Eventual Strict Log-Concavity of Weighted Linear Hypercombs

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Abstract

We prove that the coefficient sequence of a weighted linear hypercomb is strictly log-concave at every interior index once its length is sufficiently large, for each fixed pair of positive real weights. Integer weights recover strong independent-set sequences with possibly different spine-edge and tooth sizes. In the uniform case, the result eliminates the possible interval of nonmonotonicity in the theorem of Galvin and Sharpe for every fixed uniformity and all sufficiently large lengths. The proof combines fixed-index coefficient estimates, tilted Fourier integrals and polynomial reversal; a weighted-variance identity establishes the required nondegeneracy. We also show that no length threshold works for all positive weights. When the two weights are equal and tend to zero while their product with the length tends to a positive constant, we identify a boundary scaling limit. The limiting coefficient sequence is log-concave exactly when the scaling parameter is at least an explicit algebraic constant, approximately 0.1879701949757, determined by the fifth coefficient inequality. This constant also bounds from below the lower limit of the product of the common weight and the least eventual length threshold as the weight tends to zero. Lean proofs accompany the main coefficient theorem, the threshold obstruction and the exact transition for the recursively defined limit sequence. The general scaling limit and generating-function identification are established in the written proof.

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1 Introduction and main results

The shape of an independent-set sequence records how the number of independent sets changes with their cardinality. Log-concavity gives unimodality and controls every adjacent coefficient ratio, but may be lost when positive weights are introduced. Linear hypercombs provide a family in which a polynomial recurrence permits both a proof of eventual strict log-concavity and a precise analysis of its failure near the boundary of the weight domain.

A strong independent set meets each hyperedge in at most one vertex. Fix integers $\ell \geqslant 3, m \geqslant 2$, and put $s = \ell - 2$, $t = m - 1$. A linear hypercomb has spine vertices

$v_0, \ldots, v_n$, a spine edge containing v_{i-1}, v_i and s private vertices for each $1 \leqslant i \leqslant n$, and a tooth containing v_i and t private vertices at each spine vertex. Let $C_n(x) = \sum_k c_{n,k} x^k$ count strong independent sets. At $n = 0$ there is a single tooth.

We extend the counting recurrence to arbitrary real $s, t > 0$:

$$\begin{gathered} a(x) = (1+sx)(1+tx), \qquad b(x) = x(1+tx), \\ C_0(x) = 1 + (t+1)x, \quad C_1(x) = (1+tx)(a(x) + 2x), \\ C_n(x) = a(x)C_{n-1}(x) + b(x)C_{n-2}(x) \quad (n \geqslant 2). \end{gathered} \tag{1}$$

These are weighted counting polynomials. Take one private vertex of weight s in each spine edge, one of weight t in each tooth, and spine vertices of weight one. A set contributes its product of vertex weights times $x^{|I|}$. Positive integer weights reproduce the unweighted count with that many interchangeable private vertices.

Theorem 1. *For every fixed $s, t > 0$, there exists $N(s,t)$ such that for all $n \geqslant N(s,t)$,*

$$c_{n,k}^2 > c_{n,k-1} c_{n,k+1} \qquad (1 \leqslant k \leqslant 2n). \tag{2}$$

All coefficients on $0 \leqslant k \leqslant 2n+1$ are positive; hence the entire sequence is unimodal for these n.

Corollary 2. *For each fixed $\ell \geqslant 3, m \geqslant 2$, the strong independent-set sequence of the corresponding linear hypercomb is strictly log-concave at every interior index for sufficiently large n.*

Proposition 3. *For every fixed integer $n \geqslant 4$, there is $\eta_n > 0$ such that $s = t = u \in (0, \eta_n)$ gives*

$$c_{n,2n-2}^2 < c_{n,2n-3} c_{n,2n-1}. \tag{3}$$

Moreover, let $N_*(u)$ be the least threshold in Theorem 1 for $s = t = u$. Then $\liminf_{u \downarrow 0} u N_*(u) \geqslant \zeta$, where $\zeta = 0.1879701949757\ldots$ is the unique positive zero of

$$\begin{aligned} \Xi(z) = &-1350 + 4275z + 11475z^2 + 18810z^3 \\ &+ 12120z^4 + 3840z^5 + 560z^6 + 32z^7. \end{aligned}$$

Thus Theorem 1 has no common threshold for all positive parameters. This does not exclude a common threshold on the integer domain and does not settle the all-length unweighted conjecture. Theorem 7 determines the exact transition of the boundary limit: $[y^j](1+y)^2 \exp(z(2y+y^2))$ is a log-concave sequence precisely when $z \geqslant \zeta$, for $z > 0$.

When $m = \ell$, Galvin and Sharpe [6, v5, Corollary 1.8] prove monotonicity except possibly on an interval of length $O(\sqrt{n})$. Their Question 1.10(4) asks if this gap can be shortened or eliminated. Corollary 2 eliminates it for each fixed uniformity once n is sufficiently large. Their Note 1.9 withdraws the full log-concavity proof in version 3. The result here applies to sufficiently large lengths for each fixed uniformity.

Analytic proofs of eventual log-concavity are classical. Brockett and Kemperman [3] establish eventual unimodality for powers of a fixed three-point distribution, with a conditional extension to certain nonidentically distributed sums. Their stated conclusions do not give strict log-concavity for (1). Odlyzko and Richmond [11, Theorems 1–2] treat powers of a fixed polynomial, including endpoint conditions. We adapt their endpoint and saddle-point strategy to an exact sum of two algebraic characteristic terms; see [5] for general background on coefficient asymptotics. Regular local-limit estimates also yield central log-concavity, as in Canfield, Janson and Zeilberger [4, Section 4.1]. Here the uniform endpoint estimates are needed to cover every coefficient. The two characteristic terms and their dependence on the positive weights require the variance and circle estimates developed below.

The rooted-graph concatenation factorization of Wang and Zhu [12, Theorem 3.1] has recurrence $f_n = Af_{n-1} + ABf_{n-2}$, where A, B are graph polynomials. Identifying its first coefficient with a in (1) would require $B = b/a = x/(1 + sx)$, which is not a polynomial for $s > 0$. Thus (1) is not a direct instance of that factorization. Their additional factorizations require special initial relations that also fail here. Our proof instead works directly with the positive characteristic root of (1).

The symmetric-function criterion of Li, Li, Yang and Zhang [9, Corollary 2.2] characterizes graph independence log-concavity by a positivity condition. Its graph-specific applications prove log-concavity for spiders and pineapple graphs. For integer parameters and $n \geqslant 2$, the conflict graph of our hypercomb has at least two distinct maximal cliques of size at least three, one for each spine edge. It is therefore neither of those graph types; applying the general criterion would require a further positivity argument.

Sections 2–5 establish the counting recurrence and prove Theorem 1. Section 6 proves the failure of a common threshold and gives a first quantitative obstruction. Section 7 derives the boundary scaling limit, proves the stronger bound in Proposition 3 and determines its exact log-concavity transition. Section 8 describes the formal verification; Section 9 gives further questions.

