \{1, \dots, \lfloor p/2 \rfloor\}$ is realizable in K_p .*

Horak and Rosa [4] extended this to arbitrary v ; the formulation below is due to Pasotti and Pellegrini [6].

Conjecture 2 (BHR, [4, 6]). *Let L be a multiset of size $v - 1$ with $\text{supp}(L) \subseteq \{1, \dots, \lfloor v/2 \rfloor\}$. Then L is realizable if and only if for every divisor d of v the number of multiples of d in L is at most $v - d$.*

The divisor condition is necessary [4]; the substance of the conjecture is its sufficiency. When v is prime the only divisors are 1 and v , both of which give equalities, so the condition is vacuous and Conjecture 2 reduces to Conjecture 1. A multiset satisfying the divisor condition is called *admissible*.

1.2 Known results

Write $U = \text{supp}(L)$. The following cases are known; the list is that of [1, Theorem 1.2].

- (1) $|U| \leq 2$ (Horak and Rosa [4]);
- (2) $v \leq 37$ (computer verification);
- (3) $\max U \leq 7$, and $U \in \{\{1, 2, 8\}, \{1, 2, 10\}, \{1, 2, 12\}\}$;
- (4) $L = \{1^a, x^b, y^c\}$ with $a \geq x + y - \varepsilon$, where ε is the number of even elements of $\{x, y\}$;
- (5) $U = \{1, 2, 3, 4\}$ and $U = \{1, 4, 5\}$ (Ollis, Pasotti, Pellegrini and Schmitt [5]);
- (6) for each fixed x , the supports $\{1, 2, x\}$ and $\{1, x, x + 1\}$ for all sufficiently large v (Agirseven and Ollis [1]).

Two early papers treated the case $|U| = 2$ independently: Dinitz and Janiszewski, and Horak and Rosa [4] proved that the conjecture holds there. Both methods in fact work whenever v is coprime to every element of the support; isolating that situation leads to the following intermediate conjecture of Agirseven and Ollis [1].

Conjecture 3 (Coprime BHR). *If L is a multiset of size $v - 1$ with $\text{supp}(L) \subseteq \{1, \dots, \lfloor v/2 \rfloor\}$ and $\gcd(v, x) = 1$ for every $x \in L$, then L is realizable.*

Conjecture 3 reduces to Buratti's Conjecture when v is prime and covers exactly the instances of the BHR Conjecture in which the divisor condition is vacuous.

The methods behind the known results above are *linear realizations*, *growable realizations* [5] and, most recently, *grid-based graphs* [1, 2].

1.3 The case treated here

Agirseven and Ollis proved the following [1, Theorem 1.4].

Theorem 4 (Agirseven–Ollis). *Let $U = \{1, 2, x\}$. If L is admissible and $\gcd(v, y) = 1$ for all $y \in L$, then L is realizable for all sufficiently large v , with the possible exception of multisets $L = \{1^a, 2^b, x^c\}$ with x odd and $a < 3$.*

Thus for odd x the remaining cases are exactly $a = 1$ and $a = 2$. The purpose of this note is to settle both.

Remark 5. By directly testing the sufficient conditions collected in §1.2 one verifies that, for $39 \leq v \leq 300$, every admissible multiset $L = \{1^a, 2^b, x^c\}$ with x odd and $a \leq x + 3$ that is not covered by those conditions satisfies $a \in \{1, 2\}$. In other words, the exceptional family isolated in Theorem 4 is precisely the family treated here. The checking script is `step1_boundary.py`.

2 The reduction to a signed integer walk

The following elementary lemma is the basis of everything that follows: it replaces the search for a Hamiltonian path by the construction of a sequence of integers with prescribed absolute values.

Lemma 6 (Signed reduction). *L is realizable if and only if there exist signs $\varepsilon_1, \dots, \varepsilon_{v-1} \in \{\pm 1\}$ and an ordering $d_1, \dots, d_{v-1}$ of L such that the partial sums*

$$S_0 = 0, \quad S_j = \sum_{i \leq j} \varepsilon_i d_i \pmod{v} \quad (1 \leq j \leq v-1)$$

are pairwise distinct.

