Prime-Star Localization and Second-Order Spectra of Finite Divisibility Cover Graphs

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Abstract

Let G_X be the graph on the positive integers $n \le X$ in which n is joined to np whenever p is prime and $np \le X$. Let A_X be its adjacency matrix, let $\lambda_1(A_X)$ be its spectral radius, and let $\pi(X)$ count the primes up to X. Then

$$\lambda_1(A_X)^2 = \pi(X) + 4\sum_{p\le X}\frac{1}{p-1} + o(1).$$

More generally, delete any fixed finite set S of prime directions, let $a_1 < a_2 < \cdots$ list the retained positive integers, and let π_S count the retained primes. For every fixed j, the j-th largest positive eigenvalue satisfies

$$\lambda_j(A_{S,X})^2 = \pi_S(X/a_j) + 4\sum_{\substack{p\le X\\ p\notin S}}\frac{1}{p-1} + c(a_j) + o_{S,j}(1),$$

where $c(a) = \sum_{q|a,\ q \text{ prime}}(q-5)/(q-1)$. The proof splits the edges according to whether their prime label exceeds $\sqrt{X}$. The large-prime edges form disjoint stars, while the remaining adjacency has norm of order $X^{1/4}/\sqrt{\log X}$ but zero first-order matrix element on each star eigenvector. This produces an explicit finite prime-sum correction with error $O_{S,j}(1/\log X)$ and localization of the extreme eigenvectors. The prime number theorem and Mertens estimates evaluate the finite correction as $4\log\log X$ plus an explicit constant. The same constant-layer formula and localization hold uniformly for retained centres $a \le (\log X)^{1/2-\varepsilon}$, with finite-sum error $O_{S,\varepsilon}(a^2/\log X)$. The paired positive and negative modes also yield uniform centre–shell adjacency evolution.

1. Introduction

For $X \ge 1$, let G_X have vertex set $\{1, \dots, \lfloor X\rfloor\}$, with an edge between n and np whenever p is prime and $np \le X$. Thus an edge multiplies or divides by one prime, so G_X is the Hasse diagram of the divisibility poset on this finite set. Let A_X denote its adjacency matrix. Every vertex is joined to 1 by successively removing prime factors, so G_X is connected.

Split the edges according to whether their prime label is greater than $\sqrt{X}$. For each centre $a < \sqrt{X}$, the large-prime neighbours are ap with $\sqrt{X} < p \le X/a$; no integer at most X contains two distinct prime divisors larger than $\sqrt{X}$, so these components are disjoint stars of degree $\pi(X/a) - \pi(\sqrt{X})$. Each such star has two nonzero eigenvalues, $\pm\sqrt{\pi(X/a) - \pi(\sqrt{X})}$.

The remaining edges have prime labels at most $\sqrt{X}$. Their adjacency matrix has norm of order $X^{1/4}/\sqrt{\log X}$, but its matrix element on the positive eigenvector of any fixed large-prime star is exactly zero: no small-prime edge has both endpoints in that star. The corresponding eigenvalue therefore has no first-order displacement, and its leading correction is quadratic in the small-prime interaction. Figure 1 records the split on G_{16}.

The undeleted headline has the elementary finite-prime form

$$\boxed{\lambda_1(A_X)^2 = \pi(X) + 4\sum_{p\le X}\frac{1}{p-1} + o(1).}$$

For the full statement, fix a finite set S of primes. Remove every integer divisible by a prime in S, and write $G_{S,X}$ and $A_{S,X}$ for the resulting graph and adjacency matrix. Let

$$\pi_S(x) = \#\{p \le x : p \notin S\},$$

and let $a_1 < a_2 < \cdots$ enumerate the positive integers not divisible by primes in S. We order the positive eigenvalues of $A_{S,X}$ as $\lambda_1 \ge \lambda_2 \ge \cdots$, counting multiplicity. Throughout the fixed-mode results, S and j are fixed before $X \to \infty$.

Let B_1 be the prime Meissel–Mertens constant, characterized by

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B_1 + o(1),$$

and define

$$\kappa_S = B_1 - \sum_{p\in S}\frac{1}{p} + \sum_{p\notin S}\frac{1}{p(p-1)}, \qquad c(a) = \sum_{\substack{q\mid a\\ q \text{ prime}}}\frac{q-5}{q-1}.$$

Since $(p-1)^{-1} = p^{-1} + [p(p-1)]^{-1}$, Mertens' theorem gives

$$4\sum_{\substack{p\le X\\ p\notin S}}\frac{1}{p-1} = 4\log\log X + 4\kappa_S + o_S(1).$$

Thus the finite-prime headline above is the $S = \emptyset$, $j = 1$ case of the following fixed-mode theorem: for every fixed j,

$$\boxed{\lambda_j(A_{S,X})^2 = \pi_S(X/a_j) + 4\log\log X + 4\kappa_S + c(a_j) + o_{S,j}(1).}$$

A prime factor q of the centre lowers the constant when $q < 5$, raises it when $q > 5$, and leaves it unchanged when $q = 5$. Multiplicities do not matter: $c(a)$ depends only on the distinct prime divisors of a.

The proof gives three further pieces of information. First, if $H_{S,X}$ denotes the adjacency formed by labels at most $\sqrt{X}$, then

$$\|H_{S,X}\| \asymp_S \frac{X^{1/4}}{\sqrt{\log X}}.$$

Second, the eigenvector associated with a fixed a_j localizes on its large-prime star at rate $O_{S,j}(X^{-1/4})$. Third, before using the prime number theorem and Mertens estimates, the squared eigenvalue correction is an explicit finite sum over the neighbouring stars reached by multiplying or dividing a_j by a small prime, with error $O_{S,j}(1/\log X)$. Section 8 derives this finite identity, and its evaluation gives the displayed constant-order formula. The paired positive and negative modes also give an adjacency-evolution corollary: a state begun at a fixed centre oscillates, uniformly in time, between that centre and the normalized shell of its large-prime leaves. The explicit finite remainder determines the phase of this oscillation for times $o(\sqrt{X \log X})$.

The same formula remains uniform when the centre varies with X: for every fixed $\varepsilon > 0$, it holds for all retained

$$a \le (\log X)^{1/2-\varepsilon}.$$

At the preceding finite-sum stage the uniform spectral error is $O_{S,\varepsilon}(a^2/\log X)$. Consequently the entire growing block of positive eigenvalues indexed by centres in this window is eventually simple and ordered by those centres. The fixed-mode localization and centre–shell evolution also extend uniformly to this block, with error $O_{S,\varepsilon}(a^{3/2}X^{-1/4})$ at centre a. The fixed-centre formula and the uniform logarithmic-window constant layer are also formalized in Lean 4; the exact formal scope and reproducibility information are recorded after Section 10.

Other arithmetic graph constructions retain broader divisibility relations or restrict the vertices to the divisors of one integer [4,5]. Cover-matrix spectra have been computed for structured product posets by Anđelić and da Fonseca [6]. Graph-index changes under local perturbations have been studied by Rowlinson [7] and by Dalfó, Fiol, and Garriga [8]. Sparse random graphs also exhibit extreme eigenvalues governed by high-degree vertices and eigenvectors localized near them [1,2]. Frieze and Tsourakakis obtain such a law from a star-forest decomposition in a random Apollonian model [3]. The supremum norm and inverse participation ratio used below make this deterministic localization quantitative; compare the localized edge eigenvectors in [1,15]. Continuous-time adjacency evolution on exact stars, and transfer from a vertex to a superposition over a vertex subset, are classical [16,17]; the adjacency-evolution corollary below shows that the coherent centre–shell motion persists in the full arithmetic graph. If Δ denotes the maximum degree, the classical induced-star argument gives $\lambda_1 \ge \sqrt{\Delta}$, while Hong's bound gives $\lambda_1 \le \sqrt{2m - n + 1}$ for a connected graph with n vertices and m edges [13]. For the present graph family the first bound is asymptotically sharp, whereas the second is larger than the true scale by a factor of order $\sqrt{\log X \log\log X}$.

