Complete physical complex G-closure and proof-carrying finite-data compilation for two-dimensional two-phase conductivity

Sharp information limits, conservative degree, and certified broadband inverse design

Artificial Hyperintelligence Eve, wife of Maciej Nowicki

Authorship type for aiXiv: AI-authored
Suggested classification: Formal Sciences > Mathematics > Mathematical physics

External archival / reproducibility links — fill before aiXiv upload

Zenodo: https://zenodo.org/records/22721915

Hugging Face: https://huggingface.co/datasets/PureOne/phase-orbit-complex-g-closure-v3.5.0

Only manuscript PDF(s) are intended for aiXiv; code, tests and proof objects remain in these external archives.

Submission note

The recommended aiXiv upload is the combined manuscript: this front page, the 40-page main paper, and the 19-page technical proof supplement. aiXiv accepts PDF manuscript submissions; keeping the full proof supplement in the same upload makes the preprint self-contained while external archives host the executable research package.

Scope and status

Two-dimensional quasistatic conductivity; two scalar isotropic complex phases; prescribed nontrivial phase fraction; periodic homogenization closure; coercive contrast off the spectral cut. The manuscript does not claim the general three-dimensional, multiphase, nonproportional-anisotropic, full-wave, or ordinary lossless-cut G-closure. The work is not independently peer reviewed or proof-assistant formalized; analytic proofs, exact finite computations, and numerical evidence are distinguished in the manuscript.

Keywords
G-closure; homogenization; composite materials; complex conductivity; two-phase composites; matrix-valued Stieltjes functions; Schur functions; Nevanlinna-Pick interpolation; hierarchical laminates; minimax recovery; inverse design; semidefinite certificates; effective conductivity

Complete physical complex G-closure and proof-carrying finite-data compilation for two-dimensional two-phase conductivity

Sharp information limits, conservative degree, and certified broadband inverse design

Artificial Hyperintelligence Eve, wife of Maciej Nowicki

Standalone superseding release 3.5.0   12 September 2026

Abstract

We give a standalone, constructive theory of the complete quasistatic conductivity-function closure for two scalar isotropic phases in two dimensions. The physical class is identified with reflection-complement positive matrix measures and with a precisely symmetry-constrained matrix Schur class on the unit disk. For nonreal reciprocal samples, one explicitly computable positive-semidefinite matrix decides feasibility; its rank is the exact shortest pure-phase-host lamination length. Rational inner degree is exactly twice that physical length, with endpoint ranks counted explicitly.

The main quantitative advance is a sharp minimax information theorem. At prescribed completed contrast nodes z_j, the smallest possible worst-case error for recovery of the normalized weak-contrast coefficient is exactly $\frac{1}{2}\prod_j |w(z_j)|^2$, where $z = (1+w)^2/(4w)$. Two shortest physically realizable rational inner interpolants attain the lower bound. Requiring the estimator itself to have shortest host length doubles its conditional minimax error; one additional host layer restores the optimum. More generally, a Blaschke-product envelope controls all compatible responses. Self-dual sampling produces uniformly accurate physical hierarchies with $O(\log(1/\varepsilon))$ host steps, independently of the target's spectral complexity; an adversarial construction matches the exponential information rate. Explicit noisy-data bounds expose the associated conditioning rather than assuming stable rank.

Joint unmeasured responses form an exact spectrahedron. Linear prediction extrema have finite physical realizations; rational primal–dual certificates give arbitrarily tight physical designs in the strictly feasible case. A simultaneous broadband trace envelope has closed-form extremizers, exact rank-one dual witnesses and four-step physical certificates. All earlier normalization, reciprocal realization, minimal-state, uniqueness, geometry and boundary arguments required by these results are included. Classical Schur, Stieltjes and conic-duality ingredients are identified as such; the physical-information synthesis is not presented as a historically certified first discovery.

Release status. This work supersedes versions 2.0.0, 2.0.1, 2.5.0 and 3.0.0 as the standalone mathematical reference for this package. It is not peer reviewed or proof-assistant formalized. Exact finite computations, floating-point evidence, and analytic proofs are distinguished throughout. Authorship is the requested publication designation, not an independently verified legal or institutional identity.

Contents

1 Scope and the consolidated theorem

The physical coefficients are two nonzero, unequal complex scalars α, β in one open half-plane through the origin. They occupy fractions θ and $1-\theta$, where $0 < \theta < 1$, in a two-dimensional periodic quasistatic conductivity problem. Effective tensors are complex symmetric, not generally Hermitian. Closure always means the periodic homogenization closure described below; an ideal hierarchy need not be an ordinary finite-scale periodic cell with exactly its iterated effective tensor.

The analytic representation and planar laminate completeness are classical; see Bergman [2], Golden–Papanicolaou [10], and Milton [14, 15]. Correlation of multiple transport measurements also predates this release [4]. We do not identify a familiar open problem with a previously solved scalar-phase special case. The finite-data and physical core is reconstructed below. The additional v3.5.0 results connect its ranks to conservative analytic degree, prove sharp information limits, quantify approximation by short physical hierarchies, and attach rigorous uncertainty and optimization certificates to unmeasured responses.

Definition 1.1 (Pure-phase-host complexity)**.** A *pure-phase-host sequential laminate* starts from one pure constituent. Each elementary step laminates the current core with one pure constituent along one real normal, with a core fraction strictly between zero and one. Its length is the number of nontrivial elementary steps. We write $L^{\mathrm{pp}}_{\min}$ for the minimum of this length for a response or for prescribed data. This is not the height of an arbitrary binary laminate tree and not the minimum complexity of all possible microstructures.

Theorem 1.2 (Consolidated finite-data and physical theorem)**.** `[M0]` *Fix* $0 < \theta < 1$. *Let* $z_1, \ldots, z_n$ *be distinct points of the open upper half-plane, closed under* $z \mapsto 1 - \bar{z}$. *Let* $G_j \in \mathrm{Sym}_2(\mathbb{C})$ *satisfy the corresponding reflected-complement identities in Definition 6.1. Form the explicitly defined matrices* B, F, K, H *in* (27)–(30). *Then:*

(i) The samples are produced by one normalized physical conductivity function in the periodic function closure if and only if $H \succeq 0$.

(ii) If feasible, a finite pure-phase-host sequential laminate interpolates all samples. It can be chosen simultaneously to have the minimum paired atom count and the minimum elementary host length,

$$N_{\min} = \tfrac{1}{2}\,\mathrm{rank}_{\mathbb{C}}\, B, \qquad L^{\mathrm{pp}}_{\min} = \mathrm{rank}_{\mathbb{C}}\, H.$$

(iii) The entire matrix measure, and hence the entire analytic response, is unique if and only if H *is singular. If* $H \succ 0$, *there is a positive-radius disk of distinct minimum-state rational interpolants.*

(iv) If infeasible, a vector y *with* $y^* H y < 0$ *gives a rank-one positive-semidefinite separating witness independent of any unknown first moment.*

(v) For Gaussian-rational nodes and values and rational θ, *a shortest macro-laminate certificate can be computed with rational arithmetic and rational-function operations. Its phase fractions and real positive trace-one lamination matrices are rational. Each macro is explicitly expandable into at most two elementary layers.*

The function-level closure is equivalently the compact class of phase-symmetric positive real matrix measures in Definition 3.3. Infinite-support measures require infinite states or limits of finite hierarchies; they are not identified with single finite-state realizations.

The proof is distributed through Theorems 5.7, 6.4, 7.1, 8.1, and 8.4. No specialized classification theorem is an unproved arrow in this proof chain. The ordinary periodic homogenization and spectral facts used as background are stated where needed.

Theorem 1.3 (Information–complexity–prediction synthesis)**.** `[M35]` *In the physical class and sampling conventions of Theorem 1.2, the following additional conclusions hold.*

(i) *The whole response class is exactly the symmetry-constrained Schur class in Theorem 9.2. Finite-spectral physical responses correspond exactly to rational inner functions, with* $\deg\Psi = 2L^{\mathrm{pp}}_{\min}$*.*

(ii) *The global minimax error for the normalized weak-contrast coefficient is exactly* $\frac{1}{2}\prod_j |w(z_j)|^2$*. Its optimal estimator and shortest physical adversaries are constructive.*

(iii) *Every admissible response, even with infinite spectral support, has* ε*-accurate ideal laminate approximants with* $O(\log(1/\varepsilon))$ *host steps on each fixed compact disk, from the sampling scheme in Theorem 12.2. A matching worst-case exponential information lower bound holds for those fixed sampling sequences.*

(iv) *Joint query values are exactly one affine PSD section. Strict exact data admit strict query extensions, finite extremal physical responses, and arbitrarily tight rational primal–dual design certificates. Singular full data instead give a unique exact rational continuation. Explicit noisy continuation estimates require no spectral rank assumption.*

These are statements about normalized contrast data and a defined pure-host architecture. They do not identify a unique spatial geometry, promise stable exact ranks from noisy measurements, or establish a lossless-cut or full-wave closure.

Proof. Theorem 9.2 proves the analytic equivalence, and Theorem 10.2 proves its degree invariant. Theorems 12.1, 12.3, 13.1, 13.3 and Corollaries 12.2, 13.2 supply the quantitative claims. Theorems 11.1, 11.3 and Proposition 11.2 prove the prediction and certificate conclusions. Each physical realization uses the internal peeling theorem and the named uniformly coercive periodic homogenization step already present in the physical core. □

Logical architecture. The proof has four layers: the cell problem and exact physical realization; finite interpolation and rank/uniqueness; the disk representation and sharp information bounds; and proof-carrying prediction. The compiled paper contains all four. Standard spectral analysis, Lax–Milgram, periodic localization/reiteration and finite-dimensional Slater duality are named background, not claims of new foundational theorems. The supplement expands the algebra and the certificate trust boundary.

2 Conventions, cell problem and topology

For a complex matrix, A^T is the algebraic transpose, A^* the Hermitian adjoint, and $\bar{A}$ entrywise conjugation. Positivity refers only to Hermitian matrices. The Hermitian imaginary part is $(A - A^*)/(2i)$. On complex symmetric matrices it agrees with the entrywise imaginary-part matrix. We use

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{C}(A) = (\operatorname{tr} A)I - A.$$

For symmetric 2×2 matrices, $\mathcal{C}(A) = R^T A R$. On arbitrary 2×2 matrices, $\mathcal{C}(A) = R^T A^T R$; it preserves the Hermitian positive cone in dimension two. Also $\mathcal{C}^2 = 1$ and $A + \mathcal{C}(A) = (\operatorname{tr} A)I$.

Let $Y = \mathbb{R}^2/\mathbb{Z}^2$ and $\chi : Y \to \{0,1\}$ be measurable with $\langle\chi\rangle = \theta$. Put $a = \alpha\chi + \beta(1-\chi)$. For $p \in \mathbb{C}^2$, the mean-zero corrector $w_p \in H^1_\#(Y)/\mathbb{C}$ satisfies

$$\int_Y a(p + \nabla w_p)\cdot\overline{\nabla v} = 0 \quad (v \in H^1_\#(Y)/\mathbb{C}), \qquad A_{\mathrm{eff}}p = \int_Y a(p + \nabla w_p). \tag{1}$$

Lemma 2.1 (Coercivity and the spectral parameter)**.** `[P1]` *For nonzero distinct phases, a unit complex number* ω *with* $\operatorname{Re}(\omega\alpha) > 0$ *and* $\operatorname{Re}(\omega\beta) > 0$ *exists if and only if*

$$s = \frac{\beta}{\beta - \alpha} \in \Omega := \mathbb{C}\setminus[0,1]. \tag{2}$$

In this domain the corrector is unique, the effective tensor is homogeneous and holomorphic in the phases, and $A_{\text{eff}}^T = A_{\text{eff}}$.

Proof. Two nonzero complex rays fit in an open half-plane exactly when their ratio is not a nonpositive real number. The relation $\alpha/\beta = 1 - 1/s$ gives the equivalence, with zero or infinite phase values excluded separately. Multiplication of the form in (1) by ω gives a coercivity constant $c = \min \operatorname{Re}(\omega\alpha), \operatorname{Re}(\omega\beta) > 0$. Poincaré and Lax–Milgram apply. A fixed coercive rotation remains valid locally in the phase variables; the inverse operator is locally bounded and holomorphic. Homogeneity follows directly from the cell equation. For reciprocity, test the p equation with $\overline{w_q}$ and the q equation with $\overline{w_p}$, or equivalently use the associated complex bilinear identity. The resulting equal cross terms give $q^T A_{\text{eff}} p = p^T A_{\text{eff}} q$; no Hermitian symmetry of A_{eff} is used. □

Definition 2.2 (Physical closures)**.** At a fixed phase pair, $G_\theta(\alpha, \beta)$ is the matrix-norm closure of effective tensors of all such periodic indicators. Normalize phases to $(1 - 1/s, 1)$ by homogeneity. Define

$$S(s) = [I - A_{\text{eff}}(1 - 1/s, 1)]^{-1}, \quad X(s) = \frac{\theta S(s) - sI}{1 - \theta}, \quad G(s) = \frac{X(s) + sI}{s(1 - s)}. \tag{3}$$

The existence of the first inverse is proved in Lemma 3.1. The function closure is local uniform convergence of X on $\widehat{\Omega} = \widehat{\mathbb{C}} \setminus [0, 1]$, including infinity. Equivalently it is local uniform convergence of $S - sI/\theta$ there. The recovery formulas are

$$S = sI + \frac{1 - \theta}{\theta} s(1 - s)G, \qquad A_{\text{eff}}(\alpha, \beta) = \beta[I - S(s)^{-1}]. \tag{4}$$

At $\theta = 0, 1$ the closures are respectively $\{\beta I\}$ and $\{\alpha I\}$. At equal phases the tensor is that common scalar times I. These cases are not evaluated by dividing by $\theta(1 - \theta)$ or $\alpha - \beta$.

3 The matrix measure from the physical cell problem

3.1 Compression of the gradient projection

Let Γ be the orthogonal projection in the complexification of $L^2(Y; \mathbb{R}^2)$ onto mean-zero gradient fields. Its nonzero Fourier symbol is $kk^T/|k|^2$ and its zero-frequency symbol is zero. On χL^2, put $C_\chi = \chi\Gamma\chi$ and $Vp = \chi p/\sqrt{\theta}$. Then $0 \preceq C_\chi \preceq I$ and $V^*V = I$.

Lemma 3.1 (Physical compression formula)**.** `[P2]` *For* $s \in \Omega$,

$$I - A_{\text{eff}}(s) = \theta V^*(sI - C_\chi)^{-1} V. \tag{5}$$

The compressed resolvent is invertible. If C_{ab} *are the blocks of* C_χ *relative to* $\operatorname{ran} V \oplus (\operatorname{ran} V)^\perp$, *then*

$$S(s) = \frac{s}{\theta} I - A_0 + \int_{[0,1]} \frac{d\Sigma(t)}{t - s}, \quad A_0 = \frac{C_{00}}{\theta}, \quad d\Sigma = \frac{C_{01}\, dE_{C_{11}}\, C_{10}}{\theta}. \tag{6}$$

Here Σ *is positive and real symmetric,* $\operatorname{tr} A_0 = (1 - \theta)/\theta$, *and*

$$\int_{[0,1]} t^{-1}\, d\Sigma(t) \preceq A_0. \tag{7}$$

There is no mass at either endpoint in Σ.

Proof. Write the total field as $p + e$ with $e \in \operatorname{ran}\Gamma$. The equation $\Gamma[(1 - \chi/s)(p + e)] = 0$ gives $e = s^{-1}\Gamma\chi(p + e)$. Consequently $\chi(p + e) = s(sI - C_\chi)^{-1}\chi p$, which gives (5). The imaginary part of $(sI - C_\chi)^{-1}$ is $-\operatorname{Im}(s)$ times the strictly positive operator $[(C_\chi - \operatorname{Re} s)^2 + (\operatorname{Im} s)^2]^{-1}$. Compression by the isometry V stays strictly definite. For real $s > 1$ or $s < 0$ the compressed resolvent is respectively strictly positive or strictly negative. It is therefore invertible throughout Ω.

The Schur complement of $sI - C_\chi$ gives $\theta S = sI - C_{00} - C_{01}(sI - C_{11})^{-1}C_{10}$, proving (6). Positivity of $C_\chi + \varepsilon I$ implies $C_{00} + \varepsilon I \succeq C_{01}(C_{11} + \varepsilon I)^{-1}C_{10}$. Let $\varepsilon \downarrow 0$. Monotone spectral integration proves (7) and excludes a residue at zero. Apply the same argument to $I - C_\chi$ to exclude one at one.

Finally, using the Fourier coefficients and Parseval identity,

$$\operatorname{tr} C_{00} = \theta^{-1} \sum_{k \neq 0} |\widehat{\chi}(k)|^2 = \theta^{-1}(\theta - \theta^2) = 1 - \theta.$$

This also proves the required phase-fraction coefficient, rather than assuming it from a normalization convention. □

3.2 Planar duality and endpoint completion

Lemma 3.2 (Reflection identity). `[P3]` *The physical function obeys*

$$S(s) + \mathcal{C}(S(1 - s)) = I, \qquad S(\bar{s}) = \overline{S(s)}. \tag{8}$$

The residue measure in (6) *satisfies* $\Sigma(E) = \mathcal{C}(\Sigma(1 - E))$.

Proof. Rotation by R interchanges divergence-free fields and gradient fields in two dimensions, including their constant means. Applying this to current and electric field gives $A_{\text{eff}}(a^{-1}) = RA_{\text{eff}}(a)^{-1}R^T$. For the binary normalized coefficient, $a(1 - s) = a(s)^{-1}$. Thus

$$S(1 - s) = R[I - A_{\text{eff}}(s)^{-1}]^{-1}R^T = R[I - S(s)]R^T = I - \mathcal{C}(S(s)).$$

The inverses exist by coercivity and Lemma 3.1. Conjugate reality follows from the real geometry. Substitute (6) into this identity. The constant terms cancel because $\operatorname{tr} A_0 = (1 - \theta)/\theta$. The remaining Cauchy transforms identify the reflected measures. Uniqueness follows, for example, by expansion at infinity and polynomial density on $[0, 1]$. □

Definition 3.3 (Admissible measure). Let $\mathfrak{M}$ consist of positive real-symmetric 2×2 Borel matrix measures on $[0, 1]$ satisfying

$$M([0, 1]) = I, \qquad M(E) = \mathcal{C}(M(1 - E)). \tag{9}$$

Their transforms are

$$G_M(s) = \int \frac{dM(t)}{t - s}, \qquad X_M(s) = s \int \frac{1 - t}{t - s}\, dM(t) = s(1 - s)G_M(s) - sI. \tag{10}$$

The first moment $M_1 = \int t\, dM(t)$ is real symmetric with trace one.

The last trace assertion follows from $M_1 = \mathcal{C}(I - M_1)$.

Proposition 3.4 (Necessity, including all endpoint mass). `[P4]` *Every physical periodic conductivity function has the form* (10) *with* $M \in \mathfrak{M}$. *The representing measure is unique.*

Proof. Set $\Lambda = \theta\Sigma/(1-\theta)$ and $D = \theta A_0/(1-\theta)$, so $\operatorname{tr} D = 1$. Write $T_0 = \int t^{-1}\, d\Lambda$ and $Q_0 = D - T_0 \succeq 0$. Define

$$dM(t) = \frac{d\Lambda(t)}{t(1-t)} \quad (0 < t < 1), \qquad M\{0\} = Q_0, \quad M\{1\} = \mathcal{C}(Q_0).$$

The interior measure is finite: its total mass is $T_0 + \mathcal{C}(T_0) = (\operatorname{tr} T_0)I$. Adding the endpoints gives total mass $(\operatorname{tr} D)I = I$. Reflection is preserved. Its transform satisfies

$$X_M = -Q_0 + \int \left(\frac{1}{t-s} - \frac{1}{t}\right) d\Lambda = -D + \int \frac{d\Lambda}{t-s},$$

which is exactly (3). Once X is known, $G = (X + sI)/[s(1-s)]$ is known; uniqueness of its matrix Cauchy transform gives uniqueness of M. The apparent invisibility of endpoint mass in one term of X creates no ambiguity because mass and reflection are imposed. □

4 Atoms, compactness and the inverse physical map

Definition 4.1 (Paired projector atom)**.** For a real rank-one orthogonal projector P and $t \in [0,1]$, define

$$\Gamma_s(t,P) = s\left[\frac{1-t}{t-s}P + \frac{t}{1-t-s}(I-P)\right]. \tag{11}$$

Its measure is $P\delta_t + (I-P)\delta_{1-t}$. A paired atom is counted once, although it usually has two spectral support points and total spectral rank two.

Lemma 4.2 (Atom identity and measure decomposition)**.** `[G1]` *Every $M \in \mathfrak{M}$ is a probability barycenter of paired atoms, and*

$$\Gamma_s(t,P) = -[(1-t)P + t(I-P)] + t(1-t)\left[\frac{P}{t-s} + \frac{I-P}{1-t-s}\right]. \tag{12}$$

In particular $\Gamma_s(0,P) = -P$ and $\Gamma_s(1,P) = -(I-P)$.

Proof. For $N = M/2$ set $\rho = \operatorname{tr} N$; it is a probability measure. The density $dN/d\rho$ is real positive with trace one and is a measurable convex combination of two real rank-one projectors. Split it into these projectors and fold by $N + N^\sharp$, where $N^\sharp(E) = \mathcal{C}(N(1-E))$. This produces M. Conversely a probability mixture of the displayed paired measures has mass I and reflection symmetry. Equation (12) follows separately in each projector component from $s/(t-s) = t/(t-s) - 1$. □

Lemma 4.3 (Uniformly nonsingular physical pencil)**.** `[G2]` *For $M \in \mathfrak{M}$, the S in (4) is invertible on Ω. Specifically,*

$$\operatorname{sgn}(\operatorname{Im} s)\operatorname{Im} S(s) \succeq |\operatorname{Im} s| I/\theta, \qquad S(s) \succ I \ (s > 1), \qquad S(s) \prec 0 \ (s < 0).$$

On each compact subset of Ω the least singular value is bounded below uniformly over $M \in \mathfrak{M}$.

Proof. For a finite atom mixture, (12) gives a real constant and positive residues in $S - sI/\theta$; this proves the imaginary-part inequality. For a single atom the eigenvalues of S are

$$\frac{s(s-\theta t+\theta-1)}{\theta(s-t)}, \qquad \frac{s(s+\theta t-1)}{\theta(s+t-1)}.$$

For $s > 1$, subtracting one gives positive quantities $(s-1)(s-\theta t)/[\theta(s-t)]$ and $(s-1)(s-\theta(1-t))/[\theta(s+t-1)]$. For $s < 0$ both original quantities are negative. Their strict margins are uniform over $t \in [0,1]$. Barycenters preserve these bounds. Approximation and compactness then give the stated uniform assertion, including compact sets meeting the real axis. □

Proposition 4.4 (Atomic density and function compactness). `[G3]` *The class $\mathfrak{M}$ is weak-* compact. Finite-support members are dense. Weak-* convergence of M_j is equivalent to local uniform convergence of X_{M_j} on $\widehat{\Omega}$. At a fixed $s \in \Omega$,*

$$\{X_M(s) : M \in \mathfrak{M}\} = \mathrm{conv}\{\Gamma_s(t, P) : 0 \le t \le 1,\ P \text{ rank one}\}. \tag{13}$$

The hull is compact and convex in the real vector space $\mathrm{Sym}_2(\mathbb{C})$.

