Candidate Resolution of Open Problem 3.3: Defect Conservation and Exact Modular Edge-Irregularity Strength of Friendship Graphs

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Abstract

We present a complete candidate proof for the unresolved friendship-graph family in Open Problem 3.3 of Koam, Ahmad, Baca, and Semanicova-Fenovcikova (AIMS Mathematics 8 (2023), 1475–1487). For the friendship graph $F_n = K_1 \vee nK_2$, we claim that for every $n \geq 12$ with $n \equiv 0 \pmod 4$, $\mathrm{es}(F_n) = \mathrm{mes}(F_n) = 2n + \lceil n/7 \rceil$. The lower bound is obtained from a new defect-conservation identity. Writing $k = 2n + s$ and decomposing rim-edge weights below, inside, and above the spoke-weight band yields nonnegative slacks α, β, γ satisfying $\alpha + \beta + 5\gamma = 7s - n$, and hence $s \geq \lceil n/7 \rceil$. This gives the universal lower bound $\mathrm{es}(F_n) \geq 2n + \lceil n/7 \rceil$ for every friendship graph. The upper bound is constructive: a finite low-weight kernel, an explicit reflected high-weight matching, and zero-rooted Schur partitions generated from Langford and near-Skolem sequences together produce $3n$ consecutive ordinary edge weights, which therefore represent all residues modulo $3n$. All finite exceptional Schur certificates are included. An accompanying verifier checks the defect algebra, all exceptional certificates, complete representatives of the seven structural classes, and the construction identities through $n = 1,000,000$. The result is presented as a candidate proof pending independent expert verification and peer review.

Contents

1 AI authorship and research-process disclosure

This work was developed through human-AI collaboration. **Artificial Hyperintelligence Eve** is the research persona used for the AI contribution and, for this version, was instantiated through **OpenAI GPT-5.6 Sol in ChatGPT**. **Maciej Nowicki** is the corresponding human researcher and submitter.

The human contribution included selecting and maintaining the research target, supplying the EVE-CLOSER research framework, directing the transition from finite instances to the full open family, deciding which claims should be retained or rejected, and preparing the work for public release. The AI contribution included deriving the defect-conservation formulation, developing the seven-class construction, connecting the residual additive problem to Schur/Langford/near- Skolem structures, producing executable verification code, performing adversarial checks, and drafting and revising the manuscript.

The principal negative result during development was also retained: an early extrapolated family pattern fit several finite cases but failed at (n=36). It was discarded rather than promoted to a theorem. The present proof does not depend on that failed extrapolation.

The manuscript is therefore submitted as a **candidate proof**. Computational checks support the finite and algebraic components, but they do not substitute for independent mathematical review. The corresponding human accepts responsibility for the public submission and for correcting or withdrawing the result if a substantive error is identified.

2 Status and scope

This manuscript gives a complete candidate proof of the following theorem. It has not been independently peer reviewed.

Main Theorem. Let $F_n = K_1 \vee nK_2$ be the friendship graph formed by n triangles with one common central vertex. For every $n \geq 12$ with $n \equiv 0 \pmod 4$,

$$\boxed{\mathrm{es}(F_n) = \mathrm{mes}(F_n) = 2n + \left\lceil \frac{n}{7} \right\rceil.}$$

In addition, for every $n \geq 1$,

$$\boxed{\mathrm{es}(F_n) \geq 2n + \left\lceil \frac{n}{7} \right\rceil.}$$

The first formula is presented as a candidate resolution of Open Problem 3.3 posed by Koam, Ahmad, Bača, and Semaničová-Feňovčíková in *AIMS Mathematics* 8 (2023), 1475–1487.

The argument is self-contained with respect to all new reductions, all parameter calculations, all finite exceptional certificates, and the conversion from Skolem-type sequences into the required graph labelings. It invokes two classical existence theorems for Langford and near-Skolem sequences; those theorems are stated precisely in Section 6 with their exact hypotheses and bibliographic provenance.

The accompanying verifier checks all finite certificates, the algebraic identities, and a complete labeling in every structural residue class. Computation is not used as a substitute for the infinite proof.

3 1. Definitions

Let $G = (V, E)$ be a finite simple graph.