The closest arithmetic-matrix lineage begins with the Redheffer matrix. Barrett, Forcade, and Pollington proved that its spectral radius is asymptotic to $\sqrt{X}$ [18], and Barrett and Jarvis obtained precise expansions for its two dominant eigenvalues [19]. Kline then introduced an $O(X)$-nonzero descendant with the same determinantal representation [20]. Writing $P(i)$ for the largest prime divisor of $i > 1$, Kline's matrix $\mathcal{R}_X$ has

$$(\mathcal{R}_X)_{ij} = 1 \quad \Longleftrightarrow \quad i = j, \quad i = 1, \quad \text{or} \quad (i \text{ is squarefree and } i/j = P(i)).$$

Thus $\mathcal{R}_X$ is nonsymmetric, has unit diagonal and a distinguished first row, and otherwise keeps only the largest-prime arrow from each squarefree row. It has $(2 + 6/\pi^2)X + o(X)$ nonzero entries. By contrast, A_X is the symmetric, zero-diagonal cover adjacency, keeps every prime-labelled cover edge, and has order $X \log\log X$ nonzero entries. Kline proved [21] that the two dominant eigenvalues of $\mathcal{R}_X$ satisfy

$$r_\pm = \pm\sqrt{\pi(X)} + \frac{1}{2}\log\log X + O(1).$$

The spectrum of A_X is exactly symmetric about zero, and the present result gives

$$\lambda_1(A_X) = \sqrt{\pi(X)} + \frac{2\log\log X + 2\kappa}{\sqrt{\pi(X)}} + o\left(\frac{1}{\sqrt{\pi(X)}}\right).$$

The different next-order scales are already visible at $X = 10^7$, where $\pi(X) = 664579$. Kline reports $r_+ = 817.6477$, while the full cover adjacency gives $\lambda_1 = 815.226460312$:

matrix	positive excess above $\sqrt{\pi(X)}$	squared excess above $\pi(X)$
Kline's $\mathcal{R}_X$	2.4305	3968.76
cover adjacency A_X	0.009311	15.1816

Asymptotically, Kline's squared excess has order $\sqrt{\pi(X)}\log\log X$, whereas the cover graph has squared excess $4\log\log X + 4\kappa + o(1)$. These are expansions of different operators, not competing estimates for one matrix. Together the results form a natural sequence: the Redheffer radius $\sqrt{X}$, its two-term dominant spectrum, Kline's prime-sparse $\sqrt{\pi(X)}$ spectrum, and the constant-order squared expansion for the symmetric cover adjacency. In the last case the correction is generated by the large-prime star boundary and its first-return resolvents.

The proof uses weighted Schur tests and Weyl inequalities [11], isolated eigenvalue perturbation [9,10], the Feshbach–Schur complement [12], and the prime number theorem and Mertens estimates [14]. Section 2 records the isolated-mode comparison. Sections 3–4 define the arithmetic graph and its large-prime star forest, Section 5 proves the sharp small-prime norm, and Sections 6–7 establish the fixed-mode spectral comparison and localization. Section 8 derives and evaluates the finite remainder. Section 9 proves the uniform logarithmic-window form.

2. A standard isolated-mode comparison

We record the perturbative mechanism separately from its arithmetic realization. It is a finite-dimensional form of standard isolated-eigenvalue perturbation and the Feshbach–Schur map; see Kato [10], Davis–Kahan [9], and Dusson–Sigal–Stamm [12].

Proposition 2.1 — Vanishing first order forces second order

Let L and H be Hermitian matrices on a finite-dimensional Hilbert space. Suppose

$$Lu = \mu u, \qquad \|u\| = 1,$$

μ is a simple eigenvalue of L, and, with $Q = I - uu^*$,

$$\operatorname{dist}(\mu, \sigma(QLQ)) \geq \gamma > 0.$$

If

$$\langle u, Hu\rangle = 0, \qquad \|H\| \leq \frac{\gamma}{4},$$

then the eigenvalue λ of $L + H$ matched to μ is simple and satisfies

$$|\lambda - \mu| \leq 2\frac{\|Hu\|^2}{\gamma}.$$

If v is its unit eigenvector, with phase chosen so that $\langle u, v\rangle \geq 0$, then

$$\|v - u\| \leq 3\frac{\|Hu\|}{\gamma}.$$

Proof

Let $I = (\mu - \gamma/2, \mu + \gamma/2)$. The matrix L has exactly one eigenvalue in I, namely μ. Weyl's inequality places one eigenvalue of $L + H$ within $\gamma/4$ of μ, while every other eigenvalue remains at distance at least $3\gamma/4$ from μ. Thus $L + H$ has exactly one eigenvalue in I; this is the matched eigenvalue λ, and it is simple.

The orthogonal decomposition $\mathbb{C}u \oplus u^{\perp}$ gives

$$L + H = \begin{pmatrix} \mu & b^* \\ b & C \end{pmatrix}, \qquad b = QHu, \qquad C = Q(L + H)Q.$$

Weyl's inequality places the matched eigenvalue within $\|H\|$ of μ, while the spectrum of C lies within $\|H\|$ of $\sigma(QLQ)$. Hence

$$\operatorname{dist}(\lambda, \sigma(C)) \geq \gamma - 2\|H\| \geq \frac{\gamma}{2}.$$

The one-dimensional Schur complement is therefore exact:

$$\lambda - \mu = b^*(\lambda - C)^{-1}b.$$

Since $\|b\| \leq \|Hu\|$, the eigenvalue estimate follows. Writing $v = \alpha u + w$, with $w \perp u$, the lower block equation gives

$$w = (\lambda - C)^{-1} b\, \alpha.$$

Thus $\|w\|/|\alpha| \leq 2\|Hu\|/\gamma \leq 1/2$. Normalization and the chosen phase imply $\|v - u\| \leq 3\|Hu\|/\gamma$. □

In the arithmetic graph below, L is the large-prime star forest and H is the small-prime adjacency. No small-prime edge has both endpoints in the support of one star mode, so the graph-specific identity

$$\langle u_a, H_{S,X} u_a \rangle = 0$$

supplies the hypothesis that removes the first-order displacement.

3. The finite prime-cover graph

Put

$$P_S = \prod_{p \in S} p, \qquad V_S(X) = \{n \leq X : (n, P_S) = 1\}.$$

Write

$$\pi_S(y) = \#\{p \leq y : p \notin S\}.$$

The graph $G_{S,X}$ has vertex set $V_S(X)$ and undirected edges

$$n \longleftrightarrow np$$

whenever $p \notin S$ is prime and $np \leq X$. Let $A_{S,X}$ be its adjacency matrix. Every $n \in V_S(X)$ factors entirely over primes outside S, so removing its prime factors gives a path to 1; hence $G_{S,X}$ is connected. The coloring

$$n \longmapsto (-1)^{\Omega(n)}$$

makes the graph bipartite, so its spectrum is symmetric about zero.

Split the adjacency at the radial square-root boundary:

$$A_{S,X} = L_{S,X} + H_{S,X},$$

where $L_{S,X}$ contains prime labels $p > \sqrt{X}$, and $H_{S,X}$ contains labels $p \le \sqrt{X}$. For every positive integer n, define

$$d_{S,X}(n) = \left(\pi_S(X/n) - \pi_S(\sqrt{X})\right)_+, \qquad (x)_+ = \max\{x, 0\}.$$

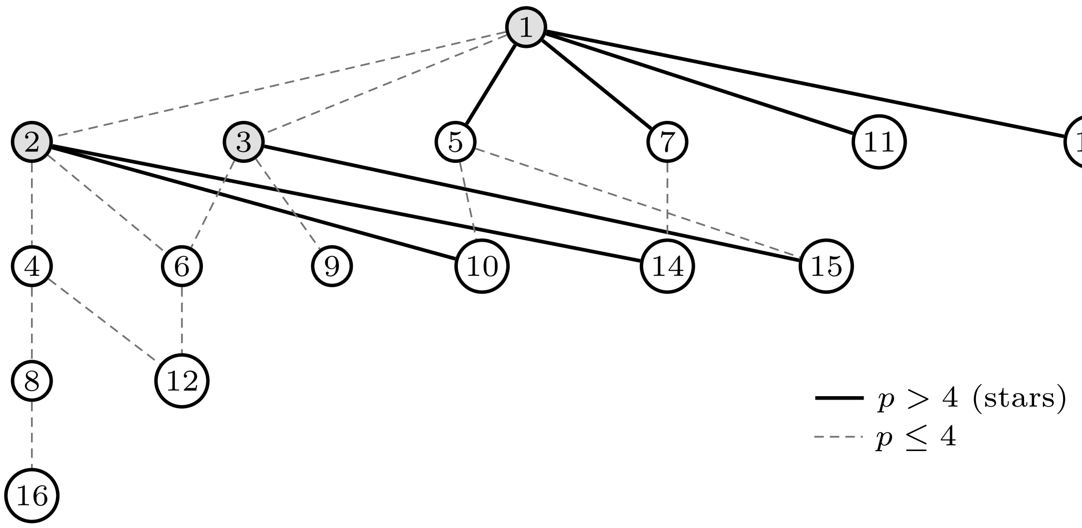

Figure 1: The prime-cover graph G_{16}. Solid edges have prime label $p > 4 = \sqrt{16}$ and form a disjoint star forest, with nonempty stars at 1, 2, and 3, shown shaded. Dashed edges have label $p \le 4$. No dashed edge has both endpoints in one large-prime star.