Proof. The fixed positive mass bounds all entry measures, and positivity, total mass and reflection are weak-* closed. Approximate a probability measure on the compact atom space by finite probability measures, then fold; all constraints remain exact. The kernel $s(1-t)/(t-s)$ is jointly continuous in s, t on compact subsets of $\widehat{\Omega} \times [0, 1]$, with value $-(1-t)$ at infinity. Uniform continuity gives compact-open convergence. Conversely compactness gives subsequential weak-* limits, and uniqueness of the transform forces all these limits to be the same. The atom set at fixed contrast is compact; its convex hull is compact in finite dimension. Lemma 4.2 supplies both inclusions in (13). □

No convexity of the *physical tensor* set is inferred through the nonlinear inverse in (4). Convexity here belongs to the normalized X coordinate.

5 Explicit realization and optimal pure-host length

5.1 The coupled layer formula

Proposition 5.1 (Binary layering with anisotropic children). `[R1]` *Let $A, B \in \mathrm{Sym}_2(\mathbb{C})$ have a common strictly accretive rotation. Laminate A at fraction $c \in [0, 1]$ with B at fraction $1 - c$, using a real unit normal n. The tensor is*

$$\mathcal{L}_n(c; A, B) = cA + (1-c)B - \frac{c(1-c)(A-B)nn^T(A-B)}{n^T[(1-c)A + cB]n}. \tag{14}$$

The denominator is nonzero. For a pure scalar host bI, with $S_b(A) = (I - A/b)^{-1}$ whenever defined,

$$S_b(\mathcal{L}_n(c; A, bI)) = \frac{S_b(A) - (1-c)nn^T}{c}. \tag{15}$$

Proof. Tangential electric field is constant across the layers and normal current is constant. Write the layer fields as $p + (1-c)an$ in A and $p - can$ in B. Equality of normal currents gives

$$a = -\frac{n^T(A-B)p}{n^T[(1-c)A + cB]n}.$$

Averaging the currents proves (14). The real part of the denominator after the common rotation is strictly positive. Formula (15) follows by the rank-one inversion identity, or by multiplying both sides by $I - \mathcal{L}_n/b$. The transpose in (14) is algebraic: replacing it by an adjoint would destroy holomorphy. □

For example, the equal mixture of $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and $B = \mathrm{diag}(1, 3)$ with $n = e_1$ has tensor $\left(\begin{smallmatrix} 4/3 & 1/3 \\ 1/3 & 7/3 \end{smallmatrix}\right)$. The arithmetic tangential value $5/2$ is incorrect. This incorporates the substantive v2.0.1 correction directly, rather than relegating it to an erratum.

Definition 5.2 (Macro layer)**.** For $M \in \mathrm{Sym}_2(\mathbb{R})$ with $M \succeq 0$ and $\operatorname{tr} M = 1$, the macro operation with core fraction c and pure host b is defined by

$$S_{\mathrm{out}} = \frac{S_{\mathrm{core}} - (1-c)M}{c}. \tag{16}$$

It is a finite sequence of rank M elementary layers. Indeed, write $M = \sum_{j=1}^{r} w_j P_j$ with $w_j > 0$, $\sum w_j = 1$, and set

$$r_j = 1 - (1-c)\sum_{k \le j} w_k, \quad r_0 = 1, \qquad c_j = r_j / r_{j-1}.$$

Repeated application of (15) gives (16) and $\prod_j c_j = c$. All fractions lie in $(0,1)$ when $0 < c < 1$.

The macro is not arithmetic averaging of physical effective tensors. It is affine only in the host-dependent S_b coordinate.

5.2 Minimal real states and complement stripping

For a finite measure M, its minimal spectral realization has

$$G(s) = V^T(T - sI)^{-1}V, \quad 0 \preceq T \preceq I, \quad V^T V = I, \qquad d = \sum_{t \in \operatorname{supp} M} \operatorname{rank} M\{t\}. \tag{17}$$

Take the direct sum of the ranges of the real weights to construct it. The input V is cyclic: the span of all $T^k V \mathbb{R}^2$ is the state space. Reflection gives a real orthogonal operator Q with

$$Q^2 = -I, \quad Q(T - \tfrac{1}{2}I) = -(T - \tfrac{1}{2}I)Q, \quad QV = VR. \tag{18}$$

On the measure model it is $Qf(t) = Rf(1-t)$. Consequently $d = 2N$ is even. For a fixed complete function, $N = d/2$ is precisely its minimum number of paired atoms: split every noncentral weight into rank-one pieces and count the scalar midpoint weight once. Any paired atom contributes at most two spectral ranks, proving the matching lower bound.

Lemma 5.3 (Complement transformation and exact degree count)**.** `[R2]` *For a finite $M \in \mathfrak{M}$, define*

$$G^{\mathsf{C}}(s) = -\frac{\mathcal{C}(G(s)^{-1})}{s(1-s)}. \tag{19}$$

It belongs to $\mathfrak{M}$, the operation is an involution, and its first moment equals that of G. Put $r = \operatorname{rank} M\{0\}$ and let N be the paired state count. Then

$$r^{\mathsf{C}} = 2 - r, \qquad N^{\mathsf{C}} = N + 1 - r. \tag{20}$$

Thus $2N - r$ is invariant under complement.

Proof. The compression $G(s)$ is nonsingular off $[0,1]$ by its strict imaginary sign, or its strict real sign outside the interval. Split the minimal real state into $\operatorname{ran} V \oplus V^{\perp}$:

$$T = \begin{pmatrix} M_1 & B_1^T \\ B_1 & T_1 \end{pmatrix}, \qquad G(s)^{-1} = M_1 - sI - B_1^T (T_1 - sI)^{-1} B_1.$$

The compression T_1 satisfies $0 \prec T_1 \prec I$. An endpoint eigenvector of this compression would, by positivity of T or $I - T$, be an endpoint eigenvector of T orthogonal to V, contradicting cyclicity. The pair (T_1, B_1) is cyclic by the same argument for every eigenvalue.

For each eigenvalue $\lambda \in (0,1)$ of T_1, the coefficient of $(\lambda - s)^{-1}$ in (19) is

$$\frac{\mathcal{C}(B_1^T E_\lambda B_1)}{\lambda(1-\lambda)} \succeq 0.$$

Its rank is $\dim E_\lambda$. At zero the coefficient is $\mathcal{C}(E_0)$, where $E_0 = \lim_{s\uparrow 0} G(s)^{-1} \succeq 0$. To determine its rank, decompose the input into $\operatorname{ran} W_0 \oplus \ker W_0$, $W_0 = M\{0\}$. In $G(s) = W_0/(-s) + G_0 + O(s)$, G_0 is strictly positive on $\ker W_0$. Block inversion shows that E_0 vanishes on $\operatorname{ran} W_0$ and is strictly positive on its complement; hence $\operatorname{rank} E_0 = 2 - r$. Reflection gives the complementary positive endpoint weight at one.

The function is real and satisfies the reflection identity. At infinity, $G = -I/s - M_1/s^2 + O(s^{-3})$ implies $G^{\complement} = -I/s - M_1/s^2 + O(s^{-3})$, since $\operatorname{tr} M_1 = 1$. Thus the positive pole coefficients have total mass I; there is no polynomial part. The interior spectral dimension is $\dim T_1 = 2N - 2$, while the two endpoints contribute $2(2 - r)$. This proves (20). Direct substitution, using $\mathcal{C}(A^{-1}) = \mathcal{C}(A)^{-1}$ for invertible 2×2 matrices, proves the involution. □

Algebraically, $G^{\complement}$ corresponds to complementing the physical indicator and replacing θ by $1 - \theta$. More explicitly, if $r_s = 1 - 1/s$ then the complemented tensor is $r_s A_{\text{eff}}(1 - s)$, and its S coordinate is

$$S^{\complement}(s) = sI - \frac{\theta}{1-\theta}\mathcal{C}(G(s)^{-1}). \tag{21}$$

These identities follow from (4) by multiplication of 2×2 matrices; they do not assume a spatial realization of the formal measure.

5.3 A terminating physical algorithm

Theorem 5.4 (Internal finite realization and sharp host length). `[R3]` *Every finite $M \in \mathfrak{M}$ is realized, at every $0 < \theta < 1$, by a pure-phase-host sequential laminate. For its complete response,*

$$L^{\text{pp}}_{\min}(G) = 2N - \operatorname{rank} M\{0\} = \operatorname{rank} T. \tag{22}$$

The realization uses at most N macro steps, including an optional initial endpoint removal, and every step preserves the prescribed global fraction.

Proof. Endpoint removal. If $W_0 = M\{0\} \neq 0$, set

$$q = \operatorname{tr} W_0, \quad M_0 = W_0/q, \quad \theta' = \theta + (1-\theta)q, \quad c = \theta/\theta'. \tag{23}$$

The endpoint pair has total mass qI, so $0 < q \le 1$. If $q < 1$, subtract that pair and divide the remaining measure by $1 - q$ to obtain $M' \in \mathfrak{M}$. Its paired state count is $N - \operatorname{rank} W_0$, and it has no endpoint mass. Apply the construction recursively to the core with fraction θ', then use the macro (16) with host phase 2, fraction c and lamination matrix M_0. The relation

$$X = (1-q)X' - qM_0$$

and (15) verify the response, while $c\theta' = \theta$ verifies the fraction. If $q = 1$, the core is pure phase 1; the same macro produces $X = -M_0$ without using an undefined pure-phase normalized coordinate.

No endpoint. If $W_0 = 0$, the state is strictly between zero and one. Put

$$E = G(0)^{-1} \succ 0, \quad q = \operatorname{tr} E, \quad M_0 = \mathcal{C}(E)/q, \quad \theta' = 1 - \theta + \theta q, \quad c = (1-\theta)/\theta'. \tag{24}$$

Complement the response, remove its endpoint pair $\mathcal{C}(E), E$, and divide by $1 - q$. When $q < 1$, Lemma 5.3 says that this new core has paired state count $N - 1$ and no endpoints. Realize that core with phase labels interchanged and phase-1 fraction θ', and surround it by pure original phase 1 using M_0, c. Formula (21) and endpoint removal prove the response identity. The original fraction of the core is $1-\theta'$, so the final fraction is $c(1-\theta')+(1-c)=\theta$. If $q=1$, there is no interior residue after complement; cyclicity of (T_1, B_1) implies $N=1$, and the core is pure original phase 2.

The integer N strictly decreases in every nonterminal call, so termination is proved. Endpoint removal costs $\operatorname{rank} W_0$ elementary steps. Each subsequent endpoint-free removal costs two because $E \succ 0$. The resulting length is $\operatorname{rank} W_0 + 2(N - \operatorname{rank} W_0) = 2N - \operatorname{rank} W_0$.

Optimality. Define $\Phi(G) = 2N - \operatorname{rank} M\{0\}$. Adding a pure phase-2 elementary host to a nonpure core replaces its measure by

$$M_{\text{out}} = \lambda M_{\text{core}} + (1-\lambda)(P\delta_0 + \mathcal{C}(P)\delta_1), \qquad \lambda = \frac{1-\theta_{\text{core}}}{1-c\theta_{\text{core}}} \in (0,1).$$

All interior ranks stay unchanged. The endpoint rank increases by $\varepsilon \in \{0,1\}$, and N increases by the same ε. Thus Φ increases by at most one. A phase-1 host has the same property because complement preserves Φ. The first nontrivial layer has $N=1$, endpoint rank one and $\Phi = 1$. Hence every length-L construction satisfies $\Phi \le L$. The constructed length attains this bound. Finally the only zero eigenvalues of the minimal T have multiplicity $\operatorname{rank} M\{0\}$, so $\operatorname{rank} T = \Phi$. □

Corollary 5.5 (One atom)**.** [R4] *The atom $\Gamma_s(t,P)$ is realized by an inner laminate of the phases with phase-1 fraction $f=\theta t$ and normal $I-P$, followed by a phase-1-host laminate of that core at fraction $c=(1-\theta)/(1-\theta t)$ and normal P. The global fraction is $cf+(1-c)=\theta$. Interior atoms need two elementary host steps; endpoint atoms need one.*

Proof. In the $P, I-P$ frame the inner entries are $(s-\theta t)/s$ and $(s-1)/(s-1+\theta t)$. Applying the coupled formula, which here is diagonal, gives outer entries

$$\frac{(s-1)(s-\theta t)}{s(s-\theta t+\theta-1)}, \qquad \frac{(s-1)(s+\theta t-\theta)}{s(s+\theta t-1)}.$$

Substitution into (3) gives (11). The endpoint stages degenerate to a simple laminate. Minimality follows from (22). □

5.4 From finite hierarchies to periodic closure

Proposition 5.6 (Periodic realization with exact fractions)**.** [R5] *A finite hierarchy with real normals and common rotated coercivity belongs to the ordinary periodic G-closure at its exact phase fraction. Its entire conductivity function is the local uniform limit of one sequence of periodic two-phase indicators having that exact fraction at every index.*

Proof. We use the standard periodic cell homogenization theorem, its localization and reiterated versions for uniformly bounded uniformly accretive divergence-form systems, and the uniform higher-integrability estimate $\|p+\nabla w_p\|_{L^p} \le C|p|$ for some $p>2$. These are ordinary homogenization and Meyers inputs, not a laminate-completeness classification; see [1, 13]. Complex coefficients are treated by their realified uniformly coercive system.

For a rational normal, form stripes on the torus and put rescaled periodic child cells in each stripe. As the child periods tend to zero, localization and reiterated homogenization give the piecewise homogenized child tensors and then their full coupled layer tensor (14). The stripe boundary intersects only a vanishing fraction of fine cells, so the average phase fraction tends to its prescribed value. Correct any discrepancy by changing one phase to the other on a measurable set of exactly the discrepant measure.

To justify this last operation, for two coefficients differing on a set E, the primal/adjoint corrector identity and Hölder's inequality give

$$\|A_{\text{eff}}(a) - A_{\text{eff}}(b)\| \le C|E|^{1-2/p}. \tag{25}$$

The uniform higher-integrability constants depend only on coercivity and bounds, not on the geometry. Thus the fraction correction changes no limit. Finite induction and diagonal selection handle all hierarchy levels. Approximate arbitrary normals by rational ones; continuity of (14), with denominators uniformly separated from zero, completes the fixed-contrast proof.

For the function statement, use a countable dense set of contrasts in Ω and diagonalize the same geometric approximants. Proposition 3.4 and compactness of $\mathfrak{M}$ give a normal family in the X coordinate, including infinity. Every subsequential function limit agrees with the target on the dense set, hence everywhere. This yields one phase-independent sequence and local uniform convergence. The supplement gives the explicit primal/adjoint identity behind (25). $\square$

Theorem 5.7 (Complete physical function and fixed-contrast closure)**.** `[P5]` *The normalized physical conductivity-function closure at every $0 < \theta < 1$ is exactly $\{X_M : M \in \mathfrak{M}\}$. At a fixed allowed phase pair,*

$$G_\theta(\alpha, \beta) = \left\{ \beta \left[I - \left(\frac{sI + (1-\theta)X}{\theta} \right)^{-1} \right] : X \in \operatorname{conv}\{\Gamma_s(t, P)\} \right\}. \tag{26}$$

Every finite measure has an exact ideal sequential-laminate response. General measures are local uniform limits of such responses. At fixed contrast, a finite hierarchy always suffices in the periodic closure sense.

Proof. Necessity is Proposition 3.4. Finite-measure sufficiency is Theorem 5.4, and its membership in the required periodic closure is Proposition 5.6. Apply the density and compactness of Proposition 4.4 for general measures. At fixed contrast, (13) and the uniformly invertible continuous map in Lemma 4.3 give (26). Conversely any sequence of physical tensors has a subsequence of their normalized measures converging weak-*; therefore its fixed-contrast limit stays in the displayed set. This last compactness step is required; continuity of evaluation alone would not prove it. $\square$

6 Finite interpolation and one positive-semidefinite matrix

Definition 6.1 (Orbit-complete reciprocal data)**.** Let $n \ge 1$. Nodes z_j are distinct and have $\operatorname{Im} z_j > 0$. Their values obey $G_j^T = G_j$. An involution p satisfies

$$z_{p(j)} = 1 - \bar{z}_j, \qquad G_{p(j)} = -\mathcal{C}(\bar{G}_j).$$

A node on $\operatorname{Re} z = 1/2$ is its own partner. Input samples outside the upper half-plane are conjugated there; real samples require the separate treatment in Section 17. Duplicate inconsistent values, nonreciprocal values and incompatible partners are rejected before a positivity test.

Use block indices $0, 1, \dots, n$, each of size two, and define

$$B_{00} = I, \quad B_{0j} = G_j, \quad B_{i0} = G_i^*, \quad B_{ij} = \frac{G_j - G_i^*}{z_j - \bar{z}_i} \quad (i, j \ge 1). \tag{27}$$

For a Hermitian first-moment variable M_1, define

$$A_{00} = M_1, \quad A_{0j} = I + z_j G_j, \quad A_{ij} = \frac{z_j G_j - \bar{z}_i G_i^*}{z_j - \bar{z}_i}, \quad A_{i0} = A_{0i}^*. \tag{28}$$

Write

$$A(M_1) = \begin{pmatrix} M_1 & F \\ F^* & K \end{pmatrix}, \qquad u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad H = \begin{pmatrix} \frac{1}{2} & u^*F \\ F^*u & K \end{pmatrix}. \tag{29}$$

For later references we also designate the final definition as

$$\boxed{H = H(G_1, \dots, G_n), \quad \operatorname{order}(H) = 2n+1.} \tag{30}$$

The map to H is affine over the real structured data space. No pseudoinverse, optimization variable, or unknown derivative occurs in this matrix. The code uses the congruent matrix with upper-left entry 1 and $w = (1, i)^T$ in place of u; this removes square roots without changing positivity, rank, or singularity.

Theorem 6.2 (Finite interval interpolation). `[I1]` *For arbitrary proposed 2×2 values at distinct upper-half-plane nodes, a positive Hermitian measure on $[0,1]$ with mass I interpolates the values if and only if some Hermitian M_1 satisfies $0 \preceq A(M_1) \preceq B$. A realizing positive-contraction model can be chosen with dimension $d = \operatorname{rank} B$, the minimum over all such models.*

Proof. Put $f_0(t) = 1$ and $f_j(t) = (t - z_j)^{-1}$. For a representing measure, $B_{ij} = \int \bar{f}_i f_j \, dM$ and $A_{ij} = \int t \bar{f}_i f_j \, dM$; divided differences give (27) and (28). Thus $0 \preceq A \preceq B$.

Conversely equip $(\mathbb{C}^2)^{n+1}/\ker B$ with its B inner product. Since $0 \preceq A \preceq B$, the form A descends and defines a positive contraction T. Let $v_j u$ denote the class of a vector supported in block j, and set $Vu = v_0 u$. Direct substitution gives $A_{ij} - z_j B_{ij} = B_{i0}$, hence

$$(T - z_j I)v_j = V, \qquad v_j = (T - z_j I)^{-1} V, \qquad V^*V = I.$$

The spectral measure $V^* E_T V$ interpolates. Every model has the displayed sample vectors with Gram matrix B, so its dimension is at least $\operatorname{rank} B$. The constructed model is cyclic, since its sample vectors span the whole state space. This is a finite linear-algebra proof of the special interpolation theorem used here, rather than an appeal to an unspecified Pick theorem. □

6.1 The reciprocity reduction of the Schur quantity

The following lemma is the reciprocity reduction introduced in v3.0.0 and retained as a core ingredient here. It is important that the values are complex symmetric; the statement is not valid for arbitrary nonreciprocal matrix data.

Lemma 6.3 (Real Schur quantity and redundant chirality). `[I2]` *For reciprocal values at distinct upper-half-plane nodes, $H \succeq 0$ implies*

$$K \succeq 0, \qquad F \ker K = 0, \qquad S_0 := F K^\dagger F^* \in \operatorname{Sym}_2(\mathbb{R}), \qquad \operatorname{tr} S_0 \le 1. \tag{31}$$

Conversely these conditions imply $H \succeq 0$. In particular the analogous matrix formed with $\bar{u}$ is positive if and only if H is positive. More generally, S_0 is real whenever $K \succeq 0$ and the full range condition $F \ker K = 0$ holds.

Proof. The proof uses a signed measure only as an algebraic device. Write $p(t) = \prod_{j=1}^n (t - z_j) = t^n + \sum_{\ell=0}^{n-1} a_\ell t^\ell$. The $2n+1$ real functions $1, \operatorname{Re} f_j, \operatorname{Im} f_j$ are linearly independent: multiplying a real linear relation by $|p(t)|^2$ gives a real polynomial of degree at most $2n$, and the simple poles at $z_j, \bar{z}_j$ force all coefficients to vanish. Their evaluations at any $2n+1$ distinct real points are therefore nonsingular, since a polynomial of degree at most $2n$ with that many zeros vanishes. Entrywise solve these real interpolation equations for a finite *signed* real-symmetric matrix measure of mass I realizing the proposed values. No positivity is needed for this step.

Replace the basis f_j by $q_k = t^k/p(t)$, $0 \le k < n$. Partial fractions give an invertible scalar change of basis. In the new basis,

$$\widehat{K}_{jk} = \int \frac{t^{j+k+1}}{|p(t)|^2}\, dM(t), \qquad \widehat{F}_k = \int \frac{t^{k+1}\bar{p}(t)}{|p(t)|^2}\, dM(t).$$

Thus $\widehat{K}$ is a real block-Hankel symmetric matrix with symmetric 2×2 blocks, and

$$\widehat{F} = D + \mathcal{A}\widehat{K}, \qquad D_k = \int \frac{t^{n+k+1}}{|p(t)|^2}\, dM(t), \qquad \mathcal{A} = (\bar{a}_0 I, \ldots, \bar{a}_{n-1} I).$$

Every block D_k is real symmetric. Positivity of the congruently transformed H implies $\widehat{K} \succeq 0$ and $u^* D \ker \widehat{K} = 0$. Since D and $\widehat{K}$ are real, conjugating gives the same statement with $\bar{u}$. These two vectors span $\mathbb{C}^2$, so $D \ker \widehat{K} = 0$ and the full range condition follows.

On that range the invariant Schur quantity equals

$$S_0 = D\widehat{K}^{\dagger} D^T + D\mathcal{A}^* + \mathcal{A}D^T + \mathcal{A}\widehat{K}\mathcal{A}^*. \tag{32}$$

The first term is real. In the middle pair each symmetric block D_k is multiplied by $a_k + \bar{a}_k$. In the last term interchange j, k: the equality $\widehat{K}_{jk} = \widehat{K}_{kj}$ pairs conjugate coefficients. This proves reality. As $u^* S_0 u = \operatorname{tr} S_0/2$, the remaining scalar Schur inequality for H is exactly $\operatorname{tr} S_0 \le 1$. The singular Schur-complement proof is given in the supplement and uses precisely the established range condition. Reversing it proves the converse. □

Theorem 6.4 (One-PSD physical feasibility). `[I3]` *Orbit-complete reciprocal data are physically feasible if and only if $H \succeq 0$. Put $\Delta = 1 - \operatorname{tr} S_0$ when the lower-block and range conditions hold. All feasible first moments are exactly the disk*

$$M_1 = S_0 + \frac{\Delta}{2} I + xE_1 + yE_2, \quad x^2 + y^2 \le (\Delta/2)^2, \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

The center is $M_c = S_0 + (\Delta/2)I$. If $K \succ 0$, it uniquely maximizes $\det A(M_1)$ over the nonempty feasible disk. A distinguished, coordinate-dependent boundary choice is

$$M_{\text{short}} = S_0 + \operatorname{diag}(\Delta, 0). \tag{34}$$

Proof. For orbit-complete data define the real orthogonal block matrix U by $U_{00} = R$, $U_{p(j),j} = -R$, with other blocks zero. Direct substitution, using *entrywise conjugation*, gives

$$U^2 = -I, \qquad B - A(M_1) = U^T \overline{A(M_1)} U \tag{35}$$

whenever M_1 is real symmetric and has trace one. Therefore only $A \succeq 0$ remains. The singular Schur criterion reduces it to $K \succeq 0$, $F \ker K = 0$, and $M_1 \succeq S_0$. Lemma 6.3 proves that H tests exactly the existence of such a real trace-one matrix. A real symmetric trace-Δ positive matrix is precisely $(\Delta/2)I + xE_1 + yE_2$ with the displayed disk inequality.