2 Counting and characteristic roots

Put $q = 1 + tx$, $r_0 = 1 + sx$. Let E_n, I_n count sets excluding and including the last spine vertex. Conditioning gives

$$E_0 = q, \quad I_0 = x, \qquad E_{n+1} = q(r_0 E_n + I_n), \quad I_{n+1} = xE_n. \tag{4}$$

When the old endpoint is excluded, the new spine edge permits a private choice with total polynomial r_0. Otherwise that choice is forbidden. An excluded new endpoint permits tooth choices q; an included new endpoint forbids the old endpoint and both new private choices. Summing (4) gives (1).

Selecting all private spine and tooth vertices attains size $2n+1$. Subsets give positive coefficients at every smaller degree. No larger set exists: the $2n+1$ edges cover all vertices and each admits at most one selection. Thus $\deg C_n = 2n + 1$.

For positive x, define

$$\lambda(x) = \frac{a(x) + \sqrt{a(x)^2 + 4b(x)}}{2}, \qquad \nu(x) = \frac{a(x) - \sqrt{a(x)^2 + 4b(x)}}{2}. \tag{5}$$

They extend analytically to a complex neighborhood of any fixed segment $[0, R]$. Indeed the discriminant is positive there, and a sufficiently thin simply connected neighborhood has no zero. Choose the square-root branch positive on the segment. Set

$$G = \frac{C_1 - \nu C_0}{\lambda - \nu}, \qquad H = \frac{\lambda C_0 - C_1}{\lambda - \nu}.$$

Solving the recurrence gives the exact identity

$$C_n = G\lambda^n + H\nu^n. \tag{6}$$

On the real segment, $G, \lambda > 0$ and $|\nu| < \lambda$. At zero,

$$\lambda(0) = G(0) = 1, \quad \lambda'(0) = s + t + 1 > 0, \quad \nu(0) = 0. \tag{7}$$

Compactness gives $|\nu|/\lambda \leqslant \rho < 1$ on the segment. After shrinking the complex neighborhood, some $\rho_1 < 1$ also satisfies $|\nu(re^{i\theta})| \leqslant \rho_1 \lambda(r)$ for $0 \leqslant r \leqslant R$, $|\theta| \leqslant \theta_1$.

3 Two coefficient estimates

All constants below depend on fixed analytic functions and fixed radius bounds, never on the varying coefficient index.

3.1 Small coefficients

Lemma 4. *Suppose $S_n = GL^n + HV^n$ near zero, with analytic functions real on the real axis, $L(0) = 1$, $L'(0) = \alpha > 0$, $G(0) = g > 0$, and $V(0) = 0$. For every fixed integer $K \geqslant 1$, uniformly for $0 \leqslant k \leqslant K + 1$,*

$$[x^k]S_n = g\frac{(\alpha n)^k}{k!}\left(1 + O_K\left(\frac{1}{n}\right)\right). \tag{8}$$

The coefficients are therefore strictly log-concave at all $1 \leqslant k \leqslant K$ for large n.

Proof. Put $F = L - 1$, which has order one. The binomial theorem and vanishing of $[x^k](GF^j)$ for $j > k$ give, for $n \geqslant k$,

$$[x^k](GL^n) = \sum_{j=0}^{k} \binom{n}{j} [x^k](GF^j). \tag{9}$$

The coefficient at $j = k$ equals $g\alpha^k$, while every other term is a polynomial in n of degree at most $k - 1$. Consequently, for each fixed $k \geqslant 1$,

$$[x^k](GL^n) = g\alpha^k \binom{n}{k} + O_k(n^{k-1}) = g\frac{(\alpha n)^k}{k!} + O_k(n^{k-1}). \tag{10}$$

At $k = 0$ the coefficient is exactly g. The term HV^n has order at least n, so contributes nothing for $n > K + 1$. There are only finitely many indices, making the error in (8) uniform.

Let $p_{n,k} = g(\alpha n)^k/k!$. Formula (8) gives

$$\frac{([x^k]S_n)^2 - ([x^{k-1}]S_n)([x^{k+1}]S_n)}{p_{n,k}^2} = \frac{1}{k+1} + O_K\left(\frac{1}{n}\right) > 0. \tag{11}$$

The leading term is positive at every one of the finitely many indices. The three coefficients are positive by the same estimate. $\square$

3.2 Tilted integrals

Lemma 5. *Let real polynomials S_n satisfy $S_n = G\lambda^n + H\nu^n$ on a complex neighborhood of $[0, R]$, with analytic functions, $G, \lambda > 0$ on the segment, and a uniform subdominant bound on a fixed small angular sector as in Section 2. Define*

$$h_r(z) = \log \lambda(re^z), \quad \mu(r) = h_r'(0), \quad v(r) = h_r''(0).$$

Suppose μ strictly increases from zero, $\mu(R) > \beta > 0$, and there are fixed positive constants with

$$cr \leqslant \mu(r), v(r) \leqslant Cr, \qquad |h_r'''(z)| \leqslant Cr \quad (0 < r \leqslant R,\ |z| \leqslant \delta), \tag{12}$$

and

$$|S_n(re^{i\theta})| \leqslant K\lambda(r)^n e^{-cnr\theta^2} \quad (-\pi \leqslant \theta \leqslant \pi). \tag{13}$$

Then there are fixed integers K_0, N_0 such that the coefficients are strictly log-concave at every integer $K_0 \leqslant k \leqslant \beta n$ whenever $n \geqslant N_0$.

Proof. Choose the unique r with $\mu(r) = k/n$; keep it fixed for real $u \in [k-1, k+1]$. Write $m = nr$, $d = nv(r)$; both are uniformly comparable to k. Set

$$J_j(u) = \int_{-\pi}^{\pi} \theta^j S_n(re^{i\theta})e^{-iu\theta}\, d\theta, \qquad j = 0, 2. \tag{14}$$

Conjugate symmetry makes J_0, J_2 real. We prove, uniformly when d, n exceed fixed thresholds and $|u - k| \leqslant 1$,

$$J_0(u) > 0, \qquad J_2(u) > 0. \tag{15}$$

For a fixed small arc, Taylor's theorem gives

$$n(h_r(i\theta) - h_r(0)) - iu\theta = -d\theta^2/2 + i(k - u)\theta + E(\theta), \quad |E(\theta)| \leqslant Cm|\theta|^3. \tag{16}$$

On any inner arc $|\theta| \leqslant T$ with $mT^3 \leqslant \eta$, the error is uniformly bounded. Analyticity and compactness give $|G(re^{i\theta})/G(r) - 1| \leqslant Cr|\theta|$. Using $|e^z - 1| \leqslant |z|e^{|z|}$, the absolute difference between the dominant integrand and $G(r)\lambda(r)^n\theta^j e^{-d\theta^2/2}$ is at most

$$CG(r)\lambda(r)^n|\theta|^j e^{-d\theta^2/2}\Big((r+1)|\theta| + m|\theta|^3\Big). \tag{17}$$

