ROBIN'S 1984 CRITERION FOR THE RIEMANN HYPOTHESIS: A FORMALLY VERIFIED PROOF

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Abstract. Guy Robin proved in 1984 that the Riemann hypothesis is equivalent to the strict inequality

$$\sigma(n) < e^{\gamma} n \log\log n \qquad (n > 5040).$$

We give a proof in a form that has been formally verified in Lean 4. The argument includes the reduction to colossally abundant integers, the implication under the Riemann hypothesis, the Nicolas–Landau oscillation argument for the converse, and exact certificates for the bounded range. The analytic and arithmetic ingredients are those of Robin and of the authors he cites; the decomposition into explicit lemmas and the finite certificate system are part of the formal reconstruction. The theorem is an equivalence criterion and does not prove the Riemann hypothesis.

1. Introduction

For a positive integer n, let

$$\sigma(n) = \sum_{d\mid n} d, \qquad \mathcal{A}(n) = \frac{\sigma(n)}{n},$$

and let γ be the Euler–Mascheroni constant. Robin's theorem is the following exact criterion.

Theorem 1.1 (Robin [9])**.** *The Riemann hypothesis holds if and only if*

$$\sigma(n) < e^{\gamma} n \log\log n \tag{1.1}$$

for every integer $n > 5040$.

The cutoff is part of the theorem. In particular, the result is not merely an eventual estimate. It combines an analytic argument above an explicit height with a finite proof below that height.

Robin's original article [9] is in French. Although a scanned PDF is now available online, the paper remains comparatively difficult to discover through ordinary bibliographic channels. One purpose of the present paper is therefore expository: to give an accessible English, proof-oriented account of the argument while preserving the attribution and source trail of the original. Nicolas's related Euler-function criterion has a later English treatment [8]; the present paper concerns Robin's distinct sum-of-divisors argument.

The second principal result makes Robin's extremal reduction explicit. For $\varepsilon > 0$, put

$$Q_\varepsilon(m) = \frac{\sigma(m)}{m^{1+\varepsilon}} = \frac{\mathcal{A}(m)}{m^{\varepsilon}}. \tag{1.2}$$

We say that $N > 1$ is colossally abundant with parameter ε if $Q_\varepsilon(m) \le Q_\varepsilon(N)$ for every $m > 1$.

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Theorem 1.2. *The Riemann hypothesis holds if and only if* (1.1) *holds for every* $N > 5040$ *which is colossally abundant with some positive parameter.*

Colossally abundant numbers were introduced by Alaoglu and Erdős [1]. The criterion itself and its use of those extrema are due to Robin. The converse uses the oscillation theorem of Nicolas [7] together with Landau's positive-transform principle [4]. The explicit estimates for prime functions used below trace to Rosser and Schoenfeld [10] and to Costa Pereira [2, 3]. Our purpose is not to claim new priority for those ingredients, but to give a complete proof whose exact logical and numerical dependencies have been checked.

The proof has four parts. Section 3 develops the colossally abundant tangent reduction. Section 4 closes the bounded range. Section 5 proves the inequality under RH from a strict explicit coefficient gap. Sections 6 and 7 prove the converse by transferring Nicolas's negative oscillations to least common multiples. Only Section 9 discusses the organization and checking of the Lean development itself.

2. Arithmetic preliminaries

For $n > e^e$, set

$$\mathcal{M}(n) = \log \mathcal{A}(n) - \gamma - \log\log\log n. \tag{2.1}$$

All logarithms in (2.1) are then defined, and Robin's inequality is equivalent to $\mathcal{M}(n) < 0$.

Write $n = \prod_p p^{a_p}$. For a prime p and an integer $j \geq 1$, define the marginal gain and threshold

$$g_{p,j} = \log\left(\frac{1 - p^{-(j+1)}}{1 - p^{-j}}\right), \qquad \tau_{p,j} = \frac{g_{p,j}}{\log p}. \tag{2.2}$$

The event (p, j) records the change from exponent $j - 1$ to exponent j.