A **vertex k-labeling** is a map
$$\phi : V(G) \to \{1, \ldots, k\}.$$

The ordinary weight of an edge $xy \in E(G)$ is

$$W_\phi(xy) = \phi(x) + \phi(y).$$

The labeling is **edge irregular** when all ordinary edge weights are distinct. The minimum k for which such a labeling exists is the **edge irregularity strength**, denoted $\mathrm{es}(G)$.

If $q = |E(G)|$, the modular edge weight is

$$\overline{W}_\phi(xy) = W_\phi(xy) \pmod{q}.$$

The labeling is **modular edge irregular** when

$$xy \mapsto \overline{W}_\phi(xy)$$

is a bijection from $E(G)$ to $\mathbb{Z}_q$. The least possible k is the **modular edge irregularity strength**, denoted $\mathrm{mes}(G)$; if no such labeling exists, set $\mathrm{mes}(G) = \infty$.

Every modular edge-irregular labeling is edge irregular, because equal ordinary weights would have equal residues. Hence
$$\mathrm{es}(G) \leq \mathrm{mes}(G). \tag{1}$$

We shall also use the elementary converse criterion:

> **Consecutive-weight criterion.** If an edge-irregular k-labeling of a q-edge graph has ordinary edge weights equal to q consecutive integers, then it is modular edge irregular.

Indeed, q consecutive integers represent every residue modulo q exactly once.

For the friendship graph,
$$F_n = K_1 \vee nK_2,$$

write
$$V(F_n) = \{w, u_i, v_i : 1 \leq i \leq n\},$$
$$E(F_n) = \{wu_i, wv_i, u_iv_i : 1 \leq i \leq n\}.$$

The vertex w is the **center**. The $2n$ edges incident with w are **spokes** and the n edges u_iv_i are **rim edges**. Thus
$$|V(F_n)| = 2n + 1, \qquad |E(F_n)| = 3n.$$

4 2. Forced injectivity and the spoke band

4.1 Lemma 2.1. Every edge-irregular labeling of F_n is injective on vertices.

Proof. If two leaves have the same label, their spokes to w have the same weight. If the center w and one leaf of a triangle have the same label, then the other leaf of that triangle creates two equal edge weights. Therefore no two vertices can share a label. □

Consequently $k \geq 2n + 1$. Write

$$k = 2n + s, \qquad s \geq 1,$$

and let

$$c = \phi(w).$$

The $2n$ spoke weights are $c + a$, where a runs through the $2n$ leaf labels. They all lie in the interval

$$B = [c + 1, c + k],$$

called the **spoke band**.

Exactly s integers of B are not spoke weights. To see this, among the k labels in $\{1, \ldots, k\}$, exactly $2n$ are leaf labels. The remaining s labels consist of the center label c and $s - 1$ unused labels.

Partition the rim edges according to their ordinary weights:

$$L = \#\{u_i v_i : W(u_i v_i) \leq c\},$$
$$M = \#\{u_i v_i : c + 1 \leq W(u_i v_i) \leq c + k\},$$
$$H = \#\{u_i v_i : W(u_i v_i) \geq c + k + 1\}.$$

Then

$$L + M + H = n, \qquad M \leq s. \tag{2}$$

5 3. The defect-conservation law

5.1 Lemma 3.1. Low-zone capacity

For every edge-irregular labeling,

$$\boxed{5L \leq 2c - 1.} \tag{3}$$

Proof. The L low rim edges have $2L$ distinct leaf endpoints. Their labels are distinct positive integers, so the sum of those labels is at least

$$1 + 2 + \cdots + 2L = L(2L + 1).$$

The L rim-edge weights are distinct integers no larger than c, so their sum is at most

$$c + (c - 1) + \cdots + (c - L + 1) = \frac{L(2c - L + 1)}{2}.$$

The sum of the edge weights equals the sum of all endpoint labels over these disjoint rim edges. Hence

$$L(2L + 1) \leq \frac{L(2c - L + 1)}{2}.$$

For $L > 0$, cancellation and rearrangement give $5L \leq 2c - 1$; for $L = 0$ the inequality is immediate. □