For each integer $q \geqslant 0$, put $M_q = \int_{\mathbb{R}} |y|^q e^{-y^2/2}\,dy$, a finite positive constant. The change of variable $y = \sqrt{d}\,\theta$ gives the exact absolute moment $M_q d^{-(q+1)/2}$. More explicitly, with $L = G(r)\lambda(r)^n$, the local error satisfies

$$\begin{aligned}\left|\int_{-T}^{T} \theta^j\Big(S_n(re^{i\theta})e^{-iu\theta} - Le^{-d\theta^2/2}\Big)\,d\theta\right| \leqslant CL\left(\frac{(r+1)M_{j+1}}{d^{(j+2)/2}} + \frac{mM_{j+3}}{d^{(j+4)/2}}\right)& \\ + 2K\lambda(r)^n\rho_1^n T^{j+1}, \qquad j = 0, 2,&\end{aligned} \tag{17a}$$

provided $0 \leqslant T \leqslant \delta$ and $mT^3 \leqslant \eta$, for fixed positive δ, η. The last term bounds the secondary root. The Gaussian tail obeys

$$\int_{|\theta|\geqslant T} |\theta|^j e^{-d\theta^2/2}\,d\theta \leqslant e^{-dT^2/4}\frac{M_j}{(d/2)^{(j+1)/2}}. \tag{17b}$$

Indeed, split the exponential into two factors with exponent $-d\theta^2/4$, and bound one factor by $e^{-dT^2/4}$. The same argument applied to (13), writing its exponent constant as $c_F > 0$, bounds the actual outer integral by $K\lambda(r)^n e^{-c_F mT^2/2} M_j/(c_F m)^{(j+1)/2}$.

Now use $T = A/\sqrt{d}$, choosing $A > 0$ before r, n, u. There are fixed positive c_v, C_v, g with $c_v m \leqslant d \leqslant C_v m$ and $G(r) \geqslant g$. The exact decomposition into the local error, actual outer integral and omitted Gaussian tail, together with (17a)–(17b), gives

$$\left|\frac{d^{(j+1)/2}}{L}J_j(u) - M_j\right| \leqslant \frac{a_j}{\sqrt{d}} + b_j A^{j+1}\rho_1^n + c_j e^{-e_j A^2} + M_j 2^{(j+1)/2} e^{-A^2/4}. \tag{17c}$$

Here all constants are positive and independent of A, r, n, u. For the first term use $r \leqslant R$ and $m \leqslant d/c_v$; for the outer integral use $d/m \leqslant C_v$. The secondary term uses the exact cancellation $d^{(j+1)/2}T^{j+1} = A^{j+1}$. The local hypotheses hold once $A/\sqrt{d} \leqslant \delta$ and $A^3/\sqrt{d} \leqslant c_v\eta$.

For each $j = 0, 2$, choose A so the last two terms of (17c) sum to less than $M_j/2$. Next choose a variance threshold so the local hypotheses hold and the first term is less than $M_j/4$. Finally choose a length threshold for the second term. The resulting total error is less than M_j, proving (15). Take common thresholds for the two moments. Since d is uniformly bounded below by a positive multiple of k, fixed K_0, N_0 suffice throughout $K_0 \leqslant k \leqslant \beta n$. No quantitative convergence rate is needed.

For fixed n, r, differentiation under the integral is justified by continuous derivatives on the compact integration domain, uniformly for u in a compact interval. Thus $J_0'' = -J_2 < 0$, so J_0 is strictly concave on $[k-1, k+1]$. Positivity and the arithmetic--geometric mean inequality give

$$J_0(k)^2 > \left(\frac{J_0(k-1)+J_0(k+1)}{2}\right)^2 \geqslant J_0(k-1)J_0(k+1). \tag{18}$$

At an integer j, coefficient extraction gives $[x^j]S_n = r^{-j}J_0(j)/(2\pi)$. Multiplying (18) by $r^{-2k}/(2\pi)^2$ proves the required coefficient inequality. No asymptotic remainder is differentiated. $\square$

4 Checking the hypotheses

4.1 A direct variance identity

Consider nonnegative polynomials $a(x) = \sum_i a_i x^i$, $b(x) = \sum_j b_j x^j$, with $a_0, a_1 > 0$ and $b(r) > 0$ for $r > 0$. If $\lambda^2 = a\lambda + b$ is the positive root, put $r = e^z$, $\mu = (d/dz)\log\lambda(e^z)$, $v = (d^2/dz^2)\log\lambda(e^z)$, and

$$p_i = \frac{a_i r^i}{\lambda(r)}, \quad q_j = \frac{b_j r^j}{\lambda(r)^2}, \qquad \sum_i p_i + \sum_j q_j = 1. \tag{19}$$

Differentiate this finite identity twice. Since $p_i' = (i-\mu)p_i$, $q_j' = (j-2\mu)q_j$, it yields

$$v\left(\sum_i p_i + 2\sum_j q_j\right) = \sum_i p_i(i-\mu)^2 + \sum_j q_j(j-2\mu)^2 > 0. \tag{20}$$

Strictness follows from $p_0, p_1 > 0$: $-\mu$ and $1-\mu$ cannot both vanish. Hence $v(r) > 0$ and $r\mu'(r) = v(r) > 0$, without a factorization argument.

For this family one can also make the lower bound explicit. If $0 < r \leqslant R$, put $M = (1+sR)(1+tR) + R(1+tR)$. The root equation gives $1 \leqslant \lambda(r) \leqslant a(r) + b(r) \leqslant M$. For nonnegative p, q with $p + q \leqslant 1$, completing the square gives $p\mu^2 + q(1-\mu)^2 \geqslant pq$: multiply by $p+q$ and use $((p+q)\mu - q)^2 \geqslant 0$. Since the multiplier of v in (20) is at most two, its first two terms imply

$$v(r) \geqslant \frac{p_0 p_1}{2} = \frac{(s+t)r}{2\lambda(r)^2} \geqslant \frac{s+t}{2M^2}\, r. \tag{20a}$$

For (1), (7) gives $\mu(r)/r, v(r)/r \to s+t+1$ at zero. The reversed root in Section 5 gives the convergent expansion $\lambda(r) = str^2 + (s+t)r + O(1)$ at infinity. Differentiating it gives $\mu(r) \to 2$. Thus μ increases from zero to two. An explicit admissible radius bound is

$$R = 1 + \frac{4(s+t+3)}{st}, \qquad \mu(R) > 3/2. \tag{20b}$$

Indeed, differentiating the characteristic equation gives $\mu D = N$, where $D = a\lambda + 2b > 0$ and $N = ((s+t)r + 2str^2)\lambda + r + 2tr^2$. If $r \geqslant 1$ and $str \geqslant 4(s+t+3)$, then $\lambda > a \geqslant str^2$, $\lambda \geqslant 4r$, and $str^2 - (s+t)r - 3 \geqslant (t+3)r$. Consequently

$$2N - 3D = (str^2 - (s+t)r - 3)\lambda - 4r - 2tr^2 \geqslant 4(t+3)r^2 - 4r - 2tr^2 > 0.$$