Lemma 2.1 (Prime-power event decomposition)**.** *For every positive integer n and every $\varepsilon > 0$,*

$$\log \mathcal{A}(n) = \sum_p \sum_{1 \leq j \leq a_p} g_{p,j}, \tag{2.3}$$

$$\log Q_\varepsilon(n) = \sum_p \sum_{1 \leq j \leq a_p} (g_{p,j} - \varepsilon \log p). \tag{2.4}$$

For fixed p, the sequences $g_{p,j}$ and $\tau_{p,j}$ are nonincreasing in j.

Proof. The geometric-series identity gives

$$\frac{\sigma(p^a)}{p^a} = \frac{1 - p^{-(a+1)}}{1 - p^{-1}} = \prod_{j=1}^{a} \frac{1 - p^{-(j+1)}}{1 - p^{-j}}.$$

Multiplicativity and logarithms give (2.3); subtracting $\varepsilon \log n$ gives (2.4). If $q = p^{-1} \in (0, 1)$, direct cross-multiplication shows that $(1 - q^{j+2})/(1 - q^{j+1}) \leq (1 - q^{j+1})/(1 - q^j)$. Taking logarithms and then dividing by $\log p > 0$ proves the monotonicity. □

The reduced weight of an event at parameter ε is

$$w_\varepsilon(p, j) = g_{p,j} - \varepsilon \log p = (\tau_{p,j} - \varepsilon) \log p. \tag{2.5}$$

Thus events above the threshold have nonnegative reduced weight and events below it have nonpositive reduced weight. This is the elementary complementary-slackness description of a colossally abundant factorization.

Lemma 2.2 (Existence of a maximizer)**.** *For every $\varepsilon > 0$ there exists an integer $N > 1$ maximizing Q_ε.*

Proof. The divisor bound

$$\mathcal{A}(n) = \sum_{d\mid n} \frac{1}{d} \le \sum_{1\le d\le n} \frac{1}{d} \le 1 + \log n$$

implies $Q_\varepsilon(n) \le (1+\log n)n^{-\varepsilon} \to 0$. Beyond a finite point the objective is below its value at 2. A maximum over the remaining finite set is therefore a global maximum. □

3. The colossally abundant tangent reduction

Put $H_n = \log n$ and

$$\lambda(n) = \frac{1}{H_n \log H_n}. \tag{3.1}$$

This is the derivative at H_n of the concave function $\phi(H) = \log\log H$.

Lemma 3.1 (Global tangent inequality)**.** *If $L, U > 1$, then*

$$\log\log U - \log\log L \le \frac{U-L}{L\log L}. \tag{3.2}$$

Proof. Apply $\log y \le y - 1$ first to $y = (\log U)/(\log L)$ and then to $y = U/L$. Division by the positive quantities L and $\log L$ gives (3.2). This proof applies whether U lies to the left or to the right of L. □

Proposition 3.2 (Tangent transfer)**.** *Let $n, q > 5040$. Suppose that*

$$\log Q_{\lambda(n)}(n) \le \log Q_{\lambda(n)}(q). \tag{3.3}$$

If Robin's inequality holds at q, then it holds at n.

Proof. Writing $L = H_n$ and $U = H_q$, inequality (3.3) becomes

$$\log \mathcal{A}(n) - \lambda(n)L \le \log \mathcal{A}(q) - \lambda(n)U.$$

By Lemma 3.1, $\phi(U) - \phi(L) \le \lambda(n)(U-L)$. Adding the two inequalities yields

$$\log \mathcal{A}(n) - \phi(L) \le \log \mathcal{A}(q) - \phi(U).$$

After subtracting γ, the left- and right-hand sides are precisely $\mathcal{M}(n)$ and $\mathcal{M}(q)$. Hence $\mathcal{M}(q) < 0$ implies $\mathcal{M}(n) < 0$. □

The remaining point is to ensure that a maximizer at the tangent parameter is itself above the exceptional cutoff. The exact startup transition is particularly useful. Let

$$\varepsilon_0 = \frac{\log(12/11)}{\log 11}. \tag{3.4}$$

The event $(11,1)$ changes 5040 into $55440 = 11 \cdot 5040$ and has threshold ε_0.

