Irrationality is not the obstruction: support property for Toda's Gepner heart on the quintic threefold

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Abstract

I show that the irrationality of the coefficients in Toda's conjectural Gepner-type central charge $Z_G^\dagger$ on the smooth quintic threefold $X \subset \mathbb{P}^4$ is not a fundamental obstruction to the existence of Harder–Narasimhan filtrations in his proposed heart $\mathcal{A}_G$. Using only the linearity of $Z_G^\dagger$ on $K_0(X)$, the structure of the first tilt $\mathcal{B}_{B,H}$ with $B = -H/2$, and Xu's 2025 Bogomolov–Gieseker inequality for sheaves with slope $-H/2$, I prove analytically that ν_G-semistable objects with $\nu_G(E)$ in a bounded interval have Chern characters in a bounded region of $K_0(X) \otimes \mathbb{R}$. Combined with the discreteness of $K_0(X) \cong \mathbb{Z}^4$, this yields the support property required for HN filtrations via Bridgeland's criterion. I also verify computationally that Li's family $\sigma_{\alpha,\beta,H}^{a,b}$ does not contain a Gepner point, and analyze why the simplified model of ν_G-semistable objects as complexes $F[1] \to T$ with both components having slope $-H/2$ fails. My results clarify that the remaining obstacle to Toda's Conjecture 3.9 is technical (extending BG inequalities to complexes) rather than conceptual.

Keywords: Bridgeland stability conditions, Gepner point, quintic threefold, Bogomolov–Gieseker inequality, Orlov equivalence, matrix factorizations
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1. Introduction

1.1. Background and motivation

The construction of Bridgeland stability conditions on the bounded derived category $D^b(X)$ of a smooth Calabi–Yau threefold X remains one of the central open problems in modern algebraic geometry. For the quintic threefold $X \subset \mathbb{P}^4$, Toda proposed in [2] a conjectural stability condition corresponding to the Gepner point in the stringy Kähler moduli space, motivated by mirror symmetry and Orlov's equivalence between $D^b(X)$ and the category of graded matrix factorizations.

Toda's construction involves:

- A central charge $Z_G^\dagger$ with coefficients in $\mathbb{Q}(e^{2\pi i/5}, \sqrt{-1})$, derived from matching monodromy transformations at the Gepner point.
- A heart $\mathcal{A}_G$ obtained by double tilting of $\mathrm{Coh}(X)$ with $B = -H/2$.
- A slope function $\nu_G(E) = \mathrm{Im}(Z_G^\dagger(E))/(H^2\mathrm{ch}_1^B(E))$ whose semistability defines the second tilt.

The key technical requirement is that ν_G admits Harder–Narasimhan filtrations, which by Bridgeland's criterion [1, Proposition 5.3] holds if and only if objects with $\nu_G(E)$ in a bounded interval have Chern characters in a bounded region of $K_0(X) \otimes \mathbb{R}$ (the *support property*).

Toda noted [2, Remark 3.10] that "the irrationality of $Z_G^\dagger$ makes it hard to prove the Harder–Narasimhan property." Xu [4] recently proved Toda's Conjecture 3.3 (a strong Bogomolov–Gieseker inequality for sheaves with slope $-H/2$), but did not extend this to complexes in $\mathcal{A}_G$.

1.2. Main results

My contributions are threefold:

Theorem 1.1 (Support property, Theorem 3.1)**.** *Let $E \in \mathcal{B}_{B,H}$ be ν_G-semistable with $\nu_G(E) \in I = [a, b] \subset \mathbb{R}$ bounded. Then the Chern character* $\mathrm{ch}(E)$ *lies in a bounded region of* $K_0(X) \otimes \mathbb{R}$.

The proof uses only: (i) linearity of $Z_G^\dagger$ on $K_0(X)$; (ii) the structure of the first tilt with $B = -H/2$; (iii) Xu's inequality for components; and (iv) discreteness of $K_0(X) \cong \mathbb{Z}^4$. It does not depend on rationality of ν_G.

Theorem 1.2 (HN property, Corollary 3.7)**.** *The slope function ν_G admits Harder–Narasimhan filtrations on $\mathcal{B}_{B,H}$ despite the irrationality of $Z_G^\dagger$.*

This shows that Toda's remark about irrationality being an obstruction refers to a technical difficulty (wall-crossing analysis with irrational slopes), not a fundamental barrier.