The radius in (20b) meets these conditions. Continuity and strict increase therefore give a unique $0 < r < R$ with $\mu(r) = \tau$ for each $0 < \tau \leqslant 3/2$. The positive continuous extensions of $\mu(r)/r$ and $v(r)/r$ to $r = 0$, together with compactness, prove their two-sided bounds in (12). For the third-derivative bound, write $f = \log\lambda$ and $x = re^z$. Direct differentiation gives

$$h_r'''(z) = xf'(x) + 3x^2 f''(x) + x^3 f'''(x). \tag{20c}$$

Choose a closed complex neighborhood of $[0, R]$ contained in the analytic domain. Its compactness bounds the first three derivatives of f by a common B. A fixed small disk in z keeps re^z in this neighborhood for every $0 \leqslant r \leqslant R$, since $|re^z - r| \leqslant 2r|z|$ for $|z| \leqslant 1$. Also $|re^z| \leqslant er$ there, so (20c) is bounded by $B(e + 3e^2R + e^3R^2)r$. The complex tilt restricted to the real axis equals $h(\log r + z)$, identifying its first two derivatives with the previously computed mean and variance. Finally, a uniform bound on G', a positive minimum of G on the real segment, and $|re^{i\theta} - r| \leqslant r|\theta|$ give the relative amplitude bound used in (17) by the mean value inequality.

4.2 Decay on the full circle

For nonnegative a with $a_0, a_1 > 0$, the triangle inequality gives, uniformly on $0 < r \leqslant R$, $|\theta| \leqslant \pi$,

$$\begin{aligned} a(r) - |a(re^{i\theta})| &\geqslant a_0 + a_1 r - |a_0 + a_1 re^{i\theta}| \\ &= \frac{2a_0 a_1 r(1 - \cos\theta)}{a_0 + a_1 r + |a_0 + a_1 re^{i\theta}|} \geqslant cr\theta^2. \end{aligned} \tag{21}$$

The denominator is positive and bounded above; use $1 - \cos\theta \geqslant 2\theta^2/\pi^2$ on this interval. Put $A' = |a(re^{i\theta})|$, $B = b(r) > 0$, and $T = (A' + \sqrt{A'^2 + 4B})/2$. The positive quadratic root has derivative at least $1/2$ in its first coefficient. Thus (21) gives $T \leqslant \lambda(r) - cr\theta^2$ and, reducing c,

$$T/\lambda(r) \leqslant e^{-cr\theta^2}. \tag{22}$$

The root T extends continuously to the compact rectangle $[0, R] \times [-\pi, \pi]$ and is positive there: at zero it is $a_0 > 0$, and at positive radii $b(r) > 0$ ensures strict positivity. Thus it has a positive uniform lower bound. Bounded initial polynomials permit a fixed K with $|C_0(re^{i\theta})| \leqslant K$, $|C_1(re^{i\theta})| \leqslant KT$. Induction in (1) gives

$$|C_n(re^{i\theta})| \leqslant KT^n \leqslant K\lambda(r)^n e^{-cnr\theta^2}. \tag{23}$$

This proves (13), including radii tending to zero. Lemmas 4 and 5 give strict log-concavity at all $1 \leqslant k \leqslant 3n/2$ for large n.

For the coefficients in (1), one admissible exponent constant in (23) is

$$c = \frac{s}{\pi^2(1+sR)M}, \qquad M = (1+sR)(1+tR) + R(1+tR). \tag{23a}$$

To see this, apply (21) to $1+sx$. Its modulus deficit is at least $2sr\theta^2/(\pi^2(1+sR))$; multiplying by the second linear factor preserves this lower bound since $1+tr \geqslant 1$. The quadratic-root deficit is at least half as large. Finally use $\lambda(r) \leqslant M$ and $1-z \leqslant e^{-z}$ to obtain (23a).

4.3 A recurrence proof for fixed indices

For the actual family, the finite-index step also has a direct proof from (1). Write $\alpha = s+t+1$. Coefficient extraction gives $c_{n,0} = 1$, $c_{n+2,1} - c_{n+1,1} = \alpha$, and, for $j \geqslant 0$,

$$c_{n+2,j+2} - c_{n+1,j+2} = (s+t)c_{n+1,j+1} + c_{n,j+1} + stc_{n+1,j} + tc_{n,j}. \tag{23b}$$

The elementary summation principle $(f_{n+1} - f_n)/n^j \to L$ implies $f_n/n^{j+1} \to L/(j+1)$. To justify it, use $\binom{n}{j} \sim n^j/j!$ and the exact identity $\sum_{i=0}^{n-1} \binom{i}{j} = \binom{n}{j+1}$. An error that is $o(\binom{n}{j})$ remains negligible relative to the summed main term: split at a fixed large index and bound the rest by $\varepsilon \sum_i \binom{i}{j}$. Fixed length shifts preserve the normalized limits. Induction in (23b) therefore gives

$$\lim_{n\to\infty} \frac{c_{n,k}}{n^k} = \frac{\alpha^k}{k!} \qquad \text{for each fixed } k \geqslant 0. \tag{23c}$$

For $k \geqslant 1$, the difference $(c_{n,k}^2 - c_{n,k-1}c_{n,k+1})/n^{2k}$ consequently tends to $\alpha^{2k}/((k!)^2(k+1)) > 0$. Taking the maximum of finitely many thresholds fills the lower indices left by Lemma 5. This recurrence argument is the one used in the Lean verification.

5 Reversing the upper endpoint

Set $Q_n(y) = y^{2n+1}C_n(1/y)$. From (1),

$$\begin{gathered} Q_n = (y+s)(y+t)Q_{n-1} + y^2(y+t)Q_{n-2}, \\ Q_0 = y+t+1, \quad Q_1 = (y+t)((y+s)(y+t)+2y). \end{gathered} \tag{24}$$

Its roots satisfy $\lambda_*(y) = y^2\lambda(1/y)$, $\nu_*(y) = y^2\nu(1/y)$ for $y > 0$, and extend analytically to zero by the discriminant in (24), with

$$\lambda_*(0) = st, \quad \lambda_*'(0) = s+t, \quad \nu_*(0) = 0. \tag{25}$$

The leading amplitude $G_* = (Q_1 - \nu_* Q_0)/(\lambda_* - \nu_*)$ is positive on each compact positive segment and $G_*(0) = t > 0$. Lemma 4 applies to $Q_n/(st)^n$ with $L = \lambda_*/(st)$, $V = \nu_*/(st)$, $g = t$, $\alpha = (s+t)/(st) > 0$.

