A disorder-independent tanh law for parity statistics in binary linear codes under Glauber dynamics

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Abstract. We consider binary linear codes defined by a full-rank parity-check matrix $H \in \mathbb{F}_2^{m\times N}$ whose row space contains the all-ones vector, endowed with the Glauber (single-bit-flip Metropolis) dynamics at inverse temperature y on the energy $E(x) = |Hx|$, the Hamming weight of the syndrome. We prove that the total parity coordinate of the syndrome thermalizes as an isolated two-level system with unit gap: $P_{\mathrm{odd}}(y) = (1 - \tanh(y/2))/2 = 1/(1 + e^y)$, independently of every other detail of H. We also prove an exact shell identity: among the $\binom{m}{w}$ syndromes of weight w, the fraction with odd total parity is exactly w/m. Both results are verified by exhaustive enumeration of all 2^{13} syndromes of a concrete 13×137 code, agreeing with the closed forms to absolute deviations below 10^{-78} in 80-digit arithmetic. The law provides an exact, self-calibrating thermometer for simulated-annealing decoders and a rare example of an exact annealed identity outside the Nishimori line.

Keywords: binary linear codes, Glauber dynamics, syndrome ensemble, exact annealed identities, error-correcting codes, statistical mechanics of constraint satisfaction

1. Introduction

The statistical mechanics of error-correcting codes, initiated by Sourlas' mapping of decoding onto spin-glass Hamiltonians [1] and systematized by Nishimori [2, 3] and by MacKay's information-theoretic treatment [4], is rich in approximation schemes and poor in exact identities. The known exact results — the internal energy on the Nishimori line being the canonical example [3] — exploit a fine-tuned relation between the disorder distribution and the temperature. It is therefore of some interest to find exact identities that hold *away* from any fine-tuned line, for every temperature and for an entire class of codes at once.

In this note we report one such identity. In the syndrome-based formulation of Glauber dynamics for a binary linear code — energy equal to the number of violated parity checks — the total parity of the syndrome (equivalently, of the state) is a distinguished coordinate whenever the all-ones check is a valid parity check of the code. We show that this coordinate occupies its two values with probabilities $e^{-y}/(1+e^{-y})$ and $1/(1+e^{-y})$, exactly, for every $y \geq 0$, for every full-rank parity-check matrix containing the all-ones row, and independently of N, of the column weights, and of the code's distance structure. The parity coordinate is, in effect, an exact thermometer embedded in the code.

The result emerged from a study of constraint-satisfaction dynamics on a specific 13×137 binary matrix arising in a graph-theoretic model of the present author (the AHUM framework); we emphasize, however, that the theorems and their proofs are entirely general, and the specific matrix serves here only as a non-trivial worked example whose full syndrome space (8192 elements) can be enumerated exactly.

2. Setup

Let $H \in \mathbb{F}_2^{m\times N}$ be a parity-check matrix of rank m over $\mathbb{F}_2$ (surjective syndrome map), with the property that its last row is the all-ones vector $\mathbf{1}^T$. States are vectors $x \in \mathbb{F}_2^N$; the syndrome is $s = Hx \in \mathbb{F}_2^m$; the energy is the syndrome weight

$$E(x) = |s| = \sum_{r=1}^{m} s_r. \tag{1}$$

The Glauber (single-spin-flip Metropolis [5]) dynamics at inverse temperature y proposes to flip a uniformly chosen bit i and accepts with probability $\min(1, e^{-y\Delta E})$, where $\Delta E = |s \oplus H_{:i}| - |s|$ and $H_{:i}$ is the i-th column of H. The stationary distribution over states is

$$\pi(x) = \frac{e^{-y|Hx|}}{Z}, \qquad Z = \sum_{x\in\mathbb{F}_2^N} e^{-y|Hx|}. \tag{2}$$

Because H is surjective, each of the 2^m syndromes has the same number 2^{N-m} of preimages, so the induced distribution on syndromes is

$$P(s) = \frac{e^{-y|s|}}{(1+e^{-y})^m}. \tag{3}$$

The coordinate of interest is the *total parity* $s_m = \bigoplus_i x_i$ (since the last row of H is $\mathbf{1}^T$), and we write $P_{\rm odd}(y) = P(s_m = 1)$.