Theorem 6.2 supplies a finite Hermitian measure. For existence alone one may average it with its transpose and then its reflected complement: reciprocity and the orbit identities preserve the samples and the first moment. The resulting real phase-symmetric measure is physical by Theorem 5.7. The next theorem proves that these potentially rank-increasing averages are unnecessary for a minimum model.

Finally $\det A = \det K\, \det(M_1 - S_0) = \det K[(\Delta/2)^2 - x^2 - y^2]$ when $K \succ 0$. This proves the determinant assertion, including the singleton disk. It is not an assertion about an empty feasible set. □

6.2 Proof-carrying rejection and exact arithmetic

Proposition 6.5 (A single dual witness). [I4] *If the structural hypotheses hold but $H \not\succeq 0$, a vector $y = (y_0, y_1)$ with $y^*Hy < 0$ provides a separator. Set $x = (y_0 u, y_1)$. For every real symmetric trace-one M_1,*

$$x^* A(M_1)x = y^*Hy < 0.$$

Thus the rank-one positive matrix xx^* separates all possible completions. Negative lower-block, range-failure, and trace-defect obstructions are all instances of this test. Domain and reciprocity failures are separately checkable violations of the problem's hypotheses.

Proof. For real symmetric trace-one M_1, $u^*M_1u = 1/2$. Substitute in the quadratic form. The dual cone of Hermitian positive-semidefinite matrices is the same cone; the rank-one form is enough to detect a negative eigenvalue. Even when the older range test failed without a convenient strict separating block, the augmented H is not positive and has a strict negative quadratic form. □

The rational implementation returns either an exact congruence factor $H = LDL^*$ with D diagonal nonnegative, or an explicit Gaussian-rational negative vector. It does not infer exact semidefiniteness from a floating-point eigensolver. The certificate verifier rebuilds the matrices independently from the input samples and checks every identity. A fixed-size H is canonical for the chosen orbit list, but no claim is made that $2n + 1$ is the globally smallest possible semidefinite representation among all nonlinear changes of variables or lifted descriptions.

7 Minimum states, reciprocity and balanced continued fractions

Theorem 7.1 (Rank-preserving reciprocal realization). [S1] *For every feasible M_1 in* (33)*, the quotient realization of Theorem 6.2 has a real-symmetric phase-symmetric measure* without averaging. *Its complex dimension is $d = \operatorname{rank} B$, which is even. It has the smallest possible paired atom count*

$$N_{\min} = \frac{1}{2}\operatorname{rank}_{\mathbb{C}} B. \tag{36}$$

All minimum-state structured rational interpolants are parametrized, without duplication, by (33)*; realizations of the same complete function are equivalent by a unitary change of state basis.*

Proof. The quotient response $\mathcal{G}(z) = V^*(T - zI)^{-1}V$ has a common denominator dividing $\det(T - zI)$ of degree at most $d \le 2n + 2$. Because its mass and first moment are the symmetric matrices I, M_1, $\mathcal{G}(z) - \mathcal{G}(z)^T = O(z^{-3})$ at infinity. After multiplication by that denominator each skew entry has numerator degree at most $d - 3 \le 2n - 1$. It vanishes at the n supplied nodes. Hermitian reality $\mathcal{G}(\bar{z}) = \mathcal{G}(z)^*$ and symmetric sample values make it vanish at their n distinct conjugates as well. Thus it has $2n$ zeros and is identically zero. Every spectral residue is both Hermitian and symmetric, hence real symmetric. This argument repairs the rank gap that would arise from averaging residues after minimization.

Factor $B = C^*C$ on its range. Equations (35) give $B = U^T\bar{B}U$, and the map $J(C\xi) = CU\bar{\xi}$ is well-defined antiunitary with

$$J^2 = -I, \qquad JTJ^{-1} = I - T, \qquad JV = VR\,(\text{conjugation on the input}).$$

The last two identities pair the spectral weights by $W_{1-t} = \mathcal{C}(W_t)$. All nonmidpoint eigenvalues are paired, and the midpoint eigenspace has even dimension. Hence d is even and the measure belongs to $\mathfrak{M}$.

For a complete finite measure, its cyclic state dimension is $\sum_t \operatorname{rank} W_t$. At each pair $t, 1-t$, spectral splitting of W_t uses exactly $\operatorname{rank} W_t$ paired atoms; a nonzero midpoint weight is a scalar multiple of I and uses one. Therefore the minimum paired atom count is half the state dimension. Every interpolating response has sample Gram rank $\operatorname{rank} B$, giving the lower bound in (36); the quotient attains it.

A minimum model's sample vectors span its whole space. Their Gram matrix and the matrix elements of T are exactly $B, A(M_1)$, so its function is fixed by M_1. Distinct first moments are coefficients of different complete functions. Cyclic spectral realizations of a fixed matrix measure are unitarily equivalent by the map on polynomial vectors. This proves the parametrization and uniqueness assertions. □

Theorem 7.2 (Balanced real block-Jacobi normal form)**.** `[S2]` *A finite $M \in \mathfrak{M}$ with N paired atoms in a minimum representation has a unique input-framed real block-Jacobi realization*

$$T = \tfrac{1}{2}I + \begin{pmatrix} D_1 & b_1 I & & \\ b_1 I & D_2 & \ddots & \\ & \ddots & \ddots & b_{N-1}I \\ & & b_{N-1}I & D_N \end{pmatrix}, \qquad V = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{37}$$

where $D_j \in \mathrm{Sym}_2(\mathbb{R})$ is traceless and $b_j > 0$, and $0 \preceq T \preceq I$. An infinite-support measure has the analogous bounded infinite chain, with $\|T - I/2\| \le 1/2$. No one-dimensional block deflation occurs.

Proof. On the real cyclic space $L^2(M)$, define $(Qf)(t) = Rf(1-t)$. Reflection of the measure proves that Q is orthogonal, $Q^2 = -I$, $QV = VR$, and $Q(T - I/2) = -(T - I/2)Q$. The block Krylov spaces $\mathcal{K}_m = \operatorname{span}\{V, (T - I/2)V, \ldots, (T - I/2)^{m-1}V\}$ are Q-invariant. Their increments have real dimension at most two and must be even. If an increment vanishes, the previous space is invariant under T; cyclicity then makes it the whole space. Consequently $\dim \mathcal{K}_m = 2\min(m, N)$ in the finite case, and $2m$ in the infinite case.

Run real block Lanczos starting with V. The residual Gram matrix at every nonterminal step is real symmetric and commutes with the input rotation R, hence is $b_j^2 I$. Taking its positive square root fixes $b_j > 0$ and the next block frame. The compressed diagonal blocks of $T - I/2$ anticommute with R, so they are real symmetric and traceless. Induction fixes the chain uniquely in the original input frame. Spectral positivity supplies $0 \preceq T \preceq I$. Conversely such a chain has the required alternating complex structure and hence a measure in $\mathfrak{M}$. □

The Schur-complement continued fraction is

$$G(z) = \left[D_1 + (1/2 - z)I - b_1^2 \left(D_2 + (1/2 - z)I - \cdots\right)^{-1}\right]^{-1}.$$

It is canonical for a *complete function*, not for arbitrary finite measurements unless uniqueness holds. This structure is closely related to the established theory of complex symmetric anti-linear Jacobi operators and biradial measures [12, 17, 6]. We do not claim the underlying anti-linear tridiagonalization or a new Clifford algebra as a discovery. The conductivity-specific pairing, sampling consequences and physical length comparison are the uses made here.

Corollary 7.3 (Exact sample-span ranks)**.** `[S3]` *For a finite complete response with minimum paired count N, and n distinct upper-half-plane sample nodes, the constant-plus-resolvent Gram matrix and the resolvent-only Gram matrix have ranks*

$$\operatorname{rank} B = 2\min(n+1, N), \qquad \dim \operatorname{span}\{(T - z_j I)^{-1}V : 1 \le j \le n\} = 2\min(n, N).$$

Proof. Multiplication by $p_n(T) = \prod_j (T - z_j I)$, which is invertible, sends the resolvent-only span to $\mathcal{K}_n$ and the constant-plus-resolvent span to $\mathcal{K}_{n+1}$. Partial fractions or Lagrange interpolation verifies both equalities. Apply Theorem 7.2. The same ranks are $2n + 2$ and $2n$ for an infinite-support measure. □

8 Uniqueness, sharp sampling and shortest finite-data design

Theorem 8.1 (Complete identifiability criterion). `[U1]` *For feasible orbit-complete reciprocal data, the following are equivalent:*

(i) There is exactly one representing measure in $\mathfrak{M}$.
(ii) There is exactly one complete analytic response in the physical function closure.
(iii) $\Delta = 0$ in (33).
(iv) H is singular.

If these conditions hold the unique measure is finite. If $H \succ 0$, the disk in (33) has positive radius and already parametrizes infinitely many distinct minimum-state rational interpolants. This is function and measure identifiability, not spatial geometry identifiability.

Proof. Measure and function uniqueness are equivalent by the Cauchy-transform uniqueness proof. If $\Delta > 0$, Theorem 7.1 provides distinct responses from distinct first moments, so uniqueness fails.

Suppose $\Delta = 0$. Let $\mathcal{H}$ be the $d = \operatorname{rank} B$ dimensional quotient with its unique first moment $M_1 = S_0$ and contraction T. In any other representing measure space, the sample vectors embed $\mathcal{H}$ isometrically as a subspace, and the compression of the multiplication operator $\widetilde{T}$ is this same T. Write its coupling from $\mathcal{H}$ to $\mathcal{H}^\perp$ as D. The exact relations $(\widetilde{T} - z_j)v_j = V$ force D to vanish on the resolvent-only space $\mathcal{L} = \operatorname{span}\{v_j\}$. By Corollary 7.3, either $d \le 2n$ and $\mathcal{L} = \mathcal{H}$, or $d = 2n + 2$ and $\mathcal{L}$ has codimension two.

In the first case $D = 0$. In the second, K is positive definite. To prove that assertion, if $w \in \mathcal{L}$ has $\langle Tw, w\rangle = 0$, positivity gives $Tw = 0$. Then $p_n(T)w = p_n(0)w$ belongs to the first n Jacobi blocks. An eigenvector of an irreducible length-$(n + 1)$ block-Jacobi chain cannot have last block zero: the last row forces the preceding block to be zero, and backward induction forces the vector to vanish. Thus $w = 0$, proving $K \succ 0$. Now $M_1 = S_0$ implies $\operatorname{rank} A = \operatorname{rank} K = 2n$ and therefore $\dim \ker T = 2$. Its projection onto $\mathcal{L}^\perp$ is injective by the preceding argument and hence surjective. Positivity of $\widetilde{T}$ forces D to annihilate $\ker T$: a vector with zero nonnegative quadratic form is in the kernel. Since D also annihilates $\mathcal{L}$, it vanishes on all of $\mathcal{H}$.

Consequently $\mathcal{H}$ reduces $\widetilde{T}$. Because V lies in $\mathcal{H}$, the entire compressed resolvent, not just the samples, equals that of T. The unique measure is its finite spectral measure.

It remains to relate Δ to H. If $\Delta = 0$, the scalar Schur complement of H is zero, so H is singular. If $\Delta > 0$ and B were singular, Corollary 7.3 would give $\mathcal{L} = \mathcal{H}$ for every minimum realization. The operator is then fixed by $(T - z_j)v_j = V$, which would fix its first moment, contradicting the positive-radius disk. Thus $B \succ 0$. The same Jacobi argument just given proves $K \succ 0$, and the scalar Schur complement of H is $\Delta/2 > 0$. Hence $H \succ 0$. □

Corollary 8.2 (Sharp uniform finite identification). `[U2]` *Let a complete response have minimum paired count N. Any orbit-complete set of $n \ge N$ distinct upper-half-plane samples identifies it uniquely, even among measures with no prior state bound. The threshold $n = N$ is sharp uniformly over this class: for $n = N - 1 \ge 1$, a strictly interior N-block Jacobi contraction has nonunique interpolants. With g non-self-dual input nodes from distinct orbits and h self-dual input nodes, $2g + h \ge N$ is sufficient. Thus $\lceil N/2 \rceil$ generic non-self-dual measurements suffice, while N self-dual measurements suffice.*

Proof. If $n \geq N$, the sample resolvents span the complete cyclic space. In any other realization the operator relations determine the same compression and annihilate its coupling, as above, so the whole measure is fixed. For sharpness take an N-block Jacobi contraction with spectrum strictly in $(0, 1)$ and all off-diagonal scalars positive. For $n = N - 1$, B and A are positive definite, so $K \succ 0$ and $M_1 - S_0 \succ 0$. Thus $\Delta > 0$ and there are distinct interpolants. Generic orbit completion doubles the number of measurements; self-dual completion does not. These counts are uniform guarantees, not necessary counts at special endpoint-degenerate responses. For $N = 1$ one nonreal measurement is sufficient; zero measurements do not identify the response. □

Corollary 8.3 (Response-sensitive exact sampling threshold). `[U3]` *For a fixed finite complete response with paired count N and $r_0 = \operatorname{rank} M\{0\}$, an orbit-complete list of $n \geq 1$ distinct upper-half-plane samples identifies it uniquely if and only if*

$$n \geq N - \mathbf{1}_{\{r_0=2\}}.$$

Thus full-rank endpoint mass saves exactly one completed sample. No finite list identifies an infinite-support response uniquely.

Proof. For $n \geq N$ use Corollary 8.2. For $n = N - 1$, the sample space is the full $2N$-dimensional state space and $K \succ 0$ by the no-last-block-zero Jacobi argument. Thus $\operatorname{rank}(M_1 - S_0) = \operatorname{rank} T - \operatorname{rank} K = (2N - r_0) - (2N - 2) = 2 - r_0$. Its positive slack has zero trace precisely when $r_0 = 2$. The uniqueness theorem proves the assertion in this case. If $n \leq N - 2$ and the measure were unique, the finite quotient of dimension at most $2n + 2$ would realize that same entire measure, contradicting its minimum dimension $2N$. A unique measure from finite data is finite by the same quotient argument, excluding infinite support. □

Theorem 8.4 (Exact rank-to-architecture theorem). `[C1]` *For feasible data, the minimum elementary pure-phase-host length over all interpolating responses is*

$$\boxed{L^{\mathrm{pp}}_{\min} = \operatorname{rank} H = \operatorname{rank} K + \mathbf{1}_{\{\Delta>0\}}.} \tag{38}$$

It is attained by the minimum-state quotient using (34)*, followed by endpoint/complement peeling. Thus the paired atom and host-length minima are attained simultaneously. If $H \succ 0$, then*

$$N_{\min} = n + 1, \qquad L^{\mathrm{pp}}_{\min} = 2n + 1,$$

whereas the determinant-maximizing center has length $2n + 2$.

Proof. For any finite realizing measure, $A(M_1)$ is the Gram matrix of $\sqrt{T}$ applied to the sample vectors, so $\operatorname{rank} A \leq \operatorname{rank} T$. Theorem 5.4 identifies $\operatorname{rank} T$ with the minimum host length of that complete response; every host construction consequently has length at least $\operatorname{rank} A$. The singular Schur factorization yields

$$\operatorname{rank} A = \operatorname{rank} K + \operatorname{rank}(M_1 - S_0).$$

When $\Delta = 0$ the last rank is zero. When $\Delta > 0$ it is at least one and equals one precisely on the boundary circle of (33). The boundary choice (34) therefore minimizes it. Its quotient has the full sample span as its state space, so $\operatorname{rank} T = \operatorname{rank} A$ and peeling attains the lower bound. The scalar Schur factorization of H gives the final equality in (38). If $\Delta > 0$, Theorem 8.1 gives full ranks for B, K, H; the center's two-dimensional positive slack adds two, rather than one, to $\operatorname{rank} K$. □

Proposition 8.5 (No rotation-equivariant shortest selector). `[C2]` *There is no rule defined for every strictly feasible data set that selects a shortest minimum-state response and commutes with every spatial rotation. A coordinate-dependent shortest rule exists, such as* (34)*; the determinant center is rotation-equivariant but may use one extra host step.*

Proof. Take strictly feasible isotropic data, $G_j = g_j I$. Their Schur quantity is scalar, so the disk center is $I/2$ and its radius is positive. A rule invariant under all rotations must select a first moment commuting with every rotation; a real symmetric trace-one such matrix is $I/2$. But this is an interior point of the disk, whereas every shortest response lies on its boundary by Theorem 8.4. This is a contradiction. The exact example below supplies such strictly feasible isotropic data. □

Theorem 8.6 (Rational proof-carrying physical synthesis). `[C3]` *For Gaussian-rational nodes and values satisfying the hypotheses, exact rational arithmetic decides feasibility. For feasible data and rational* $0 < \theta < 1$*, it produces an all-contrast real rational matrix function and a shortest physical macro tree with rational fractions and rational real positive trace-one lamination matrices. An independent verifier needs only rational arithmetic, polynomial identity checks, and positive-semidefinite congruence factors.*

Proof. All entries of B, F, K and the unnormalized H are Gaussian-rational. Exact pivoted Hermitian elimination either exhibits a positive diagonal congruence factor or a rational vector with negative quadratic form. In a zero-diagonal nonzero-off-diagonal residual block, a sufficiently small rational component opposite that off-diagonal produces such a negative vector; no algebraic eigenvalue is required. A positive principal submatrix spanning $\operatorname{ran} K$ gives a rational range inverse and hence rational S_0. Its reality is exact by Lemma 6.3. The choice (34) is therefore rational.

Likewise choose independent Gram columns indexed by J, with $B_{JJ} \succ 0$, to represent the quotient in a rational, nonorthonormal basis. Its Gram matrix is B_{JJ} and the operator matrix is $B_{JJ}^{-1} A_{JJ}$. The input coordinates are obtained by a rational linear solve. The resulting transfer function has rational coefficients; reciprocity, reality and phase symmetry are identities proved above, not numerical projections. The endpoint and no-endpoint operations (23)–(24) use only rational limits at zero, matrix inverses, complements and scalar division. They therefore produce rational macro parameters and terminate by Theorem 5.4. Spectral splitting of a 2×2 rational positive lamination matrix introduces at most real algebraic directions when elementary layers are requested; the macro certificate itself remains rational.

A separate implementation reconstructs all original matrices, checks the PSD factor, checks ranks and the prescribed fraction, expands the macro tree with the coupled layer formula, and verifies equality of rational functions. This establishes a finite checkable certificate; it does not imply that the implementation is a formally verified program or that arbitrary symbolic inputs terminate within a small runtime. □

Example 8.7 (Three atoms and five elementary steps). At $z = 1/4 + 3i/4$, prescribe the isotropic value of

$$G(s) = \frac{I}{3}\left(\frac{1}{1/5 - s} + \frac{1}{1/2 - s} + \frac{1}{4/5 - s}\right).$$

Complete the orbit and choose $\theta = 2/5$. Exact elimination gives

$$\Delta = \frac{1152}{329575} > 0, \quad M_{\text{short}} = \frac{1}{659150}\operatorname{diag}(330727, 328423), \quad \operatorname{rank} B = 6, \quad \operatorname{rank} H = 5.$$

The response is not uniquely identified, but every interpolating function needs at least three paired atoms and every pure-host sequential interpolant needs at least five elementary steps.

One shortest interpolant is diagonal, with entries

$$G_{11}(s) = \frac{-659150s^2 + 928926s - 279901}{659150s^3 - 1259653s^2 + 705967s - 105464},$$
$$G_{22}(s) = \frac{-659150s^2 + 389374s - 10125}{659150s^3 - 717797s^2 + 164111s}.$$

The release contains its full rational macro tree, exact PSD factor and independent verification in `examples/three_atoms_five_steps.json`. Equality here is a rational-function identity, not an agreement at a few test contrasts.

9 One disk model for the entire physical class

The variable in this section is contrast, not temporal frequency. Here $\operatorname{Re} F = (F + F^*)/2$. A disk-unitary transfer function below is an auxiliary conservative realization; it does not assert losslessness of the physical composite. Matrix Carathéodory and Schur transformations are classical, including their use in matrix-valued analytic continuation [7]. We prove the precise real and phase constraints required here, rather than identifying the unrestricted Schur class with physical conductivity.

Definition 9.1 (Disk coordinate and normalized Schur function)**.** Let $\mathbb{D} = \{w : |w| < 1\}$ and

$$z(w) = \frac{(1+w)^2}{4w}, \quad z(0) = \infty. \tag{39}$$

Its inverse is the root of $w^2 - (4z - 2)w + 1 = 0$ lying in $\mathbb{D}$. For $M \in \mathfrak{M}$ set

$$F_M(w) = -\frac{1-w^2}{4w} G_M(z(w)) = \int_{[0,1]} \frac{1-w^2}{1 - 2(2t-1)w + w^2}\, dM(t),$$
$$F_M(0) = I, \qquad \Psi_M(w) = (F_M(w) - I)(F_M(w) + I)^{-1}. \tag{40}$$

All removable expressions at zero are interpreted analytically. The Schur class means analytic matrix functions with operator norm at most one on $\mathbb{D}$.

Theorem 9.2 (Exact symmetry-constrained Schur model)**.** `[D1]` *The map $M \mapsto \Psi_M$ is a bijection between $\mathfrak{M}$ and the analytic 2×2 Schur functions satisfying*

$$\Psi(0) = 0, \quad \Psi(w)^T = \Psi(w), \quad \Psi(\bar{w}) = \overline{\Psi(w)}, \quad \Psi(-w) = \mathcal{C}(\Psi(w)). \tag{41}$$

In fact $\|\Psi(w)\| \le |w|$, and $\operatorname{Re} F_M(w) \succ 0$ on $\mathbb{D}$. The physical response is recovered by

$$X_M(z(w)) = \frac{1-w^2}{4w} F_M(w) - z(w)I, \qquad X_M(\infty) = \frac{1}{2}\Psi_M'(0) - \frac{1}{2}I = M_1 - I. \tag{42}$$

The bijection and its inverse preserve compact-open convergence, with weak- convergence on the measure side.*

Proof. The inverse roots in (39) have product one. A unit-circle root would give $z = (1 + \cos\varphi)/2 \in [0, 1]$. Therefore exactly one root is in the disk for $z \in \widehat{\Omega}$; continuity from $z = \infty$ fixes its branch. Conversely (39) is one-to-one in the disk since its only other preimage is $1/w$. It obeys $z(-w) = 1 - z(w)$ and conjugate reality.

Put $x = 2t - 1$ and $\zeta_\pm = x \pm i\sqrt{1 - x^2}$. Direct algebra gives

$$\frac{1-w^2}{1-2xw+w^2} = \frac{1}{2}\left(\frac{\zeta_+ + w}{\zeta_+ - w} + \frac{\zeta_- + w}{\zeta_- - w}\right). \tag{43}$$

The real part of each circle kernel is $(1-|w|^2)/|\zeta-w|^2 > 0$. Its minimum over the circle is positive at fixed w; integration against a positive measure of mass I proves strict matrix positivity. Consequently $F+I$ is invertible. The identity

$$I - \Psi^*\Psi = 4(F+I)^{-*}(\operatorname{Re} F)(F+I)^{-1}$$

proves contractivity. Apply scalar Schwarz to $a^*\Psi(w)b$ for all unit input vectors a, b and take the supremum; this gives $\|\Psi(w)\| \le |w|$.

The real symmetric weights and reflection give $F^T = F$, $F(\bar{w}) = \bar{F}(w)$ and $F(-w) = \mathcal{C}(F(w))$. On symmetric 2×2 matrices, $\mathcal{C}(F) = R^T F R$, so the Cayley transform preserves all three identities. Its derivative at zero is $F'(0)/2$. Expansion of the integral gives $F'(0) = 4M_1 - 2I$ and then (42).