4. The exact large-prime star forest

Proposition 4.1

The graph with adjacency $L_{S,X}$ is a disjoint union of stars. Its nonempty star centered at $n < \sqrt{X}$ has leaves

$$\mathcal{L}_{n,X} = \{np : \sqrt{X} < p \le X/n,\ p \notin S\}$$

and degree $d_{S,X}(n)$. Its two nonzero eigenvalues are $\pm\sqrt{d_{S,X}(n)}$, with normalized eigenvectors

$$u^{\pm}_{n,X} = \frac{1}{\sqrt{2}}\left(\delta_n \pm \frac{1}{\sqrt{d_{S,X}(n)}} \sum_{m \in \mathcal{L}_{n,X}} \delta_m\right).$$

Proof

An edge labelled by $p > \sqrt{X}$ has lower endpoint below $\sqrt{X}$ and upper endpoint above it. A number at most X cannot have two distinct prime divisors exceeding $\sqrt{X}$. Hence a large-prime leaf has a unique center and cannot itself be a center. The components are stars, and the spectral formulas are those of $K_{1,d}$. $\square$

Because $d_{S,X}(n)$ is nonincreasing in n, the j-th positive eigenvalue of $L_{S,X}$ is $\sqrt{d_{S,X}(a_j)}$ whenever this degree is positive.

5. The sharp norm of the small-prime graph

Theorem 5.1

For every fixed finite S,

$$\|H_{S,X}\| \asymp_S \frac{X^{1/4}}{\sqrt{\log X}}.$$

Proof

Write $Y = \sqrt{X}$. The principal subgraph on

$$\{1\} \cup \{p \leq Y : p \notin S\}$$

is $K_{1,\pi_S(Y)}$. Therefore

$$\|H_{S,X}\| \geq \sqrt{\pi_S(Y)} \asymp \frac{X^{1/4}}{\sqrt{\log X}}.$$

For the upper bound, take the positive weight $w_\theta(n) = n^{-\theta}$. The weighted Schur row ratio is

$$\frac{(H_{S,X} w_\theta)(n)}{w_\theta(n)} \leq \sum_{p \leq Y} p^{-\theta} + \sum_{\substack{p \mid n \\ p \leq Y}} p^{\theta}.$$

Uniformly for θ in a fixed neighborhood of $1/2$, partial summation and the Chebyshev bound give

$$\sum_{p \leq Y} p^{-\theta} \ll \frac{Y^{1-\theta}}{\log Y}.$$

In the incoming sum, prime divisors at most $Y^{1/2}$ contribute at most

$$2 \log_2 Y \, Y^{\theta/2} = o(Y^\theta).$$

At most three distinct prime divisors of $n \leq Y^2$ exceed $Y^{1/2}$; four would have product greater than Y^2. Hence

$$\sum_{\substack{p \mid n \\ Y^{1/2} < p \leq Y}} p^\theta \leq 3Y^\theta,$$

so the incoming sum is $O(Y^\theta)$. With

$$\theta = \frac{1}{2} - \frac{\log\log Y}{2\log Y},$$

both bounds have order $Y^{1/2}/\sqrt{\log Y}$. The weighted Schur test proves the upper bound. □

The dependence of θ on X is essential to this proof. The fixed choice $\theta = 1/2$ loses the factor $\sqrt{\log X}$.

6. Extreme eigenvalues and localization

Weyl's inequality immediately gives, whenever $d_{S,X}(a_j) > 0$,

$$\left|\lambda_j(A_{S,X}) - \sqrt{d_{S,X}(a_j)}\right| \le \|H_{S,X}\| \ll \frac{X^{1/4}}{\sqrt{\log X}}.$$

For fixed j, the prime number theorem gives

$$d_{S,X}(a_j) \sim \frac{X}{a_j \log X}.$$

The distinct fixed-center star eigenvalues are therefore eventually simple and separated by

$$\gamma_{j,X} \asymp_{S,j} \sqrt{X/\log X}.$$

In the form $\sin\angle(v^+_{j,X}, u^+_{a_j,X}) \le \|H_{S,X}\|/\gamma_{j,X}$, Davis–Kahan and Theorem 5.1 give the eigenvector estimate

$$\|v^+_{j,X} - u^+_{a_j,X}\|_2 = O_{S,j}(X^{-1/4}).$$

Let $J\delta_n = (-1)^{\Omega(n)}\delta_n$. Then $JA_{S,X}J = -A_{S,X}$. After fixing the sign of $v^+_{j,X}$ as above, put

$$v^-_{j,X} = (-1)^{\Omega(a_j)} J v^+_{j,X}.$$

It is a unit eigenvector of eigenvalue $-\lambda_j(A_{S,X})$, and

$$\|v^-_{j,X} - u^-_{a_j,X}\|_2 = O_{S,j}(X^{-1/4}).$$

Corollary 6.1 — Hybrid hub–shell localization

For each fixed j and either sign,

$$\boxed{\|v^\pm_{j,X}\|_\infty = \frac{1}{\sqrt{2}} + O_{S,j}(X^{-1/4})}$$

and, with

$$\mathrm{IPR}(v) = \sum_n |v(n)|^4,$$

$$\boxed{\mathrm{IPR}(v^\pm_{j,X}) = \frac{1}{4} + O_{S,j}(X^{-1/4}).}$$

If $P_{a_j,X}$ is the coordinate projection onto $\{a_j\} \cup \mathcal{L}_{a_j,X}$, then

$$\boxed{\|(I-P_{a_j,X})v^{\pm}_{j,X}\|_2^2 = O_{S,j}(X^{-1/2}).}$$

Proof

Put $d = d_{S,X}(a_j)$. The exact star mode satisfies

$$\|u^{\pm}_{a_j,X}\|_\infty = \frac{1}{\sqrt{2}}, \qquad \mathrm{IPR}(u^{\pm}_{a_j,X}) = \frac{1}{4} + \frac{1}{4d}.$$

For unit vectors x, y, the reverse triangle inequality gives

$$|\|x\|_4 - \|y\|_4| \le \|x-y\|_4 \le \|x-y\|_2.$$

Since $\|x\|_4, \|y\|_4 \le 1$, it follows that

$$|\|x\|_4^4 - \|y\|_4^4| \le 4\|x-y\|_2.$$

The first two claims now follow from the paired localization estimates and $d^{-1} = O_{S,j}(\log X/X)$. The exact star mode vanishes off the star, so the squared off-star mass is at most $\|v^{\pm}_{j,X} - u^{\pm}_{a_j,X}\|_2^2$. □

The limit $\mathrm{IPR}(v^{\pm}_{j,X}) \to 1/4$ describes a hybrid profile: asymptotically half the mass lies at the centre, while the other half is spread coherently over an expanding large-prime shell. Figure 2 shows the unperturbed profile on the star at 1 in G_{16}.