Lemma 3.3 (The startup tie)**.** *Both* 5040 *and* 55440 *maximize Q_{ε_0}. Moreover*

$$\frac{1}{34} < \varepsilon_0 < \frac{1}{22}. \tag{3.5}$$

Proof. The factorization $5040 = 2^4 3^2 5 \cdot 7$ determines its included and first excluded events. For the four included top events one checks $\tau_{p,a_p} > \varepsilon_0$, and for the first excluded event at each of $p = 2,3,5,7$ one checks $\tau_{p,a_p+1} < \varepsilon_0$. Every prime $p \ge 11$ is bounded by its first event, with equality only for $(11,1)$; monotonicity in j handles all higher events. Equation (2.5) then shows that the event packet for 5040 has at least as much reduced weight as every competitor. The tied event $(11,1)$ has weight zero, so adjoining it gives the second maximizer 55440. The bounds in (3.5) reduce, after exponentiation, to exact rational inequalities for powers of 12/11. □

Lemma 3.4. *If $n \geq 720720$ and q maximizes $Q_{\lambda(n)}$, then $q > 5040$.*

Proof. The function $\lambda(n)$ is nonincreasing above 5040. Exact logarithmic bounds give

$$\lambda(720720) < \frac{1}{34} < \varepsilon_0,$$

and hence $\lambda(n) < \varepsilon_0$. If $q \leq 5040$, maximality of 5040 at ε_0 and maximality of q at $\lambda(n)$ may be transported between the two parameters. Because 5040 and 55440 tie at ε_0, lowering the parameter strictly favors the larger logarithmic height 55440 over 5040, while no $q \leq 5040$ can gain that advantage. This contradicts the maximality of q at $\lambda(n)$. □

Proposition 3.5 (Reduction to colossally abundant integers)**.** *Assume Robin's inequality for $5040 < n < 720720$. Then the following are equivalent:*

(i) Robin's inequality holds for every $n > 5040$;
(ii) it holds for every colossally abundant $N > 5040$.

Proof. Only (ii)⇒(i) needs proof. Let $n \geq 720720$. By Lemma 2.2, choose a maximizer q for $Q_{\lambda(n)}$. Lemma 3.4 gives $q > 5040$, so (ii) gives Robin's inequality at q. Maximality gives (3.3), and Proposition 3.2 transfers the inequality to n. The assumed finite interval handles the remaining integers. □

4. The finite range

The analytic argument in Section 5 starts at $\log n = 74500$. The exact cutoff therefore requires a proof for all $5040 < n$ below that logarithmic height. We divide it into three pieces.

Proposition 4.1. *For $5041 \leq n < 7560$,*

$$35\sigma(n) \leq 127n. \tag{4.1}$$

Consequently Robin's inequality holds throughout this interval.

Proof. For each integer in the interval, an exact prime-power factorization is given. Pairwise coprimality and the identity $\sigma(p^a) = 1 + p + \cdots + p^a$ reduce (4.1) to natural-number arithmetic. To pass from (4.1) to Robin's inequality, use the exact bounds

$$e^\gamma > \frac{177}{100}, \qquad \log\log n > \frac{103}{50}, \qquad \frac{127}{35} < \frac{177}{100}\frac{103}{50}.$$

□

Proposition 4.2. *Robin's inequality holds for $7560 \leq n < 720720$.*

Proof. The maximality of 5040 at ε_0 gives

$$\log \mathcal{A}(n) - \varepsilon_0 \log n \leq \log \mathcal{A}(5040) - \varepsilon_0 \log 5040.$$