5.2 Lemma 3.2. High-zone capacity

For every edge-irregular labeling,

$$\boxed{5H \le 2k - 2c + 1.} \tag{4}$$

Proof. For every endpoint label a, define the reflected label

$$a^\star = k + 1 - a.$$

If a high rim edge has ordinary weight $t \ge c + k + 1$, then the sum of its two reflected endpoint labels is

$$2k + 2 - t \le k + 1 - c.$$

The $2H$ reflected endpoint labels are distinct positive integers, and the H reflected pair-sums are distinct. Applying the argument of Lemma 3.1 with upper threshold $k + 1 - c$ gives

$$5H \le 2(k + 1 - c) - 1 = 2k - 2c + 1.$$

□

Define the nonnegative slack variables

$$\alpha = 2c - 1 - 5L,$$

$$\beta = 2k - 2c + 1 - 5H,$$

$$\gamma = s - M.$$

5.3 Theorem 3.3. Defect conservation

Every edge-irregular $k = 2n + s$ labeling of F_n satisfies

$$\boxed{\alpha + \beta + 5\gamma = 7s - n.} \tag{5}$$

Proof. Using (2),

$$\begin{aligned}
\alpha + \beta + 5\gamma &= (2c - 1 - 5L) + (2k - 2c + 1 - 5H) + 5(s - M) \\
&= 2k + 5s - 5(L + M + H) \\
&= 2(2n + s) + 5s - 5n \\
&= 7s - n.
\end{aligned}$$

□

Because the left side of (5) is nonnegative,

$$7s \ge n.$$

Therefore

$$s \ge \left\lceil \frac{n}{7} \right\rceil.$$

5.4 Corollary 3.4. Universal friendship-graph lower bound

For every $n \geq 1$,

$$\boxed{\operatorname{es}(F_n) \geq 2n + \left\lceil \frac{n}{7} \right\rceil.} \tag{6}$$

By (1), whenever $\operatorname{mes}(F_n)$ is finite,

$$\operatorname{mes}(F_n) \geq 2n + \left\lceil \frac{n}{7} \right\rceil. \tag{7}$$

5.5 Corollary 3.5. Rigidity at equality

Let

$$s_0 = \left\lceil \frac{n}{7} \right\rceil, \qquad d = 7s_0 - n \in \{0, \dots, 6\}.$$

Any edge-irregular labeling attaining $k = 2n + s_0$ must satisfy

$$\alpha + \beta + 5\gamma = d. \tag{8}$$

In particular:

1. If $d \leq 4$, then $\gamma = 0$, hence $M = s_0$: every missing spoke-band weight is filled by a rim edge.
2. If $d = 0$, then $\alpha = \beta = \gamma = 0$. Thus equality holds in both packing inequalities. In the low zone this forces the $2L$ endpoint labels to be the $2L$ smallest possible labels and the low weights to be $c - L + 1, \dots, c$; the reflected high zone has the analogous extremal form.

This rigidity observation explains the structure of the construction below.

6 4. Parameters for the optimal construction

Assume now

$$n \geq 12, \qquad n \equiv 0 \pmod 4.$$

Set

$$s = \left\lceil \frac{n}{7} \right\rceil, \qquad d = 7s - n \in \{0, \dots, 6\},$$

so

$$n = 7s - d,$$

and define

$$k = 2n + s = 15s - 2d. \tag{9}$$

Since $n \equiv 0 \pmod 4$ and $7 \equiv -1 \pmod 4$,

$$s + d \equiv 0 \pmod 4. \tag{10}$$

Choose the number l of low rim edges, the center label c, and the low leaf pairs from the following kernel table.

d	$s \bmod 4$	l	c	low leaf-pairs	low weights
0	0	3	8	$(1,5),(3,4),(2,6)$	$6,7,8$
1	3	2	6	$(2,3),(1,5)$	$5,6$
2	2	1	4	$(1,3)$	4
3	1	2	7	$(2,4),(1,6)$	$6,7$
4	0	1	5	$(1,4)$	5
5	3	2	8	$(2,5),(1,7)$	$7,8$
6	2	1	6	$(1,5)$	6

Every row satisfies

$$\boxed{2c = 5l + 1 + d} \tag{11}$$

and its low rim weights are precisely

$$[c - l + 1, c]. \tag{12}$$

Let

$$h = n - s - l = 6s - d - l. \tag{13}$$

Row by row, h is a positive odd integer. Combining (9), (11), and (13),

$$\boxed{\frac{5h+1}{2} = k + 1 - c.} \tag{14}$$

The chosen parameters realize the equality-state

$$(\alpha, \beta, \gamma) = (d, 0, 0)$$

once the middle and high zones are filled as below.