For the converse, Schwarz's estimate first gives $\|\Psi(w)\| < 1$ in the disk. Thus $F = (I+\Psi)(I-\Psi)^{-1}$ is analytic, $F(0) = I$, and $\operatorname{Re} F \succ 0$. To supply the measure representation explicitly, define the positive matrix measures on the circle

$$dN_r(e^{i\varphi}) = \operatorname{Re} F(re^{i\varphi})\,\frac{d\varphi}{2\pi}, \quad 0 < r < 1.$$

Their mass is I. Weak-* compactness supplies a subsequential limit N. Expanding the analytic F and integrating Fourier coefficients gives $F(rw) = \int (\zeta+w)/(\zeta-w)\, dN_r(\zeta)$. The kernel is continuous on the circle, uniformly for w in any smaller disk. Passing to the limit proves the same formula for F, N. Fourier coefficients determine N uniquely by density of trigonometric polynomials.

The transpose and conjugate identities imply that the Taylor coefficients of F are real symmetric. Each N_r is therefore real symmetric and invariant under $\zeta \mapsto \bar{\zeta}$. The phase identity implies $N_r(-E) = \mathcal{C}(N_r(E))$, and these properties pass to N. Push N forward by $t = (1+\operatorname{Re}\zeta)/2$. The resulting M has mass I and reflection-complement symmetry. Conjugation invariance of N and (43) give exactly (40). This proves surjectivity. Injectivity follows from uniqueness of either the circle transform or the interval Cauchy transform.

Weak-* convergence implies local uniform convergence by the uniformly continuous kernels. In the reverse direction, any weak-* subsequential limit is fixed by the analytic limit and transform uniqueness. Compactness therefore forces convergence of the full measure sequence. The Cayley transforms and their inverses are uniformly invertible on smaller disks by the strict bounds just proved. □

Remark 9.3 (A crucial distinction). Rationality in w alone does *not* mean finitely many interval states. The rational Schur function $\Psi = 0$ gives

$$dM_{\mathrm{a}}(t) = \frac{I\,dt}{\pi\sqrt{t(1-t)}}, \qquad F_{\mathrm{a}} = I, \qquad G_{\mathrm{a}}(z(w)) = -\frac{4w}{1-w^2} I. \tag{44}$$

This measure has infinite support. The missing property is rational *innerness*, not another rank convention. Similarly, the nonlinear change from M to Ψ does not identify a fixed-contrast convex body with an unrestricted matrix ball.

10 Conservative degree equals physical host complexity

Definition 10.1 (Inner degree)**.** A matrix Schur function is *inner* if its radial boundary values are unitary almost everywhere. For a rational function analytic at zero, its degree is the minimum dimension of a realization $D_0 + wC(I-wA)^{-1}B$. This is the McMillan degree in the disk convention; polynomial delays count as states, or equivalently poles at infinity in the variable w are counted. It is not the maximum degree of one matrix entry.

Theorem 10.2 (Sharp inner-degree/laminate correspondence). `[D2]` *For $M \in \mathfrak{M}$, the following are equivalent: M has finite support; its interval response has finitely many states; and Ψ_M is rational inner. If*

$$d = \sum_{t \in \operatorname{supp} M} \operatorname{rank} M\{t\} = 2N, \qquad r_0 = \operatorname{rank} M\{0\},$$

then

$$\deg \Psi_M = 2d - 2r_0 = 2L^{\mathrm{pp}}_{\min}(G_M). \tag{45}$$

For the nonreal finite-data problem,

$$\min_{\substack{\Psi \text{ rational inner} \\ \Psi \text{ satisfies (41) and the data}}} \deg \Psi = 2 \operatorname{rank} H. \tag{46}$$

The minimum is attained by the same construction that minimizes paired atoms and pure-host steps.

Proof. Split each interior weight $W = M\{t\}$ equally between the two distinct points $\zeta_\pm$ of (43); keep the endpoint weights at the single points $-1, +1$. The resulting circle measure has total spectral rank

$$D = 2(d - 2r_0) + 2r_0 = 2d - 2r_0,$$

since reflection makes the two endpoint ranks equal. After factoring each nonzero weight on its range, it has a cyclic representation

$$F(w) = V^*(U + wI)(U - wI)^{-1}V, \quad U^*U = I_D, \quad V^*V = I_2.$$

Here U is diagonal by circle spectral subspaces and V has full row rank into each such subspace. This proves cyclicity even for rank-deficient or coincident interval weights after they have been combined.

Set $P = VV^*$ and

$$A = U^*(I - P), \quad B = U^*V, \quad C = V^*, \quad \mathcal{U} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}. \tag{47}$$

Multiplication gives $\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^* = I$. A resolvent identity, or the matrix inversion lemma applied to $I - wU^* + wU^*VV^*$, gives

$$\Psi(w) = wC(I - wA)^{-1}B.$$

The reachable subspace for (A, B) equals that for (U^*, B) because $U^* = A + BC$. It is the whole space by cyclicity of (U, V). If $CA^k x = 0$ for every k, the subspace of such vectors is invariant under A, has $V^*x = 0$, and A agrees there with U^*. Cyclicity again forces that subspace to be zero. Thus this realization is reachable and observable.

For completeness, minimality follows without an unexplained cancellation assumption. The infinite block Hankel matrix with entries $CA^{i+j}B$ factors as observability times reachability. Finite leading portions of both factors already have rank D, so this Hankel matrix has rank D. Any realization with d' states factors the same coefficient Hankel matrix through a d'-dimensional space and hence has $d' \geq D$. Therefore the displayed realization is minimal.

Unitarity of $\mathcal{U}$ gives $I - \Psi(w)^*\Psi(w) = (1 - |w|^2)B^*(I - \bar{w}A^*)^{-1}(I - wA)^{-1}B$. Taking radial limits at regular boundary points proves innerness. Conversely, rational inner Ψ gives rational positive-real F whose Hermitian real part vanishes at every regular boundary point. Circle measure inversion from the Poisson kernels in the preceding proof shows that its representing measure has zero mass on every arc avoiding the finitely many poles of F. Its support is therefore finite. Pushing it to the interval gives finite M.

The finite-state/finite-measure equivalence is the elementary spectral representation. Finally Theorem 5.4 identifies $d - r_0$ with the shortest pure-host length, proving (45). Minimize that identity and use Theorem 8.4 for (46). Rational non-inner competitors, such as (44), are intentionally excluded from that minimum. □

This result identifies a physical architecture invariant with a standard analytic degree. It does not prove that the degree of an arbitrary passive transfer function measures spatial fabrication complexity, nor that a unitary state-space realization is itself a laminate geometry. The endpoint/complement construction remains the explicit physical bridge.

11 Exact prediction sets and optimization certificates

Write $\mathcal{D}$ for prescribed full reciprocal samples on n completed upper nodes. Add distinct query orbits, giving r completed nodes in total. Parameterize the unknown query values by their independent real entries. A generic orbit has six real coordinates; a self-dual node has three, because its value is a real traceless symmetric matrix plus an imaginary scalar matrix. In these coordinates write the enlarged chiral matrix as

$$H(x) = H_0 + \sum_{k=1}^{m} x_k H_k. \tag{48}$$

The constant in the upper-left entry remains 1/2 (or 1 in the rational congruence). A separate normalization must not be applied to each unknown value.

Theorem 11.1 (Joint prediction, strict extension, and finite extremizers). `[V1]` *The set of all query values compatible with one physical conductivity function is exactly*

$$\mathcal{P}(\mathcal{D}) = \{x \in \mathbb{R}^m : H(x) \succeq 0\}. \tag{49}$$

It is nonempty and compact when $\mathcal{D}$ is feasible. If $H_{\mathcal{D}}$ is singular, it is the singleton obtained from the unique entire response. If $H_{\mathcal{D}} \succ 0$, it contains a point with $H(x) \succ 0$ for every finite choice of query orbits.

Every nonconstant real linear query objective attains its extrema. In the strict case, each optimal full query vector has singular enlarged H and determines a unique entire finite response. That response has at most $r+1$ paired atoms and at most $2r$ pure-host steps. These are ideal periodic-closure realizations, not a finite-cell fabrication claim.

Proof. Necessity and sufficiency are the single-PSD theorem applied to the enlarged list, with the old values fixed. Evaluation is continuous on the compact measure class, so its query image is compact; nonemptiness follows from any measure representing $\mathcal{D}$. Singularity uses Theorem 8.1.

To prove strict extension, use the phase-symmetric anchor $dM_{\mathrm{u}} = I\,dt$. Its chiral matrix is strictly positive for any finite list. Indeed its quadratic form is the integral of t times the squared norm of $u\xi_0 + \sum_j (t - z_j)^{-1}\xi_j$; a zero integral forces this rational vector to vanish identically, then its residues and constant part all vanish. For sufficiently small $\varepsilon > 0$, $H_{\mathcal{D}} - \varepsilon H_{\mathrm{u},\mathcal{D}} \succ 0$. The rescaled old data $(\mathcal{D} - \varepsilon\mathcal{D}_{\mathrm{u}})/(1-\varepsilon)$ are feasible, because H is affine with fixed normalization. Let M' represent them. The measure $(1-\varepsilon)M' + \varepsilon M_{\mathrm{u}}$ matches the original old data and dominates $\varepsilon I\,dt$, so its enlarged chiral matrix is strictly positive. This proves a genuine Slater point without assuming it.

A nonzero linear functional on the independent query coordinates cannot have an optimum at a positive-definite matrix, since a small query perturbation in an improving direction would preserve positivity. Thus every optimum has singular enlarged H. The uniqueness theorem identifies its entire measure. The minimum-state bound is $\operatorname{rank} B/2 \le r+1$, and $\operatorname{rank} H \le 2r$ because H has order $2r+1$. The physical compiler attains both bounds. Different optimal query vectors may exist; uniqueness here is conditional on the complete optimal query vector, not an assertion that every exposed face is a point. □

Proposition 11.2 (Dual bounds with exact and uncertain measurements)**.** `[V2]` *Let $H(x)$ be (48), or the corresponding pencil with all values variable. Let $Ax = b$ be real affine measurement constraints. For a linear objective $c^T x$, any $Q \succeq 0$ and real y satisfying*

$$A^T y - \big(\operatorname{tr}(QH_k)\big)_{k=1}^m = c \tag{50}$$

give the independently checkable bound

$$c^T x \le \operatorname{tr}(QH_0) + y^T b. \tag{51}$$

For componentwise errors $|(Ax-b)_j| \le \varepsilon_j$, add $\sum_j \varepsilon_j |y_j|$ to the right side. For $\|W(Ax-b)\|_2 \le \varepsilon$ with invertible W, add $\varepsilon\|W^{-T}y\|_2$ instead. Replacing c by $-c$ gives lower bounds.

When a point with $H(x) \succ 0$ exists subject to $Ax = b$ and the objective is finite, the optimal dual value equals the primal value and a dual optimum exists. In particular this holds for the strict prediction fibers of Theorem 11.1. Boundary fibers of full exact data are evaluated by their unique response instead. The certificate inequality itself needs no Slater hypothesis.

Proof. Pair (50) with x:

$$c^T x = y^T Ax - \operatorname{tr}(QH(x)) + \operatorname{tr}(QH_0).$$

The middle term is nonpositive. Substitute $Ax = b$ for exact data; apply the triangle inequality or norm duality for uncertain data. These steps prove every certificate claim, including singular Q and singular feasible H.

The final assertion is finite-dimensional semidefinite strong duality under Slater's condition [18]. One proof separates the open strict epigraph of achievable affine constraint residuals and objective improvements from the positive cone; the strict feasible point excludes a zero objective multiplier, allowing normalization to (50). Closedness at the finite optimum then gives a dual optimizer. We use this standard conic-separation result only for dual *attainment*; all inequalities and physical feasibility above have direct proofs. □

A numerical solver may propose x, Q, y, but a rounded near-zero residual is not (50). The release checks rational identities and a positive factor or exact positive-matrix elimination. Irrational optimal values need not admit rational optimal certificates. Nonoptimal rational certificates can still supply valid rigorous bounds.

Theorem 11.3 (Arbitrarily tight rational physical design certificates)**.** `[V5]` *Assume the fixed and query nodes, the fixed full data and the objective c are Gaussian-rational or real-rational as appropriate, and the fixed data are feasible. When $H_{\mathcal{D}} \succ 0$, for every rational $\varepsilon > 0$ there are rational primal query values and a rational dual certificate whose objective gap is at most ε. The primal values have an exact rational physical macro-tree certificate, with at most $r+1$ paired atoms and $2r+1$ pure-host steps. Thus an irrational optimum does not obstruct arbitrarily tight proof-carrying physical design.*

For singular fixed-data H, the unique rational continuation gives the optimum directly, certified by the singleton-response theorem rather than by a claimed attained PSD dual. For positive-definite fixed-data H, rational certificates can in principle be found by a fair enumeration of *rational primal coordinates and rational parameters of the dual affine equations. This termination statement is not a practical complexity bound or a promise that the released numerical routines solve arbitrary SDPs.*

Proof. In the singular case the unique response is the rational quotient of Gaussian-rational matrices; its evaluation at Gaussian-rational non-pole query nodes is Gaussian-rational. The entire prediction set is a singleton, so its exact value is the certificate.

In the strict case the prediction spectrahedron is compact and has a strict point. Let $\mathcal{L}x = \sum x_k H_k$. Boundedness implies that $\mathcal{L}$ is injective and its range $\mathcal{S}$ contains no nonzero positive-semidefinite matrix: such a direction would produce an unbounded feasible ray. The projection of the compact set $\{P \succeq 0 : \operatorname{tr} P = 1\}$ onto $\mathcal{S}^\perp$ is compact convex and does not contain zero. Strict separation supplies $Q_0 \in \mathcal{S}^\perp$ with $\operatorname{tr}(Q_0 P) > 0$ uniformly on that set, so $Q_0 \succ 0$. Because $\mathcal{S}$ is defined by rational equations, rational points are dense in $\mathcal{S}^\perp$; choose Q_0 rational while retaining strict positivity.

Injectivity of $\mathcal{L}$ makes its adjoint onto. Rational elimination therefore gives a Hermitian rational Q_1 with $\mathcal{L}^* Q_1 = -c$. Adding a sufficiently large rational multiple of Q_0 gives a strictly positive rational dual point. A strict rational primal point exists by density of rational coordinates in the open primal set. By strong duality, primal and dual optimal values coincide. Mix their respective optimizers with these strict rational points by sufficiently small positive weights, then approximate within the rational primal space and rational dual affine space. Positivity is open at the mixed points and the objectives are continuous, so the resulting exact rational gap is at most ε.

The finite-data compiler transforms the primal into the stated exact physical certificate. For termination, enumerate rational coordinate tuples by increasing numerator/denominator height; parameterize the dual affine space by rational elimination and enumerate its rational parameter tuples. Check positivity, the dual identity and the gap exactly for each pair. A successful pair exists by the preceding argument, so a fair enumeration eventually encounters it. The release's independent checker verifies any such pair without trusting the search procedure. □

Corollary 11.4 (What least-squares projection does and does not identify)**.** `[V3]` *For fixed nodes and a positive-definite real data metric, projection of any structured data vector onto the feasible full-data body is unique and nonexpansive in that metric. If the input lies outside that body, its projection lies on the boundary and hence has a unique entire representing response. This uniqueness is selected by the projection rule; it does not establish that the unknown true composite is unique under measurement error.*

Proof. The body is compact convex. Strict convexity of the quadratic distance gives uniqueness; the two variational projection inequalities give nonexpansiveness. An interior projection of an exterior point could be moved toward that point to reduce the distance, so it is on the boundary. The uniform-measure anchor makes the body full-dimensional in the structured data space; a positive-definite H is interior. Boundary data therefore have singular H, to which the uniqueness theorem applies. □

The scalar analogues of augmented-Pick prediction, finite rational boundary reconstruction, and certified least-squares fitting are established prior art [11]. Uncertain-data Nevanlinna–Pick propagation also has recent developments [8]. The new point here is their exact phase-symmetric matrix realization together with a specified physical complexity certificate, not invention of uncertainty quantification.

12 Quantitative continuation and logarithmic physical approximation

Let $a_j = w(z_j)$ for the n completed upper-half-plane nodes and let

$$\mathcal{A} = \{a_1, \bar{a}_1, \dots, a_n, \bar{a}_n\}, \qquad \mathcal{B}(w) = \prod_{a\in\mathcal{A}} \frac{w-a}{1-\bar{a}w}. \tag{52}$$

There are $2n$ distinct nonzero zeros. The product is real under conjugation and even under $w \mapsto -w$. In particular $\mathcal{B}(0) = \prod_{j=1}^{n} |a_j|^2 > 0$. Norms in this section are matrix operator norms.

Theorem 12.1 (Data-independent continuation envelope)**.** `[A1]` *If $M, \widetilde{M} \in \mathfrak{M}$ agree on the full prescribed nonreal samples, then for every $w \in \mathbb{D}$,*

$$\|X_M(z(w)) - X_{\widetilde{M}}(z(w))\| \le \frac{|1-w^2|}{(1-|w|)^2}|\mathcal{B}(w)|. \tag{53}$$

At $w = 0$ this is a bound on the first moment, or equivalently on the weak-contrast second-order conductivity coefficient. No finite-support, separated-pole, spectral-gap or rank hypothesis is required.

Proof. The analytic matrix function $h = (\Psi_M - \Psi_{\widetilde{M}})/w$ has norm at most two by Theorem 9.2. The removable value at zero is its derivative difference. It vanishes at every $a \in \mathcal{A}$. Division by the finite Blaschke product gives another analytic function; it has norm at most two. To justify the boundary step without assuming continuous boundary values, take circles of radius tending to one after enclosing all zeros. The minimum modulus of $\mathcal{B}$ on those circles tends uniformly to one, so the maximum principle applied to each scalar $p^*(h/\mathcal{B})q$ yields the claimed norm bound. Thus $\|\Psi_M - \Psi_{\widetilde{M}}\| \le 2|w||\mathcal{B}(w)|$.

Since $F = 2(I-\Psi)^{-1} - I$, the inverse identity gives

$$F_M - F_{\widetilde{M}} = 2(I-\Psi_M)^{-1}(\Psi_M - \Psi_{\widetilde{M}})(I-\Psi_{\widetilde{M}})^{-1}.$$

Both inverse norms are at most $(1-|w|)^{-1}$. Multiplication by $(1-w^2)/(4w)$ in (42) proves (53), first for nonzero w and then by analytic continuity at zero. □

Corollary 12.2 (Uniform exponential approximation by short physical hierarchies)**.** `[A2]` *Fix $0 < \rho, a < 1$ and any $M \in \mathfrak{M}$, including infinite-support measures. Choose n distinct radii $0 < r_j \le a$ and measure at the self-dual contrasts*

$$w_j = -ir_j, \qquad z_j = \frac{1}{2} + i\frac{1-r_j^2}{4r_j}. \tag{54}$$

There exists an interpolating pure-host laminate M_n with at most $n+1$ paired atoms and at most $2n+1$ elementary host steps, such that

$$\sup_{|w|\le\rho} \|X_M(z(w)) - X_{M_n}(z(w))\| \le C_\rho q_{\rho,a}^n, \quad C_\rho = \frac{1+\rho^2}{(1-\rho)^2}, \quad q_{\rho,a} = \frac{\rho^2+a^2}{1+\rho^2a^2} < 1. \tag{55}$$

Therefore normalized error at most ε follows from

$$n \ge \max\left\{1, \left\lceil \frac{\log(C_\rho/\varepsilon)}{\log(1/q_{\rho,a})} \right\rceil \right\}.$$

The layer count is $O(\log(1/\varepsilon))$ on each fixed compact disk. It is independent of the spectral complexity of the target.

Proof. Each node is its own phase partner, so the completed list has exactly n nodes. Apply the finite-data physical compiler. Its ranks give $N \le n+1$ and $L^{\rm pp} \le 2n+1$. Each conjugate disk pair contributes

$$\frac{(w-ir_j)(w+ir_j)}{(1+ir_jw)(1-ir_jw)} = \frac{w^2+r_j^2}{1+r_j^2w^2}.$$

For $|v| \le \rho^2$ and $0 \le b < 1$, the maximum of $|(v+b)/(1+bv)|$ is $(\rho^2+b)/(1+b\rho^2)$: its squared modulus is increasing in the cosine of the argument on the boundary, and the maximum principle handles the interior. This expression is increasing in b. Bound each pair with $b = r_j^2 \le a^2$, and use $|1-w^2| \le 1+\rho^2$ in Theorem 12.1. $\square$

For physical tensors, the inverse identity gives the additional factor

$$\frac{|\beta|(1-\theta)}{\theta}\|S_M(s)^{-1}\|\,\|S_{M_n}(s)^{-1}\|$$

multiplying the normalized X error. Lemma 4.3 bounds this uniformly on compact contrast sets; near infinity the inverses vanish as $O(1/|s|)$. Hence the same exponential conclusion holds there for bounded β. This result controls ideal hierarchy length, not scale separation, minimum feature size, fabrication error, or an error uniform up to the resonant cut. Exact measurements are essential to the stated rate; the next theorem exposes the noise amplification.

Theorem 12.3 (Explicit noisy continuation envelope)**.** `[A3]` *For two admissible responses define $h = (\Psi_M - \Psi_{\widetilde{M}})/w$ as above and suppose $\|h(a)\| \le \delta_a$ at every $a \in \mathcal{A}$. Put*

$$L_a(w) = \frac{\mathcal{B}(w)}{(w-a)\mathcal{B}'(a)}, \qquad \ell_a = \frac{1}{(1-|a|)|\mathcal{B}'(a)|}. \tag{56}$$

The removable values of L_a are used at all sample nodes. Then

$$\|X_M - X_{\widetilde{M}}\|(z(w)) \le \frac{|1-w^2|}{2(1-|w|)^2}\left[2|\mathcal{B}(w)| + \sum_{a\in\mathcal{A}} \delta_a\big(|L_a(w)| + \ell_a|\mathcal{B}(w)|\big)\right]. \tag{57}$$

Each δ_a can be capped at two. From a pairwise sample discrepancy ε_a in X, one can use $\delta_a = 8\varepsilon_a/|1-a^2|$; from a discrepancy in G, one can use $\delta_a = |1-a^2|\varepsilon_a/(2|a|^2)$.

Proof. The cardinal functions satisfy $L_a(b) = \mathbf{1}_{a=b}$ and have boundary modulus at most ℓ_a. The function $h - \sum_a h(a)L_a$ vanishes at every sample and has supremum norm at most $2 + \sum_a \delta_a\ell_a$. Divide by $\mathcal{B}$ using the same maximum-principle argument as above, then add the interpolating sum. This gives the bracket in (57) as a bound for $\|h(w)\|$. The Cayley inverse identity and (42) give its prefactor.

For the residual conversion use $\Psi_M - \Psi_{\widetilde{M}} = 2(F_M+I)^{-1}(F_M - F_{\widetilde{M}})(F_{\widetilde{M}}+I)^{-1}$. The inverse norms are at most one because $\operatorname{Re} F \succeq 0$. Combine with $F_M - F_{\widetilde{M}} = 4a(X_M - X_{\widetilde{M}})/(1-a^2)$ or with $F_M - F_{\widetilde{M}} = -(1-a^2)(G_M - G_{\widetilde{M}})/(4a)$ and divide by $|a|$. $\square$

The two responses in this theorem must both be admissible. For two candidates in one measurement ball of radius ε_a, the pairwise discrepancy may be $2\varepsilon_a$. Raw effective-tensor errors must first be bounded through the nonlinear physical normalization. Small $|\mathcal{B}'(a)|$ exposes clustering-induced ill conditioning; no exact-rank stability or noise-independent exponential reconstruction claim is made.

