

Finite Algebraic Non-Compressibility of the Planar Equal-Mass Newtonian Three-Body Flow

Exact polynomial realization, invariant-ideal quotient, and modular Lie-jet certification

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October 2026

Abstract

We give a reproducible finite-degree algebraic obstruction to polynomial compression of an orbit germ of the planar equal-mass Newtonian three-body flow. The question is deliberately narrower than global integrability or the existence of a closed-form solution: after quotienting the exact geometric distance constraints and the fixed energy and angular-momentum levels, can a nonzero polynomial of bounded total degree vanish identically on the orbit germ? We answer this question by an exact polynomial realization, a filtered invariant quotient, and a finite matrix of Lie derivatives evaluated at an exact rational witness. For the asymmetric witness

$$u = (3, 0), \quad v = (0, 4), \quad U = (1, 2), \quad V = (-1, 1), \quad (c_u, c_v, c_w) = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right),$$

the degree-four quotient has dimension 1207. An executable modular implementation regenerates the 1207×1207 Taylor-jet matrix directly from the polynomial vector field and obtains rank 1207 modulo both 1 000 003 and 1 000 033. Since a nonzero maximal minor modulo one good prime implies the corresponding characteristic-zero minor is nonzero, this is an exact characteristic-zero certificate. Independently, the same Lie-jet construction on the three squared mutual-distance variables has full rank through total degree 10, with dimensions 4, 10, 20, 35, 56, 84, 120, 165, 220, 286. A nonzero-minor argument yields a componentwise genericity statement for each fixed degree. The constituent mathematics is standard differential algebra, Lie differentiation, and modular linear algebra; the contribution is their explicit, invariant-aware, machine-reproducible assembly into a finite orbit-germ certificate.

Keywords. three-body problem; Newtonian dynamics; polynomialization; differential algebra; invariant ideals; orbit germs; Lie derivatives; jet matrices; modular arithmetic; algebraic non-compressibility.

1 Introduction

The Newtonian three-body problem has a long history of negative results concerning additional first integrals and Hamiltonian integrability. Bruns established a classical obstruction for algebraic first integrals [1]; later differential-Galois work gave stronger non-integrability criteria in meromorphic Hamiltonian settings, including the equal-mass problem [2, 3]. Those results answer questions about integrability of the dynamical system as a whole.

This paper asks a different and finite question. Fix a collision-free orbit germ and a chosen collection of known algebraic invariants. After quotienting by those relations, does the orbit germ

still admit a nonzero polynomial relation of total degree at most D ? Such a relation would place the orbit germ inside an additional algebraic hypersurface in the chosen state coordinates. The statement is local in time, finite in degree, and does not assert the existence or non-existence of a global solution formula.

The mathematical ingredients used below are standard. In particular, the matrix built from

$$P \mapsto (P(z_*), \mathcal{D}P(z_*), \dots, \mathcal{D}^{M-1}P(z_*))$$

is simply a finite Lie-derivative/Taylor-jet matrix. No new calculus is claimed for this operation. The point of the present construction is instead to make the obstruction exact, invariant-aware, and reproducible: the algebraic constraints are handled before the jet test, the witness is rational, the rank certificate is modular, and the full symbolic data needed to regenerate the certificate are supplied with the manuscript.

The main full-state certificate is at total degree $D = 4$. After quotienting by three distance constraints and the fixed values of energy and angular momentum, the filtered quotient has dimensions

$$1, 12, 76, 340, 1207$$

for $D = 0, 1, 2, 3, 4$. At $D = 4$ a 1207×1207 Lie-jet matrix is nonsingular modulo two independent primes. Independently, in the three-variable algebra generated by squared mutual distances, full rank holds for every degree $D \leq 10$.

The results are intentionally finite. They do not prove non-integrability in the Morales-Ramis sense, do not exclude transcendental parametrizations, do not exclude higher-degree algebraic relations, and do not claim that every three-body orbit has the same algebraic behavior. What is proved is a finite-degree certificate at one exact witness together with a fixed-degree componentwise genericity consequence.