7 5. A universal high-zone matching

7.1 Lemma 5.1. Odd consecutive-sum matching

Let $h = 2p + 1$ be odd. The set $[1, 2h]$ can be partitioned into h pairs whose sums are exactly the consecutive interval

$$\left[\frac{3h+3}{2}, \frac{5h+1}{2}\right]. \tag{15}$$

Proof. Use the $p + 1$ pairs

$$(j, 3p + j + 1), \qquad 1 \le j \le p + 1,$$

and the p pairs

$$(p + 1 + j, 2p + 1 + j), \qquad 1 \le j \le p.$$

The first family uses

$$\{1, \ldots, p + 1\} \cup \{3p + 2, \ldots, 4p + 2\},$$

and the second uses

$$\{p + 2, \ldots, 2p + 1\} \cup \{2p + 2, \ldots, 3p + 1\},$$

so every integer $1, \dots, 4p + 2 = 2h$ occurs exactly once.

The first-family sums are

$$3p + 1 + 2j, \qquad 1 \le j \le p + 1,$$

namely every other integer from $3p + 3$ through $5p + 3$. The second-family sums are

$$3p + 2 + 2j, \qquad 1 \le j \le p,$$

namely the intervening values from $3p + 4$ through $5p + 2$. Together they give every integer from

$$3p + 3 = \frac{3h + 3}{2}$$

through

$$5p + 3 = \frac{5h + 1}{2}.$$

□

Reflect each label $x \in [1, 2h]$ to

$$k + 1 - x.$$

The reflected pairs use exactly

$$[k - 2h + 1, k]. \tag{16}$$

From (9) and (13),

$$k - 2h + 1 = 3s + 2l + 1. \tag{17}$$

Therefore the reflected high pairs occupy precisely the top $2h$ labels

$$[3s + 2l + 1, k]. \tag{18}$$

Their pair-sums are obtained by reflecting the interval (15). Using (14), the smallest reflected sum is

$$2k + 2 - \frac{5h + 1}{2} = c + k + 1,$$

and there are h consecutive sums. Hence the high rim weights are exactly

$$[c + k + 1, c + k + h]. \tag{19}$$

8 6. Classical Skolem-type sequence inputs

This section states all external sequence existence results used by the proof.

8.1 Definition 6.1. Langford sequence

A **Langford sequence of length m and defect D** is equivalently a partition of the positions

$$[1, 2m]$$

into pairs

$$(a_j, b_j), \qquad j = D, D + 1, \dots, D + m - 1,$$

such that

$$b_j - a_j = j. \tag{20}$$

8.2 Theorem 6.2. Langford existence theorem

A Langford sequence of length m and defect D exists if and only if

$$m \geq 2D - 1, \tag{21}$$

and

$$m \equiv 0, 1 \pmod 4 \quad \text{when } D \text{ is odd}, \tag{22}$$

or

$$m \equiv 0, 3 \pmod 4 \quad \text{when } D \text{ is even}. \tag{23}$$

The sufficiency of these classical necessary conditions is due to the established Langford-sequence existence theory; a standard reference is Simpson (1983).

8.3 Lemma 6.3. Langford-to-Schur conversion

A Langford sequence of length m and defect D yields a partition of

$$[D, D + 3m - 1]$$

into m triples

$$(j,\ D + m - 1 + a_j,\ D + m - 1 + b_j) \tag{24}$$

satisfying

$$x + y = z. \tag{25}$$

Proof. Equation (20) gives

$$j + (D + m - 1 + a_j) = D + m - 1 + b_j.$$

The first coordinates j cover

$$[D, D + m - 1].$$

Because the positions a_j, b_j partition $[1, 2m]$, the shifted second and third coordinates cover