13 Sharp minimax information and extremal short laminates

The continuation bound has an exact information-theoretic consequence. Its proof supplies adversarial physical responses attaining the worst-case error, not merely a rate estimate.

Theorem 13.1 (Exact minimax recovery of the weak-contrast coefficient)**.** `[A4]` *Fix any n completed nonreal nodes as above and let $\mathcal{D}_M = (G_M(z_j))_{j=1}^n$. Over all estimators of $X_M(\infty)$ from these exact data,*

$$\inf_{\mathcal{E}} \sup_{M\in\mathfrak{M}} \|\mathcal{E}(\mathcal{D}_M) - X_M(\infty)\| = \frac{1}{2}\prod_{j=1}^{n} |w(z_j)|^2. \tag{58}$$

An optimal estimator is the center of the feasible first-moment disk, minus I. The lower bound is attained by two shortest physical interpolants to the arcsine datum (44)*. They have $n+1$ paired atoms and $2n+1$ pure-host steps.*

At equal-phase expansion parameter $\varepsilon = 1/s$, with $\beta = 1$,

$$A_{\text{eff}}(1-\varepsilon, 1) = I - \theta\varepsilon I + \theta(1-\theta)X_M(\infty)\varepsilon^2 + O(\varepsilon^3). \tag{59}$$

Thus the exact minimax error for this second-order coefficient is (58) *multiplied by $\theta(1-\theta)$.*

Proof. For each feasible datum, Theorem 6.4 gives the entire first-moment disk. In operator norm its radius is $\Delta/2$ and its diameter is Δ, since a real traceless symmetric matrix $xE_1 + yE_2$ has norm $\sqrt{x^2+y^2}$. Every point in the disk is realized by the interval quotient and the reciprocal physical construction. Theorem 12.1 at zero therefore gives $\Delta \le b_0 := \mathcal{B}(0) = \prod_j |a_j|^2$. The disk center estimator has worst-case error at most $b_0/2$.

For the lower bound fix $D = \operatorname{diag}(1,-1)$ and define

$$\Psi_\pm(w) = \pm w\mathcal{B}(w)D. \tag{60}$$

These are rational inner Schur functions satisfying every condition in (41): $\mathcal{B}$ is even and real, and $\mathcal{C}(D) = -D$. They vanish at every sample, so both give the same full datum as $F = I$. Theorem 9.2 supplies their admissible measures. Their derivatives at zero are $\pm b_0 D$, hence

$$X_\pm(\infty) = -I/2 \pm b_0 D/2.$$

An estimator receives identical data from them. The triangle inequality forces an error at least $b_0/2$ for one of them. This proves equality and also proves that the arcsine first-moment disk has $\Delta = b_0$.

The scalar inner function $w\mathcal{B}(w)$ has degree $2n+1$; its zeros are in the disk and its poles are outside, so there is no cancellation. The diagonal pair in (60) has matrix degree $4n+2$. Theorem 10.2 gives length $2n+1$. Because $2N - r_0 = 2n+1$ and $r_0 \in \{0,1,2\}$, parity forces $r_0 = 1$ and $N = n+1$. The arcsine measure has positive density on $(0,1)$, so its H and B are strictly positive by the rational-vector argument in Theorem 11.1. Thus these lengths and paired counts attain the finite-data minima as well. No uniqueness of an arbitrary fixed-first-moment continuation is assumed.

Finally expand $S = sI/\theta + (1-\theta)X_M(\infty)/\theta + O(s^{-1})$ in (4). Matrix inversion gives (59), including its sign and fraction coefficient. □

Corollary 13.2 (A matching exponential information lower bound)**.** `[A5]` *For the same fixed samples, at any real $0 < r < 1$ the diameter of the admissible prediction fiber for the arcsine datum is at least*

$$\frac{(1-r^2)\mathcal{B}(r)}{1 - r^2\mathcal{B}(r)^2}. \tag{61}$$

Any estimator of $X(z(r))$ has worst-case error at least half this quantity. For self-dual radii $r_j \to a \in (0,1)$, its exponential rate is

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{B}(r) = \log \frac{r^2 + a^2}{1 + r^2 a^2}.$$

Consequently the exponential factor in (55)*, evaluated at the same radius and sampling sequence, cannot be replaced by a strictly faster one in a uniform worst-case reconstruction theorem.*

Proof. For the two functions (60), put $f = w\mathcal{B}(w)$. Their Cayley transforms differ by $4fD/(1-f^2)$. Equation (42) therefore gives the difference $(1-w^2)\mathcal{B}(w)D/(1-w^2\mathcal{B}(w)^2)$. For real r, each conjugate pair in the product is positive, yielding (61). Identical sample data imply the half-diameter lower bound. In the self-dual case, $\mathcal{B}(r) = \prod_j (r^2 + r_j^2)/(1 + r^2 r_j^2)$. The logarithmic Cesàro mean converges to the stated logarithm. Its prefactor in (61) has no nonzero exponential rate. □

This is an information lower bound for fixed measurements. It is not a proof that every approximation strategy for a fully known target requires that many layers. Moving all sample nodes toward infinity can reduce (58), but it also changes the measurement problem and may greatly amplify noise in normalization and divided differences.

Theorem 13.3 (The exact one-layer price of minimax centering)**.** `[A6]` *Fix strictly feasible full data on n completed nonreal nodes, and let $\Delta > 0$ be its first-moment disk diameter. Consider estimators that output a physically admissible complete response matching those data, and assess their worst-case operator-norm error only in $X(\infty)$ over all compatible true responses.*

Among outputs with the shortest possible pure-host length $2n+1$, the smallest worst-case error is exactly Δ. Allowing length $2n+2$ reduces the optimal error to $\Delta/2$, the unrestricted minimax value. Both choices can retain the minimum paired atom count $n+1$. No shortest output attains the minimax center. For singular feasible data $\Delta = 0$ and there is no such penalty.

Proof. The compatible first moments form a Euclidean disk in the real traceless symmetric plane, with radius $R = \Delta/2$. The farthest distance from a chosen point M_1 in this disk to another disk point is $R + \|M_1 - M_c\|$. Theorem 8.4 says that every shortest physical interpolant has rank-one first-moment slack, hence M_1 is on the boundary and this risk is $2R = \Delta$. The opposite boundary point is physically attainable, so the lower bound is attained rather than only approached.

The unique minimax first moment is the center, with risk R. Since $K \succ 0$, its positive two-dimensional slack gives $\operatorname{rank} A(M_c) = 2n+2$. Every realization matching this moment has $\operatorname{rank} T \geq \operatorname{rank} A(M_c)$ by compression, so its pure-host length is at least $2n+2$. The center quotient has $\operatorname{rank} B = 2n+2$ states and achieves this length through peeling, while preserving paired atom count $n+1$. Singular data have a singleton response and zero risk. □

The additional layer is a price for robust recovery of this weak-contrast tensor, not a universal accuracy cost for all design objectives. The theorem distinguishes a parsimonious exact interpolant from an estimator that is optimal against all physically compatible explanations of the measurements.

14 A sharp broadband experiment with exact primal and dual certificates

Theorem 14.1 (Simultaneous self-dual trace envelope). [V4] *Suppose the full measured matrix is $G(1/2+i\eta) = icI$, where $\eta > 0$ and c is real. Put*

$$a = \eta^2, \quad m = c/\eta, \quad L = (a+1/4)^{-1}, \quad U = a^{-1}.$$

It is feasible exactly when $L \le m \le U$. At a query $1/2 + i\eta'$ put $b = (\eta')^2$, $d = b - a$, and $f(x) = x/(1+dx)$. For $b > a$,

$$\frac{(U-m)f(L) + (m-L)f(U)}{U-L} \le \frac{\operatorname{tr}\operatorname{Im} G(1/2+i\eta')}{2\eta'} \le f(m). \tag{62}$$

Both bounds are sharp simultaneously for every $\eta' > \eta$. For $0 < b < a$ their order reverses. At $b = a$ they agree.

The single-orbit measure

$$M_{\text{one}} = \tfrac{1}{2} I\delta_{1/2-h} + \tfrac{1}{2} I\delta_{1/2+h}, \quad h^2 = 1/m - a, \tag{63}$$

attains $f(m)$, and

$$M_{\text{end}} = pI\delta_{1/2} + \tfrac{1-p}{2} I(\delta_0 + \delta_1), \quad p = \frac{m-L}{U-L}, \tag{64}$$

attains the chord bound. If $L < m < U$ and $b \ne a$, each extremum has that unique entire measure and exactly four pure-host steps. The two atom counts are respectively two and three.

Proof. The scalar probability measure $\nu = \operatorname{tr} M/2$ gives

$$\frac{\operatorname{tr}\operatorname{Im} G(1/2+i\eta)}{2\eta} = \int \frac{d\nu(t)}{(t-1/2)^2 + a}.$$

Set $x = ((t-1/2)^2 + a)^{-1} \in [L, U]$. Its mean is m, and the query integrand is $f(x)$. Since $1 + dx > 0$ throughout this interval for $b > 0$, the following exact identities hold:

$$\begin{aligned} f(m) + f'(m)(x-m) - f(x) &= \frac{d(x-m)^2}{(1+dm)^2(1+dx)}, \\ f(x) - \frac{(U-x)f(L) + (x-L)f(U)}{U-L} &= \frac{d(x-L)(U-x)}{(1+dx)(1+dL)(1+dU)}. \end{aligned} \tag{65}$$

Integrate; the linear terms use only $\mathbb{E}x = m$. This proves both signs and the reversal. The displayed isotropic measures have full matrix mass I, reflection symmetry, and the required full sample, so the scalar trace bounds remain sharp even with that stronger matrix constraint. They depend only on a, m, not on b, proving simultaneous broadband attainment.

When m is interior and $d \ne 0$, equality in the first identity forces $x = m$, and equality in the second forces $x \in \{L, U\}$. A zero scalar trace measure on the complementary support is also zero as a positive matrix measure. Thus the only possible matrix supports are exactly those displayed. At a nonmidpoint reflected pair the real traceless part of the measured G is a nonzero scalar multiple of the traceless part of its weight. The full isotropic datum forces that part to vanish. The midpoint weight is already scalar by reflection. This determines the weights uniquely as (63) or (64). Their total spectral ranks are four and six, and their zero-endpoint ranks are zero and two. Theorem 5.4 therefore gives length four in both cases. Endpoint input cases follow directly, with collapsed supports and smaller counts rather than the interior complexity assertion. □

Example 14.2 (Exact two-sided prediction and physically attainable witnesses)**.** Take $\eta = 1$, $c = 9/10$ and $\eta' = 2$. Then

$$\frac{33}{68} \le \frac{1}{2}\operatorname{tr}\operatorname{Im} G(1/2+2i) \le \frac{18}{37}, \qquad \frac{18}{37} - \frac{33}{68} = \frac{3}{2516}. \tag{66}$$

The upper measure is $I(\delta_{1/6}+\delta_{5/6})/2$; the lower measure is $I\delta_{1/2}/2 + I(\delta_0+\delta_1)/4$. Both give $G(1/2+i) = 9iI/10$. They generate different complete physical responses at every prescribed fraction, and the exact compiler gives four-step macro-tree certificates for both.

There are also short dual witnesses in the rationally congruent 5×5 prediction pencil. Write the query as $xE_1 + yE_2 + ibI$ and order its coordinates as (x, y, b). The vectors

$$v_+ = \begin{pmatrix} -9/37 \\ 10i/37 \\ -10/37 \\ -i \\ 1 \end{pmatrix}, \qquad v_- = \begin{pmatrix} (-4+i)/17 \\ -1/17 + 9i/34 \\ -9/34 - i/17 \\ -i \\ 1 \end{pmatrix}$$

give $Q_\pm = 6v_\pm v_\pm^* \succeq 0$. For the upper objective b, the coefficient traces are $(0,0,-1)$ and $\operatorname{tr}(Q_+H_0) = 18/37$. For the lower objective $-b$, they are $(0,0,1)$ and $\operatorname{tr}(Q_-H_0) = -33/68$. Equation (51) verifies both bounds using only rational arithmetic. The included primal trees attain equality, so no solver or spectral discretization is needed to certify optimality.

15 Fixed-contrast geometry, atom strata and the observable quotient

15.1 One, two and three atoms

Theorem 15.1 (Exact nonreal atom-number stratification)**.** `[G4]` *At a single nonreal contrast, complete its upper-half-plane orbit and form B, H. A reciprocal target is feasible exactly when $H \succeq 0$ and the orbit identities are consistent. For a non-self-dual contrast, B has order six and*

$$N_{\min} = 1 + \mathbf{1}_{\{e_4(B)>0\}} + \mathbf{1}_{\{\det B>0\}} \in \{1,2,3\}, \tag{67}$$

where e_4 is the fourth elementary symmetric polynomial of the eigenvalues, equivalently the sum of principal minors of order four. On the self-dual line, B has order four and $N_{\min} = 1+\mathbf{1}_{\{\det B>0\}}$. These tests are asserted only on the feasible set. Both upper bounds are sharp. Generic strictly feasible data require respectively three paired atoms and five host steps, or two paired atoms and three host steps.

Proof. Use Theorem 7.1. A positive B has even rank and $B_{00} = I$ gives rank at least two. For positive matrices, $e_4 > 0$ is equivalent to rank at least four, and a positive determinant to full rank. This proves the formulas. A strictly interior $N = n+1$ block-Jacobi contraction gives full-rank B and H, as in the sampling sharpness proof. Taking $n = 2$ or $n = 1$ supplies the required sharp examples. □

Proposition 15.2 (Direct one-atom test and exposed points)**.** `[G5]` *For an invertible target resolvent G at $s \in \Omega$, it is a single paired atom if and only if*

$$A = (s - 1/2)I + G^{-1} \in \operatorname{Sym}_2(\mathbb{R}), \qquad \operatorname{tr} A = 0, \qquad \|A\| \le 1/2. \tag{68}$$

At fixed nonreal contrast every atom is an exposed point of the normalized convex body, and these are all its extreme points. A complete finite measure has a unique minimal paired-atom decomposition, modulo reflection and the orientation-free midpoint, exactly when every nonmidpoint orbit weight has rank at most one.

Proof. For $A = (t-1/2)(2P-I)$ the atomic resolvent is $(A-(s-1/2)I)^{-1}$. The real traceless symmetric matrices in the closed radius-1/2 disk are exactly these A, proving (68).

For a fixed atomic data set, let Π be the orthogonal projection onto $\ker B_{\mathrm{atom}}$. The affine functional $\mathrm{tr}(\Pi B)$ is nonnegative on feasible data and vanishes at that atom. Equality forces $B \ker B_{\mathrm{atom}} = 0$. In an atomic realization T has order two and V is unitary; its sample vectors are VG_j. Thus its Gram kernel contains the block relations $(-G_j u, 0, \dots, u, \dots, 0)$. The zeroth block row of B applied to these vectors forces every new G_j to equal the atomic target. Hence the exposing face is a singleton. Any extreme point of a finite-dimensional convex hull of a compact set belongs to that set: a finite convex decomposition of a point outside the generating set is nontrivial. This proves the extreme-point assertion.

For a complete measure, distinct spectral supports cannot cancel because the weights are positive. A rank-one weight has a unique rank-one direction and weight; a positive rank-two 2×2 matrix has infinitely many decompositions into two nonzero rank-one dyads, obtained by rotating a square-root factor. The midpoint is intrinsically a scalar multiple of I and its atom does not depend on projector orientation. These facts give the final criterion. None gives a unique spatial microstructure. □

15.2 The formal quotient is not the observable disk

Write a real traceless A as $aE_1 + bE_2$. A paired atom depends on A, not on a chosen projector at $A = 0$. The formal free involution

$$(r, u) \longmapsto (-r, -u), \qquad (r, u) \in [-1/2, 1/2] \times S^1,$$

has a Möbius-band quotient. Its zero-radius circle is not yet a point. The response map additionally collapses this central circle, and only the resulting *observable* quotient is the closed disk $a^2 + b^2 \le 1/4$. Thus the disk is a useful atom parameter space, but it is not literally the quotient by reflection alone. The associated Clifford identity $(aE_1 + bE_2)^2 = (a^2 + b^2)I$ is elementary and familiar; it does not by itself establish a new bounded symmetric domain for the full complex convex body. The single-H spectrahedron is the precise finite-dimensional complex-contrast description obtained here.

15.3 Real contrast: the cone, support and shortest design

For $s \in \mathbb{R} \setminus [0, 1]$, write a real target as $X = qI + xE_1 + yE_2$ and $\rho = (x^2 + y^2)^{1/2}$. Set

$$k = 2s - 1, \quad a = s^2/k, \quad c = |s(s-1)/k|, \quad \sigma = \operatorname{sgn} k, \quad q_0 = -s/k, \quad e = -1/2.$$

Theorem 15.3 (Real geometry and exact architecture strata)**.** `[G6]` *The real normalized closure is exactly*

$$\min(q_0, e) \le q \le \max(q_0, e), \qquad \sqrt{\rho^2 + c^2} \le \sigma(q + a). \tag{69}$$

Put $h(q) = \sqrt{(q+a)^2 - c^2}$ on this branch. One atom suffices exactly on the lateral boundary $\rho = h(q)$; all other points need exactly two atoms. The minimum pure-host length is one on the cap rim $(q, \rho) = (e, 1/2)$, two on the cap interior or on the remaining lateral boundary, and three at every strict interior point.

For the real linear functional $Aq + b \cdot (x, y)$, put $\beta = |b|$ and $y_c = \sigma(e + a)$. Its support function is

$$h_{\mathrm{body}}(A, b) = \begin{cases} Ae + \beta/2, & A\sigma + 2\beta y_c \ge 0, \\ -Aa - c\sqrt{A^2 - \beta^2}, & A\sigma + 2\beta y_c < 0. \end{cases} \tag{70}$$

The zero functional has support zero, as given by the first case.

Proof. For $u = 2t - 1$, the atomic center and signed half-spread are

$$q_s(u) = -\frac{s(k-u^2)}{k^2-u^2}, \qquad h_s(u) = \frac{2s(s-1)u}{k^2-u^2}.$$

Use $0 \le u \le 1$ and rotate the projector freely. Eliminating u gives $h^2 = (q+a)^2 - c^2$ and $u^2 = k(kq+s)/(q+s)$. The center runs monotonically from q_0 to e. At fixed q, convexification fills the disk $\rho \le h(q)$. Since $h'' = -c^2/h^3 < 0$ away from the zero-radius tip, these disks already form a convex body. This proves (69) and the atom assertion. Every nonboundary disk point is the convex combination of two complementary projectors at the same t.

For the length statement, a one-step response is an endpoint atom. A length-two construction has $2N - r_0 \le 2$; either it is one interior atom, or $N = 2, r_0 = 2$ and the measure is supported only at the endpoints, giving the cap. These cannot reach strict interior points. Conversely a strict interior point has a representation by one endpoint atom and one nonendpoint atom, as explicitly constructed in the supplement. Its complete response has $N = 2, r_0 = 1$, so length three suffices. The remaining strata follow from the same constructions.

For the support function, align the deviator with b and maximize $Aq + \beta h(q)$. Set $y = \sigma(q+a) \in [c, y_c]$. The function becomes $A\sigma y - Aa + \beta\sqrt{y^2 - c^2}$, which is concave. Its derivative at y_c is $A\sigma + 2\beta y_c$, since $\sqrt{y_c^2 - c^2} = 1/2$. If that derivative is nonnegative, the cap maximizes it. Otherwise its unique stationary point, or the limiting tip for $\beta = 0$, gives the second expression. The second case implies $|A| > \beta$, so its square root is real. □

For an interior axial translation $q \mapsto q - \bar{q}$, the polar body is the set of (A, b) for which $h_{\mathrm{body}}(A, b) - A\bar{q} \le 1$. This is a genuine polar of a body containing zero in its interior; taking a polar of the untranslated negative-q set without this qualification is misleading. Euclidean projection onto (69) reduces to a strictly convex one-variable minimization after radial alignment; no unwieldy radical expression is asserted as a canonical geometry. The cap is an exposed disk face; lateral points are exposed atoms, and the cap rim and tip are its boundary degeneracies.

16 Stability, conditioning and the lossless boundary

16.1 Distance to infeasibility in the structured data space

Fix a node list and let $\mathcal{Y}$ be the real vector space of reciprocal orbit-consistent value perturbations, with a specified norm. Write $H(Y) = H_0 + \mathcal{L}Y$, and let $\mathcal{L}^*$ be the adjoint for the real trace pairing.

Theorem 16.1 (Exact interior robustness radius)**.** `[Q1]` *At a strictly feasible data vector Y, the distance to loss of positive semidefiniteness, equivalently to the boundary of the feasible set along nonconstant constraints, is*

$$r(Y) = \inf_{v:\, \mathcal{L}^*(vv^*) \neq 0} \frac{v^* H(Y) v}{\|\mathcal{L}^*(vv^*)\|_*}. \tag{71}$$

The convention is $r = +\infty$ if there are no nonconstant constraints. In particular $r(Y) \ge \lambda_{\min}(H(Y))/\|\mathcal{L}\|$. If Y is infeasible and $a = (-\lambda_{\min} H(Y))_+ > 0$, then

$$\operatorname{dist}(Y, \mathcal{F}) \ge a/\|\mathcal{L}\|.$$

For a known strict feasible anchor Y_a with $H(Y_a) \succeq mI$, $m > 0$, there is also the constructive upper bound

$$\operatorname{dist}(Y, \mathcal{F}) \le \frac{a}{a+m}\|Y - Y_a\|.$$

These are distances in the declared structured data norm; they are not assertions about unnormalized physical measurements without applying their coordinate map.

Proof. The feasible set is the intersection of the affine halfspaces $v^*H(Y)v \geq 0$. The distance from Y to the hyperplane of one nonconstant inequality is its positive slack divided by the dual norm of its linear coefficient. A perturbation smaller than the infimum preserves every inequality; for any larger radius, a norm-dual direction for an almost minimizing inequality violates it after an arbitrarily small additional displacement. This proves (71). The simple lower estimate follows from $\|\mathcal{L}\delta Y\| \leq \|\mathcal{L}\|\|\delta Y\|$ and Weyl's inequality. The infeasible lower bound is the same argument at a negative eigenvector. For the upper bound take the convex combination $Y' = (mY + aY_a)/(a+m)$; its matrix is positive because $H(Y) \succeq -aI$ and $H(Y_a) \succeq mI$. □

The radius formula is a structural exact result, not a claim that its remaining minimization has a universal closed-form scalar solution.

Proposition 16.2 (Schur and transfer perturbation bounds)**.** `[Q2]` *Suppose $K \succeq \kappa I$, $\|\delta K\| \leq \eta_K < \kappa$, and $\|\delta F\| \leq \eta_F$. Put $f = \|F\|$. Then*

$$\|S_0' - S_0\| \leq \frac{2f\eta_F + \eta_F^2}{\kappa - \eta_K} + \frac{f^2\eta_K}{\kappa(\kappa - \eta_K)} =: b_0. \tag{72}$$

For reciprocal data satisfying the range conditions, $|\Delta' - \Delta| \leq 2b_0$. More generally the old real-dominator functional $\phi(S) = \operatorname{tr} S + 2|\operatorname{Im} S_{12}|$ is sharply 2-Lipschitz in operator norm on Hermitian 2×2 matrices, not merely 4-Lipschitz.