As a function of $y = \log n$, the difference between the Robin bound and the right-hand tangent is concave. A concave function on an interval is bounded below by the smaller endpoint value. Exact rational estimates show that the gap is positive at both $y = \log 7560$ and $y = \log 720720$. It is therefore positive throughout the interval. □

For the much larger interval above 720720, a compact rational certificate replaces integer-by-integer factorization. A certificate row consists of a log-height interval $[L, U]$, a rational slope λ, prime-layer cutoffs, an upper bound for the corresponding product, a lower height bound, and lower bounds for $\log L$ and $\log U$. The layer inequalities certify which prime-power events can have positive reduced weight. The product inequality bounds the maximum possible value of $\log \mathcal{A}(n) - \lambda \log n$. Finally, concavity reduces the Robin comparison on $[L, U]$ to two endpoint inequalities.

Lemma 4.3 (Soundness of a finite row)**.** *Suppose a row* $[L, U]$ *satisfies its layer-sign, product, height, and endpoint inequalities. If* $n > 5040$ *and* $L \le \log n \le U$*, then* $\mathcal{M}(n) < 0$.

Proof. The sign checks and (2.5) bound the actual factorization packet by the finite threshold box. The exact product bounds convert that box into an upper bound B for $\log \mathcal{A}(n) - \lambda \log n$. The two endpoint checks state

$$B < \gamma + \log\log L - \lambda L, \qquad B < \gamma + \log\log U - \lambda U.$$

Concavity of $y \mapsto \log\log y - \lambda y$ gives the same strict inequality at $y = \log n$. Rearrangement is $\mathcal{M}(n) < 0$. □

Proposition 4.4 (Certified finite cover)**.** *Robin's inequality holds whenever*

$$720720 \le n, \qquad \log n < 74500. \tag{4.2}$$

Proof. Thirty-six rational rows form a gap-free chain from 336/25 to 37283397387/500000, an endpoint strictly larger than 74500. Since $336/25 < \log 720720$, they cover (4.2). Every row satisfies the hypotheses of Lemma 4.3. In the final retained row the first-layer product includes every prime at most 70849; it is split into 105 contiguous blocks, whose exact products concatenate to the required prefix. All comparisons are integer or rational inequalities. Thus the list of rows and their checks constitute a finite proof of the proposition. □

Combining Propositions 4.1, 4.2, and 4.4 gives the following.

Theorem 4.5 (Finite completion)**.** *If* $5040 < n$ *and* $\log n < 74500$*, then Robin's inequality holds.*

In particular, Proposition 3.5 is now unconditional.

5. The implication under the Riemann hypothesis

Let

$$\vartheta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{p^k \le x} \log p.$$

The analytic estimate is most transparent after setting $H = \log n$ and

$$S(H) = H^{-1/2} (\log H)^{-1}. \tag{5.1}$$

We first record the weighted explicit-formula input. For $m \ge 1$, define

$$w_m(t) = t^{-m-1} \frac{m \log t + 1}{(\log t)^2}, \qquad I_m(x) = \int_x^\infty (\psi(t) - t) w_m(t)\, dt. \tag{5.2}$$

For a nontrivial zero ρ of ζ, counted with multiplicity, put

$$K_m(\rho, x) = \int_x^\infty t^{\rho - m - 1} \frac{m \log t + 1}{(\log t)^2}\, dt.$$

Lemma 5.1 (Robin's weighted explicit formula)**.** *Assume* RH. *For* $m \ge 1$ *and* $x \ge 2$,

$$I_m(x) = -\sum_\rho \frac{K_m(\rho, x)}{\rho} - T_m(x), \tag{5.3}$$

where $T_m(x)$ *is the correction from the pole, the gamma factor, and the trivial zeros. If*

$$c_0 = \gamma + 2 - \log(4\pi),$$

then

$$-B_m(x) - \frac{\log(2\pi) x^{-m}}{\log x} \le I_m(x) \le B_m(x), \tag{5.4}$$

where

$$B_m(x) = c_0 x^{1/2-m} (\log x)^{-1} \left(m + (\log x)^{-1} + \frac{4}{(2m-1)(\log x)^2} \right). \tag{5.5}$$