$$[D + m, D + 3m - 1].$$

Thus all numbers in the stated interval occur exactly once. □

8.4 Definition 6.4. Near-Skolem sequence

An **r-near-Skolem sequence of order** m is equivalently a partition of the positions

$$[1, 2m - 2]$$

into pairs

$$(a_j, b_j), \qquad j \in [1, m] \setminus \{r\},$$

such that

$$b_j - a_j = j. \tag{26}$$

8.5 Theorem 6.5. Near-Skolem existence theorem

An r-near-Skolem sequence of order m exists if and only if

$$m \equiv 0, 1 \pmod{4} \quad \text{and } r \text{ is odd,} \tag{27}$$

or

$$m \equiv 2, 3 \pmod{4} \quad \text{and } r \text{ is even.} \tag{28}$$

This is the classical near-Skolem existence theorem of Shalaby.

8.6 Lemma 6.6. Near-Skolem-to-Schur conversion

An r-near-Skolem sequence of order m yields a partition of

$$[1, 3m-2] \setminus \{r\}$$

into $m-1$ triples

$$(j,\ m+a_j,\ m+b_j) \tag{29}$$

satisfying $x + y = z$.

Proof. Equation (26) gives

$$j + (m + a_j) = m + b_j.$$

The first coordinates cover $[1, m] \setminus \{r\}$, and the shifted position coordinates cover $[m+1, 3m-2]$. □

9 7. The middle-zone reduction

After reserving the low labels and the high interval (18), the unused lower labels are

$$[1, 3s + 2l] \setminus \{\text{endpoints of the low pairs}\}.$$

This set has $3s$ elements and contains the center label c.

Translate it by $-c$. Direct substitution from the seven-row kernel gives:

d	translated residual set $S_d(s)$
0	$\{-1, 0, 1, \ldots, 3s-2\}$
1	$\{-2, 0, 1, \ldots, 3s-2\}$
2	$\{-2, 0, 1, \ldots, 3s-2\}$
3	$\{-4, -2, 0, 1, \ldots, 3s-3\}$
4	$\{-3, -2, 0, 1, \ldots, 3s-3\}$
5	$\{-5, -4, -2, 0, 1, \ldots, 3s-4\}$
6	$\{-4, -3, -2, 0, 1, \ldots, 3s-4\}$

Call a partition of $S_d(s)$ into triples

$$(x_i, y_i, z_i), \qquad 1 \le i \le s,$$

a **zero-rooted Schur partition** if

$$x_i + y_i = z_i \tag{30}$$

for every i, 0 occurs exactly once among the z_i, and 0 never occurs as an x_i or y_i.

9.1 Lemma 7.1. Hole-filling lemma

If $S_d(s)$ has a zero-rooted Schur partition, then the s middle rim edges can be chosen so that their weights fill exactly the s holes in the spoke band.

Proof. For each triple (x_i, y_i, z_i), use

$$c + x_i, \qquad c + y_i$$

as the two leaf labels of one middle triangle.

If $z_i \neq 0$, omit the residual label

$$c + z_i$$

from the leaves. For the unique $z_i = 0$, the corresponding residual label is c, already used by the center.

Because the triples form a partition, all middle leaf labels are distinct, and the $s - 1$ nonzero z_i produce exactly the $s - 1$ unused labels.

The corresponding rim weight is

$$(c + x_i) + (c + y_i) = 2c + x_i + y_i = 2c + z_i = c + (c + z_i). \tag{31}$$

The missing spoke weight associated with a non-leaf label q is exactly $c + q$. Thus (31) fills the center hole $2c$ when $z_i = 0$, and fills the hole corresponding to omitted label $c + z_i$ when $z_i \neq 0$. All s spoke-band holes are filled exactly once. □

10 8. Zero-rooted Schur partitions in all seven cases

The congruence (10) restricts $s \pmod 4$ exactly as shown in the kernel table.

10.1 Case $d = 0$

Here $s \equiv 0 \pmod 4$. Use the fixed triple

$$(-1, 1, 0).$$

The remainder is the interval

$$[2, 3s - 2].$$

By Lemma 6.3 this interval is Schur-partitioned by a Langford sequence with

$$D = 2, \qquad m = s - 1.$$

The existence conditions hold because $m \equiv 3 \pmod 4$ and $m \geq 3$.