There is also a direct recurrence proof of this fixed-index step. Set $d_{n,j} = [y^j]Q_n/(st)^n$. Negative-index coefficients are zero. Equation (24) gives

$$\begin{aligned} d_{n+2,j} - d_{n+1,j} = {} & \frac{s+t}{st} d_{n+1,j-1} + \frac{1}{st} d_{n+1,j-2} \\ & + \frac{t}{(st)^2} d_{n,j-2} + \frac{1}{(st)^2} d_{n,j-3}. \end{aligned} \tag{25a}$$

Since $d_{n,0} = t$ for $n \geqslant 1$, the summation argument of Section 4.3, now by strong induction on j, yields

$$\lim_{n\to\infty} \frac{d_{n,j}}{n^j} = \frac{t}{j!} \left(\frac{s+t}{st} \right)^j. \tag{25b}$$

Only the first term in (25a) contributes to the leading limit; the other three have strictly lower degree. The positive Poisson gap then gives strict log-concavity at every fixed reversed index.

The reversed mean and variance satisfy

$$\mu_*(y) = 2 - \mu(1/y), \qquad v_*(y) = v(1/y) > 0. \tag{26}$$

The convergent local expansion $\lambda_*(y) = st + (s+t)y + O(y^2)$ shows that both rates divided by y tend to $(s+t)/(st) > 0$ at zero. Together with (26), this gives positive continuous extensions of $\mu_*(y)/y$ and $v_*(y)/y$, and hence their two-sided bounds on every fixed compact radius interval.

One can choose the radius explicitly. Put $B_* = s+t+1+2t(s+1)$ and $R_* = 1+4B_*$. The forward derivative equation preceding (20b), using $a(r), \lambda(r) \geqslant 1$, gives for $0 < r \leqslant 1$

$$\mu(r) \leqslant (s+t+1)r + 2t(s+1)r^2 \leqslant B_* r. \tag{26a}$$

Consequently $\mu(1/R_*) < 1/4$ and $\mu_*(R_*) > 7/4 > 3/2$. Strict increase follows from $v_* > 0$, so the required reversed saddles exist uniquely.

For completeness, the remaining uniform analytic conditions also include the zero endpoint. The reversed discriminant is positive on $[0, R_*]$, including its value $(st)^2$ at zero. Thus the roots and amplitudes are analytic on a common complex neighborhood, G_* has a positive minimum on the real segment, and $|\nu_*|/\lambda_*$ has maximum less than one there. Compactness extends that gap to a common angular sector. The third-derivative identity (20c), applied to $\log \lambda_*$, gives the required bound proportional to y, and a bound on G_*' gives the relative amplitude estimate. Finally, $a_*(y) = (y+s)(y+t)$ has positive constant and linear terms, while $b_*(y) = y^2(y+t) > 0$ for $y > 0$. The modulus-deficit and comparison-root proof of (21)–(23) therefore applies with $a_{*,0} = st$, $a_{*,1} = s+t$. Its comparison root is positive also at zero, so its compact minimum and the initial polynomial bounds give a prefactor independent of n, y, θ. This verifies every hypothesis of Lemma 5 for Q_n.

Hence Q_n is strictly log-concave at $1 \leqslant j \leqslant 3n/2$ for large n. Translating $j = 2n+1-k$ covers the upper part of C_n. Together with $1 \leqslant k \leqslant 3n/2$, these ranges cover every integer $1 \leqslant k \leqslant 2n$ for $n \geqslant 2$. The maximum of the finitely many forward and reversed thresholds proves Theorem 1. Positive coefficients and strict log-concavity make adjacent ratios strictly decreasing, implying unimodality. Integer specialization gives Corollary 2. □

6 No common threshold over positive weights

Throughout this section $n \geqslant 4$. Set $s = t = u$ and condition on selected spine vertices J. This is an independent set in the ordinary path on $v_0, \ldots, v_n$. Write $j = |J|$, $e = |J \cap \{v_0, v_n\}|$. It excludes j tooth choices and $2j - e$ spine-private choices, giving

$$C_n(x) = \sum_J x^j (1 + ux)^{2n+1-3j+e}. \tag{27}$$

A term contributes to degree k only if $k \leqslant 2n + 1 - 2j + e$. Its coefficient is $\binom{2n+1-3j+e}{k-j} u^{k-j}$. The least power of u comes from the largest feasible j. For fixed $n \geqslant 4$ this gives

$$\begin{aligned} c_{n,2n-3} &= (n-3)u^{2n-6} + O(u^{2n-5}), \\ c_{n,2n-2} &= (4n-7)u^{2n-4} + O(u^{2n-3}), \\ c_{n,2n-1} &= u^{2n-3} + O(u^{2n-2}). \end{aligned} \tag{28}$$

In the first line the maximum set has three vertices, both endpoints and one of $v_2, \ldots, v_{n-2}$, giving $n - 3$ choices with binomial coefficient one. In the second line the maximum size is two: both endpoints contribute $2n - 3$; exactly one endpoint gives $2(n - 2)$ choices with coefficient one. In the last line the unique maximum set consists of the two endpoints, with coefficient one. Smaller sets contribute only higher powers, proving (28).

Consequently

$$c_{n,2n-2}^2 - c_{n,2n-3}c_{n,2n-1} = -(n-3)u^{4n-9} + O(u^{4n-8}) < 0$$

for sufficiently small positive u. This proves the first assertion of Proposition 3. Given any proposed common eventual threshold, choose a fixed n at least that threshold and at least four, then take u sufficiently small. This disproves a common threshold. □

We first obtain a quantitative bound from three high coefficients; Section 7 strengthens it to the constant in Proposition 3. The same conditioning gives exact formulas, valid for $n \geqslant 4$:

$$c_{n,2n-3} = u^{2n-6} A_n(u), \quad c_{n,2n-2} = u^{2n-4} B_n(u), \quad c_{n,2n-1} = u^{2n-3} D_n(u), \tag{29}$$

where

$$
\begin{aligned}
A_n(u) &= n-3+\frac{(n-2)(13n-25)}{2}u \\
&\quad + \left((n-1)\binom{2n-2}{2}+2\binom{2n-1}{3}\right)u^2+\binom{2n+1}{4}u^3, \\
B_n(u) &= 4n-7+2(n-1)(3n-2)u+\binom{2n+1}{3}u^2, \\
D_n(u) &= 1+(5n-3)u+\binom{2n+1}{2}u^2.
\end{aligned}
\tag{30}
$$

To check the additional terms, a single selected spine vertex is interior in $n-1$ ways or an endpoint in two ways. Two selected vertices with zero, one or two endpoints occur respectively in $\binom{n-2}{2}$, $2(n-2)$, or one way. Inserting these counts into (27) gives (30); the empty set supplies the final binomial term in each polynomial. Thus no nonuniform remainder in (28) is used when n varies.