Proof. Apply the explicit formula to the continuous cutoff whose Mellin transform is $K_m(s,x)/s$. Absolute convergence justifies pairing the nontrivial zeros and interchanging their sum with the integral. Under RH every nontrivial zero has real part $1/2$, and Hadamard factorization of the completed zeta function gives the exact multiplicity-counted identity

$$\sum_\rho |\rho|^{-2} = \gamma + 2 - \log(4\pi) = c_0.$$

Two integrations by parts bound the paired zero kernel by the scalar in (5.5). Direct evaluation of the gamma and trivial-zero terms gives the one-sided correction in (5.4). This is Robin's Lemma 2, including the endpoint $m = 1$. □

Costa Pereira's estimates for $\psi - \vartheta$, with the corrected constants from [3], control the prime-power part separated from (5.3). Abel summation supplies the complementary lower bound for a complete block of prime squares.

Define the complete prime cost

$$\mathcal{C}(H) = \sum_{p \le H/2} \min\left\{ \frac{\log p}{H \log H}, \frac{1}{p^2} \right\}. \tag{5.6}$$

Also define

$$\begin{aligned} \mathcal{E}(H) = (2 + c_0)\frac{H^{-1/2}}{\log H} + (c_0 - 2)\frac{H^{-1/2}}{(\log H)^2} \\ + (8 + 4c_0)\frac{H^{-1/2}}{(\log H)^3} + 2\frac{H^{-2/3}}{\log H} + \log(2\pi)\frac{H^{-1}}{\log H}. \end{aligned} \tag{5.7}$$

Proposition 5.2 (Complete exponent budget)**.** *Assume* RH. *If $n \neq 0$ and $H = \log n \ge 20000$, then*

$$\mathcal{M}(n) \le \mathcal{E}(H) - \mathcal{C}(H). \tag{5.8}$$

Proof. For a prime p occurring in n with exponent a, the telescoping identity from Lemma 2.1 and $\log y \le y - 1$ give

$$\log \frac{1 - p^{-(a+1)}}{1 - p^{-1}} \le \frac{1}{p} - p^{-(a+1)} - m_p, \qquad m_p = \log(1 - p^{-1}) + p^{-1} \le 0. \tag{5.9}$$

Subtract $a \log p/(H \log H)$ from each prime contribution. Completing the sum to all primes $p \le H$ does not lose the omitted primes: if $a = 0$ and $p \le H/2$, the benchmark contribution pays at least the minimum in (5.6). Summation therefore yields a prime-reciprocal expression minus $\mathcal{C}(H)$.

Partial summation writes the prime-reciprocal expression as $\log \log H$ plus the Meissel–Mertens constant and a weighted ϑ-error. Replacing ϑ by ψ separates the nonnegative prime-power tail. Lemma 5.1 and the Costa Pereira bounds show that the remaining weighted error is at most (5.7). The Mertens constants cancel with the sum of the m_p, giving exactly (5.8). □

The numerical heart of the large-height proof is the following strict gap.

Proposition 5.3 (Explicit coefficient gap)**.** *Assume* RH. *For $H \ge 74500$,*

$$\mathcal{E}(H) < \frac{113}{50} S(H), \qquad \mathcal{C}(H) \ge \frac{113}{50} S(H). \tag{5.10}$$

Proof. For the first inequality, divide (5.7) by $S(H)$. The exact identity becomes

$$2 + c_0 + (c_0 - 2)(\log H)^{-1} + (8 + 4c_0)(\log H)^{-2} + 2H^{-1/6} + \log(2\pi)H^{-1/2}.$$

Uniform rational bounds valid for every $H \geq 74500$ control $(\log H)^{-1}$, $H^{-1/6}$, $H^{-1/2}$, c_0, and $\log(2\pi)$. Substitution reduces the assertion to an exact rational inequality.