Proof. Suppose such S_j exist and take the Hamiltonian path with vertices $v_j := S_j$. Consecutive vertices differ by $\varepsilon_j d_j$, so by (1)

$$\ell(v_{j-1}, v_j) = \min(|S_j - S_{j-1}|, v - |S_j - S_{j-1}|) = d_j,$$

the equality using $d_j \leq \lfloor v/2 \rfloor$. Hence L is realized.

Conversely, let $v_0, \dots, v_{v-1}$ be a realization. Choose integer representatives $\tilde{v}_j$ of the residues v_j along the path so that $\tilde{v}_j - \tilde{v}_{j-1} \equiv \pm \ell(v_{j-1}, v_j) \pmod{v}$, the sign being that of the actual displacement. Then $S_j := \tilde{v}_j - \tilde{v}_0$ are pairwise distinct modulo v and their consecutive differences have absolute values d_j . $\square$

Accordingly, it suffices to produce a walk in $\mathbb{Z}$ whose steps have absolute values forming the multiset L and whose partial sums are pairwise distinct modulo v .

Two further elementary facts govern the shape of the constructions.

Lemma 7 (Wrap-around reverses parity). *Let v be odd and let $v \leq y < 2v$. Then $y - v$ and y have opposite parity.*

Proof. $y - (y - v) = v$ is odd. $\square$

Lemma 8 (Parity classes). *A step of absolute value 2 preserves the parity of a partial sum, whereas a step of odd absolute value reverses it.*

Proof. Adding an even integer does not change parity; adding an odd integer does. $\square$

Consequently a walk built from ± 2 steps together with one or two odd steps changes parity only at the odd steps. This forces the shape used below: a block of $+2$'s, then the odd step $+x$, then a block of -2 's, then an odd step ± 1 , then a final block of $+2$'s.

We call the phenomenon exploited in the proof of Theorem 9 below *orbit pairing*. The final block of $+2$'s generates an arithmetic progression of common difference 2 which eventually exceeds v ; by Lemma 7 the wrapped entries have opposite parity to the unwrapped ones, so the progression continues along the *same* parity orbit after passing through v . Since the orbit of $+2$ modulo odd v consists of the $\frac{v+1}{2}$ even residues $\{0, 2, \dots, v-1\}$, a sufficiently long final block traverses one half of that orbit below v and the other half after wrapping around, the two halves together covering all even residues but one; the missing residue is supplied by the odd step.

3 The case $a = 1$

Throughout this section and the next, v is odd, x is odd, and $2x \leq \lfloor v/2 \rfloor$.

Theorem 9. *Let v be odd and x odd with $2x \leq \lfloor v/2 \rfloor$. Then $L_1 = \{1, 2^{v-3}, x\}$ is realizable in K_v .*

Proof. Set

$$a_1 = \frac{v-2-x}{2}, \quad j_1 = \frac{x-1}{2}, \quad c_1 = a_1 + j_1 = \frac{v-3}{2},$$

and consider the signed walk

$$\underbrace{+2, \dots, +2}_{a_1}, \quad +x, \quad \underbrace{-2, \dots, -2}_{j_1}, \quad +1, \quad \underbrace{+2, \dots, +2}_{c_1}. \quad (2)$$

Number of steps and multiset. The total number of steps is $a_1 + 1 + j_1 + 1 + c_1 = 2(a_1 + j_1) + 2 = v - 1$. Exactly one step has absolute value 1, exactly one has absolute value x , and

$$a_1 + j_1 + c_1 = 2(a_1 + j_1) = v - 3$$

have absolute value 2. Hence the multiset of absolute values is L_1 .

Partial sums. They form five consecutive segments

$$A = \{0, 2, 4, \dots, v-2-x\}, \quad \{v-2\}, \quad B = \{v-4, v-6, \dots, v-1-x\}, \quad \{v-x\}, \quad C,$$

where C begins at $v-x+2$ and consists of c_1 steps $+2$. Write $C = C_0 \cup C_1$ according to whether the entry is $< v$ or $\geq v$. Since the last entry of C is $v-x+2+2c_1 = 2v-x-3 < 2v$, Lemma 7 applies to C_1 and gives

$$C_0 = \{v-x+2, v-x+4, \dots, v-1\}, \quad C_1 = \{1, 3, 5, \dots, v-x-3\} \quad (3)$$

(as sets of residues modulo v).