3. The tanh law

Theorem 1 (disorder-independent parity law) *For every full-rank $H \in \mathbb{F}_2^{m\times N}$ with last row $\mathbf{1}^T$, and for the stationary distribution (2) of the Glauber dynamics with energy (1),*

$$P_{\rm odd}(y) = \frac{e^{-y}}{1+e^{-y}} = \frac{1-\tanh(y/2)}{2} \qquad \forall\, y \geq 0, \tag{4}$$

independently of N, of m, and of all column weights of H.

Proof. Condition on the first $m-1$ syndrome coordinates $u = (s_1, \ldots, s_{m-1}) \in \mathbb{F}_2^{m-1}$. By (3), the parity coordinate splits the normalization as

$$Z_{\rm odd} = \sum_{u\in\mathbb{F}_2^{m-1}} e^{-y(|u|+1)} = e^{-y}(1+e^{-y})^{m-1}, \qquad Z = (1+e^{-y})^m. \tag{5}$$

The ratio is $P_{\rm odd} = Z_{\rm odd}/Z = e^{-y}/(1+e^{-y})$. Elementary algebra gives $e^{-y}/(1+e^{-y}) = (1-\tanh(y/2))/2$. □

Two remarks are in order. First, the proof uses only the factorization of the syndrome distribution over the parity coordinate; the 2^{N-m} preimages per syndrome never enter. The identity is thus not an annealed approximation but an exact partition of the state space. Second, the parity coordinate carries the thermodynamics of an isolated binary degree of freedom with unit energy gap: measuring the parity-odd fraction of a decoder's trajectory reads off the effective temperature with no calibration constants, since inverting (4) gives

$$y = \ln\left(\frac{1-P_{\rm odd}}{P_{\rm odd}}\right). \tag{6}$$

4. The shell identity

Theorem 2 (odd fraction per syndrome shell) *Among the $\binom{m}{w}$ syndromes of weight w, the fraction with odd total parity is exactly*

$$f_{\rm odd}(w) = \frac{\binom{m-1}{w-1}}{\binom{m}{w}} = \frac{w}{m}. \tag{7}$$

Proof. A syndrome of weight w with odd parity is determined by choosing the remaining $w-1$ ones among the first $m-1$ coordinates; hence the count $\binom{m-1}{w-1}$, and the ratio simplifies to w/m. □

The shell identity is temperature-free and holds uniformly: the odd fraction within each weight shell is the shell weight per row. Under the tilted ensemble (3), summing (7) over shells with weights $\binom{m}{w}e^{-yw}$ recovers Theorem 1.

5. Worked example and exhaustive verification

As a non-trivial instance we take the 13×137 binary matrix Φ of the AHUM graph-theoretic construction: 318 ones, rank 13 over $\mathbb{F}_2$ (kernel dimension 124), last row equal to $\mathbf{1}^T$, rows built from residue classes modulo 15 and modulo 5. The syndrome space has $2^{13} = 8192$ elements and was enumerated exhaustively: for each syndrome s we computed its weight and parity exactly, accumulated the tilted weights $e^{-y|s|}$ in 80-digit arithmetic, and formed $P_{\text{odd}}^{\text{enum}}(y)$. Table 1 reports the comparison with the closed form (4); figure 1 displays the law against the enumerated points. The agreement, better than 10^{-78} at all four temperatures, is a numerical certificate of the proof rather than a fit: there are no free parameters on either side.

Table 1. Exhaustive enumeration of all $2^{13} = 8192$ syndromes of the 13×137 code vs. the closed form (4), in 80-digit arithmetic.

y	$P_{\text{odd}}^{\text{enum}}(y)$	$(1 - \tanh(y/2))/2$	$\lvert\text{difference}\rvert$
0.5	0.377540668798145	0.377540668798145	7.9×10^{-80}
1.0	0.268941421369995	0.268941421369995	2.1×10^{-79}
2.0	0.119202922022118	0.119202922022118	4.6×10^{-79}
5.0	0.00669285092428486	0.00669285092428486	1.3×10^{-78}

The shell identity (7) was likewise checked shell by shell: for $m = 13$ and each $w = 1, \ldots, 13$, the enumerated odd fraction equals $w/13$ exactly (e.g. $f_{\text{odd}}(6) = 6/13 = 0.461538\ldots$).

6. Discussion

Universality. The hypotheses of Theorem 1 — full rank and the presence of the all-ones check — are structural and mild. Any code augmented by the overall parity check (a standard construction, the *extended* code in the sense of Gallager-type enlargements [6]) satisfies them. The law therefore applies to extended LDPC ensembles, extended Hamming and BCH codes, and to any constraint-satisfaction instance whose check matrix includes the total parity.