For the second inequality, take the complete prime-square block between $\sqrt{2H}$ and $H/2$. Abel summation and the RH bounds for ϑ give a lower bound whose normalization by $S(H)$ is

$$\begin{aligned}
&2\sqrt{2}\frac{\log H}{\log H + \log 2}\left(1 - \frac{1}{\log H + \log 2}\right) \\
&- \frac{9}{4}\,2^{1/4}H^{-1/4}\frac{\log H}{\log H + \log 2} - \log(2\pi)H^{-1/2}\frac{\log H}{\log H + \log 2} \\
&- 4H^{-1/2}\frac{\log H}{\log H - \log 2} \\
&- 2\sqrt{2}\,c_0H^{-1}\frac{\log H}{\log H - \log 2}\left(2 + \frac{1}{\log H - \log 2} + \frac{4}{3(\log H - \log 2)^2}\right).
\end{aligned}$$

The last line retains the RH error term at the far endpoint instead of bounding the entire ϑ tail by a global multiple of $\log 4$. The nonnegative prime-power contribution is discarded only after this cancellation has been preserved. Exact rational bounds for $\sqrt{2}$, $2^{1/4}$, and the logarithmic ratios show that the result is at least $113/50$. □

Theorem 5.4 (The RH implication). *If* RH *holds, then Robin's inequality holds for every* $n > 5040$.

Proof. If $\log n < 74500$, use Theorem 4.5. Otherwise combine Propositions 5.2 and 5.3:

$$\mathcal{M}(n) < \left(\frac{113}{50} - \frac{113}{50}\right) S(\log n) = 0.$$

This is equivalent to Robin's inequality. □

6. The Nicolas–Landau oscillation

Define Nicolas's finite product and normalized function by

$$\mathfrak{P}(x) = \prod_{p \leq x}\left(1 - \frac{1}{p}\right), \qquad F(x) = e^{\gamma}\log\vartheta(x)\,\mathfrak{P}(x). \tag{6.1}$$

The signed quantity used below is $\log F(x)$.

Let

$$k(t) = \frac{(\log t)^{-1} + (\log t)^{-2}}{t^2},$$

and put

$$K(x) = \int_x^{\infty}(\vartheta(t) - t)k(t)\,dt, \qquad J(x) = \int_x^{\infty}(\psi(t) - t)k(t)\,dt. \tag{6.2}$$

For functions $g, h : \mathbb{R} \to \mathbb{R}$, write $g = \Omega_-(h)$ at infinity if there is $c > 0$ such that, for every X, some $x \geq X$ satisfies $g(x) \leq -ch(x)$.

Lemma 6.1 (Finite Nicolas comparison). *For all sufficiently large* x,

$$\log F(x) \leq J(x) + \frac{4}{x}. \tag{6.3}$$

Proof. Mertens' convergent summand series and partial summation give the exact decomposition of $\log F(x)$ into a summand remainder, a concave height remainder, and $K(x)$. The first is at most $4/x$, and $\log y \le y - 1$ makes the height remainder nonpositive. Finally $\psi - \vartheta \ge 0$ and $k(t) \ge 0$ imply $K(x) \le J(x)$. □

Proposition 6.2 (Nicolas–Landau negative oscillation)**.** *If* RH *is false, then there is* b *with*

$$0 < b < \frac{1}{2} \tag{6.4}$$

such that

$$J(x) = \Omega_-(x^{-b}). \tag{6.5}$$

Proof. Failure of RH, together with the functional equation and conjugation symmetry, gives a nontrivial zero ρ of ζ with $1/2 < \Re\rho < 1$. On its horizontal line choose a zero ρ_* with maximal real part. Choose

$$1 - \Re\rho_* < b < \frac{1}{2}.$$

Suppose (6.5) fails. Then, beyond some X,

$$J(x) + x^{-b} > 0. \tag{6.6}$$

Regard $x^{-1}(J(x)+x^{-b})\,dx$ on (X,∞) as a positive measure and pass to the logarithmic coordinate. Its Mellin transform becomes a moment-generating function. Landau's principle says that if such a positive transform has analytic continuation at its abscissa of convergence, then the true convergence domain crosses strictly to the right.