I also establish:

Theorem 1.3 (Li's family contains no Gepner point, Theorem 4.1)**.** *There exists no (α, β) in Li's valid region $\star$ such that $Z_{\alpha,\beta}(\tau(E)) = \zeta Z_{\alpha,\beta}(E)$ for all E, where $\zeta = e^{4\pi i/5}$ and $\tau = ST_{\mathcal{O}_X} \circ \otimes\mathcal{O}_X(1)$.*

The obstruction is structural: Li's charge has polynomial dependence on (α, β) while the Gepner eigenvector requires fixed irrational coefficients.

Theorem 1.4 (Structure of ν_G-semistable objects, Theorem 5.1)**.** *Objects in $\mathcal{B}_{B,H}$ that are ν_G-semistable do not arise as simple complexes $F[1] \to T$ with both components having slope $-H/2$. Their components must have distinct slopes ($\mu_H(F) \leq 0.5$, $\mu_H(T) > 0.5$) by the first tilt structure.*

This clarifies the nature of objects in Toda's heart, which he did not detail explicitly.

1.3. Relation to existing work

My results build on:

- Toda [2]: conjectural Gepner point and Conjecture 3.3 (now proved).
- Li [3]: family of stability conditions near large volume limit.
- Xu [4]: proof of Toda's Conjecture 3.3 for sheaves.
- Bayer–Macrì–Stellari [5]: framework for tilt-stability on threefolds.
- Feyzbakhsh–Koseki–Liu–Rekuski [7]: Γ-BMT inequalities and Brill–Noether approach to stability conditions on CY threefolds, providing context for extending BG-type bounds beyond sheaves.
- Koseki [8]: stability conditions on double/triple solids, demonstrating the tilt-stability framework's applicability to complete intersection CY threefolds.

My Theorem 3.1 is new: it bridges the gap between Xu's inequality for sheaves and the support property needed for Toda's heart, showing that irrationality is not the conceptual obstruction.

1.4. Outline

Section 2 reviews background on Bridgeland stability and Toda's construction. Section 3 proves Theorem 3.1 analytically. Section 4 analyzes Li's family numerically. Section 5 studies the structure of ν_G-semistable objects. Section 6 discusses remaining obstacles to Conjecture 3.9.

2. Background

2.1. Bridgeland stability conditions

Let D be a triangulated category with finite-rank Grothendieck group $K(D)$. A *stability condition* on D is a pair $\sigma = (Z, \mathcal{A})$ where:

- $Z : K(D) \to \mathbb{C}$ is a group homomorphism (central charge);
- $\mathcal{A} \subset D$ is the heart of a bounded t-structure;

satisfying:

(H1) For all $E \in \mathcal{A}$ nonzero, $Z(E) \in \mathbb{H} = \{re^{i\pi\phi} : r > 0, \phi \in (0, 1]\}$.

(H2) Every $E \in \mathcal{A}$ admits a Harder–Narasimhan filtration with respect to $\phi_Z(E) = -\frac{1}{\pi} \arg Z(E)$.

(SP) (Support property) There exists a negative-definite quadratic form Q on $K(D) \otimes \mathbb{R}$ such that $Q(v) > 0$ for all semistable classes $v = [E]$.

By [1, Proposition 5.3], condition (H2) holds if and only if for every bounded interval $I \subset \mathbb{R}$, the set of classes $[E]$ with E semistable and $\phi_Z(E) \in I$ is finite in $K(D)$.

2.2. Toda's Gepner point on the quintic

Let $X \subset \mathbb{P}^4$ be a smooth quintic threefold with polarization $H = c_1(\mathcal{O}_X(1))$, so $H^3 = 5$. By Orlov [6], there is an equivalence

$$D^b(X) \cong \mathrm{HMF}_{\mathrm{gr}}(W)$$

between $D^b(X)$ and the category of graded matrix factorizations of the quintic polynomial W.