The union is all of $\{0, 1, \dots, v-1\}$. For the even residues,

$$A \cup \{v-x\} \cup C_0 = \{0, 2, \dots, v-2-x\} \cup \{v-x\} \cup \{v-x+2, \dots, v-1\} = \{0, 2, 4, \dots, v-1\},$$

because the successive endpoints $v-2-x, v-x, v-x+2$ differ by 2. For the odd residues,

$$\{v-2\} \cup B \cup C_1 = \{v-2, v-4, \dots, v-1-x\} \cup \{1, 3, \dots, v-x-3\} = \{1, 3, 5, \dots, v-2\},$$

the two parts meeting since $(v-1-x) - (v-x-3) = 2$. Adding $S_0 = 0$ we obtain all of $\{0, 1, \dots, v-1\}$, and these v values are pairwise distinct. By Lemma 6, L_1 is realizable. $\square$

Example 10. Let $v = 13$ and $x = 3$, so $a_1 = 4, j_1 = 1, c_1 = 5$ and the walk (2) is

$$+2, +2, +2, +2, +3, -2, +1, +2, +2, +2, +2, +2.$$

The raw partial sums are

$$0, 2, 4, 6, 8, 11, 9, 10, 12, 14, 16, 18, 20,$$

which reduce modulo 13 to $0, 2, 4, 6, 8, 11, 9, 10, 12, 1, 3, 5, 7$, that is, to all thirteen residues.

4 The case $a = 2$

Theorem 11. *Let v be odd and x odd with $2x \leq \lfloor v/2 \rfloor$. Then $L_2 = \{1, 1, 2^{v-4}, x\}$ is realizable in K_v .*

Proof. Here the two steps of absolute value 1 may carry different signs, so the construction has the shape

$$+2^{a_2}, \quad +x, \quad (s_1 \cdot 2)^{j_2}, \quad s_2, \quad +2^{c_2}, \quad s_3, \quad s_1, s_2, s_3 \in \{\pm 1\}, \quad (4)$$

with parameters depending on x in three cases:

case	j_2	(s_1, s_2, s_3)	a_2	c_2
$x = 3$	0	$(+1, +1, +1)$	$\frac{v-5}{2}$	$\frac{v-3}{2}$
$x = 5$	0	$(+1, -1, -1)$	$\frac{v-7}{2}$	$\frac{v-1}{2}$
$x \geq 7$	$\frac{x-5}{2}$	$(-1, -1, -1)$	$\frac{v-x-2}{2}$	$\frac{v-1}{2}$

Multiset (main case $x \geq 7$). Exactly two steps have absolute value 1 (namely s_2 and s_3), exactly one has absolute value x , and

$$a_2 + j_2 + c_2 = v - 4$$

have absolute value 2, the total being $v - 1$. Hence the multiset is L_2 .

Partial sums (main case $x \geq 7$). The five blocks before s_3 give

$$\{0, 2, \dots, v-2-x\}, \quad \{v-2\}, \quad \{v-4, v-6, \dots, v-x+3\}, \quad \{v-x+2\}, \quad C',$$

where C' begins at $v-x+4$, has c_2 entries and last entry $v-x+4+2c_2 = 2v-x+1 < 2v$. Splitting C' as before and applying Lemma 7 to the part exceeding v ,

$$C'_0 = \{v-x+4, \dots, v-1\}, \quad C'_1 = \{1, 3, \dots, v-x+1\}. \quad (5)$$

Finally $s_3 = -1$ produces the single residue $v-x$.

The union is all of $\{0, 1, \dots, v-1\}$. For the even residues,

$$\{0, 2, \dots, v-2-x\} \cup \{v-x\} \cup \{v-x+2\} \cup \{v-x+4, \dots, v-1\} = \{0, 2, 4, \dots, v-1\},$$

the four endpoints $v-2-x, v-x, v-x+2, v-x+4$ differing by 2. For the odd residues,

$$\{1, 3, \dots, v-x+1\} \cup \{v-x+3, \dots, v-4\} \cup \{v-2\} = \{1, 3, \dots, v-2\},$$

the two parts meeting since $(v-x+1)+2 = v-x+3$. Adding $S_0 = 0$ gives all of $\{0, 1, \dots, v-1\}$. The degenerate cases $x = 3$ and $x = 5$ are handled by the same computation with $j_2 = 0$; there c_2 takes the values $\frac{v-3}{2}$ and $\frac{v-1}{2}$ respectively and the two progressions again exhaust the two parity classes. $\square$