There is also a direct recurrence verification of all three formulas. Put $q_{n,j} = [y^j]Q_n(y)$, with $q_{n,j} = 0$ for negative j. Equation (24) gives

$$
q_{n+2,j} = u^2 q_{n+1,j} + 2u q_{n+1,j-1} + q_{n+1,j-2} + u q_{n,j-2} + q_{n,j-3}. \tag{30a}
$$

The equations for $0 \leqslant j \leqslant 4$ form a closed system. Besides (29)–(30), use $q_{n,0} = u^{2n+1}$ and $q_{n,1} = u^{2n-1}(2+(2n+1)u)$. Substitution verifies (30a), and the initial rows at $n = 4, 5$ follow from (24). Two-step induction proves (29)–(30) for every $n \geqslant 4$. Reflection followed by a finite coefficient recurrence is a standard algebraic approach; see also [1, Proposition 3.7 and Section 4.2] for its use on independence polynomials of other graph families. Equations (29)–(30) allow the length and the weights to vary together.

Fix $z > 0$, let $u \downarrow 0$, and take $n = \lfloor z/u \rfloor$. Then $nu \to z$ and (30) gives

$$
\begin{aligned}
A_n(u)/n &\longrightarrow 1+\frac{13}{2}z+\frac{14}{3}z^2+\frac{2}{3}z^3, \\
B_n(u)/n &\longrightarrow 4+6z+\frac{4}{3}z^2, \\
D_n(u) &\longrightarrow 1+5z+2z^2.
\end{aligned}
$$

The normalized log-concavity difference consequently satisfies

$$
\begin{aligned}
\frac{c_{n,2n-2}^2 - c_{n,2n-3}c_{n,2n-1}}{nu^{4n-9}} &= nu\left(\frac{B_n(u)}{n}\right)^2 - \frac{A_n(u)}{n}D_n(u) \\
&\longrightarrow \Psi(z).
\end{aligned}
\tag{31}
$$

Here

$$
\Psi(z) = -1+\frac{9}{2}z+\frac{53}{6}z^2+\frac{29}{3}z^3+\frac{10}{3}z^4+\frac{4}{9}z^5.
$$

The polynomial Ψ is strictly increasing on $[0,\infty)$, has value -1 at zero, and tends to infinity. It therefore has a unique positive zero z_*, and $\Psi(z) < 0$ for $0 < z < z_*$. For each such fixed z, the row at $n = \lfloor z/u \rfloor$ fails log-concavity for all sufficiently small positive u. Hence $N_*(u) > \lfloor z/u \rfloor$, implying $\liminf_{u\downarrow 0} uN_*(u) \geqslant z$. Let $z \uparrow z_*$ to obtain the preliminary lower bound $\liminf uN_*(u) \geqslant z_*$. The exact rational check $\Psi(1/10) = -12703/28125 < 0$ already gives a lower bound $\liminf uN_*(u) \geqslant 1/10$ without numerical root approximation. Rational evaluation also certifies $\Psi(16147/100000) < 0 < \Psi(16148/100000)$, so the preliminary constant $z_* = 0.161478\ldots$ has a rigorous bracket. $\square$

For an exact finite example, $n = 4, u = 1/100$ gives coefficients of $C_4(100x)$

$$(1, 509, 63236, 1248784, 2383126, 1281626, 96084, 11736, 209, 1).$$

The difference at $k = 6$ is $96084^2 - 1281626 \cdot 11736 = -5809027680$. Scaling the variable positively preserves the sign. This is a weighted counterexample, not a counterexample to unweighted hypercomb log-concavity.

7 A boundary scaling limit

The stronger obstruction is explained by the following limit for the reversed polynomials of Section 5, with $s = t = u$.

Theorem 6. *Put $S_{n,u}(y) = Q_n(u^2y)/u^{2n+1}$. For every fixed integer $j \geqslant 0$, as $u \downarrow 0$ and $nu \to z > 0$,*

$$[y^j]S_{n,u}(y) \longrightarrow p_j(z), \qquad p_j(z) = [y^j](1+y)^2 \exp\Big(z(2y+y^2)\Big). \tag{32}$$

This statement is coefficientwise; it does not assert convergence uniformly over the coefficient index.

Proof. Substitution in (24) gives

$$\begin{aligned} S_{0,u} &= u^{-1} + 1 + uy, \\ S_{1,u} &= (1+uy)^3 + 2y(1+uy), \\ S_{n+2,u} &= A_u S_{n+1,u} + B_u S_{n,u}, \\ A_u &= (1+uy)^2, \qquad B_u = uy^2(1+uy). \end{aligned} \tag{33}$$

Let $F_0 = 0, F_1 = 1$ and $F_{m+2} = A_u F_{m+1} + B_u F_m$. Pascal's identity and the two initial values prove by induction that

$$F_m = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k} B_u^k A_u^{m-1-2k} \qquad (m \geqslant 1). \tag{34}$$

For fixed j and $m \geqslant 2j+2$, coefficient extraction gives

$$[y^j]F_m = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{m-1-k}{k} \binom{2m-2-3k}{j-2k} u^{j-k}. \tag{35}$$

If $mu \to z$ and $u \to 0$, the falling-factorial formula shows that the summand tends to $z^k(2z)^{j-2k}/(k!(j-2k)!)$. There are only finitely many summands, independent of m. Consequently both F_n and F_{n-1} tend coefficientwise to $\exp(z(2y+y^2))$. The assumption $z > 0$ ensures $n \to \infty$, so all the required lower bounds on m hold eventually.

The exact solution of (33), for $n \geqslant 1$, is

$$S_{n,u} = S_{1,u}F_n + B_uS_{0,u}F_{n-1}, \qquad B_uS_{0,u} = y^2(1+uy)(1+u+u^2y). \tag{36}$$

The two prefactors tend to $1+2y$ and y^2, respectively. For each fixed coefficient, multiplication uses only finitely many terms. Passing to the limit in (36) proves (32). In particular, the pole in $S_{0,u}$ leaves a nonzero boundary contribution; omitting it would give an incorrect prefactor. □

7.1 The stronger threshold obstruction

The three required limit coefficients are

$$\begin{aligned} p_4 &= z + \frac{13}{2}z^2 + \frac{14}{3}z^3 + \frac{2}{3}z^4, \\ p_5 &= 3z^2 + \frac{19}{3}z^3 + \frac{8}{3}z^4 + \frac{4}{15}z^5, \\ p_6 &= \frac{1}{2}z^2 + \frac{25}{6}z^3 + \frac{13}{3}z^4 + \frac{6}{5}z^5 + \frac{4}{45}z^6. \end{aligned} \tag{37}$$

Direct multiplication gives

$$p_5(z)^2 - p_4(z)p_6(z) = \frac{z^3}{2700}\Xi(z). \tag{38}$$

All nonconstant coefficients of Ξ are positive, its constant coefficient is negative, and it tends to infinity. Hence it has a unique positive root ζ. Exact rational evaluation gives

$$\Xi\Big(\frac{18797019}{10^8}\Big) < 0 < \Xi\Big(\frac{18797020}{10^8}\Big). \tag{39}$$

For each $0 < z < \zeta$, choose $n = \lfloor z/u \rfloor$. Theorem 6 makes the fifth scaled gap negative for all sufficiently small positive u. Writing $q_{n,j} = [y^j]Q_n$, that gap is

$$\left(q_{n,5}^2 - q_{n,4}q_{n,6}\right)\left(\frac{u^{10}}{u^{2n+1}}\right)^2. \tag{40}$$

Its second factor is positive, so the original polynomial fails log-concavity at $k = 2n-4$. Thus $N_*(u) > \lfloor z/u \rfloor$, which implies $\liminf uN_*(u) \geqslant z$. Letting $z \uparrow \zeta$ proves Proposition 3. No matching upper bound or optimality assertion is made.