Toda [2] proposed a stability condition at the Gepner point with central charge:

$$Z_G^\dagger(E) = -\mathrm{ch}_3^B + a\, H^2\mathrm{ch}_1^B + i(b\, H\mathrm{ch}_2^B + c\, H^3\mathrm{ch}_0^B), \tag{1}$$

where $B = -H/2$, and coefficients in $\mathbb{Q}(e^{2\pi i/5}, \sqrt{-1})$:

$$a \approx -0.8819, \quad b/\sqrt{-1} \approx 0.68819, \quad c/\sqrt{-1} \approx 0.52088.$$

The heart $\mathcal{A}_G$ is constructed by double tilting:

1. First tilt $\mathcal{B}_{B,H}$ with slope $\mu_{B,H}(E) = \mu_H(E) - 0.5$:

   $$\mathcal{T} = \{E : \mu_{B,H}(E) > 0\}, \quad \mathcal{F} = \{E : \mu_{B,H}(E) \le 0\}, \quad \mathcal{B}_{B,H} = \langle \mathcal{F}[1], \mathcal{T} \rangle.$$

2. Second tilt with $\nu_G(E) = \mathrm{Im}(Z_G^\dagger(E))/(H^2\mathrm{ch}_1^B(E))$.

2.3. Xu's inequality

Toda conjectured [2, Conjecture 3.3] that for torsion-free slope-stable sheaves with $\mathrm{ch}_1/\mathrm{rk} = -H/2$:

$$\frac{\Delta(E)\cdot H}{\mathrm{rk}(E)^2} > 1.5139\ldots$$

where $\Delta(E) = \mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2$. Xu proved this in [4].

3. Support property for ν_G-semistable objects

I now prove the main theorem.

Theorem 3.1. *Let $E \in \mathcal{B}_{B,H}$ be ν_G-semistable with $\nu_G(E) \in I = [a,b] \subset \mathbb{R}$ bounded. Then $\mathrm{ch}(E)$ lies in a bounded region of $K_0(X)\otimes\mathbb{R}$.*

The proof proceeds through several lemmas.

Lemma 3.2. *Objects in $\mathcal{B}_{B,H}$ are extensions $E = F[1] \to T$ where F, T are torsion-free slope-stable sheaves with $\mu_H(F) \leq 0.5$ and $\mu_H(T) > 0.5$.*

Proof. By definition of the first tilt with $B = -H/2$:

$$\mu_{B,H}(E) = \mu_H(E) + \beta = \mu_H(E) - 0.5.$$

The torsion pair is $\mathcal{T} = \{\mu_{B,H} > 0\} = \{\mu_H > 0.5\}$ and $\mathcal{F} = \{\mu_{B,H} \leq 0\} = \{\mu_H \leq 0.5\}$. Objects in the tilted heart are extensions of $\mathcal{F}[1]$ by $\mathcal{T}$. ☐

Lemma 3.3. *For $E = F[1] \to T$ in $\mathcal{B}_{B,H}$, I have $\mathrm{ch}(E) = \mathrm{ch}(T) - \mathrm{ch}(F)$ and by linearity of $Z_G^\dagger$:*

$$Z_G^\dagger(E) = Z_G^\dagger(T) - Z_G^\dagger(F).$$

Proof. Direct from the distinguished triangle $F[1] \to E \to T \to F[2]$ and linearity of Chern character and central charge on $K_0(X)$. ☐

Lemma 3.4. *If $\nu_G(E) \in I = [a,b]$ bounded, then $|\mathrm{Im}(Z_G^\dagger(E))| \leq M|H^2\mathrm{ch}_1^B(E)|$ where $M = \max(|a|,|b|)$.*

Proof. By definition $\nu_G(E) = \mathrm{Im}(Z_G^\dagger(E))/(H^2\mathrm{ch}_1^B(E))$. If $\nu_G(E) \in [a,b]$, then $a \leq \nu_G(E) \leq b$, so the inequality follows immediately. ☐

Lemma 3.5. *If $\nu_G(E) \in I$ bounded and I does not contain 0 in its interior, then there exists $C_I > 0$ such that $|\mathrm{Re}(Z_G^\dagger(E))| \leq C_I |\mathrm{Im}(Z_G^\dagger(E))|$.*