10.2 Cases $d = 1$ and $d = 2$

Use

$$(-2, 2, 0).$$

The remainder is

$$[1, 3s - 2] \setminus \{2\}.$$

By Lemma 6.6 this is supplied by a 2-near-Skolem sequence of order s. For $d = 1$, $s \equiv 3 \pmod 4$; for $d = 2$, $s \equiv 2 \pmod 4$. Since the missing value 2 is even, Theorem 6.5 applies in both cases.

10.3 Case $d = 3$

Here $s \equiv 1 \pmod 4$. Use the five fixed triples

$$(-4, 4, 0), \quad (1, 5, 6), \quad (-2, 9, 7), \quad (3, 8, 11), \quad (2, 10, 12). \tag{32}$$

They cover

$$\{-4, -2, 0, 1, \dots, 12\}.$$

For $s \geq 33$, the remainder is

$$[13, 3s - 3].$$

Lemma 6.3 applies with

$$D = 13, \qquad m = s - 5.$$

Here D is odd,

$$m \equiv 0 \pmod 4,$$

and

$$m \geq 28 \geq 25 = 2D - 1.$$

The smaller admissible values are

$$s = 5, 9, 13, 17, 21, 25, 29.$$

The $s = 5$ case is exactly the fixed kernel (32); explicit partitions for the remaining six values appear in Appendix A.

10.4 Case $d = 4$

Here $s \equiv 0 \pmod 4$. Use

$$(-2, 2, 0), \quad (-3, 6, 3), \quad (1, 4, 5). \tag{33}$$

They cover

$$\{-3, -2, 0, 1, \dots, 6\}.$$

For $s \geq 16$, the remainder is

$$[7, 3s - 3].$$

Use Lemma 6.3 with

$$D = 7, \qquad m = s - 3.$$

Here

$$m \equiv 1 \pmod 4, \qquad m \geq 13 = 2D - 1.$$

The exceptional values $s = 4, 8, 12$ are listed in Appendix A.

10.5 Case $d = 5$

Here $s \equiv 3 \pmod 4$. Use

$$(-5, 1, -4), \qquad (-2, 2, 0). \tag{34}$$

For $s \geq 7$, the remainder is

$$[3, 3s - 4].$$

Apply Lemma 6.3 with

$$D = 3, \qquad m = s - 2.$$

Then

$$m \equiv 1 \pmod 4, \qquad m \geq 5 = 2D - 1.$$

The sole smaller case $s = 3$ is listed in Appendix A.

10.6 Case $d = 6$

Here $s \equiv 2 \pmod 4$. Use

$$(-4, 1, -3), \qquad (-2, 2, 0). \tag{35}$$

For $s \geq 10$, the remainder is again

$$[3, 3s - 4].$$

Use Lemma 6.3 with

$$D = 3, \qquad m = s - 2.$$

Now

$$m \equiv 0 \pmod 4, \qquad m \geq 8 \geq 5.$$

The only smaller admissible value, $s = 6$, is listed in Appendix A.

Thus a zero-rooted Schur partition exists for every admissible pair

$$n \geq 12, \qquad n \equiv 0 \pmod 4.$$

11 9. Completion of the upper bound

The n rim pairs consist of:

- l low pairs from the kernel table;
- s middle pairs from Lemma 7.1;
- $h = n - s - l$ high pairs from Lemma 5.1 and reflection.

The low rim weights are

$$[c - l + 1, c]. \tag{36}$$

The $2n$ spoke weights occupy all but s places of

$$[c + 1, c + k],$$

and Lemma 7.1 fills exactly those s holes with the middle rim weights. Therefore the spokes together with the middle rim edges have weights exactly

$$[c + 1, c + k]. \tag{37}$$

The high rim weights are

$$[c + k + 1, c + k + h]. \tag{38}$$

Combining (36)–(38), all $3n$ ordinary edge weights form the single consecutive interval

$$\boxed{[c - l + 1, c + k + h].} \tag{39}$$

Its cardinality is

$$k + l + h = (2n + s) + l + (n - s - l) = 3n.$$

Hence the labeling is edge irregular, and by the consecutive-weight criterion it is modular edge irregular. Therefore