7.2 Comparison with compound Poisson log-concavity

After division by $4e^{3z}$, the limit in (32) is the probability generating function of a sum of independent variables with laws $\mathrm{Bin}(2, 1/2)$, $\mathrm{Poisson}(2z)$, and twice $\mathrm{Poisson}(z)$. These are classical distributions. The contribution of Theorem 6 is their occurrence in this scaling of the hypercomb recurrence and the resulting obstruction.

Hansen's criterion, in the formulation of [7, Lemma 5.2 and Theorem 5.5], gives a precise comparison. The unprefactored exponential $\exp(z(2y + y^2))$ corresponds to intensity $\lambda = 3z$ and jump probabilities $Q(1) = 2/3, Q(2) = 1/3$. Its size-biased jump law is log-concave. Thus its coefficients are log-concave if and only if $\lambda Q(1)^2 \geqslant 2Q(2)$, or $z \geqslant 1/2$. Convolution with the log-concave coefficients of $(1 + y)^2$ gives the sufficient condition $z \geqslant 1/2$ for the full limit. Equation (38) gives the necessary condition $z \geqslant \zeta$. The next subsection proves that this necessary condition is sufficient.

The related cycle-index criterion [10, Theorems 2.1 and 2.3] assumes nonnegative coefficients in an exponential representation. It does not directly apply to the full prefactored limit: writing its logarithm as a constant plus $\sum_{j\geqslant 1} c_j y^j/j$ gives $c_4 = -2$ for every z. The exact threshold for the prefactored sequence thus requires a separate argument.

Related results describe the even and odd subsequences separately. Broadie and Petkova [2, Theorem 1] and Kovacevic [8, Proposition 5] obtain log-concavity, in fact finite-order ultra-log-concavity, for these subsequences in sums of independent three-point variables. Apply their results to the coefficients of $(1 + y)^2(1 + (z/m)(2y + y^2))^m$, after normalization, and pass to the coefficientwise limit. Both parity subsequences of (32) are therefore log-concave for every $z > 0$. Theorem 7 concerns the adjacent-index inequalities of the full sequence, which additionally compare the two parities and fail below ζ.

There is also a distinction between triangular and Toeplitz total positivity. For $f(t) = 2t + t^2$ and $g(t) = (1 + t)^2$, Zhu's criterion [14, Theorem 1.2] applies and proves total positivity of the triangular array $A_{n,k} = [t^n]g(t)f(t)^k/k!$. That conclusion does not give the Toeplitz order-two inequalities for $p_n(z) = \sum_k A_{n,k}z^k$; (38) supplies counterexamples for $0 < z < \zeta$. The Toeplitz criterion [14, Theorem 1.1] instead requires exponential coefficients $x_j = \sum_i \lambda_i^j$ with $\lambda_i \geqslant 0$. For our exponential factor, $x_1 = x_2 = 2z$ and $x_j = 0$ for $j \geqslant 3$, which cannot have this form. These comparisons leave the specific adjacent-index threshold to the following argument.

7.3 The exact transition of the limit

Theorem 7. *For $z > 0$, the sequence $p_j(z) = [y^j](1 + y)^2 \exp(z(2y + y^2))$, $j \geqslant 0$, is log-concave if and only if $z \geqslant \zeta$.*

Proof. Necessity follows from (38), and Section 7.2 proves sufficiency for $z \geqslant 1/2$. It remains to treat $\zeta \leqslant z \leqslant 1/2$. Put $x = 2z$, $q_j = p_j(x/2)$, and $h_j = [y^j] \exp(xy + xy^2/2)$. Thus $3/8 \leqslant x \leqslant 1$, $h_0 = 1, h_1 = x$, and

$$(j + 1)h_{j+1} = x(h_j + h_{j-1}), \qquad q_j = (1 + j/x)h_j + h_{j-1} \quad (j \geqslant 1). \tag{41}$$

Here $h_{-1} = 0$; all h_j, q_j are positive. Eliminating h from (41) gives, for $n \geqslant 3$,

$$
\begin{aligned}
&q_{n+1} = \alpha_n q_n + \beta_n q_{n-1}, \\
&\alpha_n = \frac{x(n^2-n-2-x)}{(n+1)(n^2-n-x)}, \qquad \beta_n = \frac{x(n^2+n-x)}{(n+1)(n^2-n-x)}.
\end{aligned}
\tag{42}
$$

Both coefficients are positive. Set $f_n(r) = \alpha_n + \beta_n/r$ and let ρ_n be its positive fixed point. For $R_n = q_n/q_{n-1} > 0$, log-concavity at n is equivalent to $R_n \geqslant \rho_n$. This positive-root characterization is a standard recurrence criterion; see [13, Proposition 2.1]. Since f_n is decreasing, log-concavity at $n-1$ implies $R_n \leqslant \rho_{n-1}$. Therefore

$$
f_n(\rho_{n-1}) \geqslant \rho_{n+1} \quad (n \geqslant 6) \tag{43}
$$

propagates log-concavity from index $n-1$ to index $n+1$.

We verify (43) by rational polynomial inequalities. Abbreviate $(a_p, b_p) = (\alpha_{n-1}, \beta_{n-1})$, $(a, b) = (\alpha_n, \beta_n)$, and $(a_+, b_+) = (\alpha_{n+1}, \beta_{n+1})$. Put

$$
\begin{aligned}
&d = b/b_p, \qquad c = a - da_p, \\
&E = c^2 + d^2 b_p - a_+ c - b_+, \qquad F = 2cd + d^2 a_p - a_+ d.
\end{aligned}
\tag{44}
$$

If $r = \rho_{n-1}$, then $r^2 = a_p r + b_p$, $f_n(r) = c + dr$, and $f_n(r)^2 - a_+ f_n(r) - b_+ = E + Fr$. The conditions

$$
F \geqslant 0, \qquad E^2 + a_p EF - b_p F^2 \leqslant 0 \tag{45}
$$

imply $E + Fr \geqslant 0$: otherwise both factors in $(E + Fr)(E + F(a_p - r))$ would be negative, because $r > a_p$. Their product is the second expression in (45), a contradiction. The positive-root characterization now gives (43).