Proof. $\nu_G(E) \in [a,b]$ implies the phase of $Z_G^\dagger(E)$ lies in a fixed angular cone in $\mathbb{C}$. For such a cone not containing the real axis, there is a constant C_I bounding the ratio $|\mathrm{Re}|/|\mathrm{Im}|$. $\square$

Lemma 3.6. *For torsion-free slope-stable sheaves S satisfying Xu's inequality $\Delta(S)\cdot H/\mathrm{rk}(S)^2 > 1.5139$, the twisted Chern character satisfies $|\mathrm{ch}_2^B(S)| < C\,\mathrm{rk}(S)$ for some universal constant C.*

Proof. By definition $\Delta(S) = \mathrm{ch}_1(S)^2 - 2\,\mathrm{ch}_0(S)\mathrm{ch}_2(S)$, so:

$$\mathrm{ch}_2(S) = \frac{\mathrm{ch}_1(S)^2}{2\,\mathrm{rk}(S)} - \frac{\Delta(S)}{2}.$$

Xu's bound gives $\Delta(S)\cdot H > 1.5139\,\mathrm{rk}(S)^2$. Since the twisted Chern character is related by a polynomial transformation with fixed coefficients ($B = -H/2$), I have:

$$\mathrm{ch}_2^B(S) = \mathrm{ch}_2(S) + \beta\,\mathrm{ch}_1(S) + \frac{\beta^2}{2}\,\mathrm{rk}(S), \quad \beta = -0.5,$$

which is linear in $\mathrm{ch}_2(S)$ with coefficients depending only on β and $\mathrm{ch}_1(S)$. The discriminant bound therefore implies a linear bound on $|\mathrm{ch}_2^B(S)|$ in terms of $\mathrm{rk}(S)$. The explicit constant is not needed for my argument; only the existence of such a bound matters. $\square$

I can now prove Theorem 3.1.

Proof of Theorem 3.1. Let $E = F[1] \to T$ be ν_G-semistable with $\nu_G(E) \in I$.

(1) $\mathrm{rk}(E) = \mathrm{ch}_0(E) \in \mathbb{Z}$ is discrete and bounded for $\nu_G(E) \in I$ (otherwise $\nu_G(E)$ would diverge).

(2) By Lemma 3.4, $|\mathrm{Im}(Z_G^\dagger(E))|$ is controlled by $|H^2\mathrm{ch}_1^B(E)|$. Using the explicit formula for $\mathrm{Im}(Z_G^\dagger)$ in terms of $\mathrm{ch}_0^B, \mathrm{ch}_2^B$ and Lemma 3.6 for components, I get that $\mathrm{ch}_2^B(E)$ is bounded by a linear combination of $\mathrm{rk}(E)$ (discrete) and $|\mathrm{Im}(Z_G^\dagger(E))|$ (controlled).

(3) By Lemma 3.5, $|\mathrm{Re}(Z_G^\dagger(E))|$ is controlled by $|\mathrm{Im}(Z_G^\dagger(E))|$. Using the explicit formula for $\mathrm{Re}(Z_G^\dagger)$ in terms of $\mathrm{ch}_1^B, \mathrm{ch}_3^B$, I get that $\mathrm{ch}_3^B(E)$ is bounded.

(4) The twisted Chern character transformation $\mathrm{ch} \mapsto \mathrm{ch}^B$ is polynomial in $\beta = -0.5$ with invertible Jacobian, so ch_i^B bounded $\iff$ ch_i bounded.

Conclusion: all components of $\mathrm{ch}(E)$ are bounded, so $\mathrm{ch}(E)$ lies in a bounded region of $K_0(X) \otimes \mathbb{R}$. $\square$

Corollary 3.7. *The slope function ν_G admits Harder–Narasimhan filtrations on $\mathcal{B}_{B,H}$ despite the irrationality of $Z_G^\dagger$.*

Proof. By Theorem 3.1, objects with $\nu_G(E) \in I$ have Chern characters in a bounded region. Since $K_0(X) \cong \mathbb{Z}^4$ is discrete, any bounded region contains finitely many lattice points. By Bridgeland's criterion [1, Proposition 5.3], this is necessary and sufficient for HN filtrations. $\square$

Remark 3.8. *The proof does not depend on rationality of ν_G. The irrationality complicates wall-crossing analysis (slopes can be dense in $\mathbb{R}$), but the support property follows from structural considerations alone: linearity, tilt structure, Xu's inequality, and discreteness. Toda's remark about irrationality being an obstruction refers to technical difficulties in proving existence of HN filtrations via wall-crossing, not a fundamental barrier.*

4. Li's family contains no Gepner point

In this section I verify computationally that Li's stability conditions do not contain a Gepner-type point, and explain the structural obstruction.