2 Exact polynomial realization

Let three equal unit masses move in the plane with $G = 1$. Introduce relative coordinates

$$u = r_2 - r_1, \quad v = r_3 - r_1, \quad w = v - u, \tag{1}$$

and velocities

$$U = \dot{u}, \quad V = \dot{v}. \tag{2}$$

Write

$$c_u = |u|^{-1}, \quad c_v = |v|^{-1}, \quad c_w = |w|^{-1}. \tag{3}$$

The collision-free Newtonian relative equations are

$$\mathcal{D}u = U, \tag{4}$$

$$\mathcal{D}v = V, \tag{4}$$

$$\mathcal{D}U = -2c_u^3 u + c_w^3 w - c_v^3 v, \tag{5}$$

$$\mathcal{D}V = -2c_v^3 v - c_w^3 w - c_u^3 u. \tag{6}$$

Differentiating the inverse-distance variables gives

$$\mathcal{D}c_u = -c_u^3(u \cdot U), \tag{7}$$

$$\mathcal{D}c_v = -c_v^3(v \cdot V), \tag{8}$$

$$\mathcal{D}c_w = -c_w^3(w \cdot (V - U)). \tag{9}$$

Let

$$R = \mathbb{Q}[u_1, u_2, v_1, v_2, U_1, U_2, V_1, V_2, c_u, c_v, c_w]$$

and define

$$g_u = c_u^2(u \cdot u) - 1, \quad (10)$$

$$g_v = c_v^2(v \cdot v) - 1, \quad (11)$$

$$g_w = c_w^2(w \cdot w) - 1. \quad (12)$$

The differential operator $\mathcal{D}$ is the polynomial derivation determined by the displayed first-order equations.

Proposition 2.1 (Exact differential realization). *The ideal*

$$I_{\text{dist}} = \langle g_u, g_v, g_w \rangle$$

is differential, with

$$\mathcal{D}g_u = -2c_u^2(u \cdot U)g_u, \quad (13)$$

$$\mathcal{D}g_v = -2c_v^2(v \cdot V)g_v, \quad (14)$$

$$\mathcal{D}g_w = -2c_w^2(w \cdot (V - U))g_w. \quad (15)$$

Proof. For the first generator,

$$\mathcal{D}(c_u^2 u \cdot u) = 2c_u(\mathcal{D}c_u)(u \cdot u) + 2c_u^2(u \cdot U).$$

Substituting $\mathcal{D}c_u = -c_u^3(u \cdot U)$ gives

$$\mathcal{D}g_u = 2c_u^2(u \cdot U)(1 - c_u^2 u \cdot u) = -2c_u^2(u \cdot U)g_u.$$

The other identities are identical after replacing (u, U) by (v, V) and $(w, V - U)$. The ideal inclusion follows from the Leibniz rule. $\square$

Proposition 2.2 (Equivalence on the physical branch). *If $g_u = g_v = g_w = 0$ and $c_u, c_v, c_w > 0$, then*

$$c_u = |u|^{-1}, \quad c_v = |v|^{-1}, \quad c_w = |w|^{-1},$$

and the polynomial vector field is exactly the collision-free Newtonian vector field in relative coordinates. Conversely, every collision-free Newtonian state on the positive inverse-distance branch satisfies the three polynomial constraints.

Proof. From $c_u^2|u|^2 = 1$ one has $|c_u||u| = 1$. Positivity gives $c_u = |u|^{-1}$, and likewise for c_v, c_w . Substitution recovers the Newtonian equations. The converse is immediate from the definitions of the inverse distances. $\square$

Remark 2.3 (Collision and localization). *The physical branch excludes $u = 0$, $v = 0$, and $w = 0$. The identities $c_u^2(u \cdot u) = 1$, etc., already force the three squared separations to be nonzero in the constrained algebra. Thus no extra localization is required merely to state the polynomial system. Localization is useful only when one changes to a rational reduced coordinate description or isolates a particular semialgebraic branch.*

3 Energy, angular momentum, and the invariant quotient

In the center-of-mass frame, the energy is

$$E = \frac{1}{3} (|U|^2 + |V|^2 - U \cdot V) - c_u - c_v - c_w, \quad (16)$$

and the scalar angular momentum is

$$L = \frac{2}{3} (u \times U + v \times V) - \frac{1}{3} (u \times V + v \times U), \quad (17)$$

where $(a_1, a_2) \times (b_1, b_2) = a_1 b_2 - a_2 b_1$.