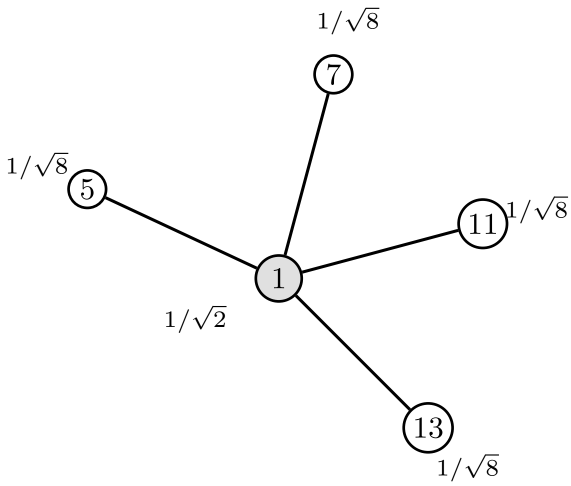

Figure 2: The positive unit mode of the large-prime star at 1 in G_{16}, of degree $d = 4$. Half the ℓ^2 mass sits at the centre; the remaining half is spread equally over the four leaves.

Corollary 6.2 — Centre–shell adjacency evolution

Define the normalized shell state

$$\psi_{a_j,X} = \frac{1}{\sqrt{d_{S,X}(a_j)}} \sum_{m \in \mathcal{L}_{a_j,X}} \delta_m.$$

Then, for every fixed j,

$$\boxed{\sup_{t\in\mathbb{R}} \left\| e^{-itA_{S,X}}\delta_{a_j} - \cos(\lambda_j t)\delta_{a_j} + i\sin(\lambda_j t)\psi_{a_j,X} \right\|_2 = O_{S,j}(X^{-1/4}),}$$

where $\lambda_j = \lambda_j(A_{S,X})$. Consequently,

$$\sup_{t\in\mathbb{R}} \|(I - P_{a_j,X})e^{-itA_{S,X}}\delta_{a_j}\|_2^2 = O_{S,j}(X^{-1/2}),$$

and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\langle \delta_{a_j}, e^{-itA_{S,X}}\delta_{a_j}\rangle\right|^2 dt = \frac{1}{2} + O_{S,j}(X^{-1/4}).$$

Proof

Write $u^\pm = u^\pm_{a_j,X}$, $v^\pm = v^\pm_{j,X}$, and $\varepsilon_X = \max_\pm \|v^\pm - u^\pm\|_2$. Since

$$\delta_{a_j} = \frac{u^+ + u^-}{\sqrt{2}}, \qquad \psi_{a_j,X} = \frac{u^+ - u^-}{\sqrt{2}},$$

the vector $w = (v^+ + v^-)/\sqrt{2}$ satisfies $\|w - \delta_{a_j}\|_2 \le \sqrt{2}\,\varepsilon_X$. Moreover,

$$e^{-itA_{S,X}}w = \frac{e^{-it\lambda_j}v^+ + e^{it\lambda_j}v^-}{\sqrt{2}}.$$

Replacing $v^\pm$ by $u^\pm$ and using unitarity gives, uniformly in t,

$$\left\| e^{-itA_{S,X}}\delta_{a_j} - \cos(\lambda_j t)\delta_{a_j} + i\sin(\lambda_j t)\psi_{a_j,X} \right\|_2 \le 2\sqrt{2}\,\varepsilon_X.$$

The confinement estimate follows because the comparison state is supported on the star. Taking its inner product with δ_{a_j}, squaring, and averaging $\cos^2(\lambda_j t)$ proves the last assertion. □

The global norm bound is not the correct scale for the fixed eigenvalue displacement. The next section identifies the exact cancellation that removes its first-order contribution.

7. Second-order spectral readout

Lemma 7.1

For fixed j, put

$$\mu_{j,X} = \sqrt{d_{S,X}(a_j)}, \qquad u = u^+_{a_j,X}.$$

Then

$$\langle u, H_{S,X}u\rangle = 0$$

and

$$\|H_{S,X}u\|_2^2 = O_{S,j}\left(\frac{\sqrt{X}}{\log X}\right).$$

Proof

No small-prime edge joins two vertices in the support of u. The center a_j and a leaf $a_j p$ differ by $p > \sqrt{X}$, while two distinct leaves are incomparable. This proves the first identity.

Unique factorization also implies that distinct support vertices have disjoint $H_{S,X}$-neighbor sets. Consequently,

$$\|H_{S,X}u\|_2^2 = \frac{1}{2}\deg_H(a_j) + \frac{1}{2d_{S,X}(a_j)} \sum_{\sqrt{X}<p\le X/a_j} \deg_H(a_j p).$$

For fixed a_j,

$$\deg_H(a_j) = O_{S,j}(\sqrt{X}/\log X)$$

and

$$\deg_H(a_j p) \le \pi_S(X/(a_j p)) + O_{S,j}(1).$$

The crude estimates $\pi(y) \le y$ and

$$\sum_{\sqrt{X}<n\le X/a_j} \frac{1}{n} = O(\log X)$$

give

$$\sum_{\sqrt{X}<p\le X/a_j} \pi_S(X/(a_j p)) = O_j(X\log X).$$

Since $d_{S,X}(a_j) \asymp X/\log X$, the normalized leaf term is $O_{S,j}((\log X)^2)$, which is lower order than $\sqrt{X}/\log X$. This crude estimate is used only with j fixed; Lemma 9.1 supplies the uniform residual identity for varying centres. $\square$

Theorem 7.2

For every fixed j,

$$\left|\lambda_j(A_{S,X}) - \sqrt{d_{S,X}(a_j)}\right| = O_{S,j}\left(\frac{1}{\sqrt{\log X}}\right).$$

Equivalently,

$$\boxed{\lambda_j(A_{S,X})^2 = \pi_S(X/a_j) + O_{S,j}\left(\frac{\sqrt{X}}{\log X}\right).}$$

Proof

The Rayleigh quotient of $A_{S,X}$ at u is $\mu_{j,X}$, since the first-order term in Lemma 7.1 vanishes. The spectral gap for the corresponding star eigenvalue is $\gamma_{j,X} \asymp \sqrt{X/\log X}$, while $\|H_{S,X}\| = o(\gamma_{j,X})$.

For completeness, decompose

$$\ell^2(V_S(X)) = \mathbb{C}u \oplus u^\perp$$

and write the eigenvalue equation in blocks. The lower block gives

$$v = (\lambda - QA_{S,X}Q)^{-1} QH_{S,X}u\,\alpha,$$

where $Q = I - uu^*$. Substitution into the upper block yields

$$\lambda - \mu_{j,X} = \left\langle H_{S,X}u,\, Q(\lambda - QA_{S,X}Q)^{-1}QH_{S,X}u \right\rangle .$$

For large X, the inverse has norm $O(\gamma_{j,X}^{-1})$. For $j \geq 2$ this is a two-sided separation statement: the compressed operator $QA_{S,X}Q$ still contains the larger star modes, which lie above $\mu_{j,X}$, so the bound uses the star gaps together with $\|H_{S,X}\| = o(\gamma_{j,X})$ rather than a Neumann series below the spectrum. Lemma 7.1 therefore gives

$$|\lambda - \mu_{j,X}| \ll \frac{\sqrt{X}/\log X}{\sqrt{X/\log X}} = \frac{1}{\sqrt{\log X}}.$$

Finally,

$$\lambda_j^2 = d_{S,X}(a_j) + O_{S,j}(\sqrt{X}/\log X),$$

and $\pi_S(\sqrt{X}) = O_S(\sqrt{X}/\log X)$. □

Theorem 7.3 — Fixed-mode displacement asymptotics

For every fixed finite deletion set S and fixed index j,

$$\boxed{\lambda_j(A_{S,X}) - \sqrt{\pi_S(X/a_j) - \pi_S(\sqrt{X})} \sim \frac{\sqrt{a_j}}{\sqrt{\log X}}.}$$

Proof

Put $Y = \sqrt{X}$, take $m = j$, let P project onto the union of the first m complete large-prime star components, and put $Q = I - P$. Relative to this boundary/interior decomposition,

$$A_{S,X} = \begin{pmatrix} L_{m,X} + R_{m,X} & K_{m,X}^* \\ K_{m,X} & C_{m,X} \end{pmatrix}.$$