The Mellin transform of J is computed from the logarithmic derivative of ζ. Removing the pole at 1 gives an analytic numerator, and the added x^{-b} contributes the elementary term $X^{s-b}/(b-s)$. Compact and tail pieces provide analytic continuation up to the ray ending at $s = 1 - \rho_*$. Maximality of $\Re\rho_*$ keeps the continuation regular along the open ray, while the zero at ρ_* forces its norm to diverge at the endpoint. On the other hand, (6.6) and Landau's principle make the moment-generating function analytic there, hence locally bounded. This contradiction proves (6.5). □

The proof just given is the strict form of the oscillation argument used by Nicolas and Robin. Its quantifiers are important: the excursions occur arbitrarily far out, rather than at one selected numerical frontier.

Corollary 6.3. *If* RH *is false, there exists* $0 < b < 1/2$ *such that*

$$\log F(x) = \Omega_-(x^{-b}). \tag{6.7}$$

Proof. Use Lemma 6.1 and $x^{-1} = o(x^{-b})$, which holds because $b < 1$. □

7. Least common multiples and the converse

Let

$$L_P = \operatorname{lcm}(1, 2, \dots, P).$$

The classical identity $\log L_P = \psi(P)$ connects this finite integer with the analytic frontier. Define the logarithmic height loss

$$D(P) = \log\log\psi(P) - \log\log\vartheta(P). \tag{7.1}$$

The saturated Euler product differs from the actual abundancy of L_P by the nonnegative prime-power reserve

$$T(P) = \sum_{p \le P} -\log\left(1 - p^{-a_p - 1}\right), \qquad a_p = \max\{a : p^a \le P\}. \tag{7.2}$$

Lemma 7.1 (Exact LCM identity)**.** *For all sufficiently large P,*

$$\mathcal{M}(L_P) = -\log F(P) - D(P) - T(P). \tag{7.3}$$

Proof. The finite Euler product gives

$$\sum_{p \le P} -\log(1 - p^{-1}) = -\log \mathfrak{P}(P),$$

where $\mathfrak{P}(P)$ denotes the product in (6.1). The prime-power factorization of L_P gives

$$-\log \mathfrak{P}(P) = \log \mathcal{A}(L_P) + T(P).$$

Now expand $\log F(P)$ and use $\log L_P = \psi(P)$. Rearrangement is exactly (7.3). □

Lemma 7.2 (The transfer losses)**.** *For every $b < 1/2$,*

$$D(P) + T(P) = o(P^{-b}). \tag{7.4}$$

Proof. The elementary bound

$$0 \le \psi(P) - \vartheta(P) \le 2\sqrt{P} \log P$$

and a linear lower bound for $\vartheta(P)$ give

$$0 \le D(P) \ll \frac{\log P}{\sqrt{P}}.$$

Splitting the reserve at $\sqrt{P}$ and summing the geometric tails gives $T(P) \ll P^{-1/2}$. Since $P^{-1/2} \log P = o(P^{-b})$ whenever $b < 1/2$, the claim follows. □

Theorem 7.3 (The converse implication)**.** *If Robin's inequality holds for every $n > 5040$, then* RH *holds.*

Proof. Suppose RH is false. Choose b as in Corollary 6.3. Negative excursions of $\log F(P)$ give positive excursions of $-\log F(P)$ of order P^{-b}. By Lemma 7.2, subtracting $D(P) + T(P)$ preserves such excursions. Lemma 7.1 therefore gives arbitrarily large P with $\mathcal{M}(L_P) > 0$. Since $L_P \ge P$, these least common multiples eventually exceed 5040, and $\mathcal{M}(L_P) > 0$ is a literal failure of Robin's inequality. This contradicts the hypothesis. □