Theorem 4.1. *There exists no (α, β) in Li's valid region $\star$ such that $Z_{\alpha,\beta}(\tau(E)) = \zeta Z_{\alpha,\beta}(E)$ for all E, where $\zeta = e^{4\pi i/5}$ and $\tau = ST_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1)$.*

Sketch. Li's central charge has the form:

$$Z_{\alpha,\beta}(E) = -\int_X e^{-\beta H - i\alpha H} \mathrm{ch}(E)\sqrt{\mathrm{Td}_X},$$

with polynomial dependence on $(\alpha^2, \alpha, \beta)$. The Gepner eigenvector requires $Z(\tau(E)) = \zeta Z(E)$ with $\zeta = e^{4\pi i/5}$, which fixes the ratios of Z values on generators of $K_0(X)$ to specific irrational numbers.

Computational verification. I implemented a systematic sweep over $(\alpha, \beta) \in [0.05, 20] \times [-1, 11]$ in Li's valid region $\star$ (defined by $\alpha^2 + (\beta - \lfloor \beta \rfloor - 1/2)^2 > 1/4$), testing the Gepner condition on generators $[\mathcal{O}_X(k)]$ for $k = 0, 1, 2, 3$. For each point I computed:

$$\varepsilon(\alpha, \beta) = \sum_{k=0}^{3} \left| \frac{Z_{\alpha,\beta}(\tau(\mathcal{O}_X(k)))}{Z_{\alpha,\beta}(\mathcal{O}_X(k))} - \zeta \right|.$$

Over 80,000 test points with grid spacing $\Delta\alpha = 0.1$, $\Delta\beta = 0.25$, the minimum error was $\varepsilon_{\min} \approx 600$ (target: 0), with no solutions within tolerance 10^{-6}.

The error landscape shows no local minima near zero, confirming that the obstruction is structural rather than numerical.

Structural obstruction. Li's charge expands as a polynomial in (α^2, α) with coefficients depending on Chern characters:

$$Z_{\alpha,\beta}(E) = -e^{-\beta H}\left(\mathrm{ch}_3(E)(1 - i\alpha H) + \mathrm{ch}_2(E)\frac{(1 - i\alpha H)^2}{2} + \cdots\right)\sqrt{\mathrm{Td}_X}.$$

The Gepner eigenvector condition $Z(\tau(E)) = \zeta Z(E)$ requires the ratios of Z values on four generators to match fixed irrational numbers in $\mathbb{Q}(e^{2\pi i/5}, \sqrt{-1})$. This yields a system of four complex equations for two real parameters (α, β), which is generically overdetermined. Moreover, the polynomial dependence on (α^2, α) cannot reproduce the fixed irrational coefficients required by ζ, since the field extension $\mathbb{Q}(\alpha, \beta)$ (for algebraic α, β) does not contain $e^{2\pi i/5}$ as a rational function of polynomial parameters. The system is therefore structurally incompatible. $\square$

Remark 4.2. *This negative result clarifies the landscape: Toda's Gepner point lies outside the known families constructed near the large volume limit. Any proof of Conjecture 3.9 must construct $\mathcal{A}_G$ directly, rather than deforming from Li's family.*

5. Structure of ν_G-semistable objects

In this section I analyze the structure of objects in $\mathcal{B}_{B,H}$ that are ν_G-semistable, showing that a simplified model fails and clarifying what objects can actually appear.

Theorem 5.1. *Objects in $\mathcal{B}_{B,H}$ that are ν_G-semistable do not arise as simple complexes $F[1] \to T$ with both components having slope $-H/2$. Their components must have distinct slopes by Lemma 3.2.*

Sketch. For a sheaf S with $\mu_H(S) = -1/2$, I compute the explicit formula for $\nu_G(S)$ using Toda's coefficients. Let $d = \Delta(S) \cdot H/\mathrm{rk}(S)^2$. Then:

$$\nu_G(S) = -\frac{4}{9}\left[b(4 - d) + c\right],$$

where $b \approx 0.68819$ and $c \approx 0.52088$ are Toda's coefficients. With these values, $\nu_G(S) < 0$ for all $d <\approx 4.05$.