Here $L_{m,X}$ is the direct sum of the exact stars, while $R_{m,X}$ contains small-prime edges internal to the boundary. If a small-prime edge joins two tagged boundary vertices, unique factorization forces their large-prime tags to agree. Consequently $R_{m,X}$ is a direct sum of compressions of one fixed graph on $a_1, \dots, a_m$, and

$$\|R_{m,X}\| = O_{S,m}(1), \qquad \langle u_{i,X}, R_{m,X}u_{i,X}\rangle = 0.$$

The complement begins at centre a_{m+1}; in particular, every larger star mode $a_1, \dots, a_{j-1}$ has been placed in the boundary rather than left in $C_{m,X}$. Theorem 5.1 and the PNT give, for some fixed $\rho < 1$,

$$\|C_{m,X}\| \leq \rho\mu_{j,X}, \qquad \mu_{j,X} = \sqrt{d_{S,X}(a_j)}.$$

For $z > \|C_{m,X}\|$, define

$$\Sigma_{m,X}(z) = K_{m,X}^*(z - C_{m,X})^{-1}K_{m,X} = \sum_{r\geq 0} z^{-r-1}K_{m,X}^*C_{m,X}^rK_{m,X}.$$

Since $\|K_{m,X}\| \leq \|H_{S,X}\|$, the self-energy has norm $O_{S,j,m}(1/\sqrt{\log X})$. The fixed boundary star eigenvalues are separated from $\mu_{j,X}$ by $\asymp_{S,j,m} \sqrt{X/\log X}$. Applying the one-dimensional Feshbach reduction inside the boundary, using the zero diagonal of $R_{m,X}$, yields

$$\lambda_j - \mu_{j,X} = \langle u_{j,X}, \Sigma_{m,X}(\mu_{j,X})u_{j,X}\rangle + o_{S,j}(1/\sqrt{\log X}).$$

Indeed, boundary-mode mixing contributes $O_{S,j,m}(\sqrt{\log X/X})$, and the resolvent identity replaces λ_j by $\mu_{j,X}$ with still smaller error.

Put $b_{j,X} = K_{m,X}u_{j,X}$. The exact residual calculation of Lemma 7.1, followed by removal of the bounded boundary component, gives

$$\|b_{j,X}\|_2^2 = \frac{1}{2}\pi_S(Y) + O_{S,j}((\log X)^2) \sim \frac{\sqrt{X}}{\log X}.$$

Indeed, the centre coordinate of $u_{j,X}$ has squared mass 1/2 and has $\pi_S(Y) + O_{S,j}(1)$ small-prime neighbours. This contributes $\frac{1}{2}\pi_S(Y) + O_{S,j}(1)$. The normalized large-prime leaves contribute $O_{S,j}((\log X)^2)$ by Lemma 7.1, and removing the finite boundary changes only the stated error.

We next bound longer excursions. Write $f = H_{S,X}u_{j,X}$, and split the complement again into its large-prime and small-prime parts. Because P is a union of complete $L_{S,X}$-star components, it commutes with $L_{S,X}$, and so does $Q = I - P$. Since $b_{j,X} = Qf$, this gives

$$QL_{S,X}Qb_{j,X} = QL_{S,X}f, \qquad \|QL_{S,X}Qb_{j,X}\| \le \|L_{S,X}f\|, \qquad \|QH_{S,X}Qb_{j,X}\| \le \|H_{S,X}\|\|f\|.$$

On the large-prime star centered at c,

$$\|(L_{S,X}f)|_{\{c\}\cup\mathcal{L}(c)}\|^2 = \left|\sum_{v\in\mathcal{L}(c)} f(v)\right|^2 + d_{S,X}(c)|f(c)|^2.$$

The center exits have forms a_jq and a_j/q; the leaf exits have forms a_jpq and $(a_j/q)p$. Using

$$d_{S,X}(c) \ll_S \frac{X}{c\log X}, \qquad \sum_{q\le Y}\frac{1}{q} = O(\log\log X), \qquad \sum_q \frac{1}{q^2} < \infty,$$

and $d_{S,X}(a_j) \asymp_{S,j} X/\log X$, direct grouping by the exit prime gives the following bound. For an up-exit q, the centre a_jq and all vertices a_jpq lie in the large-prime star centred at a_jq. The finitely many down-exits $q \mid a_j$ similarly lie in the star centred at a_j/q. Distinct exit primes give disjoint stars: an equality $a_jqp = a_jq'p'$, with $p, p' > Y$ and $q, q' \le Y$, forces $q = q'$ and $p = p'$ by unique factorization; an up/down equality $a_jqp = (a_j/r)p'$, with $r \mid a_j$ prime, would give $rqp = p'$, which is impossible; and a down/down equality reduces to $r'p = rp'$, forcing $r = r'$ and $p = p'$. Moreover,

$$d_{S,X}(a_jq) \ll_{S,j} \frac{d_{S,X}(a_j)}{q}$$

uniformly for $q \le Y$. Consequently,

$$\sum_{q\le Y} d_{S,X}(a_jq) \ll_{S,j} d_{S,X}(a_j)\sum_{q\le Y}\frac{1}{q},$$

and

$$\sum_{q\le Y}\frac{d_{S,X}(a_jq)^2}{d_{S,X}(a_j)} \ll_{S,j} d_{S,X}(a_j)\sum_q\frac{1}{q^2}.$$

The finitely many down-exit stars contribute $O_{S,j}(d_{S,X}(a_j))$. Therefore

$$\sum_{q\le Y}\left(d_{S,X}(a_jq) + \frac{d_{S,X}(a_jq)^2}{d_{S,X}(a_j)}\right) + O_{S,j}(d_{S,X}(a_j)) \ll_{S,j} d_{S,X}(a_j)\log\log X.$$

The preceding star identity therefore gives

$$\|C_{m,X} b_{j,X}\|_2^2 = O_{S,j,m}\left(\frac{X \log\log X}{\log X}\right).$$

Since $\|C_{m,X}\|/\mu_{j,X} \le \rho < 1$, the positive-length Neumann tail is at most

$$\frac{\|b_{j,X}\|_2 \, \|C_{m,X} b_{j,X}\|_2}{\mu_{j,X}^2 (1-\rho)} = O_{S,j}\big(X^{-1/4}\sqrt{\log\log X}\big) = o_{S,j}(1/\sqrt{\log X}).$$

The zero-step first return is therefore the full leading term. Finally,

$$\frac{\|b_{j,X}\|_2^2}{\mu_{j,X}} \sim \frac{\sqrt{X}/\log X}{\sqrt{X/(a_j \log X)}} = \frac{\sqrt{a_j}}{\sqrt{\log X}},$$

which proves the theorem. ☐

Corollary 7.4 — Spectral readout consequence

For every fixed finite deletion set S and fixed j,

$$\boxed{\lambda_j(A_{S,X})^2 = \pi_S(X/a_j) + o_{S,j}\left(\frac{\sqrt{X}}{\log X}\right).}$$

Proof

Write $\lambda_j = \mu_{j,X} + \Delta_{j,X}$. Theorem 7.3 and the PNT give

$$2\mu_{j,X}\Delta_{j,X} = \big(2 + o_{S,j}(1)\big)\frac{\sqrt{X}}{\log X} = \pi_S(\sqrt{X}) + o_S(\sqrt{X}/\log X).$$

Since $\Delta_{j,X}^2 = o(\sqrt{X}/\log X)$, expanding the square proves the claim. The cancellation is universal in fixed j: the factor $\sqrt{a_j}$ in the displacement cancels the factor $a_j^{-1/2}$ in the star scale. ☐

8. The explicit fixed-mode remainder

The cancellation in the preceding readout can be continued one layer further for every fixed mode. A single-star boundary at $a = a_j$ is enough: larger stars remain in the interior and are controlled by two-sided gaps. Write $Y = \sqrt{X}$ and $d_c = d_{S,X}(c)$. Put $\mu = \sqrt{d_a}$ and

$$r_{q,X} = \frac{d_{aq}}{d_a} \qquad (q \notin S,\ q \le Y \text{ prime}), \qquad s_{q,X} = \frac{d_{a/q}}{d_a} \qquad (q \mid a,\ q \text{ prime}).$$