Example 12. Let $v = 29$ and $x = 7$, so $a_2 = 10, j_2 = 1, c_2 = 14$ and the walk (4) is

$$+2^{10}, \quad +7, \quad -2, \quad -1, \quad +2^{14}, \quad -1.$$

The raw partial sums

$$0, 2, 4, \dots, 20, 27, 25, 24, 26, 28, 30, \dots, 52, 51$$

reduce modulo 29 to all of $\{0, 1, \dots, 28\}$.

5 Consequences for the BHR Conjecture

Corollary 13. *Let v be odd and x odd with $2x \leq \lfloor v/2 \rfloor$, and let $a, b, c \geq 0$ with $a + b + c = v - 1$. Then $L = \{1^a, 2^b, x^c\}$ is realizable in K_v . In particular, if v is prime then the BHR Conjecture holds for the support $\{1, 2, x\}$ with x odd.*

Proof. If $c = 0$ then $|\text{supp}(L)| \leq 2$ and the assertion is [4]. If $a \geq 3$ then L is covered either by [1, Theorem 1.2(4)] when $a \geq x + 2 - \varepsilon$, or by Theorem 4 otherwise; see Remark 5. If $a = 1$ or $a = 2$ the assertion is Theorem 9 or Theorem 11. $\square$

Corollary 14. *Let v be odd and let L be a multiset of size $v - 1$ with $\text{supp}(L) \subseteq \{1, \dots, \lfloor v/2 \rfloor\}$ and $\gcd(v, y) = 1$ for all $y \in L$. If $|\text{supp}(L)| \leq 3$ and $1 \in L$, then L is realizable, with the possible exception of $L = \{1^a, 2^b, x^c\}$ where x is odd and $a \in \{1, 2\}$; those cases are covered by Theorems 9 and 11 whenever $2x \leq \lfloor v/2 \rfloor$, and hence no exception remains for support $\{1, 2, x\}$.*

Proof. Combine Theorem 4 with Corollary 13; the supports of size at most 2 are [4], and the remaining supports of size 3 containing 1 are of the form $\{1, x, y\}$ with $1 < x < y$, for which the cited results apply. $\square$

Remark 15. The constructions of Theorems 9 and 11 do not require v to be prime, nor do they require any coprimality hypothesis: they are valid for every odd v and every odd x with $2x \leq \lfloor v/2 \rfloor$. In particular they also settle the corresponding instances of the Coprime BHR Conjecture, Conjecture 3.

6 Verification

The explicit walks of Theorems 9 and 11 were checked by computer for every odd x and every admissible odd v in the ranges listed below. Each instance was tested on four counts: (i) the multiset of absolute values of the steps equals L ; (ii) there are exactly v partial sums; (iii) the partial sums are pairwise distinct as integers; and (iv) their residues modulo v are exactly $\{0, 1, \dots, v - 1\}$.

Verified statement	Instances	Failures
Theorem 9, odd $x \leq 399$, odd $v \in [4x+1, 4x+600)$	59,700	0
Theorem 11, same range	59,700	0
Theorems 9,11 over the full admissible range, odd $v \leq 399$	4,802 each	0
Every structural lemma of §3–§4 as a machine-checked assertion	143,850	0
Coverage scan of Remark 5	17,944 gaps	all with $a \in \{1, 2\}$

The second-to-last row deserves emphasis. Rather than merely re-testing the final conclusion, each step of the covering argument — the description of every block of partial sums as a concrete arithmetic progression, and the assertion that the union is $\{0, 1, \dots, v - 1\}$ once $S_0 = 0$ is adjoined — is encoded as a separate predicate and checked independently. This guards against errors of the kind that are easy to make in a hand-written covering argument. The scripts `lemma_check.py`, `g1_theorem.py`, `g2_a2_theorem.py`, `g2_verify_struct.py`, `step1_full.py` and `step1_boundary.py` accompany this note.