Proposition 3.1 (Conservation).

$$\mathcal{D}E = 0, \quad \mathcal{D}L = 0.$$

Hence, for fixed constants E_0, L_0 ,

$$I_{E,L} = \langle g_u, g_v, g_w, E - E_0, L - L_0 \rangle \quad (18)$$

is a differential ideal.

Proof. Differentiation of the kinetic energy using the relative accelerations produces the negative of the potential derivative. Specifically,

$$\mathcal{D}(-c_u - c_v - c_w) = c_u^3(u \cdot U) + c_v^3(v \cdot V) + c_w^3(w \cdot (V - U)),$$

which cancels the work terms. For L , differentiating the bilinear expression leaves only internal torque terms, and each Newtonian pair force is parallel to its separation vector, so the internal torques cancel. Therefore both quantities are conserved. Together with Proposition 1, this gives differential invariance of $I_{E,L}$. $\square$

For the exact rational witness below it is convenient to clear denominators only in the computational representation. Define

$$e = 20(|U|^2 + |V|^2 - U \cdot V) - 60(c_u + c_v + c_w) - 73, \quad (19)$$

$$\ell = 2(u_1 U_2 - u_2 U_1 + v_1 V_2 - v_2 V_1) \quad (20)$$

$$- (u_1 V_2 - u_2 V_1 + v_1 U_2 - v_2 U_1) - 21. \quad (21)$$

Then

$$e = 60(E - 73/60), \quad \ell = 3(L - 7).$$

The exact Groebner computation used the five generators g_u, g_v, g_w, e, ℓ .

Remark 3.2 (First integrals versus orbit relations). *A first integral is constant along every trajectory in its domain. The relation studied here is weaker and different: a polynomial P may vanish on one orbit germ without being constant on neighboring trajectories. Consequently a result about the non-existence of additional first integrals does not by itself settle the present finite orbit-relation question, and the present certificate should not be described as a restatement of Bruns's theorem.*

Remark 3.3 (Prime and radical issues). *No assertion that $I_{E,L}$ is prime or radical is required. The finite certificate uses the filtered vector-space quotient*

$$\mathcal{N}_D = F_D R / (F_D R \cap I_{E,L}),$$

where $F_D R$ denotes the polynomials of total degree at most D . Componentwise genericity is formulated on irreducible components of the algebraic zero set, avoiding any implicit prime-ideal assumption.

4 Lie-jet obstruction

Let z denote the eleven polynomial state variables. Away from collision, the polynomial vector field has a local analytic flow Φ_t . For $P \in R$,

$$\frac{d}{dt}P(\Phi_t(z)) = (\mathcal{D}P)(\Phi_t(z)), \quad (22)$$

and inductively

$$\frac{d^n}{dt^n}P(\Phi_t(z)) = (\mathcal{D}^n P)(\Phi_t(z)). \quad (23)$$

Definition 4.1 (Filtered quotient and anchored Lie-jet matrix). *Let*

$$F_D R = \{P \in R : \deg P \leq D\}, \quad \mathcal{N}_D = F_D R / (F_D R \cap I_{E,L}).$$

Choose a basis $[f_1], \dots, [f_M]$ of $\mathcal{N}_D$. The anchored Lie-jet matrix at z_ is*

$$J_D(z_*) = [\mathcal{D}^n f_j(z_*)]_{0 \leq n < M, 1 \leq j \leq M}. \quad (24)$$

Lemma 4.2 (Taylor coefficients have the same rank). *Let $a_{n,j}$ be the coefficient of t^n in $f_j(\Phi_t(z_*))$. Then*

$$a_{n,j} = \frac{1}{n!} \mathcal{D}^n f_j(z_*).$$

If the coefficient field has characteristic $p > M$, the Taylor-coefficient matrix $A_D = [a_{n,j}]$ has the same rank as J_D .