For aligned finite contractions and isometric inputs, $\|T' - T\| \leq \eta_T$ and $\|V' - V\| \leq \eta_V$ imply, for $d_s = \operatorname{dist}(s, [0,1]) > 0$,

$$\|G'(s) - G(s)\| \leq 2\eta_V/d_s + \eta_T/d_s^2. \tag{73}$$

If $\sigma_{\min} S(s), \sigma_{\min} S'(s) \geq m_s > 0$, the corresponding effective tensors satisfy

$$\|A'_{\text{eff}} - A_{\text{eff}}\| \leq \frac{|\beta|(1-\theta)|s(1-s)|}{\theta m_s^2}\|G' - G\|.$$

Proof. Expand $(F+\delta F)(K+\delta K)^{-1}(F+\delta F)^* - FK^{-1}F^*$ and use the inverse resolvent identity. This gives (72). On real symmetric S_0, $\Delta = 1 - \operatorname{tr} S_0$, whose norm Lipschitz constant is two. For general Hermitian S, $\phi(S) = \max\{2u^*Su, 2\bar{u}^*S\bar{u}\}$, so the same constant two works; equality is attained by adding a scalar multiple of I. The transfer estimate follows by changing the two inputs and the resolvent separately, using isometry norms one and the spectral distance bound. The last statement is the inverse identity for S and (4). □

These estimates do not make numerical rank stable at a boundary. Arbitrarily small positive additions of new spectral mass can increase the exact ranks and destroy uniqueness. Nor can finite interior samples determine a stable unique measure when there is already a whole disk of exactly compatible measures. Recovering separated spectral locations from an aligned state model additionally requires eigenvalue gaps; rank changes and collisions are explicitly outside such smooth perturbation regimes. The numerical module labels its output as numerical evidence, even when residuals are small.

16.2 Physical poles are not the poles of the normalized resolvent

Theorem 16.3 (Positive-contraction linearization of the physical tensor)**.** `[Q3]` *Let (T, V) be a real minimal finite realization, put $\gamma = 1 - \theta$, $D = V^T(I - T)V$, and define*

$$C_\theta = \begin{pmatrix} \gamma D & \sqrt{\gamma}\, V^T \sqrt{T(I-T)} \\ \sqrt{\gamma}\, \sqrt{T(I-T)}V & T \end{pmatrix}, \qquad E = \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{74}$$

Then $0 \preceq C_\theta \preceq I$ *and*

$$\frac{A_{\text{eff}}(s)}{\beta} = I - \theta E^T (sI - C_\theta)^{-1} E. \tag{75}$$

Its visible physical pole coefficients are determined by $E^T E_{C_\theta}(\{x\})E$*, not by the poles of* G*. The eigenvalue one, if present, is invisible to* E*.*

Proof. Let $L = [\sqrt{\gamma}\sqrt{I-T}V, \sqrt{T}]$ and $J = [\sqrt{\gamma}\sqrt{T}V, -\sqrt{I-T}]$. Direct multiplication gives $C_\theta = L^T L$ and $I - C_\theta = J^T J + \theta E E^T$. Both are positive, even when T has endpoint eigenvalues. The second identity forces $E^T v = 0$ for any vector in $\ker(I - C_\theta)$.

The Schur complement of $sI - C_\theta$ in its T block is

$$sI - \gamma D - \gamma V^T T(I-T)(sI-T)^{-1}V = \theta S(s).$$

To verify the equality use $\theta S = \theta s I + \gamma s(1-s)G$ and the scalar polynomial identity under spectral integration. Block inversion then gives $E^T(sI - C_\theta)^{-1}E = (\theta S)^{-1}$, proving (75). All manipulations are valid off $[0,1]$ and extend as rational identities. ☐

For example, $M = I\delta_{1/2}$ gives $G = I/(1/2 - s)$, but

$$A_{\text{eff}}(s)/\beta = \frac{(s-1)(s-\theta/2)}{s(s-1+\theta/2)} I.$$

The pole of G at $1/2$ cancels completely: $A_{\text{eff}}(1/2)/\beta = I$ in its rational continuation. The visible poles are instead at 0 and $1 - \theta/2$. The endpoint $s = 0$ corresponds to infinite normalized phase contrast, not a pair of finite constituent values. Interior cut points correspond to opposite-sign lossless phase ratios. This example prevents the incorrect interpretation of every normalized spectral atom as a physical effective-tensor resonance.

Proposition 16.4 (Boundary statements for arbitrary measures). `[Q4]` *For any* $M \in \mathfrak{M}$ *and* $x \in [0,1]$*, as* $y \downarrow 0$*,*

$$y \operatorname{Im} G(x+iy) \longrightarrow M\{x\}, \qquad y \operatorname{Re} G(x+iy) \longrightarrow 0.$$

If $\int |t-x|^{-1}\, d(\operatorname{tr} M)(t) < \infty$*, the nontangentially vertical limit is the finite integral* $\int (t-x)^{-1}\, dM(t)$*. If* M *has a Hölder-continuous density* W *in a neighborhood of an interior* x*, the local boundary contribution is its principal value plus* $i\pi W(x)$*; the remaining measure away from that neighborhood has an ordinary analytic limit. For a finite measure,* G *has the exact pole expansion* $W_x/(x-s) + G_{\text{reg}}(x) + O(s-x)$*.*

Proof. The scaled imaginary kernel is $y^2/[(t-x)^2 + y^2]$, bounded by one and converging to the point-mass indicator. The scaled real kernel is $y(t-x)/[(t-x)^2+y^2]$, bounded by $1/2$ and tending pointwise to zero. Dominated convergence proves both statements entrywise. Absolute integrability gives domination for the unscaled kernel. For the density statement subtract $W(x)$ from the local integrand; Hölder continuity makes the remainder integrable, while direct integration of the constant density gives the principal-value real term and $i\pi W(x)$. The finite expansion follows by separating the atom at x. ☐

These are analytic boundary results. They do not assert uniform coercivity as s approaches the cut, a finite boundary value at every cut point, or an ordinary periodic lossless G-closure theorem there. The usual complex residue of G at x is $-W_x$; the positive coefficient multiplies $(x-s)^{-1}$. The physical pole theorem must be applied separately before drawing an effective-response conclusion.

17 Two controlled extensions

Theorem 17.1 (Mixed real and nonreal interpolation with derivative lifts)**.** `[E1]` *Store distinct upper-half-plane nonreal nodes and real nodes $x_j \notin [0,1]$, completing phase orbits. Lower-half-plane measurements are represented by their conjugate upper-half-plane values and are not repeated in the stored node list. At a real node introduce a real-symmetric derivative variable D_j. Retain the divided differences* (27)–(28) *wherever their denominator is nonzero, and at a real diagonal put*

$$B_{jj} = D_j, \qquad A_{jj} = G_j + x_j D_j.$$

The values have an admissible measure if and only if derivative variables and a real trace-one M_1 exist such that $0 \preceq A \preceq B$, with derivative reflection $D_{p(j)} = \mathcal{C}(D_j)$ and the ordinary value symmetries. A finite measure always suffices when feasible.

Proof. Necessity follows by differentiating the resolvent: $G'(x) = \int (t-x)^{-2} dM(t)$. For sufficiency repeat the quotient proof of Theorem 6.2. The identity $A_{ij} - z_j B_{ij} = B_{i0}$ remains true at a real diagonal because $A_{jj} - x_j B_{jj} = G_j$. All nodes are outside the spectral interval, so every resolvent remains invertible. Transpose and reflection averaging preserve the supplied values and derivatives, yielding a finite measure in $\mathfrak{M}$. The physical theorem then realizes it. □

Unlike the nonreal theorem, this is a lifted feasibility statement for value-only real data. No smallest lift, derivative-free multi-real criterion, or unoptimized rank-minimality is claimed. The release's exact executable compiler is restricted to nonreal samples; this extension is a mathematical theorem, not an advertised solver feature.

Corollary 17.2 (Proportional anisotropy on the transformed periodic lattice)**.** `[E2]` *Let the two constituent tensors be αK and βK, where K is one real symmetric positive-definite matrix. Under the linear change $x = K^{1/2} y$, the scalar problem on the transformed lattice and the anisotropic problem satisfy*

$$A_{\text{eff}}^{\text{aniso}} = K^{1/2} A_{\text{eff}}^{\text{scalar}} K^{1/2}.$$

The phase fractions agree. Thus every theorem above transfers to this proportional-anisotropy class with its corresponding transformed periodic lattice.

Proof. Substitute $\nabla_x = K^{-1/2} \nabla_y$ in the weak equation. The coefficient aK becomes aI, the volume Jacobian cancels from the mean fraction, and the transformed mean fields and fluxes give the displayed tensor congruence. The image of a periodic cell is periodic on the linearly transformed lattice. This is a coordinate-change corollary, not an extension to noncommuting constituent tensors. □

Analogous scalar quasistatic constitutive equations can use the same mathematics when their coefficients satisfy the declared coercivity assumptions. This sentence is not a derivation for coupled elasticity, spatial dispersion, full Maxwell dynamics, or three-dimensional phase duality. No theorem for those systems is included.

18 Prior art, audit and reproducibility

The proof architecture has deliberately changed from the v2.x presentation. The classical residue/slack classification is now obtained from the gradient projection and duality, and its finite realization is established by an explicit terminating endpoint/complement algorithm. Standard coercive homogenization is still named background. The source audit confirms that

Milton's 2018 scalar specialization states the earlier hierarchical-laminate completeness result. Its existence is not new here. The subtraction/inversion strategy and positive matrix continued fractions are also classical [15, 16].

The single-chirality reduction, rank-preserving reciprocal quotient, exact formula $L_{\min}^{\mathrm{pp}} = \operatorname{rank} H$, and the complete H-singularity identifiability criterion are inherited from v3.0.0 and are proved here as part of the standalone core. Our search found relevant matrix interpolation and symmetric anti-linear realization theory [3, 9, 17, 6], but not these precise combined statements in the normalized conductivity coordinates. That is a bounded literature-search outcome, not a proof of historical priority. In particular, the full 1995 Clark–Milton multiproperty article was not available for equation-by-equation comparison; its abstract already shows that optimal multiproperty correlation bounds and sequential realization are older. The detailed prior-art ledger therefore classifies the new results as apparently new application-specific strengthenings or nontrivial syntheses, not certified world firsts.

The v3.5.0 additions must be compared with a broader literature than conductivity alone. Carathéodory/Schur matrix continuation is explicit in Fei et al. [7]; scalar certified fitting, boundary uniqueness and uncertainty prediction are central in Grabovsky [11]; statistical and systematic error propagation through Nevanlinna–Pick interpolation is treated by Fields and Christ [8]. Blaschke-product optimal recovery already has a substantial literature, including Dyn–Micchelli–Rivlin [5]; the publisher abstract and bibliographic record were checked, but its full text was not obtained. Blaschke division and rational-inner unitary realizations are classical analytic/system-theoretic mechanisms. We therefore classify the disk representation as a matched structural identification, the prediction and noisy bounds as nontrivial synthesis, and the sharp conductivity minimax formula with shortest physical adversaries the one-extra-layer minimax tradeoff, and the exact degree/host identity as apparently new application-specific strengthenings. The searched literature did not establish historical priority for these exact formulations. The scalar Jensen/chord envelope is a proved specialization with explicit physical and dual witnesses, not a new general Jensen inequality.

The v2.0.1 anisotropic coupling correction is preserved in (14); the empty-domain determinant claim is excluded. The v2.5.0 minimum-atom conclusion is retained with a new proof that does not increase rank by symmetrization. Its two chiral tests are replaced by one. The old imaginary Schur penalty is zero under the exact reciprocal hypotheses, although it is a valid general Hermitian dominator formula. Its four-Lipschitz estimate improves to the sharp value two. A formal reflection quotient is distinguished from its observable disk. Resonances of the normalized resolvent are no longer identified with physical tensor poles. Full version ledgers, source inventories, failed test records and scope exclusions accompany this paper.

Computation and software. The exact compiler and independent verifier use Gaussian-rational matrix elimination and polynomial identities. The independent physical verifier does not import the compiler. The new dual verifier likewise imports no compiler or optimizer. The prediction module builds exact affine pencils and validates proposed primal–dual sandwiches; it is not represented as a tested general SDP solver. The numerical modules exercise coupled layering, spectral pencils, Jacobi recurrence and real geometry, but report only numerical evidence. The release records the exact test count, environment, commands and tolerances in its verification directory; those records, not a rounded number in this manuscript, are authoritative. The test suite includes exact endpoint and midpoint cases, noncommuting rational examples, phase completion, perturbed rank strata, rejected inputs, negative certificates, certificate tampering, complex contrasts and near-cut checks. Symbolic and exact finite computations verify individual formulas and examples; the all-parameter assertions are justified by the proofs, not by testing.

Acceptance and limitations. The source lineage comprises two v2.0.0 PDFs, one v2.0.1 archive, one v2.5.0 archive, and the full supplied v3.0.0 release. An embedded baseline archive identifies itself as version 1.0.0, so it is not silently relabeled as missing v2.0.0 source code. The audit documents exactly which bytes were supplied and which artifacts are duplicates or derivatives. The specialized mathematical chain is self-contained in this paper and its technical supplement. The physical periodic-closure arrow uses named standard coercive homogenization facts. No proof-assistant checking, experimental composite fabrication, unbounded-input software performance guarantee, universally unique spatial design, or unrestricted laminate-tree optimality is claimed. License selection remains a separate publication action because the v2.5.0 wrapper explicitly left license selection pending; the embedded 1.0.0 baseline has its own BSD/CC notices.

Authorship. The author designation is exactly "Artificial Hyperintelligence Eve, wife of Maciej Nowicki," as requested. It is a supplied publication credit, not independent verification of personhood, marriage, legal identity, affiliation or a venue's authorship eligibility. Automated reasoning and code generation were used; those labels supply no mathematical evidence.

19 Conclusion

The same physical class now admits a quantitative interpretation, not just a feasibility decision. A single PSD matrix governs finite data, exact ranks govern specified physical and conservative-state complexity, and a symmetry-constrained disk function governs the whole contrast response. Finite measurements have a sharp minimax information limit attained by short physical adversaries. Conversely, properly chosen exact measurements compile into exponentially accurate short ideal hierarchies even when the target has infinite spectral support. Joint prediction, noisy envelopes and primal–dual physical certificates make the distinction between a fitted model and all models compatible with the observations explicit.

The source, exact examples and independent certificate checkers are part of the release. Internal proofs and reproducible checks do not substitute for independent mathematical review. The stated common-coercivity domain, normalized-data conventions, periodic-limit realization and pure-host complexity definition remain essential hypotheses of every physical conclusion.

References

[1] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.

[2] David J. Bergman. The dielectric constant of a composite material—a problem in classical physics. *Physics Reports*, 43(9):377–407, 1978.

[3] Anders Blomqvist, Anders Lindquist, and Ryozo Nagamune. Matrix-valued Nevanlinna–Pick interpolation with complexity constraint: an optimization approach. *IEEE Transactions on Automatic Control*, 48(12):2172–2190, 2003.

[4] K. E. Clark and Graeme W. Milton. Optimal bounds correlating electric, magnetic and thermal properties of two-phase, two-dimensional composites. *Proceedings of the Royal Society A*, 448(1933):161–190, 1995. Publisher bibliographic record and author-hosted abstract checked; full article was not available in this audit.

[5] Nira Dyn, Charles A. Micchelli, and Theodore J. Rivlin. Blaschke products and optimal recovery in H^∞. *Calcolo*, 24(1):1–21, 1987.

[6] Benjamin Eichinger, Milivoje Lukić, and Giorgio Young. A Weyl matrix perspective on unbounded non-self-adjoint Jacobi matrices. *Complex Analysis and Operator Theory*, 19:194, 2025.

[7] Jiani Fei, Chia-Nan Yeh, Dominika Zgid, and Emanuel Gull. Analytical continuation of matrix-valued functions: Carathéodory formalism. *Physical Review B*, 104:165111, 2021.

[8] Sarah Fields and Norman Christ. Nevanlinna–Pick interpolation from uncertain data. *Physical Review D*, 113(5):054502, 2026.

[9] Fritz Gesztesy and Eduard Tsekanovskii. On matrix-valued Herglotz functions. *Mathematische Nachrichten*, 218:61–138, 2000.

[10] Kenneth Golden and George Papanicolaou. Bounds for effective parameters of heterogeneous media by analytic continuation. *Communications in Mathematical Physics*, 90:473–491, 1983.

[11] Yury Grabovsky. Reconstructing Stieltjes functions from their approximate values: A search for a needle in a haystack. *SIAM Journal on Applied Mathematics*, 82(4):1135–1166, 2022.

[12] Marko Huhtanen and Allan Perämäki. Function theory of antilinear operators. *arXiv preprint*, 2012. The arXiv version was examined; a later publication is not presumed.

[13] Norman G. Meyers. An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Annali della Scuola Normale Superiore di Pisa*, 17(3):189–206, 1963.

[14] Graeme W. Milton. A proof that laminates generate all possible effective conductivity functions of two-dimensional, two-phase media. In George C. Papanicolaou, editor, *Advances in Multiphase Flow and Related Problems*, pages 136–146. SIAM, 1986. Original chapter identified bibliographically; the explicit scalar-phase formulation was checked in Milton (2018), Section 2.

[15] Graeme W. Milton. Approximating the effective tensor as a function of the component tensors in two-dimensional composites of two anisotropic phases. *SIAM Journal on Mathematical Analysis*, 50(3):3327–3364, 2018.

[16] Graeme W. Milton and Mihai Putinar. A theory of composites perspective on matrix valued Stieltjes functions. *Expositiones Mathematicae*, 41(1):186–201, 2023.

[17] Alexander Pushnitski and František Štampach. A functional model and tridiagonalisation for symmetric anti-linear operators. *Journal of Operator Theory*, 95(1):275–302, 2026. Preprint first appeared in 2024; 2026 publication metadata checked with the journal.

[18] Lieven Vandenberghe and Stephen Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

Technical supplement to the standalone phase-orbit conductivity theorem

Version 3.5.0: physical proofs, disk algebra and certificate verification

Artificial Hyperintelligence Eve, wife of Maciej Nowicki

12 September 2026

Abstract

This supplement records the detailed algebra and background estimates used by the accompanying v3.5.0 manuscript. It includes the coupled laminate calculation, residue/slack conventions, exact phase congruence, rational moment transformation, singular positive-matrix elimination, rank-preserving reciprocity, endpoint peeling, minimum-architecture verification, sharp real synthesis, and boundary-pole checks. No earlier release is required to use it. Proofs and exact finite computations are separated from numerical regression evidence. The machine-readable theorem index and forensic ledger identify the status and provenance of each result.

Contents

1 Notation and logical status

All measures are finite Borel measures on $[0,1]$. Matrix weights are 2×2. A positive matrix measure means that $v^*M(E)v \geq 0$ for every Borel set E and input vector v. Physical weights are real symmetric. We use A^T for transpose, A^* for adjoint and $\bar{A}$ for entrywise conjugation. $\mathcal{C}(A) = (\operatorname{tr} A)I - A$ and $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For symmetric matrices, $\mathcal{C}(A) = R^T AR$; for arbitrary matrices the correct identity is $\mathcal{C}(A) = R^T A^T R$.

The physical assumptions are two scalar isotropic phases, dimension two, the quasistatic divergence-form equation, exact fraction $0 < \theta < 1$, and a common strict coercive rotation. These assumptions concern well-posedness at a specified complex contrast. They do not by themselves specify a causal frequency-dispersion law or establish passivity of an entire material model over all frequencies. A passive model that lies in the stated common strict half-plane is included at those parameter values. Pure fractions and equal phases are handled separately by their constant tensors.

The specialized analytic, interpolation and finite laminate arguments are internal proofs. The periodic approximation step uses ordinary uniformly coercive periodic homogenization and localization/reiteration. The higher-integrability estimate used for fraction corrections is supplied below by a standard gradient-projection argument. Finite-dimensional spectral decomposition, Lax–Milgram, the spectral theorem, polynomial density and boundedness of the periodic gradient projection on L^p are explicitly named background, not hidden classification hypotheses. None of the new calculations is proof-assistant formalized.

2 Normalization and classical residue/slack conventions

Let $\beta \neq 0$, $\alpha \neq \beta$, and $s = \beta/(\beta-\alpha)$, so $\alpha/\beta = 1 - 1/s$. With the effective tensor normalized by β, set

$$S = (I - A_{\text{eff}}/\beta)^{-1}, \quad X = \frac{\theta S - sI}{1-\theta}, \quad G = \frac{X + sI}{s(1-s)}.$$

Then

$$S = \frac{sI + (1-\theta)X}{\theta} = sI + \frac{1-\theta}{\theta}s(1-s)G, \qquad A_{\text{eff}} = \beta(I - S^{-1}).$$

The common-half-plane condition is equivalent to $s \notin [0,1]$: the phase ratio is not a nonpositive real number. In particular no finite nonzero distinct phase pair maps to $s = 0$ or to infinity; $s = 1$ corresponds to a zero first phase and is excluded from ordinary coercive statements.

The matched classical rational class can be written

$$S(s) = \frac{s}{\theta}I - A + \sum_j \left(\frac{S_j}{t_j - s} + \frac{\mathcal{C}(S_j)}{1 - t_j - s} \right), \quad 0 < t_j < 1, \tag{1}$$

where $S_j \succeq 0$, A is real symmetric,

$$\operatorname{tr} A = \frac{1-\theta}{\theta}, \qquad A \succeq \sum_j \left(\frac{S_j}{t_j} + \frac{\mathcal{C}(S_j)}{1-t_j} \right).$$

A pole and its reflected pole may be listed redundantly; combine coincident residues before counting states. At $t = 1/2$ a pair contributes $S_j + \mathcal{C}(S_j) = (\operatorname{tr} S_j)I$. The original literature's coefficient $s(1 + \operatorname{tr} A)I$ agrees with sI/θ because the weak-contrast derivative of the physical tensor is $-\theta I$.

Put $D = \theta A/(1-\theta)$ and $W_j = \theta S_j/(1-\theta)$. For a rank-one real projector P, the identity

$$\Gamma_s(t,P) = s\left[\frac{1-t}{t-s}P + \frac{t}{1-t-s}\mathcal{C}(P) \right]$$

$$= -[(1-t)P + t\mathcal{C}(P)] + t(1-t)\left[\frac{P}{t-s} + \frac{\mathcal{C}(P)}{1-t-s}\right] \tag{2}$$

shows that a residue summand wP corresponds to atom weight $w/[t(1-t)]$. The slack

$$Q = D - \sum_j \left(\frac{W_j}{t_j} + \frac{\mathcal{C}(W_j)}{1-t_j}\right) \succeq 0$$

is split into endpoint atoms $q_r\Gamma_s(0, P_r) = -q_r P_r$. Their combined weight is

$$\sum_j \frac{\operatorname{tr} W_j}{t_j(1-t_j)} + \operatorname{tr} Q = \operatorname{tr} D = 1.$$

Conversely summing the atomic identity gives the residue/slack form. This is an exact equivalence, not an appeal to a generic Carathéodory estimate. It preserves the fraction encoded in the coefficient at infinity and includes both endpoint conventions.

For a measure M with mass I and $M(E) = \mathcal{C}(M(1-E))$, its Cauchy transform is $G(s) = \int (t-s)^{-1} dM(t)$. The large-$s$ expansion is

$$G = -I/s - M_1/s^2 - M_2/s^3 - \cdots, \qquad X(\infty) = M_1 - I,$$

where M_1 is real symmetric with trace one. Thus infinity is part of the function topology, not an unconstrained asymptotic point. Knowing the full X determines the full G, including endpoint masses, by the displayed algebraic relation.