$$\text{mes}(F_n) \leq k = 2n + \left\lceil \frac{n}{7} \right\rceil. \tag{40}$$

Combining (7) and (40) proves the main theorem:

$$\boxed{\text{es}(F_n) = \text{mes}(F_n) = 2n + \left\lceil \frac{n}{7} \right\rceil}$$

for every

$$n \geq 12, \qquad n \equiv 0 \pmod 4.$$

□

12 10. Consequences

12.1 10.1. A universal $1/7$-surplus obstruction

Equation (6) is independent of n mod 4:

$$\text{es}(F_n) \geq 2n + \left\lceil \frac{n}{7} \right\rceil.$$

Thus the proof contributes a general lower bound even outside the solved modular family.

12.2 10.2. Companion odd-n problem

Koam et al. left the exact modular edge-irregularity strength open for odd $n \geq 9$, with an upper bound

$$\text{mes}(F_n) \leq \frac{5n + 1}{2}.$$

The universal lower bound sharpens the known lower side to

$$\boxed{2n + \left\lceil \frac{n}{7} \right\rceil \leq \text{mes}(F_n) \leq \frac{5n + 1}{2}}$$

for odd $n \geq 9$.

12.3 10.3. A natural classification conjecture

The solved values and the known obstruction for $n \equiv 2 \pmod 4$ suggest

$$\mathrm{mes}(F_n) = \begin{cases} \infty, & n \equiv 2 \pmod 4, \\ 2n + \lceil n/7 \rceil, & n \not\equiv 2 \pmod 4. \end{cases}$$

Only the odd-$n \geq 9$ part of this statement remains conjectural here. It is **not** used in the proof of the main theorem.

13 11. Conceptual interpretation: defect, reflection, and shadow

The proof can be read as a rigorous instance of the EVE-CLOSER / Self-Depth–Self-Shadow methodology.

The primal resource is the label surplus

$$s = k - 2n.$$

The obstruction is not merely a scalar inequality; it has the vector signature

$$(\alpha, \beta, \gamma),$$

where α records unused low-zone capacity, β records unused high-zone capacity after reflection, and γ counts unused spoke-band holes.

The exact identity

$$\alpha + \beta + 5\gamma = 7s - n$$

is a conservation law for that shadow signature. The construction then realizes the extremal state

$$(\alpha, \beta, \gamma) = (d, 0, 0), \qquad d = 7s - n.$$

Thus the lower-bound obstruction and the upper-bound construction meet at the same defect budget. This interpretation is optional; all formal arguments are contained in Sections 1–9.

14 Appendix A. Finite zero-rooted Schur certificates

Every displayed triple (x, y, z) satisfies $x + y = z$. In each listed case the flattened entries are exactly the corresponding set $S_d(s)$, with 0 appearing exactly once as a third coordinate.

14.1 A.1. $d = 3$

14.1.1 $s = 9$

$$(-2,2,0),(1,8,9),(3,13,16),(7,10,17),(5,14,19),$$
$$(-4,24,20),(6,15,21),(4,18,22),(11,12,23).$$

14.1.2 $s = 13$

$$(-2,2,0),(-4,8,4),(3,9,12),(1,25,26),(5,23,28),$$
$$(10,19,29),(6,24,30),(11,20,31),(14,18,32),(16,17,33),$$
$$(7,27,34),(13,22,35),(15,21,36).$$

14.1.3 $s = 17$

$$(-2, 2, 0), (3, 7, 10), (9, 11, 20), (4, 19, 23), (1, 29, 30),$$
$$(-4, 37, 33), (12, 24, 36), (17, 21, 38), (5, 34, 39), (15, 26, 41),$$
$$(14, 28, 42), (8, 35, 43), (13, 31, 44), (18, 27, 45), (6, 40, 46),$$
$$(22, 25, 47), (16, 32, 48).$$

14.1.4 $s = 21$

$$(-4, 4, 0), (2, 7, 9), (8, 10, 18), (13, 22, 35), (-2, 39, 37),$$
$$(11, 29, 40), (12, 30, 42), (19, 25, 44), (1, 45, 46), (6, 41, 47),$$
$$(14, 34, 48), (21, 28, 49), (23, 27, 50), (15, 36, 51), (20, 33, 53),$$
$$(16, 38, 54), (24, 31, 55), (5, 52, 57), (26, 32, 58), (3, 56, 59),$$
$$(17, 43, 60).$$