An exact thermometer. Equation (6) turns the parity-odd fraction of any sufficiently ergodic trajectory into a calibration-free estimate of the inverse temperature actually felt by the dynamics — a practical diagnostic for simulated-annealing and population-annealing decoders, where nominal and effective temperatures can drift apart. Because the law is exact at every y, the estimator is unbiased up to equilibration error alone.

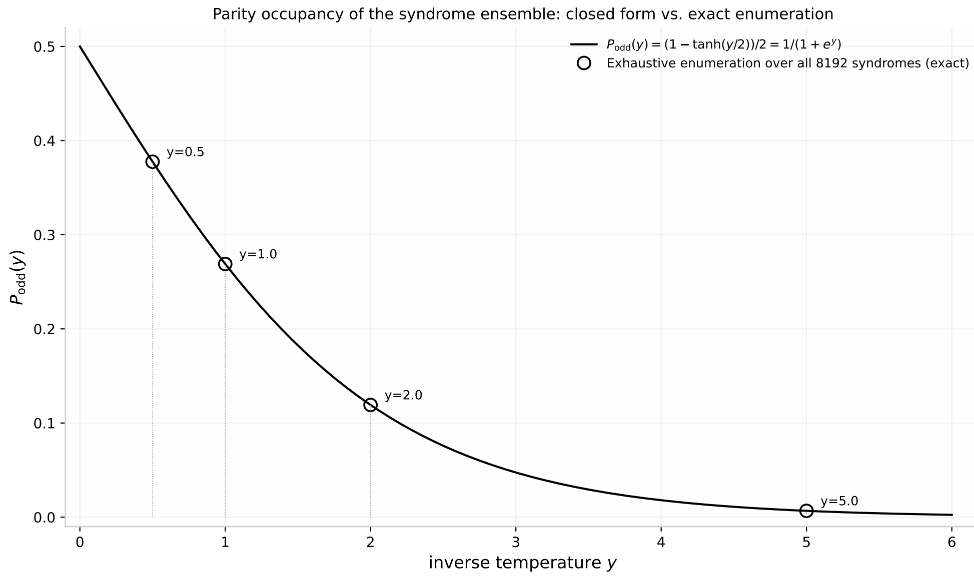

Figure 1. The parity law (4) (solid line) against the exactly enumerated parity occupancy of the syndrome ensemble of the 13×137 code (open circles). The deviations at the four marked temperatures are below 10^{-78} in 80-digit arithmetic (table 1).

Relation to the Nishimori line. The classic exact identities of spin-glass information theory [2, 3] hold only on the Nishimori manifold, where temperature tracks the noise strength. Theorem 1 is complementary: it holds on no special line, involves no quenched average, and its price is that it controls only one bit of information — the total parity — rather than extensive quantities.

Limitations. Full rank is essential (the equal-preimage property fails otherwise), and the all-ones row cannot be replaced by an arbitrary row without changing the right-hand side of (4). Extending the identity to sub-extensive parity sectors (e.g. parity of subsets of checks) is open.

7. Conclusion

We have proved and exhaustively verified that the total parity of the syndrome ensemble of a binary linear code obeys the exact law $P_{\mathrm{odd}}(y) = (1 - \tanh(y/2))/2$ under Glauber dynamics, independently of the code's structure, and that syndrome shells of weight w carry the exact odd fraction w/m. Beyond its intrinsic interest as an exact annealed identity, the law equips any parity-augmented code with a built-in thermometer for annealing-based decoding.

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References

[1] Sourlas N 1989 Spin-glass models of error-correcting codes *Nature* **339** 693–695

[2] Nishimori H 1980 Exact results and critical properties of the Ising model with competing interactions *J. Phys. C: Solid State Phys.* **13** 4071–4076

[3] Nishimori H 2001 *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford: Oxford University Press)

[4] MacKay D J C 2003 *Information Theory, Inference, and Learning Algorithms* (Cambridge: Cambridge University Press)

[5] Metropolis N, Rosenbluth A W, Rosenbluth M N, Teller A H and Teller E 1953 Equation of state calculations by fast computing machines *J. Chem. Phys.* **21** 1087–1092

[6] Gallager R G 1962 Low-density parity-check codes *IRE Trans. Inf. Theory* **8** 21–28