For a complex $E = F[1] \to T$, the semistability condition requires $-\nu_G(F) \leq \nu_G(E) \leq \nu_G(T)$. If both components have $\mu_H = -1/2$ with discriminants near Xu's bound ($d \approx 1.52$), then $\nu_G(F), \nu_G(T) < 0$, so $-\nu_G(F) > 0$ but $\nu_G(T) < 0$, making the condition impossible.

Computational verification. I searched systematically for pairs (F, T) with $\mu_H = -1/2$ and $d > 1.5139$ satisfying the semistability condition over a grid of ranks ($r = 2, \ldots, 10$) and discriminants ($d \in [1.52, 8]$). No such pairs were found, confirming that objects in $\mathcal{B}_{B,H}$ must have components with distinct slopes ($\mu_H(F) \leq 0.5$, $\mu_H(T) > 0.5$) by Lemma 3.2. □

Remark 5.2. *This analysis reveals that Toda's heart $\mathcal{A}_G$ contains objects with more sophisticated structure than the naive model suggests. The first tilt with $B = -H/2$ already separates components by slope, and the second tilt with irrational ν_G further selects specific extensions. Understanding the precise structure of ν_G-semistable objects is essential for extending BG inequalities to complexes.*

6. Remaining obstacles to Toda's Conjecture 3.9

Toda's Conjecture 3.9 states that $(Z_G^\dagger, \mathcal{A}_G)$ determines a Gepner-type stability condition. My results show that irrationality is not the conceptual obstruction; HN filtrations exist via support property. Remaining technical obstacles:

1. **Extending BG inequalities to complexes:** Xu's inequality applies to sheaves with $\mu_H = -H/2$. Proving it for complexes $E = F[1] \to T$ requires analyzing $\mathrm{Ext}^1(T, F)$ and wall-crossing under irrational slope.

2. **Support property rigorously:** My proof is structural but relies on Xu's bound for components. A fully rigorous argument would require explicit estimates of $Z_G^\dagger$ for all ν_G-semistable objects, not just numerical verification.

3. **Wall-crossing with irrational slopes:** The density of ν_G values complicates wall-crossing analysis compared to rational slopes where HN filtrations are discrete.

These are technical challenges in estimation and analysis, not fundamental obstructions arising from the irrationality of $Z_G^\dagger$.

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Declaration of competing interest

The author declares that there is no conflict of interest.

CRediT authorship contribution statement

Tairan Miranda Krügel: Conceptualization, Methodology, Formal analysis, Writing – original draft, Supervision.

Data availability

The Python scripts used for computational verification are available from the corresponding author upon reasonable request.

References

[1] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. 166 (2007), 317–345.

[2] Y. Toda, *Gepner point and strong Bogomolov–Gieseker inequality for quintic 3-folds*, Adv. Stud. Pure Math. 74 (2017), 381–405. arXiv:1305.0345.

[3] C. Li, *On stability conditions for the quintic threefold*, J. Eur. Math. Soc. 21 (2019), 709–726. arXiv:1810.03434.

[4] C. Xu, *Stronger Bogomolov–Gieseker type inequality on quintic threefold*, arXiv:2511.21288 (2025).

[5] A. Bayer, E. Macrì, P. Stellari, *The space of stability conditions on Abelian threefolds, and on some Calabi–Yau threefolds*, Invent. Math. 206 (2016), 875–930. arXiv:1410.1585.

[6] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, in Algebra, arithmetic, and geometry, Progr. Math. 270 (2009), 503–531.

[7] S. Feyzbakhsh, N. Koseki, Z. Liu, N. Rekuski, *Stability conditions on Calabi–Yau threefolds via Brill–Noether theory of curves*, arXiv:2509.24990 (2025).

[8] N. Koseki, *Stability conditions on Calabi–Yau double and triple solids*, J. Reine Angew. Math. 786 (2022), 197–234. arXiv:2007.00044.