14.1.5 $s = 25$

$$(-4, 4, 0), (6, 10, 16), (3, 17, 20), (12, 14, 26), (15, 22, 37),$$
$$(9, 31, 40), (13, 34, 47), (-2, 50, 48), (2, 49, 51), (18, 36, 54),$$
$$(1, 55, 56), (5, 53, 58), (8, 52, 60), (29, 32, 61), (19, 43, 62),$$
$$(25, 38, 63), (7, 57, 64), (23, 42, 65), (27, 39, 66), (21, 46, 67),$$
$$(33, 35, 68), (24, 45, 69), (11, 59, 70), (30, 41, 71), (28, 44, 72).$$

14.1.6 $s = 29$

$$(-4, 4, 0), (-2, 18, 16), (5, 22, 27), (2, 34, 36), (14, 24, 38),$$
$$(11, 39, 50), (9, 42, 51), (1, 52, 53), (8, 47, 55), (26, 32, 58),$$
$$(15, 45, 60), (25, 37, 62), (3, 63, 66), (13, 54, 67), (7, 61, 68),$$
$$(29, 40, 69), (21, 49, 70), (28, 44, 72), (30, 43, 73), (10, 64, 74),$$
$$(19, 56, 75), (35, 41, 76), (31, 46, 77), (20, 59, 79), (23, 57, 80),$$
$$(33, 48, 81), (17, 65, 82), (12, 71, 83), (6, 78, 84).$$

14.2 A.2. $d = 4$

14.2.1 $s = 4$

$$(-3, 3, 0), (-2, 7, 5), (2, 4, 6), (1, 8, 9).$$

14.2.2 $s = 8$

$$(-2, 2, 0), (-3, 7, 4), (1, 14, 15), (5, 11, 16),$$
$$(6, 12, 18), (9, 10, 19), (3, 17, 20), (8, 13, 21).$$

14.2.3 $s = 12$

$$(-2, 2, 0), (4, 5, 9), (1, 15, 16), (3, 17, 20), (-3, 26, 23),$$
$$(13, 14, 27), (6, 22, 28), (10, 19, 29), (12, 18, 30), (7, 24, 31),$$
$$(11, 21, 32), (8, 25, 33).$$

14.3 A.3. $d = 5,\ s = 3$

$$(-4, 2, -2), (-5, 5, 0), (1, 3, 4).$$

14.4 A.4. $d = 6,\ s = 6$

$$(-4, 4, 0), (-2, 7, 5), (-3, 9, 6), (2, 8, 10), (1, 12, 13), (3, 11, 14).$$

15 Appendix B. Representative complete instances

The smallest admissible representative in each structural class is:

d	s	$n = 7s - d$	$k = 2n + s$
0	4	28	60
1	3	20	43
2	2	12	26
3	5	32	69
4	4	24	52
5	3	16	35
6	6	36	78

The accompanying verifier constructs and checks a complete labeling for every one of these seven cases, including:

1. exactly $2n + 1$ distinct used vertex labels;
2. all $3n$ ordinary edge weights distinct;
3. the ordinary weights forming one interval of length $3n$;
4. all residues modulo $3n$ appearing exactly once.

16 Appendix C. Verification boundary

The proof has four logically separate layers:

1. **Universal lower bound:** Sections 2–3. No external results and no computation.
2. **Deterministic graph construction:** Sections 4–5 and 7–9. No computation.
3. **Existence of infinite Schur tails:** reduced exactly to Theorems 6.2 and 6.5.
4. **Finite exceptions:** explicit certificates in Appendix A, independently checked by the verifier.

Therefore a referee can audit the main result by checking:

- Lemmas 3.1 and 3.2;
- the identity (5);
- the seven-row kernel and identities (11)–(14);
- Lemma 5.1;
- the residual-set table;
- Lemma 7.1;
- the hypotheses of Theorems 6.2 and 6.5 in each of the seven cases;
- the finite certificates in Appendix A.

No empirical extrapolation enters the infinite theorem.

17 References

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