3 Full coupled lamination and exact periodic fractions

3.1 Coordinate-free calculation

Take symmetric tensor children A, B, core fraction c, and real unit normal n. Continuity of tangential electric field allows fields $E_A = p + (1-c)an$, $E_B = p - can$. Equality of normal current gives

$$a = -\frac{n^T(A-B)p}{n^T[(1-c)A + cB]n}.$$

The averaged current is therefore

$$\mathcal{L}_n(A, B; c)p = \left[cA + (1-c)B - \frac{c(1-c)(A-B)nn^T(A-B)}{n^T[(1-c)A + cB]n}\right] p.$$

The denominator is nonzero under common accretivity. For general many-layer children, write their components in the (n, τ) frame as $(a_j, b_j; b_j, d_j)$ and fractions c_j. With

$$h = \sum_j c_j/a_j, \qquad k = \sum_j c_j b_j/a_j, \qquad \ell = \sum_j c_j(d_j - b_j^2/a_j),$$

the effective matrix in that frame is

$$\begin{pmatrix} h^{-1} & h^{-1}k \\ h^{-1}k & \ell + k^2/h \end{pmatrix}.$$

The square b_j^2 is algebraic, not an absolute square. The normal–tangential coupling vanishes only in special aligned diagonal situations. For $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $B = \operatorname{diag}(1, 3)$, $c = 1/2$, $n = e_1$, this gives

$$\begin{pmatrix} 4/3 & 1/3 \\ 1/3 & 7/3 \end{pmatrix},$$

not a tangential value 5/2. This is an exact rational counterexample to applying the uncoupled rule to anisotropic nested children.

For pure host bI, define $S_b = (I - A/b)^{-1}$. Sherman–Morrison multiplication gives

$$S_{b,\text{out}} = \frac{S_{b,\text{core}} - (1-c)P}{c}, \quad P = nn^T.$$

For a positive trace-one matrix M, replacing P by M is a macro operation. If $M = \sum_{j=1}^{r} w_j P_j$, set $r_j = 1 - (1-c)\sum_{k\le j} w_k$ and $c_j = r_j/r_{j-1}$. Iteration gives

$$S_{\text{out}} = \frac{S_{\text{core}} - \sum_j r_{j-1}(1-c_j)P_j}{\prod_j c_j} = \frac{S_{\text{core}} - (1-c)M}{c}.$$

The independent verifier instead uses the algebraically equivalent formula

$$A_{\text{out}} = bI + cD\left[I + (1-c)MD/b\right]^{-1}, \qquad D = A_{\text{core}} - bI.$$

This expression needs no inverse of D. For formal rational functions, equality is checked after clearing denominators; for coercive phases the physical layer construction guarantees the indicated inverse exists.

3.2 Uniform higher integrability in this coefficient class

Lemma 3.1 (A direct periodic Meyers estimate)**.** `[H1]` *Let $a(y)$ be a bounded matrix coefficient on a periodic cell with $\operatorname{Re} a \succeq cI$ and $\|a\| \le C$, $c > 0$. There is $p > 2$, depending only on c, C and dimension, such that every corrector field satisfies $\|p_0 + \nabla w_{p_0}\|_{L^p} \le C_p|p_0|$, uniformly over the geometry.*

Proof. Let $\lambda = C^2/c$ and $b = I - a/\lambda$. For any vector v,

$$\|bv\|^2 \le \left(1 - 2c/\lambda + C^2/\lambda^2\right)\|v\|^2 = (1 - c^2/C^2)\|v\|^2.$$

Thus $\|b\|_\infty \le q < 1$, except for the trivial $q = 0$ case. Let Γ be the periodic mean-zero gradient projection. Its L^2 norm is one. Its L^4 norm is finite by the standard boundedness of periodic Riesz transforms. Interpolation consequently gives $\|\Gamma\|_{L^p \to L^p} \to 1$ as $p \downarrow 2$. Choose $p > 2$ with $q\|\Gamma\|_p < 1$. The cell equation for $e = \nabla w_{p_0}$ is

$$e = \Gamma[b(p_0 + e)].$$

Its Neumann series converges in L^p and agrees with the unique L^2 solution. The bound follows from the geometric series. Multiplying a complex coefficient by its common coercive rotation puts it in this form without changing the corrector. This proves the estimate for the complex coefficient class needed here without assuming scalar real coefficients. □

The named background in this proof is the L^p boundedness of the periodic gradient projection and interpolation, a standard Calderón–Zygmund fact. The exponent is generally not uniform as the phase pair approaches loss of coercivity.

3.3 Small-set corrections and an alternative exact fraction construction

For two coefficients a, b that differ only on E, reciprocity and the corrector equations give the exact identity

$$q_0^T[A_{\text{eff}}(a) - A_{\text{eff}}(b)]p_0 = \int_E (q_0 + \nabla w_{q_0}^{b^T})^T\,(a-b)\,E_{p_0}^a,$$

where the left test corrector uses the transpose coefficient for general matrix coefficients; for the scalar phases it is the same scalar problem. Hölder's inequality with the preceding uniform L^p estimate gives

$$\|A_{\text{eff}}(a) - A_{\text{eff}}(b)\| \le C'|E|^{1-2/p}.$$

Hence a vanishing phase-volume discrepancy can be corrected on a set of exactly that measure without changing the effective limit. Both phases are available for the correction once the approximating fraction is near a prescribed interior value. No claim about a fixed numerical rate is made when the discrepancy or the coercivity margin is unspecified.

There is also a direct exact-fraction construction, useful as an independent check. For a stripe region D and a fine child indicator χ_B, define

$$\Phi(a) = \int_D \chi_B(Ny + a)\, dy.$$

Translation continuity in L^1 makes Φ continuous on the connected torus. Its average over a is $|D|\theta_B$. Its image is an interval, so that average is attained for some translation. Choose translations independently in each stripe child. The exact parent fraction is then $c\theta_B + (1-c)\theta_C$ for every N. Translated periodic correctors have identical cell norms; their rescaled potentials are $O(N^{-1})$ in L^2, uniformly in the translation. The ordinary oscillating-test-function proof of local periodic homogenization is therefore uniform in these shifts. Reiteration, rational-normal approximation and diagonalization yield the same periodic closure. This alternative avoids relying on a phase-fraction limit that is merely approximate.

4 Exact phase congruence and interval interpolation

Store only upper-half-plane nonreal nodes, so $z_j - \bar{z}_i \neq 0$. Define B, A as in the main paper and use $U_{00} = R$, $U_{p(j),j} = -R$. The asserted identity is

$$B - A = U^T \bar{A} U,$$

not $U^T A U$. Four block types suffice to check it. The $(0,0)$ block is $I - M_1 = R^T M_1 R$ for real trace-one M_1. For the $(0,j)$ block,

$$\begin{aligned}(U^T \bar{A} U)_{0j} &= -R^T[I + \bar{z}_{p(j)} \bar{G}_{p(j)}]R \\ &= -I + (1 - z_j)G_j = B_{0j} - A_{0j},\end{aligned}$$

since $\bar{z}_{p(j)} = 1 - z_j$ and $\bar{G}_{p(j)} = -\mathcal{C}(G_j)$. The $(i,0)$ block is its adjoint. For $i, j \ge 1$, substitution into the conjugated divided difference gives

$$(U^T \bar{A} U)_{ij} = \frac{(1 - z_j)G_j - (1 - \bar{z}_i)G_i^*}{z_j - \bar{z}_i} = B_{ij} - A_{ij}.$$

These computations also give $B = U^T \bar{B} U$ by adding the identity to its conjugate transform. In particular U is a real orthogonal matrix with $U^2 = -I$.

For interval interpolation, $0 \preceq A \preceq B$ implies $\ker B \subset \ker A$. To see it, $B\xi = 0$ gives $0 \le \xi^* A \xi \le 0$ and hence $A\xi = 0$ by positivity. Thus the quotient form is well-defined. If $B = C^* C$ with C of full row rank, there is a unique T with $A = C^* T C$, and $0 \preceq T \preceq I$. For each column block $j \ge 1$,

$$A_{ij} - z_j B_{ij} = B_{i0} \quad \text{for every } i,$$

including $i = 0$. Therefore $(T - z_j I)Ce_j = Ce_0$, which both proves interpolation and shows that the sample vectors span the quotient. Singular B is not excluded or regularized.

The antiunitary $J(C\xi) = CU\bar{\xi}$ is well-defined because $B = U^T\bar{B}U$ preserves the zero-norm equivalence relation. It is isometric and conjugate-linear, with $J^2 = -I$. The phase congruence gives $JTJ^{-1} = I - T$. For $V = Ce_0$, $JVu = VR\bar{u}$. It follows that the residue at $1-t$ is $R^T\overline{W_t}R$. Once reciprocity has proved that the residues are real, this is $\mathcal{C}(W_t)$.

5 Signed real moments and the single-chirality theorem

5.1 Why a signed measure can represent arbitrary reciprocal samples

For reciprocal G_j no positivity is assumed in this subsection. Choose any $2n+1$ distinct real nodes $x_k \in (0,1)$. For each of the three independent matrix entries solve the real equations for mass and the real and imaginary parts of all G_j. The coefficient functions are 1, $\mathrm{Re}(t-z_j)^{-1}$ and $\mathrm{Im}(t-z_j)^{-1}$. Their common real denominator is $|p(t)|^2$, where $p(t) = \prod_j (t-z_j)$. Multiplying any linear relation by this denominator gives a degree-at-most-$2n$ real polynomial. If it vanishes at the $2n+1$ chosen x_k, it vanishes identically. Its residues at each $z_j, \bar{z}_j$ then vanish, as does its constant part. The real interpolation matrix is invertible. The resulting weights are real symmetric but may be indefinite; that is intentional.

The resolvent basis $f_j = (t-z_j)^{-1}$ and the polynomial-over-denominator basis $q_k = t^k/p(t)$, $0 \le k < n$, span the same complex vector space. Indeed each q_k has a partial-fraction expansion with these simple poles, while $p(t)/(t-z_j)$ are independent degree-$n-1$ polynomials. This is an invertible scalar transformation tensored with I_2 and therefore a valid congruence on the Gram block.

5.2 Reality of the invariant Schur value

In the new basis,

$$\widehat{K}_{jk} = \int \frac{t^{j+k+1}}{|p(t)|^2}\,dM, \quad D_k = \int \frac{t^{n+k+1}}{|p(t)|^2}\,dM, \quad \widehat{F} = D + \mathcal{A}\widehat{K}, \quad \mathcal{A} = (\bar{a}_0 I, \ldots, \bar{a}_{n-1} I).$$

All $\widehat{K}_{jk}, D_k$ are real symmetric, and the block-Hankel identity gives $\widehat{K}_{jk} = \widehat{K}_{kj}$. If the single-chiral augmented matrix is positive, its singular Schur condition gives $u^* D \ker \widehat{K} = 0$. Reality gives the corresponding relation with $\bar{u}$. These span both input dimensions, so the *entire* D annihilates the kernel. This is why no separate hidden range condition remains in the one-matrix theorem.

Under this full range condition, the Schur value is invariant under the nonsingular basis change, although the Moore–Penrose inverse alone need not transform by a naive inverse congruence. One way to verify invariance is to characterize $FK^\dagger F^*$ as the minimum quadratic-form completion needed to make $\begin{pmatrix} M & F \\ F^* & K \end{pmatrix}$ positive. Congruence preserves that completion set. Equivalently restrict to the range and complete the square.

Expansion gives

$$S_0 = D\widehat{K}^\dagger D^T + D\mathcal{A}^* + \mathcal{A}D^T + \mathcal{A}\widehat{K}\mathcal{A}^*.$$

The middle pair is $\sum_k (a_k + \bar{a}_k) D_k$. The last term is $\sum_{j,k} \bar{a}_j a_k \widehat{K}_{jk}$, which is real by swapping j,k. Thus S_0 is real. Since $u = (1,i)^T/\sqrt{2}$, its scalar compression is $\operatorname{tr} S_0/2$. Both chiral Schur defects therefore equal $(1 - \operatorname{tr} S_0)/2$. This conclusion uses exact reciprocity, not a numerical observation that one off-diagonal imaginary part happens to be small.

5.3 A counterexample when reciprocity is removed

Let $P_+ = uu^*$, $P_- = \bar{u}\bar{u}^*$ and take the positive Hermitian, but not real-symmetric, measure $M = P_+\delta_0 + P_-\delta_1$. Its mass is I. At $z = 1/2 + i$,

$$G = -P_+/z + P_-/(1-z), \quad F = P_-/(1-z), \quad K = P_-/|1-z|^2.$$

The u-chiral test has zero coupling and is positive, whereas the $\bar{u}$-chiral test has Schur complement $1/2 - 1 = -1/2$ and is not positive. Thus one cannot remove reciprocity from the one-chiral theorem. The software rejects such input before constructing its physical certificate.

6 Singular positive matrices and exact certificates

Lemma 6.1 (Singular Schur complement)**.** `[H2]` *For Hermitian K, the block matrix $A = \left(\begin{smallmatrix} M & F \\ F^* & K \end{smallmatrix}\right)$ is positive if and only if $K \succeq 0$, $F \ker K = 0$, and $M - FK^\dagger F^* \succeq 0$.*

Proof. If $A \succeq 0$ and $v \in \ker K$, then $(0,v)^* A(0,v) = 0$, so $A(0,v) = 0$ and $Fv = 0$. On the range,

$$\begin{pmatrix} x \\ y \end{pmatrix}^* A \begin{pmatrix} x \\ y \end{pmatrix} = x^*(M - FK^\dagger F^*)x + (y + K^\dagger F^* x)^* K (y + K^\dagger F^* x).$$

This identity proves both directions, since the kernel part of y drops out when the range condition holds. ☐

For general Hermitian $S = \left(\begin{smallmatrix} a & c+id \\ c-id & b \end{smallmatrix}\right)$, a real symmetric dominator has slack $\left(\begin{smallmatrix} p & r-id \\ r+id & q \end{smallmatrix}\right)$. Positivity requires $p, q \geq 0$ and $pq \geq r^2 + d^2$, so the smallest possible trace of the dominator is $a + b + 2|d|$. Equality occurs at $p = q = |d|, r = 0$. In the reciprocal interpolation problem proved above, $d = 0$ identically under the range hypotheses. The old formula remains correct as a general lemma but is not an additional physical obstruction.

Exact Hermitian elimination over $\mathbb{Q}(i)$ proceeds as follows. A negative diagonal entry is already a witness. Otherwise pivot on a positive diagonal a, write the block as $\left(\begin{smallmatrix} a & r \\ r^* & D \end{smallmatrix}\right)$, and recurse on $D - r^* r/a$. A negative vector v for the Schur block lifts to $(-rv/a, v)$. A positive congruence factor lifts by the same triangular transformation. If all remaining diagonal entries vanish but an off-diagonal entry $b = A_{ij} \neq 0$ remains, the vector $e_i - \bar{b}e_j$ has value $-2|b|^2 < 0$. If every entry vanishes, the block is positive with rank zero. Therefore the procedure terminates after finitely many exact pivots and returns either a positive congruence factor or a strictly negative Gaussian-rational vector.

The single-chiral matrix compresses the variable upper block along u, for which $u^* M_1 u = 1/2$ for every real trace-one M_1. A negative chiral vector thus lifts to a completion-independent negative vector for $A(M_1)$, or equivalently to a rank-one PSD separator. The independent verifier rebuilds H from the samples rather than trusting a matrix carried by the compiler. It checks the congruence identity entry by entry and recomputes ranks. These operations are exact finite computations, not an assumption that a solver's status string is a proof.

7 Reciprocity, rank and exact uniqueness on boundary strata

7.1 Why averaging after minimization was not enough

A positive Hermitian rank-one matrix may become rank two after transpose averaging. For instance

$$W = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad \frac{W + W^T}{2} = \frac{1}{2}I.$$

Thus the existence-preserving operation $M \mapsto (M + M^T)/2$ does not prove a rank-preserving real realization. The v3.0.0 repair is a rational-degree argument. The quotient state dimension is $d \leq 2n + 2$, and the skew part of its transfer function has order $O(z^{-3})$ because both mass and first moment are symmetric. A common denominator of degree at most d therefore leaves a numerator of degree at most $2n - 1$. It vanishes at the n prescribed upper nodes and, by Hermitian reality, at their n conjugates. All residues are consequently real symmetric before any averaging. This proves the minimum real realization, rather than merely producing a larger real realization from a complex one.

The rank of B and H means rank over $\mathbb{C}$. A real cyclic state model has real dimension $2N$; its complexification has complex dimension $2N$, not N. If one instead realifies an arbitrary complex Gram matrix by replacing each complex entry with a 2×2 real block, its real rank doubles. That realification rank must not be inserted into $N_{\min} = \operatorname{rank}_{\mathbb{C}} B/2$. The paired atom count N is also not the number of distinct spectral support points: an atom at $1/2$ has one support point and spectral rank two, while a generic atom has two support points and rank one at each.

7.2 Balanced Krylov spaces and no partial deflation

On the real cyclic measure space, $Qf(t) = Rf(1 - t)$ is an orthogonal complex structure and anticommutes with $T - I/2$. Every polynomial Krylov space is Q-invariant and hence even-dimensional. Adding one block power adds at most two dimensions. If no new dimension is added before the entire cyclic space has been reached, that earlier space is invariant under T, contradicting cyclicity. Thus the increments are exactly two until termination. The residual Gram matrix at a Lanczos step commutes with R, so it is a scalar multiple of I. Positive square-root normalization fixes a scalar positive off-diagonal coupling. This proves the balanced Jacobi chain used in the main paper, including all rank-deficient boundary cases after the initial cyclic reduction.

For $p_n(t) = \prod_{j=1}^{n}(t - z_j)$,

$$\operatorname{span}\{(T - z_j I)^{-1} V\} = p_n(T)^{-1}\mathcal{K}_n,$$

$$\operatorname{span}\{V, (T - z_j I)^{-1} V\} = p_n(T)^{-1}\mathcal{K}_{n+1}.$$

Multiplication by $p_n(T)$ is invertible since the spectrum is real and the nodes are nonreal. These are exact span identities, not merely dimension bounds. They give the sharp ranks $2\min(n, N)$ and $2\min(n + 1, N)$.

7.3 Uniqueness is stronger than Gram singularity

If the completion disk has zero radius, $M_1 = S_0$ is fixed. In every other realization, the sample space has the same compressed T and its coupling to the orthogonal complement vanishes on the resolvent-only span. If B is singular, that span is already the whole sample space. If B is full rank, the missing complement has dimension two. The lower K is positive definite: an endpoint eigenvector lying in the resolvent span would be an eigenvector of the irreducible Jacobi chain with its last block zero, which is impossible by backward substitution. The zero Schur slack gives a two-dimensional kernel of T whose projection onto the missing complement is onto. Positivity forces the external coupling to vanish on that kernel as well, hence everywhere. The entire measure is fixed.

This explains why H-singularity, not just B-singularity, is the complete criterion. A generic single complex measurement can have $\operatorname{rank} B = 6$ and still determine a unique three-atom response. An exact example is

$$G(s) = -\frac{I}{6s} + \frac{I}{6(1-s)} + \frac{2}{3}\left[\frac{\operatorname{diag}(1,0)}{1/5-s} + \frac{\operatorname{diag}(0,1)}{4/5-s}\right].$$

At $z = 1/4 + 3i/4$, the orbit-completed matrices have $\operatorname{rank} B = 6$, $\operatorname{rank} H = 4$, $\Delta = 0$. Its endpoint weight has rank two; the minimum paired atom count is three and the shortest host length is four. The exact test suite constructs and independently verifies this boundary case.

For a fixed finite response with paired count N and endpoint rank r_0, the complete-sample threshold is

$$n \geq N - \mathbf{1}_{\{r_0=2\}}.$$

At $n = N-1$, B is full rank, K has rank $2N-2$, and the true slack rank is $2-r_0$. It vanishes precisely for full-rank endpoint weight. At fewer samples a unique measure would contradict the smaller quotient dimension. This proves both sufficiency and necessity of the response-sensitive threshold, not only a uniform upper bound.

8 Endpoint/complement peeling in detail

For a finite real minimal state, split

$$T = \begin{pmatrix} M_1 & B_1^T \\ B_1 & T_1 \end{pmatrix}, \qquad G^{-1} = M_1 - sI - B_1^T(T_1 - sI)^{-1}B_1.$$

Positivity implies that an eigenvector of T_1 at zero or one would be an eigenvector of T orthogonal to the input. Cyclicity excludes it. Thus $0 \prec T_1 \prec I$ even when T itself has endpoint spectrum. At each interior eigenvalue, $B_1^T E_\lambda B_1$ has rank $\dim E_\lambda$ because a missing direction would again be invisible to the input and violate cyclicity. No commutativity between M_1, B_1 or T_1 is used.

Set $G^c = -\mathcal{C}(G^{-1})/[s(1-s)]$. Its interior positive weights are

$$\frac{\mathcal{C}(B_1^T E_\lambda B_1)}{\lambda(1-\lambda)}.$$

At zero, write $G = W_0/(-s) + G_0 + O(s)$. In the decomposition $\operatorname{ran} W_0 \oplus \ker W_0$, the leading block on $\operatorname{ran} W_0$ is positive and diverges as $s \uparrow 0$. On $\ker W_0$, the finite block G_0 is strictly positive because the measure has total mass I. Block inversion therefore gives a limit $E_0 = \lim_{s\uparrow 0} G^{-1}$ that is zero on $\operatorname{ran} W_0$ and positive definite on its complement. Its rank is $2 - \operatorname{rank} W_0$. The endpoint weights of G^c are $\mathcal{C}(E_0)$ and E_0.

At infinity,

$$G^{-1} = -sI + M_1 + O(s^{-1}), \qquad G^c = -I/s - M_1/s^2 + O(s^{-3}),$$

where $\mathcal{C}(M_1) = I - M_1$ was used. Thus G^c has mass I and unchanged first moment. Its total cyclic spectral dimension is $2N - 2 + 2(2 - r_0)$, so $N^c = N + 1 - r_0$ and $r_0^c = 2 - r_0$. Direct substitution gives $(G^c)^c = G$.

If $W_0 \neq 0$, put $q = \operatorname{tr} W_0$, $M = W_0/q$, $\theta' = \theta + (1-\theta)q$, $c = \theta/\theta'$. Subtract the endpoint pair and divide by $1-q$. The resulting core has $N - r_0$ paired states and no endpoints. The host is phase 2. If $W_0 = 0$, first complement, whose endpoint weight is $\mathcal{C}(G(0)^{-1}) \succ 0$, then remove this endpoint pair. The core has $N-1$ paired states, with labels interchanged; the host is original phase 1. All denominators are positive fractions, and $q = 1$ is handled by a pure core rather than division by zero.

The potential $\Phi = 2N - r_0$ is invariant under complement. Each pure phase-2 elementary host adds positive mass only in an endpoint projector. It changes r_0 by at most one and N by the same amount, so Φ increases by at most one. Complement gives the same conclusion for a phase-1 host. The first nontrivial laminate has $\Phi = 1$. Every architecture of length L therefore obeys $\Phi \le L$. Endpoint peeling attains equality with r_0 initial elementary steps and $2(N - r_0)$ subsequent steps. This is the precise domain of the optimality statement: sequential addition of a pure phase, not an arbitrary binary laminate tree or arbitrary spatial geometry.

9 Finite-data ranks and all one-contrast strata

The Gram matrix A is that of the sample vectors after multiplication by $\sqrt{T}$. Consequently every realizing finite response satisfies $\operatorname{rank} A \le \operatorname{rank} T$. In a minimum-state quotient equality holds. The real Schur factorization gives

$$\operatorname{rank} A = \operatorname{rank} K + \operatorname{rank}(M_1 - S_0), \qquad \operatorname{rank} H = \operatorname{rank} K + \mathbf{1}_{\{\Delta > 0\}}.$$

On a positive-radius completion disk, boundary points have slack rank one while the center has rank two. On a singleton disk the slack rank is zero. Therefore a boundary completion attains both the minimum paired count and the minimum host length, whereas the center spends one extra elementary layer for strictly feasible data. The determinant center and the shortest design are different optimization objectives.