Proof. The first identity is the Taylor formula. Multiplying row n by the nonzero scalar $n!$ converts A_D into J_D . When $p > M$, all factorials $0!, \dots, (M-1)!$ are nonzero modulo p . $\square$

Theorem 4.3 (Finite orbit-germ obstruction). *If*

$$\text{rank } J_D(z_*) = M,$$

then there is no polynomial $P \in F_D R$ whose class in $\mathcal{N}_D$ is nonzero and for which

$$P(\Phi_t(z_*)) \equiv 0$$

for all sufficiently small t .

Proof. If $P(\Phi_t(z_*)) \equiv 0$, analyticity gives $\mathcal{D}^n P(z_*) = 0$ for every n . Writing $[P] = \sum_j a_j [f_j]$ with $a \neq 0$ gives

$$J_D(z_*)a = 0,$$

contradicting full rank. $\square$

Remark 4.4 (Finite differential-polynomial scope). *The phase system is autonomous and first order: $\dot{q} = X(q)$. Thus $q^{(n)} = X^n(q)$, so a finite differential polynomial in phase derivatives reduces to an algebraic/rational expression in the state variables after imposing the vector-field graph relations. The present theorem therefore addresses finite polynomial relations in the exact polynomialized state, not an independent hierarchy of derivative orders.*

5 Exact asymmetric rational witness

Take

$$u = (3, 0), \quad v = (0, 4), \quad U = (1, 2), \quad V = (-1, 1). \quad (25)$$

Then

$$w = (-3, 4), \quad (c_u, c_v, c_w) = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right).$$

For the squared shape variables

$$x = u \cdot u, \quad y = v \cdot v, \quad z = u \cdot v,$$

the witness has

$$(x, y, z) = (9, 16, 0), \quad (\dot{x}, \dot{y}, \dot{z}) = (6, 8, 5).$$

The exact invariants are

$$E = \frac{73}{60}, \quad L = 7. \quad (26)$$

The triangle is nondegenerate because

$$16A^2 = 4(xy - z^2) = 576, \quad (27)$$

so $A = 6$.

The Jacobian of

$$(g_u, g_v, g_w, E - 73/60, L - 7)$$

with respect to $(u_1, u_2, v_1, v_2, U_1)$ has exact determinant

$$\frac{2}{25} \neq 0. \quad (28)$$

Thus the implicit-function theorem gives a six-dimensional smooth real constraint manifold through the witness. This check is independent of the later jet-rank computation.

The witness is intentionally asymmetric. It is not selected from a symmetry-reduced periodic orbit, and it avoids the collinear locus. Its role is purely to provide an exact admissible point at which a maximal Lie-jet minor can be certified.

6 Degree-four quotient certificate

For eleven variables, the number of raw monomials of total degree at most D is

$$\dim F_D R = \binom{11 + D}{D}.$$

Using graded reverse lexicographic order in the variable order

$$(u_1, u_2, v_1, v_2, U_1, U_2, V_1, V_2, c_u, c_v, c_w),$$

the exact rational Groebner basis of the five-generator invariant ideal contains 47 polynomials and has maximum total degree 11. The complete basis is included in the reproducibility artifact `certificate_data/groebner_basis.txt`; the corresponding degree-four standard-monomial list is `certificate_data/standard_monomials_D4.txt`.

Table 1: Exact filtered quotient dimensions.

D	$\dim F_D R$	$\dim \mathcal{N}_D$	$\dim(F_D R \cap I_{E,L})$
0	1	1	0
1	12	12	0
2	78	76	2
3	364	340	24
4	1365	1207	158

The degree-four deficit of 158 can be audited independently. The two degree-two invariant relations e, ℓ contribute two relations at $D = 2$. At $D = 3$, their products with the 12 monomials of degree at most one give 24 independent relations. At $D = 4$, their degree-at-most-two multiples contribute 156 before the single scalar syzygy $\ell e - e\ell = 0$, leaving 155 independent invariant relations; the three degree-four distance constraints add three more. Thus the predicted deficit is 158, matching the exact Groebner calculation. The Groebner basis, rather than this hand count, is the definitive computation.

Let $M = 1207$. The quotient basis consists of the 1207 standard monomials of degree at most four. The jet orders are $n = 0, 1, \dots, 1206$.

Table 2: Modular rank certificate for the full degree-four quotient.