8. Proofs of the main theorems

Proof of Theorem 1.1. The implication RH ⇒(1.1) is Theorem 5.4. The reverse implication is Theorem 7.3. □

Proof of Theorem 1.2. If RH holds, Theorem 1.1 gives Robin's inequality for every integer above 5040, hence in particular for every colossally abundant one. Conversely, assume it for every colossally abundant integer above 5040. The finite startup theorem and Proposition 3.5 extend it to every integer above 5040. Theorem 1.1 then gives RH. □

9. Formal verification and reproducibility

The preceding proof has been formalized in Lean 4.33.1 using Mathlib [5, 6]. The two public declarations have the mathematical types

$$(\forall n \in \mathbb{N},\ 5040 < n \Rightarrow \sigma(n) < e^{\gamma} n \log\log n) \iff \text{RH},$$
$$\text{RH} \iff (\forall n, \varepsilon,\ 5040 < n \Rightarrow n \text{ is colossally abundant at } \varepsilon \Rightarrow \sigma(n) < e^{\gamma} n \log\log n).$$

They are named

`Robin1984.robin_inequality_iff_riemannHypothesis`
`Robin1984.riemannHypothesis_iff_colossallyAbundant_robin`.

The small Palomar challenge file fixes these statements using Mathlib only; the solution file supplies the proof terms, so statement comparison is independent of the internal proof graph.

The formal development separates arithmetic identities, colossally abundant extrema, the RH estimates, the Nicolas–Landau argument, finite certificates, and final assembly. Reusable order, logarithm, Mellin-transform, and prime-power lemmas are exposed in their mathematical families. Each owned Lean file carries a content-reviewed provenance label distinguishing direct source formalization, another published source, standard mathematics without a claimed individual originator, and primarily project-authored formalization work. The complete reviewed ledger is distributed with the repository.

The finite proof contains no floating-point oracle. The startup factorizations and the 36 interval rows reduce to exact natural-number and rational inequalities. The large final retained prime product is split into 105 blocks only to keep individual kernel computations manageable. The certificate checkers are proved sound once; each data row is then accepted by kernel reduction.

The repository contains no project-local axiom declarations. Outside the two deliberate statement holes in the Palomar challenge, project proof sources contain no `sorry`, `admit`, or `native_decide`. Diagnostic commands such as `#print axioms` occur neither in repository-owned Lean sources nor in the timed external import closure. A completed kernel audit of both advertised declarations returned exactly the standard principles

`propext`, `Classical.choice`, `Quot.sound`.

The ordinary `lake build` command is the full formalization build. Its Lake-native default target reads a checked-in topological order of 245 modules, schedules one module job at a time, bounds each Lean process to one internal task thread, and ends with `Solution`. Thus the submitted project itself controls the memory-intensive finite-certificate order; no external runner or local patching step is part of the proof build.

All dependencies are pinned. The required Lean 4.33.1 changes to `PrimeNumberTheoremAnd` are published in the public fork `https://github.com/kimihiro64/PrimeNumberTheoremAnd` at commit `f8f58c7`, based on upstream commit `47fa486`. The full hashes are recorded in the Lake manifest. Lake reconstructs this dependency directly from public Git history; no local patching step is required. The added multiplicity-aware xi-divisor source retains Matteo Cipollina's authorship and Apache-2.0 notice. The upstream project is due to Alex Kontorovich, Terence Tao, and its contributors.

The formalization was developed with human mathematical direction and AI-assisted proof engineering. Lean's kernel checks the resulting proof terms; that mechanical guarantee does not replace mathematical review of the chosen statements, source fidelity, or attribution.

Acknowledgements and attribution

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