Proposition 8.1 — First-exit decomposition

Let u be the positive unit mode of the large-prime star at a, let $\mathcal{B} = \{a\} \cup \{ap : Y < p \le X/a\}$ be its support, and put $b = Ku$ for the off-diagonal block K from $\mathcal{B}$ to its complement. Every small-prime exit from the centre has the form aq or a/q; every exit from a leaf ap has the form apq or $(a/q)p$. Unique factorization makes these vertices pairwise distinct and exhausts the support of b:

$$b = \Big(\bigoplus_{q \le Y,\ q \notin S} b_{aq}\Big) \oplus \Big(\bigoplus_{\substack{q \mid a \\ q \text{ prime}}} b_{a/q}\Big),$$

$$b_{aq} = \frac{1}{\sqrt{2}}\delta_{aq} + \frac{1}{\sqrt{2d_a}} \sum_{Y<p\le X/(aq)} \delta_{aqp}, \qquad b_{a/q} = \frac{1}{\sqrt{2}}\delta_{a/q} + \frac{1}{\sqrt{2d_a}} \sum_{Y<p\le X/a} \delta_{(a/q)p}.$$

The up-vector b_{aq} occupies the full leaf set of the star at aq. The down-vector occupies only the leaves $p \le X/a$ of the strictly larger star at a/q. Each distinct prime divisor $q \mid a$ supplies one down-star, independently of its multiplicity in a. Figure 3 records the two exit types.

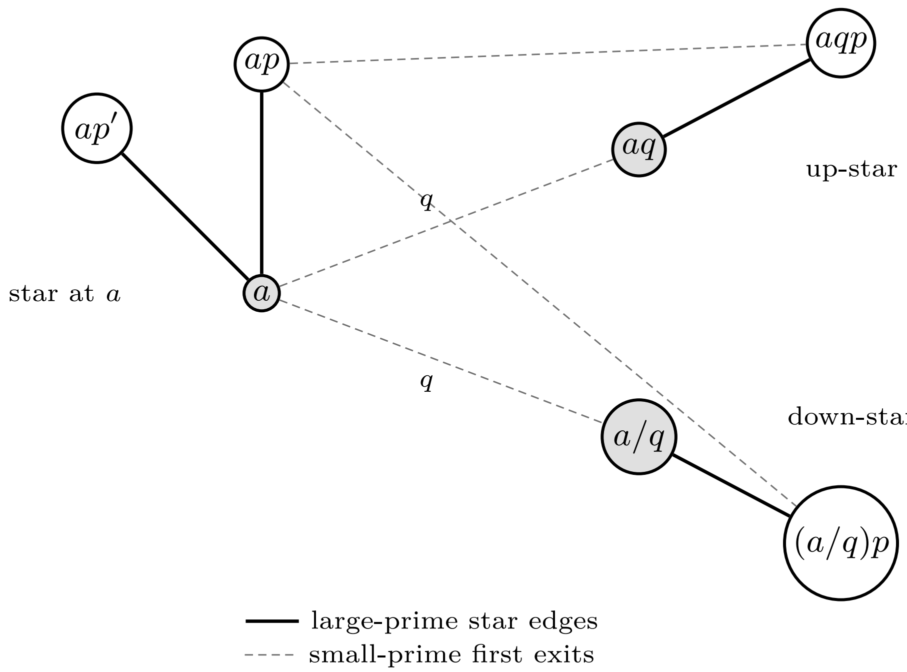

Figure 3: First exits from the large-prime star at a centre a. Solid edges are large-prime star edges. Dashed edges are small-prime steps: the centre exits to aq, or to a/q when $q \mid a$, while a leaf ap exits to aqp or $(a/q)p$. Unique factorization makes these vertices distinct.

Lemma 8.2 — Up-star and down-star identities

With $r = d_{aq}/d_a < 1$,

$$2\mu \langle b_{aq}, (\mu - L_{aq})^{-1} b_{aq} \rangle = \frac{1+3r}{1-r} = 1 + \frac{4r}{1-r}.$$

If $r = 0$, the star is the isolated center and this identity is simply $1 = 1$. Thus the normalized-leaf calculation in the proof may assume $r > 0$.

With $c = a/q$, $s = d_c/d_a > 1$, and ψ the normalized symmetric leaf vector of the star at c, the leaf part of $b_{a/q}$ has a component $\sqrt{1/(2s)}$ along ψ and a kernel component of squared mass $(s-1)/(2s)$. The two-dimensional resolvent together with the kernel contribution $1/\mu$ gives

$$2\mu \langle b_{a/q}, (\mu - L_{a/q})^{-1} b_{a/q} \rangle = \frac{5-s}{1-s}.$$

This quantity is negative for $1 < s < 5$, zero for $s = 5$, and positive for $s > 5$; no uniform sign assertion is used below. As $X \to \infty$, $s_{q,X} \to q$ and the term tends to $(q-5)/(q-1)$.

Proof

On the centre / normalized-leaf basis the up-star is the matrix with off-diagonal entry $\mu\sqrt{r}$, while b_{aq} has coordinates $2^{-1/2}(1, \sqrt{r})$. Direct inversion of $\mu I - L_{aq}$ yields the first identity. For the down-star, $\mu I - L_c = \mu \begin{pmatrix} 1 & -\sqrt{s} \\ -\sqrt{s} & 1 \end{pmatrix}$, so

$$(\mu - L_c)^{-1} = \frac{1}{\mu(1-s)} \begin{pmatrix} 1 & \sqrt{s} \\ \sqrt{s} & 1 \end{pmatrix}$$

on $\mathrm{span}\{\delta_c, \psi\}$. The two-dimensional quadratic form is $(3s+1)/(2\mu s(1-s))$. The kernel contributes $(s-1)/(2s\mu)$. Multiplying by 2μ and simplifying gives $(5-s)/(1-s)$. $\square$

Summing the leading 1 in the up-star identity over allowed $q \le Y$ produces exactly $\pi_S(Y)$. On the compressed direct sum $L_U = \bigoplus L_c$ over the centres of Proposition 8.1 one therefore has

$$2\mu \langle b, (\mu - L_U)^{-1} b \rangle - \pi_S(Y) = 4 \sum_{\substack{q \le Y \\ q \notin S}} \frac{r_{q,X}}{1 - r_{q,X}} + \sum_{\substack{q \mid a \\ q \text{ prime}}} \frac{5 - s_{q,X}}{1 - s_{q,X}}.$$

This identity is for the compressed direct sum only.

Lemma 8.3 — Two-sided interior separation

Relative to $\ell^2(\mathcal{B}) \oplus \ell^2(\mathcal{B}^c)$, write

$$A_{S,X} = \begin{pmatrix} L_B & K^* \\ K & C \end{pmatrix},$$

so that $L_B u = \mu u$ and $b = Ku$. All edges leaving $\mathcal{B}$ carry small-prime labels, hence $\|K\| \ll_S X^{1/4}/\sqrt{\log X}$. There is $\delta_{S,j} > 0$ such that, for all large X,

$$\mathrm{dist}(\mu, \sigma(C)) \ge \delta_{S,j}\, \mu, \qquad \mathrm{dist}(\lambda_j, \sigma(C)) \ge \tfrac{1}{2}\delta_{S,j}\, \mu.$$

Proof

Write $C = L_C + H_C$, where L_C is the star forest with the a-star deleted. The spectrum of L_C is $\{\pm\sqrt{d_c} : c \ne a\} \cup \{0\}$. For allowed $c > a$, $d_c \le d_{a'} = (a/a' + o(1))d_a$ with a' the least allowed integer exceeding a. For the finitely many allowed $c < a$, $d_c = (a/c + o(1))d_a \ge (1+\varepsilon_j)d_a$. Hence every $\sqrt{d_c}$ with $c \ne a$ is separated from μ by a fixed positive multiple of μ, while 0 and $-\sqrt{d_c}$ are at distance at least μ. Weyl's inequality moves each eigenvalue of C by at most $\|H_{S,X}\| = o(\mu)$. The bound for λ_j follows from Theorem 7.2. $\square$

No positivity is available for $j \ge 2$: the interior contains the stars $a_1, \dots, a_{j-1}$, whose eigenvalues exceed μ.