Prime p	Matrix size	Rank modulo p	Result
1,000,003	1207×1207	1207	PASS
1,000,033	1207×1207	1207	PASS

Theorem 6.1 (Degree-four characteristic-zero certificate). *At the exact witness, the degree-four quotient Lie-jet map has full rank 1207 over $\mathbb{Q}$. Hence no nonzero class in*

$$F_4 R / (F_4 R \cap I_{E,L})$$

vanishes identically on the corresponding orbit germ.

Proof. The reproducibility implementation constructs the Taylor-coefficient matrix modulo each stated prime and performs exact modular Gaussian elimination. By the previous lemma, this coefficient matrix has the same rank as the derivative matrix because $1206!$ is nonzero modulo either prime. A rank of 1207 supplies a nonzero maximal minor modulo the prime. If the corresponding characteristic-zero determinant were zero, every good-prime reduction of that determinant would also be zero. Therefore the determinant is nonzero over $\mathbb{Q}$, and the characteristic-zero rank is 1207. The second prime is an independent implementation check rather than an additional logical requirement. $\square$

Remark 6.2 (No rank semicontinuity argument). *The modular implication is a direct determinant argument. No statement about rank semicontinuity under specialization is needed.*

7 Witness-to-genericity for fixed degree

Let

$$V = V(I_{E,L}) \subseteq \mathbb{A}_{\mathbb{Q}}^{11}$$

and fix an irreducible component C on which the algebraic determinant under consideration is not identically zero. For a fixed degree D , choose a quotient basis and write

$$\Delta_D(z) = \det J_D(z).$$

Theorem 7.1 (Componentwise generic fixed-degree obstruction). *Suppose $\Delta_D(z_*) \neq 0$ at a point $z_* \in C$. Then Δ_D is not identically zero on C , and*

$$\mathcal{E}_D = \{z \in C : \Delta_D(z) = 0\}$$

is a proper Zariski-closed subset of C . Every point of $C \setminus \mathcal{E}_D$ has no nonzero polynomial orbit-germ relation of total degree at most D modulo $I_{E,L}$.

Proof. Every entry $\mathcal{D}^n f_j(z)$ is polynomial in the polynomialized state variables because $\mathcal{D}$ is a polynomial derivation. Therefore Δ_D is a polynomial function. Since it is nonzero at z_* , it is not the zero function on C . Its zero set is consequently proper and Zariski closed. At points where $\Delta_D \neq 0$, the finite orbit-germ obstruction theorem applies. $\square$

Corollary 7.2 (Real exceptional set). *On any smooth real component where Δ_D is not identically zero, the real exceptional set has empty relative interior and zero induced volume.*

Proof. A nonzero polynomial restricted to a smooth real manifold cannot vanish on a nonempty open set. Its zero locus is lower-dimensional away from singular points and therefore has zero induced volume. $\square$

Remark 7.3 (Exactly what is generic). *The statement is fixed-degree and componentwise. It does not assert that*

$$\bigcup_{D \geq 0} \mathcal{E}_D$$

is a proper algebraic subset, nor does it imply an all-degree non-compressibility theorem.

8 Independent shape-only certificate through degree ten

Define the squared mutual-distance variables

$$x = u \cdot u, \quad y = v \cdot v, \quad z = u \cdot v. \tag{29}$$

A shape polynomial is an element of $\mathbb{Q}[x, y, z]$. The number of monomials of total degree at most D is

$$N_D = \binom{D+3}{3}.$$

Using the full eleven-variable polynomial vector field to generate the exact Taylor series of $x(t), y(t), z(t)$, and then evaluating the shape monomial basis, gives an independent three-variable certificate.

Proposition 8.1 (Shape non-compressibility through degree ten). *At the exact witness, there is no nonzero $H \in \mathbb{Q}[x, y, z]$ of total degree at most 10 such that*

$$H(x(t), y(t), z(t)) \equiv 0$$

near the witness.

Table 3: Shape-space ranks. Every rank below was regenerated modulo both stated primes.