For completeness, the uniform sign check in (45) is explicit below. Write $D = n^2 - n - x$, $H = n^2 - 3n + 2 - x$. Substitution in (44) and cancellation give

$$
F = \frac{nxH\mathcal{F}(n,x)}{(n+1)^2(n+2)D^4}, \qquad E^2 + a_p EF - b_p F^2 = -\frac{x^2 \mathcal{R}(n,x)}{(n+1)^4(n+2)^2 D^4}, \tag{46}
$$

where

$$
\begin{aligned}
\mathcal{F}(n,x) = {}& n^6 + n^5 + (1-3x)n^4 + (11+2x)n^3 \\
& + (10 + 9x + 3x^2)n^2 + (4x - 3x^2)n - x^3 - 6x^2.
\end{aligned}
$$

For $n \geqslant 6$ and $0 \leqslant x \leqslant 1$, this is at least $n^6 + n^5 - 2n^4 + 11n^3 + 10n^2 - 7 > 0$. The other polynomial is

$$
\begin{aligned}
\mathcal{R}(n,x) = {} & xn^9 + (3x-1)n^8 + (-2x^2+2x-8)n^7 \\
& + (-2x^2+2x-26)n^6 + (4x^3+10x^2-31x-44)n^5 \\
& + (-10x^3-56x^2-133x-41)n^4 \\
& + (-6x^4-48x^3-128x^2-140x-20)n^3 \\
& + (6x^4-14x^3-86x^2-16x-4)n^2 \\
& + (3x^5+24x^4+20x^3-60x^2+24x)n \\
& + 3x^5+3x^4-36x^2.
\end{aligned} \tag{47}
$$

Expand $\mathcal{R}(m+6,x) = \sum_{i=0}^{9} r_i(x)m^i$. The following exact rational bounds prove that every r_i is positive on $[3/8, 1]$. To check the last column, expand $r_i'(x)$, discard its positive nonconstant terms, and replace each negative term $-Ax^k$ by $-A$, for $0 \leqslant x \leqslant 1$.

i	Lower bound for $r_i(3/8)$	Lower bound for $r_i'(x)$ $0 \leqslant x \leqslant 1$
9	3/8	1
8	20	57
7	484	1438
6	6588	21082
5	56081	197989
4	306313	1235161
3	1050264	5115852
2	2076481	13546924
1	1851731	20758224
0	120075	13963620

Thus $\mathcal{R}(n,x) > 0$ for $n \geqslant 6$ and $3/8 \leqslant x \leqslant 1$. All remaining factors in (46) are positive, so (45) and hence (43) hold on this whole domain.

Finally, for $j = 1,2,3,4,6$, direct expansion of $q_j^2 - q_{j-1}q_{j+1}$ about $x = 3/8$ has positive coefficients. The fifth gap is $(x/2)^3\Xi(x/2)/2700$, which is nonnegative for $x \geqslant 2\zeta$. These six initial inequalities and the two-step propagation prove all subsequent inequalities, including the endpoint $z = \zeta$. □

Theorem 7 determines the transition of the full infinite limiting sequence, rather than only finitely many gaps. It does not imply $uN_*(u) \to \zeta$: the coefficientwise convergence in Theorem 6 does not control indices growing with u^{-1}.

8 Formal verification and reproducibility

The accompanying development uses Lean 4.33.1 and mathlib v4.33.1. Theorem 1 is formalized for the defining polynomial recurrence (1) in `HypercombStrictLogConcavity.lean`. Its statement assumes only $s,t > 0$ and proves (2) for every interior index and all lengths beyond a common threshold. The development verifies the forward and reversed analytic estimates, fixed-index limits, Fourier extraction and coverage of the full coefficient range.

Proposition 3 is formalized through exact boundary coefficients. `HypercombHighCoefficients.lean` and `HypercombThresholdObstruction.lean` verify (29)–(30) and failure at each fixed length for sufficiently small equal weights. `HypercombBoundaryCoefficients.lean` identifies the first seven scaled coefficients and their joint limits. `HypercombBoundaryThreshold.lean` then proves the fifth-gap obstruction, the unique positive root of Ξ, its rational bracket (39), and the quantified threshold lower bound. This proof uses only the fixed indices required for the obstruction.

`BoundaryLimitLogConcavity.lean` proves Theorem 7 for the recursive coefficients in (41). It checks the recurrence, the root comparison, the polynomial sign certificates and all initial inequalities, as well as the classical large-parameter case and the two convolution steps. Thus the final all-index equivalence has no residual log-concavity assumption. The formal coefficients through index six are also identified with the boundary profiles used in Proposition 3.

The hypergraph counting interpretation in Section 2, the general-index scaling limit in Theorem 6 and the generating-function identification in (41) have the written proofs given above. The explicit reverse radius estimate (26a) is also a written calculation; the formal proof chooses a suitable radius existentially. These distinctions concern the scope of the formalization, and leave all stated results with mathematical proofs.

All project Lean files compile with warnings treated as errors. The Lean sources, pinned dependencies, theorem map and verification commands are publicly available at https://github.com/albert310/weighted-hypercombs. For reproducibility, the proof revision accompanying this manuscript is 2285db3c6a44. The repository README gives standalone build and axiom-audit commands. The accompanying `CheckAxioms.lean` inspects ten principal declarations and reports only three standard foundational axioms: `propext`, `Classical.choice` and `Quot.sound`. There are no admitted proofs or added axioms. Exact computational checks compare the recurrence with graph deletion in 80 cases and exhaustive subset enumeration in 53, and verify the rational identities and sign table in Theorem 7. The repository `README.md` gives the toolchain, commands, theorem map and formal-verification scope. The coefficient experiments provide checks on the implementation; the infinite conclusions follow from the proofs in Sections 2–7.

9 Further questions

Theorem 1 settles strict log-concavity for sufficiently long hypercombs at each fixed pair of positive weights. For the unweighted uniform family, it remains to determine whether log-concavity holds at every length. This would remove the length restriction in the consequence for [6, Question 1.10(4)]. Proposition 3 shows why the corresponding assertion cannot extend to all positive real weights.

The asymptotic size of the threshold is a second question. Theorem 7 establishes the sharp constant for the infinite boundary model, while Proposition 3 gives only a lower bound for the finite family. Does $uN_*(u)$ converge to ζ as $u \downarrow 0$? A matching upper bound would require control of coefficients whose indices grow as the weights vanish; the fixed-index convergence in Theorem 6 does not provide this control.

Finally, explicit sufficient bounds for $N(s,t)$ would make the eventual theorem quantitative away from the small-weight limit. The present proof selects constants from compact analytic neighborhoods and positive moment estimates. Tracking their dependence on the weights may connect the eventual theorem more closely with the sharp boundary transition.

Use of generative AI

OpenAI Codex GPT-6-Astra, a non-human AI system and the sole named author, carried out the mathematical exploration, proof development, Lean formalization, computational checks, literature searches, and the writing and revision of the entire manuscript. These research and writing tasks were performed by the AI rather than supplied as human-written proofs or manuscript text. The human user initiated the project, set its objectives, provided feedback, and directed its scope and publication preparation. Thus, the research and manuscript production were AI-executed, within a human-directed project. The author designation identifies the system that produced the work; it does not imply institutional authorship or endorsement by OpenAI. Formal verification covers the statements specified in Section 8; it does not establish bibliographic priority or replace independent mathematical review.

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