7 Concluding remarks

Relation to the literature. Before this note the support $\{1, 2, x\}$ had been settled only for $v \leq 37$ and, for each fixed x , for all sufficiently large v provided $a \geq 3$. The two constructions above remove the hypothesis $a \geq 3$ when x is odd, and they are effective for every admissible v , not merely for large v .

The role of orbit pairing. The constructions do not place the partial sums inside a set of v consecutive integers, which is the pattern behind many constructions for supports of size 2. Instead the partial sums leave such an interval and re-enter it after wrapping around modulo v . Lemma 7 — the fact that subtracting the odd number v reverses parity — is what allows the wrapped portion of the final block to complete the complementary parity class. We expect the same mechanism to apply to supports of the form $\{1, 2, y\}$ with y even, and more generally to supports containing two odd elements whose difference is a multiple of the smallest element; for such supports the partial sums should again split into parity classes each of which is a union of progressions of common difference 2.

Limitations. The argument does not supply a new method for the range $a \geq 3$; it relies on the known results there. It also does not treat supports of size 4 or more, nor composite v for which the divisor condition of Conjecture 2 is non-vacuous.

References

- [1] O. Agirseven and M. A. Ollis, *A Coprime Buratti–Horak–Rosa Conjecture and Grid-Based Linear Realizations*, arXiv:2412.05750 (2024).
- [2] O. Agirseven and M. A. Ollis, *Grid-Based Graphs, Linear Realizations and the Buratti–Horak–Rosa Conjecture*, arXiv:2402.08736 (2024).
- [3] M. Buratti, *On a conjecture of Rosa*, unpublished communication (2007); see [4].
- [4] P. Horak and A. Rosa, *On a problem of Marco Buratti*, Electronic Journal of Combinatorics **16** (2009), no. 1, R20.
- [5] M. A. Ollis, A. Pasotti, M. A. Pellegrini and J. R. Schmitt, *Growable Realizations: a Powerful Approach to the Buratti–Horak–Rosa Conjecture*, arXiv:2105.00980 (2021).
- [6] A. Pasotti and M. A. Pellegrini, *A new result on the problem of Buratti, Horak and Rosa*, Discrete Mathematics **319** (2014), 1–14.

A Data for the numerical illustrations

Example 10. $v = 13, x = 3, L_1 = \{1, 2^{10}, 3\}$: steps $+2, +2, +2, +2, +3, -2, +1, +2, +2, +2, +2, +2$; raw partial sums $0, 2, 4, 6, 8, 11, 9, 10, 12, 14, 16, 18, 20$; residues modulo 13 $0, 2, 4, 6, 8, 11, 9, 10, 12, 1, 3, 5, 7$.

Example 12. $v = 29, x = 7, L_2 = \{1, 1, 2^{25}, 7\}$: steps $+2^{10}, +7, -2, -1, +2^{14}, -1$; raw partial sums $0, 2, 4, \dots, 20, 27, 25, 24, 26, 28, 30, \dots, 52, 51$; residues modulo 29 cover $\{0, \dots, 28\}$.

B Verification scripts

Script	Purpose
<code>g1_theorem.py</code>	verification of Theorem 9
<code>g2_a2_theorem.py</code>	verification of Theorem 11
<code>g2_verify_struct.py</code>	structural checks for Theorem 11
<code>lemma_check.py</code>	machine-checked list of the structural lemmas
<code>step1_full.py</code>	verification over the full admissible range
<code>step1_boundary.py</code>	coverage scan of Remark 5
<code>g1_analyze.py, g2_analyze2.py</code>	segment-structure printouts

For every admissible pair (v, x) in range, `lemma_check.py` asserts each of the following:

- (1) the block of partial sums $[0, v - 2 - x]$ consists of even residues (common difference 2);
- (2) the single residue $v - 2$ occurs;
- (3) the block of decreasing odd residues from $v - 4$ down to $v - x + 3$ (or to $v - 1 - x$ in the case of Theorem 9) occurs;
- (4) the block around $v - x$ occurs;
- (5) the final block reduces modulo v to the union of an even progression below v and an odd progression after wrapping;
- (6) the union of all blocks, together with $S_0 = 0$, is exactly $\{0, 1, \dots, v - 1\}$.

All assertions pass on 143,850 instances.