D	N_D	rank mod 1,000,003	rank mod 1,000,033
1	4	4	4
2	10	10	10
3	20	20	20
4	35	35	35
5	56	56	56
6	84	84	84
7	120	120	120
8	165	165	165
9	220	220	220
10	286	286	286

Proof. For each $D \leq 10$, write H in the shape monomial basis. Vanishing of the orbit germ forces all Taylor coefficients to vanish. The shape matrices have full modular rank at a prime larger than the largest required factorial, so the corresponding characteristic-zero matrices are full rank. Hence all coefficients of H are zero. $\square$

The shape certificate and the full invariant-quotient certificate should be regarded as complementary. The shape algebra is smaller, which permits a higher degree, while the full quotient removes the exact distance and first-integral relations and therefore tests a structurally richer class of state polynomials.

9 Reproducible computation

The computational contribution is now distributed as an executable certificate rather than described only in prose. The submission bundle contains:

1. `certificate_data/groebner_basis.txt`: the complete exact rational Groebner basis (47 polynomials, maximum total degree 11);
2. `certificate_data/standard_monomials_D4.txt`: the complete 1207-element standard-monomial quotient basis used for $D = 4$;
3. `reproducibility/build_groebner.py`: exact symbolic construction of the polynomial system, Groebner basis, and filtered quotient dimensions;
4. `reproducibility/verify_witness.py`: exact witness, invariant, and Jacobian-minor checks;
5. `reproducibility/verify_shape_mod.py`: independent modular shape-rank calculation through degree 10; and
6. `reproducibility/full_d4_cert.cpp`: the degree-four Taylor-jet generator and exact modular Gaussian elimination program.

The reference build is:

```
python3 verify_witness.py
python3 build_groebner.py
```

```
python3 verify_shape_mod.py
```

```
g++ -O3 -march=native -fopenmp -std=c++17 full_d4_cert.cpp -o full_d4_cert
./full_d4_cert 1000003
./full_d4_cert 1000033
```

The verified outputs are:

Item	Output
Jacobian minor	2/25 (PASS)
Groebner basis	47 polynomials, max degree 11
Filtered dimensions	1, 12, 76, 340, 1207
Shape ranks	full rank for every $D = 1, \dots, 10$ at both primes
Full quotient rank	1207 at both primes

On the reference runtime used to prepare this revision, the exact Groebner computation completed in approximately 18 seconds, while the degree-four modular rank computation completed in approximately 4.5 seconds per prime. These timings are machine-dependent and are included only as reproducibility guidance.

The modular programs use Taylor coefficients rather than raw derivatives. By Lemma 4.2, this is an exact row rescaling for the primes used here. The implementation therefore does not rely on floating-point differentiation, numerical trajectory integration, or threshold-based numerical rank decisions.

10 Relation to established three-body results

The present finite obstruction is not intended to supersede classical integrability theory. Bruns’s result concerns algebraic first integrals; Morales-Ruiz’s differential-Galois framework and the later N -body analysis of Morales-Ruiz and Simon concern meromorphic integrability and variational obstructions [1, 2, 3]. The present orbit relation is a different object: a polynomial can vanish on a single trajectory germ without being a globally conserved quantity.

Special periodic solutions also do not contradict the finite certificate. The equal-mass figure-eight choreography was established by Chenciner and Montgomery through a variational argument [4], with later computer-assisted work establishing local uniqueness and convexity properties [5]. The existence of such special trajectories is compatible with the fact that a generic asymmetric witness need not lie on an additional low-degree algebraic hypersurface.

The computational algebra terminology is likewise deliberately standard. Groebner bases, filtered polynomial spaces, Lie derivatives, Taylor jets, and modular rank certificates are established tools in computational algebra and dynamical systems. Standard references such as [6] place the algebraic elimination component in its usual setting. The present paper’s contribution is the concrete certificate protocol and its application to the invariant quotient of the equal-mass planar three-body flow, not a claim to have invented any of these constituent operations.

11 Fixed-degree algorithmic boundary

The degree-four result should not be described as a theoretically privileged cutoff. It is the largest degree for which the present full invariant-quotient computation was carried out in this certificate set. The correct general statement is an algorithmic one.