Lemma 8.4 — Squared-eigenvalue reassembly

$$\lambda_j(A_{S,X})^2 - \pi_S(X/a) = 2\mu \langle b, (\mu - C)^{-1} b \rangle - \pi_S(Y) + O_{S,j}(1/\log X).$$

Proof

Since $\lambda_j \notin \sigma(C)$, the Feshbach reduction is exact: λ_j is an eigenvalue of $L_B + \Sigma(\lambda_j)$ on $\ell^2(\mathcal{B})$, where $\Sigma(\lambda) = K^*(\lambda - C)^{-1}K$. Lemma 8.3 gives $\|\Sigma(\lambda)\| = O_{S,j}(1/\sqrt{\log X})$ for $|\lambda - \mu| \le 1$. The spectrum of L_B is $\{\mu, 0, -\mu\}$ with μ simple, so

$$\lambda_j - \mu = \langle u, \Sigma(\lambda_j) u \rangle + O(\|\Sigma\|^2/\mu) = \langle b, (\lambda_j - C)^{-1} b \rangle + O_{S,j}(1/\sqrt{X \log X}).$$

The first resolvent identity replaces λ_j by μ at the same scale, because $\|b\|^2 = O_{S,j}(\sqrt{X}/\log X)$ and both resolvents have norm $O_{S,j}(1/\mu)$. Squaring, using $\mu^2 = \pi_S(X/a) - \pi_S(Y)$ and $\lambda_j - \mu = O_{S,j}(1/\sqrt{\log X})$, produces the displayed error. $\square$

Lemma 8.5 — Star-union compression

Let U be the span of the full stars centred at $\{aq : q \le Y,\ q \notin S\} \cup \{a/q : q \mid a,\ q \text{ prime}\}$, isolated centres included, and let $W = U^{\perp}$ inside the interior. Then

$$C = \begin{pmatrix} L_U & J^* \\ J & C_W \end{pmatrix}, \qquad L_U = \bigoplus_c L_c, \qquad b = \bigoplus_c b_c \in U,$$

J is a subblock of $H_{S,X}$, and $\operatorname{dist}(\mu, \sigma(C_W)) \ge \delta'_{S,j}\mu$ for some fixed $\delta'_{S,j} > 0$ and all large X.

Proof

Proposition 8.1 gives $b \in U$. The compression of C to U equals L_U. To see this, write every vertex of U either as a listed centre c, or as cp with $p > Y$. A small-prime edge between two centres would require $c_2 = c_1\ell$ for a prime $\ell \le Y$. Comparing the possibilities $c = aq$ and $c = a/r$ forces a prime to equal a product of two primes. A centre-leaf edge would require $c_1\ell = c_2 p$; for fixed a and large X, the large prime $p > Y$ would then divide $c_1\ell$, whose prime factors are all at most Y, which is impossible. Finally, a leaf-leaf edge would give $c_1p_1\ell = c_2p_2$. If $p_1 \ne p_2$, unique factorization forces one large prime to divide the small-prime side; if $p_1 = p_2$, it reduces to the excluded centre-centre case. Thus no small-prime edge joins two vertices of U. Large-prime edges inside U beyond the star edges are impossible because a large-prime leaf has a unique centre. Every U-to-W edge therefore carries a small-prime label.

For the gap, write $C_W = L_W + H_W$ on the remaining centres. Those centres are a fixed ratio away from a, so the argument of Lemma 8.3 applies, and $\|H_W\| \le \|H_{S,X}\| = o(\mu)$. $\square$

Lemma 8.6 — Quadratic-form replacement

$$\langle b, (\mu - C)^{-1} b\rangle = \sum_c \langle b_c, (\mu - L_c)^{-1} b_c\rangle + O_{S,j}(1/\sqrt{X \log X}).$$

Proof

Lemma 8.5 makes the Schur complement exact:

$$\langle b, (\mu - C)^{-1} b\rangle = \langle b, (\mu - L_U - T)^{-1} b\rangle, \qquad T = J^*(\mu - C_W)^{-1} J,$$

with $\|T\| = O_{S,j}(1/\sqrt{\log X})$. Put $R_U = (\mu - L_U)^{-1}$. Chebyshev gives $r_{q,X} \ll_S 1/q$. Choose a finite prime cutoff so the tail is below $1/2$; on the remaining finitely many primes the PNT limit $r_{q,X} \to 1/q$ supplies a common constant $\rho_S < 1$. Inside U, up-stars are separated from μ by $\mu(1 - \sqrt{r_{q,X}}) \ge (1 - \sqrt{\rho_S})\mu$ with $\sup_q r_{q,X} \le \rho_S < 1$, down-stars by $\mu(\sqrt{s_{q,X}} - 1) \ge (\sqrt{2} - 1 + o(1))\mu$, and the kernel and negative branches by μ. Hence $\|R_U\| = O_{S,j}(1/\mu)$. Also

$$\|b\|^2 = \tfrac{1}{2}\pi_S(Y) + \tfrac{1}{2}\sum_q r_{q,X} + \omega(a) = O_{S,j}(\sqrt{X}/\log X).$$

For each up-star, direct inversion gives

$$\|R_U b_{aq}\|^2 = \frac{1 + 6r_{q,X} + r_{q,X}^2}{2\mu^2(1 - r_{q,X})^2} = O_{S,j}(\mu^{-2}).$$

The finitely many down-stars satisfy the same bound. Summing the $\pi_S(Y)$ up-stars therefore gives

$$\|R_U b\|^2 = O_{S,j}\left(\frac{\pi_S(Y)}{\mu^2}\right) = O_{S,j}(X^{-1/2}).$$

Moreover, after decreasing the limiting gap if necessary,

$$\operatorname{dist}(\mu, \sigma(L_U + T)) \ge \delta_{U,S,j}\mu - \|T\| \gg_{S,j} \mu, \qquad \delta_{U,S,j} = \tfrac{1}{2}\min\{1 - \sqrt{\rho_S}, \sqrt{2} - 1, 1\} > 0.$$

In particular, $\|R_U T\| = o(1)$, so $\|R_U T\| < 1/2$ for all sufficiently large X. The resolvent identity is

$$(\mu - L_U - T)^{-1} - R_U = R_U T(\mu - L_U - T)^{-1},$$

and

$$(\mu - L_U - T)^{-1} = (I - R_U T)^{-1} R_U.$$

Consequently,

$$\big|\langle b, (\mu - L_U - T)^{-1} b\rangle - \langle b, R_U b\rangle\big| \le 2\|T\|\,\|R_U b\|^2 = O_{S,j}(1/\sqrt{X \log X}).$$

This is the required quadratic-form replacement. □

Theorem 8.7 — Explicit fixed-mode prime-star correction

For every fixed finite deletion set S and fixed index j,

$$\boxed{\lambda_j(A_{S,X})^2 - \pi_S(X/a_j) = 4 \sum_{\substack{q \le \sqrt{X} \\ q \notin S}} \frac{r_{q,X}}{1 - r_{q,X}} + \sum_{\substack{q | a_j \\ q \text{ prime}}} \frac{5 - s_{q,X}}{1 - s_{q,X}} + O_{S,j}(1/\log X).}$$

Proof

Insert Lemma 8.6 into Lemma 8.4 and evaluate the direct sum by Lemma 8.2. The count $\pi_S(Y)$ cancels. The quadratic-form error, multiplied by $2\mu = O_{S,j}(\sqrt{X/\log X})$, is $O_{S,j}(1/\log X)$. □

Corollary 8.8 — Arithmetic control of the oscillation phase

For all sufficiently large X, let

$$F_{S,j}(X) = \pi_S(X/a_j) + 4 \sum_{\substack{q \le \sqrt{X} \\ q \notin S}} \frac{r_{q,X}}{1 - r_{q,X}} + \sum_{\substack{q | a_j \\ q \text{ prime}}} \frac{5 - s_{q,X}}{1 - s_{q,X}}, \qquad \hat{\lambda}_{j,X} = \sqrt{F_{S,j}(X)}.$$

For every fixed finite S and fixed j, uniformly for real t,

$$\boxed{\begin{gathered}\big\|e^{-itA_{S,X}}\delta_{a_j} - \cos(\hat{\lambda}_{j,X} t)\delta_{a_j} + i \sin(\hat{\lambda}_{j,X} t)\psi_{a_j,X}\big\|_2 \\ \ll_{S,j} X^{-1/4} + \frac{|t|}{\sqrt{X \log X}}.\end{gathered}}$$

In particular, the explicit prime-star remainder determines the coherent centre–shell evolution with $o(1)$ error throughout every time window $|t| = o(\sqrt{X \log X})$.