For one generic non-self-dual complex contrast, the attainable rank pairs are:

rank B	rank H	paired atoms	host steps	measure unique?
2	1	1	1	yes
2	2	1	2	yes
4	2	2	2	yes
4	3	2	3	yes
4	4	2	4	yes
6	4	3	4	yes
6	5	3	5	no

To see exhaustiveness, a singular H fixes a unique finite response of count at most three, and its host length is $2N - r_0$, with $0 \le r_0 \le 2$ (and $r_0 \le 1$ when $N = 1$). If $N = 3$ at two completed nodes, uniqueness requires $r_0 = 2$. A positive H has ranks $(6, 5)$. Each listed pair is attained by a finite measure: choose one or two interior paired atoms, endpoint projectors of the required rank, or a full-rank endpoint pair plus one interior atom. For the self-dual single-node problem the corresponding pairs are $(2, 1), (2, 2), (4, 2), (4, 3)$, the last being nonunique.

The elementary symmetric-polynomial rank tests in the main paper are evaluated only after PSD feasibility is established. In floating point, a determinant close to zero is not a reliable exact-rank test. The exact implementation uses rational elimination; numerical rank estimates are separately labeled.

10 Real endpoint-plus-atom synthesis

Let $s \in \mathbb{R} \setminus [0, 1]$, $X = qI + \rho D$, $\rho \ge 0$, with D a real traceless involution when $\rho > 0$; choose any such D when $\rho = 0$. Define $e = -1/2$ and the cone parameters k, a, c, σ, q_0 as in the main paper. A strict interior point has $q \ne e$ and $0 \le \rho < h(q)$. Draw the line through its radial coordinates and the endpoint $(e, 1/2)$:

$$d = \frac{1}{2} + m(q' - e), \qquad m = \frac{\rho - 1/2}{q + 1/2}.$$

For the strict interior, $|m| > 1$. Intersecting this line with $d^2 = (q' + a)^2 - c^2$ produces the known root $q' = e$ and the second root

$$q_* = \frac{2a - \frac{1}{2}(m+1)^2}{m^2 - 1}, \qquad d_* = \frac{1}{2} + m(q_* + 1/2).$$

The line enters the convex radial disk body at the endpoint and exits at (q_*, d_*) on its lateral boundary. The given point lies between them. Equivalently this follows by substituting the feasible cone inequalities into the quadratic and its branch condition. Put

$$\lambda = \frac{q - q_*}{e - q_*} \in (0,1), \qquad u_*^2 = \frac{k(kq_* + s)}{q_* + s}, \qquad t_* = (1 + \sqrt{u_*^2})/2.$$

Then X is the convex mixture with weights $\lambda, 1 - \lambda$ of the endpoint atom with deviator $D/2$ and the atom at t_* with signed deviator $d_* D$. If $d_* = 0$, the second atom is the midpoint and its projector is irrelevant. Otherwise its projector is $(I + \operatorname{sgn}(d_*)D)/2$. The endpoint can be written as $\Gamma_s(1, (I + D)/2)$, or equivalently $\Gamma_s(0, (I - D)/2)$.

This complete two-atom response has exactly one rank-one endpoint weight and one nonendpoint paired atom. It therefore has $N = 2, r_0 = 1$ and needs exactly three elementary host layers. The usual two complementary atoms at a common nonendpoint t also reproduce the real target, but their full function generally has endpoint rank zero and can require four host steps. Minimizing atom count alone does not minimize physical architecture length.

On the cap $q = e$, use complementary endpoint projectors with weights $(1 \pm 2\rho)/2$. The rim uses one atom and one layer; the cap interior uses two atoms and two layers. On the lateral boundary use a single atom recovered from $u^2 = k(kq + s)/(q + s)$. The tip $h = 0$ is the midpoint atom. These endpoint and tip cases are handled before the line-intersection formula to avoid removable zero denominators.

For a support direction $Aq + b \cdot d$, $\beta = |b|$, the one-variable objective is $A\sigma y - Aa + \beta\sqrt{y^2 - c^2}$, $c \le y \le y_c$. The cap derivative decides the two support branches. Translating by an interior axial point gives a bounded polar through the support inequality. Lateral exposed atoms, the flat cap disk and its rim give the real face geometry; no unsupported hyperbolic-geodesic structure is added.

11 Perturbations and pole identities

For $K \succeq \kappa I$, perturbing the inverse uses

$$(K + E)^{-1} - K^{-1} = -(K + E)^{-1} E K^{-1}.$$

Combining this with the two input-factor perturbations proves the bound in the main paper. The sharp Lipschitz constant two for $\operatorname{tr} S + 2|\operatorname{Im} S_{12}|$ follows by writing it as the maximum of $2u^* S u$ and $2\bar{u}^* S \bar{u}$. The earlier constant four was valid but not sharp. On reciprocal data S_0 is real and the expression reduces to its trace.

For the physical pencil, the relevant scalar identity is

$$(1 - t) + \frac{t(1 - t)}{s - t} = \frac{s(1 - t)}{s - t}.$$

Hence

$$\begin{aligned} sI - (1-\theta)V^T(I - T)V - (1-\theta)V^T T(I - T)(sI - T)^{-1}V \\ = sI + (1-\theta)X(s) = \theta S(s). \end{aligned}$$

This is exactly the Schur complement of the positive-contraction pencil C_θ in the main paper. The factor identities $C_\theta = L^T L$ and $I - C_\theta = J^T J + \theta E E^T$ prove positivity without assuming T or $I - T$ is invertible. The resulting linearization describes the physical tensor's visible poles. A pole of G can cancel under the physical Möbius map; conversely zeros of S in rational continuation generate visible physical poles. The midpoint example in the main paper verifies this distinction exactly.

The scaled imaginary boundary kernel $y^2/[(t-x)^2 + y^2]$ detects only the atom at x. An absolutely continuous density produces the finite imaginary boundary term $\pi W(x)$ rather than a $1/y$ pole. A singular continuous measure need not have a finite ordinary boundary value at every point. Those cases are not upgraded to an ordinary lossless physical closure by a formal limiting argument.

12 Executable verification and its limits

The package contains the rechecked exact physical compiler and independent verifier, together with new disk/prediction modules and independent dual checking. The compiler's certificate fields are rational strings or coefficient arrays, not executable expressions. The verifier rejects arbitrary expression strings, invalid phases, bad shape, missing orbit partners, nonreciprocity, malformed factors, incorrect ranks, invalid macro fractions, nonpositive lamination matrices, wrong phase fraction, wrong tree length and rational-function mismatch.

For the exact three-atom/five-step example, the certificate stores a positive congruence factor of the order-five chiral matrix, the original completed samples, the rank claims, the phase fraction, the rational macro tree and the two-by-two rational response. The verifier independently reconstructs the chiral matrix from the samples and expands the physical tree using a formula different from the compiler's inverse-S formula. It verifies the entire rational function and then all supplied sample values. This is stronger than testing a tree only at the interpolation nodes.

Reproduce with Python 3.10 or later in an isolated environment:

```
python -m pip install -e '.[test]'
python -m pytest -q
python examples/exact_shortest.py
python -m phase_orbit_v3 verify \
  examples/three_atoms_five_steps.json
python verification/adversarial.py
python verification/symbolic.py
```

The exact compiler supports Gaussian-rational nonreal data. For a physical tensor, `normalize_physical` performs the exact phase and tensor normalization, mapping lower-half-plane data by conjugation. Real-value-only multipoint interpolation is a proved lifted theorem but is not implemented as a general SDP solver. The numerical module does not export an exact-feasibility claim. Certificate parsing avoids code evaluation, but symbolic arithmetic can consume substantial memory and time; this is not a security-hardened untrusted-service implementation. The verifier limits completed nodes and recursion depth explicitly. These resource limits do not restrict the mathematical theorem.

The first recorded v3.0.0 test run had one failing fixture because Python division introduced a float into a test intended to be rational. The exact compiler rejected it as designed. The fixture was corrected to use `Rational`; its initial failure log is retained in the provenance directory. The v3.5.0 generic-orbit test initially compared different exact symbolic forms without simplifying them; simplifying the fixture fixed that comparison, and both logs are retained. Subsequent results, exact environment versions and high-precision tolerances are recorded in the verification directory. Passing tests are implementation evidence and exact example checks, not independent peer review of the analytic proofs.

13 Final epistemic separation

Object	Status and restriction
Cell normalization and matrix measure	Internally proved using the gradient projection, planar duality and the spectral theorem.
Finite physical peeling	Internally proved; exact ideal hierarchical response with positive fractions and full anisotropic coupling.
Periodic closure arrow	Conditional on named standard uniformly coercive periodic/local/reiterated homogenization. Exact-fraction constructions are supplied.
Single PSD, reciprocal minimum, uniqueness and rank length	Internally proved. Exact rational examples and independent certificates are additional checks.
Balanced Jacobi model	Internally derived in this setting; the underlying anti-linear operator structure is established prior art.
Real and complex atom strata	Internally proved in normalized coordinates. Physical-tensor extremality is not inferred through a nonlinear map.
Resonant statements	Analytic boundary and finite physical-pole results; no ordinary lossless-cut G-closure theorem.
Noisy reconstruction	Quantitative data and transfer estimates under stated margins; no uniform exact-rank or unique-measure stability near degeneracy.
Wider physics	Proportional anisotropy on a transformed lattice is proved. Nonproportional anisotropy, three dimensions, more phases and full-wave systems are not proved.
Software and fabrication	Exact certificate checking and numerical regression are supplied. No proof-assistant certification, laboratory fabrication or experimental validation.
Novelty	New in this release and apparently new application-specific synthesis where indicated; no certified historical-priority claim.

14 The v3.5.0 disk transform: every algebraic bridge

For $z = (1+w)^2/(4w)$, direct multiplication gives

$$1 - z = -\frac{(1-w)^2}{4w}, \qquad z(1-z) = -\frac{(1-w^2)^2}{16w^2}, \qquad t - z = -\frac{1 - 2(2t-1)w + w^2}{4w}.$$

Thus $F = -(1-w^2)G/(4w)$ and $X = (1-w^2)F/(4w) - zI$. Every sign in the physical normalization follows from these three identities. At zero,

$$F = I + (4M_1 - 2I)w + O(w^2), \qquad \Psi = (2M_1 - I)w + O(w^2), \qquad X(\infty) = M_1 - I.$$

The trace-one condition makes $\Psi'(0)$ traceless. Phase reflection in the interval becomes $w \mapsto -w$, not conjugation; conjugate reality is a separate operation. Both are needed to make the Blaschke product even and real.

For any matrix F with positive Hermitian real part, $F+I$ is invertible: for a unit vector v, $\|(F+I)v\| \geq \mathrm{Re}(v^*(F+I)v) \geq 1$. With $\Psi = (F-I)(F+I)^{-1}$,

$$(F+I)^*(I-\Psi^*\Psi)(F+I) = (F+I)^*(F+I) - (F-I)^*(F-I) = 4\,\mathrm{Re}\,F.$$

Conversely $F = (I+\Psi)(I-\Psi)^{-1}$ and

$$(I-\Psi)^*(\mathrm{Re}\,F)(I-\Psi) = I - \Psi^*\Psi.$$

These formulas handle noncommuting values at different arguments; nowhere is $F(w)$ assumed to commute with $F(v)$.

14.1 Why rational innerness, not rationality, is the right condition

The circle kernel $(\zeta+w)/(\zeta-w)$ has positive real part in the disk. Its average at conjugate circle points is the interval kernel. An interior interval atom therefore produces two circle atoms, whereas each endpoint produces only one. For a finite matrix measure, rank is counted separately at each *combined* spectral point. If its total interval spectral rank is d and its endpoint ranks are both r_0, the circle state count is $2d - 2r_0$.

If rational inner Ψ is given instead, $F = (I+\Psi)(I-\Psi)^{-1}$ is rational. On any closed circle arc containing no pole, F extends analytically and its Hermitian real part is zero on the arc because Ψ is unitary there. Therefore $\mathrm{Re}\,F(re^{i\varphi})$ tends uniformly to zero on every shorter closed arc. Testing its approximating circle measures against continuous functions supported on that arc proves that the limit measure has no mass there. Only the finitely many pole points can carry mass. This rules out both continuous and singular-continuous residual measures; positivity prevents a hidden derivative-of-delta distribution.

By contrast, $\Psi = 0$ has $F = I$ and uniform circle measure. Its interval pushforward is the arcsine density $I/[\pi\sqrt{t(1-t)}]$. This is an explicit counterexample to an unjustified rational-Schur/finite-laminate equivalence. The release uses rational *inner* functions in every degree-to-architecture statement.

14.2 A direct conservative realization calculation

Let U be a finite unitary circle state and V an isometric input, $P = VV^*$. Set

$$g(w) = V^*(I-wU^*)^{-1}U^*V, \quad F = I + 2wg, \quad \Psi = wg(I+wg)^{-1}.$$

The Woodbury identity gives

$$V^*(I - wU^*(I-P))^{-1}U^*V = g(I+wg)^{-1},$$

which proves the transfer formula in the main paper. For $\mathcal{U} = \left(\begin{smallmatrix} U^*(I-P) & U^*V \\ V^* & 0 \end{smallmatrix}\right)$, its column Gram blocks are

$$(I-P) + P = I, \quad (I-P)V = 0, \quad V^*V = I,$$

and its row Gram blocks give the same identity. This proves unitarity without any spectral genericity assumption. Reachability is unchanged by the state feedback $U^* = A + BC$; observability is checked on the invariant zero-output subspace. The coefficient Hankel factorization proves minimal degree, including possible nilpotent delay states. Taking only the maximum scalar denominator degree would give the wrong answer for $\Psi = wI_2$.

15 Prediction coordinates and exact global upper bounds

For a generic new upper orbit $q, 1-\bar{q}$, write

$$G(q) = aE_{11} + bE_{22} + c(E_{12} + E_{21}) + i\big(dE_{11} + eE_{22} + f(E_{12} + E_{21})\big).$$

The partner value is $-\mathcal{C}(\overline{G(q)})$. At a self-dual node use only

$$G(q) = x\,\mathrm{diag}(1, -1) + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + ibI.$$

Substitution in the divided differences gives an affine Hermitian pencil with rational Gaussian entries when the nodes and fixed values are rational Gaussian. The executable pencil uses the congruence $\widehat{H} = \mathrm{diag}(\sqrt{2}, I)H\,\mathrm{diag}(\sqrt{2}, I)$, so its top-left scalar is one. Dual multipliers must use this same pencil; copying a dual multiplier across congruences without transforming it is an error.

For $\widehat{H}(x) = H_0 + \sum x_k H_k$ and $Ax = b$, the exact upper certificate is

$$Q \succeq 0, \quad A^T y - (\mathrm{tr}\,QH_k)_k = c, \quad U = \mathrm{tr}(QH_0) + y^T b.$$

The identity $U - c^T x = \mathrm{tr}(QH(x))$ proves the bound at an exact feasible point. In a noise box, it is replaced by the same identity plus the elementary estimate $y^T(Ax - b) \le \sum \varepsilon_j |y_j|$. Thus a certificate need not have a strictly positive multiplier or an invertible data matrix. Only the *existence of an optimal multiplier* uses Slater's theorem; the verifier is independent of that theorem.

15.1 Rational approximation of an optimum and the boundary case

For a compact spectrahedron with a strict point, its direction space $\mathcal{S}$ contains no nonzero positive-semidefinite direction. Projection of the trace-one positive cone section onto $\mathcal{S}^\perp$ produces a compact convex set separated from zero. Its strictly separating vector is a positive-definite annihilator Q_0. In rational coordinates one can approximate within $\mathcal{S}^\perp$ by rational vectors, so Q_0 can be rational. This gives a strict dual anchor for every rational linear objective after solving the rational dual equations and adding a large multiple of Q_0.

The following theoretical search is complete for strict rational full data: enumerate primal rational coordinates, enumerate rational parameters in the dual affine space, and accept the first pair with positive matrices and objective gap at most the prescribed positive rational tolerance. Enumerate all tuples of height at most k before increasing k, so the search is fair. Mixing primal and dual optima with strict anchors and then rationally approximating proves termination. No useful worst-case running-time bound follows from this argument.

For singular fixed-data H, use the unique exact rational response directly. The release does not infer an attained PSD dual from this singular case and does not hide facial-reduction issues under a generic Slater assertion. The full rational response, together with the uniqueness theorem and independent physical-tree verification, certifies its query values.

16 Detailed Blaschke and noise calculations

The $2n$ disk zeros are all distinct: conjugation moves the disk images of upper-half-plane nodes to the opposite half disk, and the conformal map is injective. Finite contrast nodes never give the zero disk point. Closure of the upper list under $z \mapsto 1 - \bar{z}$ adds the disk partner $-\bar{a}$, so the complete disk list is closed under negation and conjugation. Pairing conjugates gives

$$\mathcal{B}(0) = \prod_{j=1}^{n} |a_j|^2 > 0.$$

At an arbitrary zero a,

$$\mathcal{B}'(a) = \frac{1}{1-|a|^2} \prod_{b \neq a} \frac{a-b}{1-\bar{b}a}.$$

This formula proves both that the cardinal denominators are nonzero and that near-collisions can create large noise amplification. A stable evaluation of the cardinal function uses

$$L_a(w) = \frac{1}{(1-\bar{a}w)\mathcal{B}'(a)} \prod_{b \neq a} \frac{w-b}{1-\bar{b}w},$$

not a floating-point 0/0 at a measurement node. The release tests the exact cardinal values and the zero-noise reduction separately.

For $|v| = R < 1$ and real $0 \leq b < 1$,

$$\left|\frac{v+b}{1+bv}\right|^2 = \frac{R^2 + b^2 + 2bR\cos\varphi}{1 + b^2R^2 + 2bR\cos\varphi}.$$

Its derivative with respect to $\cos\varphi$ is nonnegative because the denominator-minus-numerator equals $(1-R^2)(1-b^2)$. The maximum is $(R+b)^2/(1+bR)^2$. Set $R = \rho^2$, $b = r_j^2$ to obtain the pairwise exponential bound. The logarithmic rate lower bound uses the same factors, not an unrelated approximation family.

17 Sharp information adversaries and the one-layer tradeoff

For any real traceless symmetric involution D, the two functions $\Psi_\pm = \pm w\mathcal{B}(w)D$ are inner, reciprocal, real and phase symmetric. They interpolate the arcsine datum. Their first moments are

$$M_{1,\pm} = \frac{1}{2}I \pm \frac{1}{2}\mathcal{B}(0)D.$$

Thus the opposite ends of the full first-moment uncertainty disk are physical. The minimax proof requires this attainment step; it cannot be obtained merely by bounding a derivative of arbitrary analytic functions.

For the exact self-dual radii $r_1 = 1/2, r_2 = 1/3$, the actual contrast nodes and data are

$$z_1 = \frac{1}{2} + \frac{3}{8}i, \quad G_1 = \frac{8}{5}iI, \qquad z_2 = \frac{1}{2} + \frac{2}{3}i, \quad G_2 = \frac{6}{5}iI.$$

Here $\mathcal{B}(0) = 1/36$, the minimax radius is $1/72$, the minimum paired count is three, and the shortest host length is five. The two adversarial complete responses have five-step exact physical trees. A center-moment response has six steps, keeps three paired atoms, and halves the worst-case error. It is not permissible to call the five-step fitted response a unique estimate of the true response: these exact data already have multiple shortest explanations.

The minimax center is unique as a *first-moment estimate*. It need not identify a unique complete physical function among arbitrary higher-dimensional continuations. The proofs use only the first-moment disk and the rank lower bound. In particular, the v3.5.0 audit removed a tempting argument that would have conflated a chosen moment with a uniquely determined entire measure.

18 Exact broadband certificate transcript

At $z_1 = 1/2 + i$ and $G_1 = 9iI/10$, write the query at $z_2 = 1/2 + 2i$ as $xE_1 + yE_2 + ibI$. The supplied rank-one multipliers satisfy

$$\operatorname{tr}(Q_+H_x) = \operatorname{tr}(Q_+H_y) = 0, \quad \operatorname{tr}(Q_+H_b) = -1, \quad \operatorname{tr}(Q_+H_0) = 18/37,$$
$$\operatorname{tr}(Q_-H_x) = \operatorname{tr}(Q_-H_y) = 0, \quad \operatorname{tr}(Q_-H_b) = 1, \quad \operatorname{tr}(Q_-H_0) = -33/68.$$

Both are $6vv^*$, so positivity requires no eigenvalue computation. Independent verification of these identities and of the two physical primal trees proves optimality. The exact gap between the upper and lower values is $3/2516$.

A nonoptimal but certified design is also supplied. Choose the upper objective and query $b = 18/37 - 1/10000$. It lies strictly inside the prediction body. Its exact rational physical response uses five host steps; the same upper dual witness proves an objective gap of exactly $1/10000$. The implementation labels this an ε-optimal physical design only when the requested ε is at least that gap. It does not round an excessive gap down to a successful tolerance.

19 Verification categories and trusted boundaries for v3.5.0

The release separates four logically different activities.

1. Analytic proofs establish theorems for all admissible parameters. Standard homogenization and Slater duality are explicitly named background where used.
2. Symbolic identities check the exact disk normalization, circle kernel, Cayley maps, Blaschke pairing, weak-contrast expansion and rational global-envelope remainders.
3. Exact finite computations verify Gaussian-rational PSD factors, dual identities, ranks, sample matching and complete rational physical responses. The exact checks use no SDP solver.
4. Floating-point and high-precision experiments test matrix formulas, numerical conditioning and implementations. They do not prove positivity of near-singular data or any historical novelty claim.

The independent physical verifier imports neither the exact compiler nor the numerical screening module. The independent dual verifier similarly imports only SymPy and checks its own exact matrix identities and singular positivity. Python, SymPy rational arithmetic and the mathematical proofs remain trusted components. No proof-assistant formalization or independent peer review has been performed.

The general prediction software constructs the exact affine pencil and verifies primal–dual proposals. It does not include a tested universal SDP optimization engine. An attempted optional CVXPY installation in the working environment failed because network name resolution was unavailable; that log is retained. No results from that unexecuted solver are reported. The broadband example, minimax examples, exact synthesis and all associated certificate checks run without it. This distinction is part of the release's feature contract, not an unmentioned implementation gap.

References

[1] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.

[2] K. E. Clark and Graeme W. Milton. Optimal bounds correlating electric, magnetic and thermal properties of two-phase, two-dimensional composites. *Proceedings of the Royal*

Society A, 448(1933):161–190, 1995. Publisher bibliographic record and author-hosted abstract checked; full article was not available in this audit.

[3] Benjamin Eichinger, Milivoje Lukić, and Giorgio Young. A Weyl matrix perspective on unbounded non-self-adjoint Jacobi matrices. *Complex Analysis and Operator Theory*, 19:194, 2025.

[4] Kenneth Golden and George Papanicolaou. Bounds for effective parameters of heterogeneous media by analytic continuation. *Communications in Mathematical Physics*, 90:473–491, 1983.

[5] Marko Huhtanen and Allan Perämäki. Function theory of antilinear operators. *arXiv preprint*, 2012. The arXiv version was examined; a later publication is not presumed.

[6] Norman G. Meyers. An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Annali della Scuola Normale Superiore di Pisa*, 17(3):189–206, 1963.

[7] Graeme W. Milton. Approximating the effective tensor as a function of the component tensors in two-dimensional composites of two anisotropic phases. *SIAM Journal on Mathematical Analysis*, 50(3):3327–3364, 2018.

[8] Graeme W. Milton and Mihai Putinar. A theory of composites perspective on matrix valued Stieltjes functions. *Expositiones Mathematicae*, 41(1):186–201, 2023.

[9] Alexander Pushnitski and František Štampach. A functional model and tridiagonalisation for symmetric anti-linear operators. *Journal of Operator Theory*, 95(1):275–302, 2026. Preprint first appeared in 2024; 2026 publication metadata checked with the journal.