Proposition 11.1 (Finite-degree decidability of the certificate test). *Fix D . Given a polynomial vector field, a finitely generated invariant ideal, and an algebraic witness, the construction*

$$F_DR \longrightarrow \mathbb{Q}^M, \quad [P] \longmapsto (P(z_*), \mathcal{D}P(z_*), \dots, \mathcal{D}^{M-1}P(z_*))$$

is finite and exactly computable whenever the filtered quotient dimension $M = \dim(F_DR/(F_DR \cap I))$ is finite.

Proof. The filtered polynomial space F_DR is finite-dimensional. A Groebner basis or an equivalent exact linear-algebra computation determines a quotient basis and therefore M . Each iterated Lie derivative is a polynomial, and evaluation at the algebraic witness is exact. Only the first M functionals are needed for the rank obstruction theorem. Thus the construction terminates as a finite exact computation. $\square$

The proposition gives a well-defined degree ladder: D can be increased and the same certificate protocol rerun. What is not supplied here is an asymptotic theorem asserting full rank for every degree. Establishing such a theorem would be a substantially stronger result and is not needed for the finite statement proved in this paper.

12 What is and is not proved

The paper proves the following finite claims.

1. The collision-free planar equal-mass Newtonian equations admit the stated exact polynomial realization after adjoining inverse-distance variables.
2. The three geometric distance relations form a differential ideal.
3. Energy and angular momentum enlarge this to a differential invariant ideal.
4. At the exact asymmetric rational witness, the filtered invariant quotient has dimensions 1, 12, 76, 340, 1207 for $D = 0, 1, 2, 3, 4$.
5. The degree-four quotient Lie-jet matrix has characteristic-zero rank 1207, certified by exact modular computations at two primes.
6. Therefore no nonzero quotient class of total degree at most four gives a polynomial orbit-germ relation at the witness.
7. For each fixed degree covered by a nonzero determinant, the same obstruction holds off a proper algebraic exceptional set on the corresponding irreducible component.
8. Independently, no nonzero rational shape polynomial in (x, y, z) of total degree at most ten vanishes identically on the witness orbit germ.

The paper does not prove an all-degree non-compressibility theorem, a global impossibility theorem for closed forms, a new Morales-Ramis non-integrability theorem, or the absence of transcendental or higher-degree relations. It also does not claim that the degree-four result alone captures the full dynamical complexity of the three-body problem. Its claim is narrower: after exact quotienting by specified known invariants, a bounded-degree polynomial orbit relation would have to satisfy a finite exact jet dependence, and the certified matrix is full rank.

13 Conclusion

The planar equal-mass Newtonian three-body flow can be represented exactly by a polynomial vector field in eleven variables. Once the three inverse-distance constraints and fixed energy and angular momentum levels are incorporated into an invariant ideal, the bounded-degree orbit-relation question becomes a finite Lie-jet rank problem.

At the exact asymmetric 3-4-5 witness, the degree-four quotient has dimension 1207, and the associated 1207×1207 Lie-jet matrix has full rank over $\mathbb{Q}$, certified by modular computations at 1 000 003 and 1 000 033. The independent shape calculation reaches degree ten. The witness-to-minor argument turns each finite witness certificate into a componentwise generic statement for that fixed degree.

The principal result should therefore be understood as a reproducible finite certificate, not as a new global theory of three-body solvability. The immediate mathematical extensions are clear: raise the degree, vary the invariant quotient, compare different algebraic coordinate systems, and eventually couple the algebraic certificate to separately validated temporal closure problems. None of those extensions is assumed in the present theorem set.

A Certificate protocol in compact form

For a fixed degree D , the reproducible protocol is:

1. Construct the exact polynomial vector field and invariant ideal.
2. Compute an exact Groebner basis or equivalent basis of the filtered ideal intersection.
3. Enumerate a quotient basis of $\mathcal{N}_D$.
4. Generate the Taylor coefficients of each quotient basis monomial along the polynomial flow at the algebraic witness.
5. Form the $M \times M$ coefficient matrix.
6. Reduce the matrix modulo a prime $p > M$ that does not annihilate the cleared denominators.
7. Compute exact modular rank.
8. If the rank is M , the characteristic-zero determinant is nonzero and the orbit-germ obstruction follows.

The protocol is independent of the particular programming language. The supplied implementation uses SymPy for the exact symbolic stage and a C++17/OpenMP implementation for the large modular degree-four calculation.

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