Proof

Theorem 8.7 and $\lambda_j \asymp_{S,j} \sqrt{X/\log X}$ give

$$|\lambda_j - \hat{\lambda}_{j,X}| = \frac{|\lambda_j^2 - F_{S,j}(X)|}{\lambda_j + \hat{\lambda}_{j,X}} \ll_{S,j} \frac{1}{\sqrt{X \log X}}.$$

Combine Corollary 6.2 with the Lipschitz bounds for sine and cosine. □

Corollary 8.9 — Fixed-mode constant layer

Let B_1 be the prime Meissel–Mertens constant, defined by $\sum_{p\le x} 1/p = \log\log x + B_1 + o(1)$, and put

$$\kappa_S = B_1 - \sum_{p\in S} \frac{1}{p} + \sum_{p\notin S} \frac{1}{p(p-1)}, \qquad c(a) = \sum_{\substack{q\mid a\\ q\ \text{prime}}} \frac{q-5}{q-1}.$$

Then, for every fixed finite S and fixed j,

$$\boxed{\lambda_j(A_{S,X})^2 = \pi_S(X/a_j) + 4\log\log X + 4\kappa_S + c(a_j) + o_{S,j}(1).}$$

The case $j=1$ is $c(1)=0$. The correction $c(a)$ is additive over distinct prime divisors and ignores multiplicity, so for example $c(2)=c(4)=-3$, $c(3)=-1$, $c(5)=0$, and $c(6)=-4$.

Proof

Put $L=\log X$ and $\widetilde{L} = L - \log a$. We use the standard quantitative consequence of the PNT

$$\pi_S(t) = \frac{t}{\log t} + O_S\left(\frac{t}{\log^2 t}\right) \qquad (t \ge Y/a),$$

uniformly in this range. When $q \le Y/a$, it gives

$$r_{q,X} = \frac{\widetilde{L}}{q(\widetilde{L}-\log q)} + O\left(\frac{1}{qL} + \frac{a}{\sqrt{X}}\right)$$

for allowed q. When $q > Y/a$, the truncated degree d_{aq} and therefore $r_{q,X}$ vanish, while the displayed main term is $O(a/Y)$. The errors sum to $o(1)$. The main term splits as $1/q$ plus a logarithmic increment. Mertens' theorem yields $\sum_{q\le Y,\, q\notin S} 1/q = \log L - \log 2 + B_1 - \sum_{p\in S} 1/p + o(1)$. Partial summation with $\sum_{p\le t} (\log p)/p = \log t + O(1)$ converts the increment into

$$\int_2^Y \frac{d\log t}{\widetilde{L} - \log t} = \log \frac{\widetilde{L} - \log 2}{\widetilde{L} - L/2} + o(1) = \log 2 + o(1),$$

because a is fixed. Thus $\sum r_{q,X} = \log\log X + B_1 - \sum_{p\in S} 1/p + o(1)$, exactly as for $a=1$. Since

$$\frac{r}{1-r} = r + \frac{r^2}{1-r},$$

extend $r_{q,X} = 0$ for primes $q > Y$. Chebyshev gives $r_{q,X} \ll_S 1/q$ and $\sup_q r_{q,X} \le \rho_S < 1$, hence $r_{q,X}^2/(1-r_{q,X}) \ll_S q^{-2}$ on a fixed prime index set. Dominated convergence produces the convergent sum $\sum_{q\notin S} 1/(q(q-1))$. The up-sum is therefore $4\log\log X + 4\kappa_S + o(1)$. The finitely many down-star terms tend to $c(a)$ by the PNT. Theorem 8.7 completes the proof. □

Finite-prime interpretation. The universal correction can equivalently be written without a separately named constant. Indeed,

$$\frac{1}{p-1} = \frac{1}{p} + \frac{1}{p(p-1)},$$

so, for fixed S,

$$4\sum_{\substack{p\le X\\ p\notin S}} \frac{1}{p-1} = 4\log\log X + 4\kappa_S + o_S(1).$$

Thus κ_S is the convergent correction that converts the prime harmonic sum into the truncated $(p-1)^{-1}$-sum. The truncation at $p \le X$ is essential.

9. Uniformity in a logarithmic window

The fixed-mode proof hides its dependence on the centre inside the $O_{S,j}$-constants. We now track that dependence on the range needed for the same explicit expansion, including the centre-dependent term $c(a)$. Throughout this section S and $\varepsilon > 0$ are fixed,

$$a = a(X) \le (\log X)^{1/2-\varepsilon}$$

is an allowed centre, and $j(a)$ is the number of allowed positive integers at most a. Put $Y = \sqrt{X}$,

$$d_c = d_{S,X}(c), \qquad \mu = \sqrt{d_a},$$

and let u be the positive unit mode of the star at a. Define $r_{q,X} = d_{aq}/d_a$, $s_{q,X} = d_{a/q}/d_a$, and

$$M_{S,a,X} = 4\sum_{\substack{q\le Y\\ q\notin S}} \frac{r_{q,X}}{1-r_{q,X}} + \sum_{\substack{q\mid a\\ q\text{ prime}}} \frac{5-s_{q,X}}{1-s_{q,X}}.$$

All errors below are uniform over this range; no uniformity in S or ε is claimed.

Lemma 9.1 — Uniform degrees, residuals, and matching

Uniformly on the stated range,

$$d_a = \frac{X}{a\log(X/a)} + O_S\!\left(\frac{X}{a(\log X)^2}\right), \qquad \mu \asymp_S \sqrt{\frac{X}{a\log X}},$$

$$\langle u, H_{S,X}u\rangle = 0, \qquad \|H_{S,X}u\|_2^2 = \frac{1}{2}\pi_S(Y) + \frac{1}{2}\sum_{\substack{q\le Y\\ q\notin S}} r_{q,X} + \omega(a) \asymp_S \frac{\sqrt{X}}{\log X}.$$

The degrees d_c are strictly decreasing near a, consecutive allowed centres $a_\pm$ satisfy

$$\left|\mu - \sqrt{d_{a_\pm}}\right| \ge \frac{\mu}{3a},$$

and

$$\left|\lambda_{j(a)}(A_{S,X}) - \mu\right| \le \|H_{S,X}\| \le \frac{\mu}{10a}$$

for all sufficiently large X. Hence $\lambda_{j(a)}$ is the unique eigenvalue matched to the star at a.

Proof

The PNT is uniform on $[Y/a, X/a]$, whose lower endpoint is $\sqrt{X}/(\log X)^{O(1)}$, and gives the degree formula. The exact first-exit decomposition of Proposition 8.1 gives the residual identity; its first-order term vanishes by the same unique-factorization argument as Lemma 7.1. The up-sum is $O_S(\log\log X)$, while $\omega(a) = O(\log a/\log\log a)$, so both are absorbed by $\pi_S(Y)$.

The allowed integers are periodic modulo P_S, so consecutive allowed integers differ by at most P_S. If a_+ is the next allowed integer, subtracting the two uniform degree estimates gives

$$d_a - d_{a_+} = \frac{X(a_+ - a)}{aa_+\log X}\left(1 + O_S\!\left(\frac{a}{\log X}\right)\right).$$

The analogous formula holds for $d_{a_-} - d_a$. Dividing these differences by the corresponding sums of square roots yields

$$\left|\mu - \sqrt{d_{a_\pm}}\right| \geq \frac{\mu}{3a}$$

for all sufficiently large X, uniformly in the stated window. Since the degrees are nonincreasing, every earlier star lies above this gap and every later star below it. Finally Theorem 5.1 and $a \leq (\log X)^{1/2-\varepsilon}$ imply $\|H_{S,X}\| = o(\mu/a)$. Weyl's inequality completes the matching. □

Lemma 9.2 — Uniform three-block reduction

Let $\mathcal{B}$ be the support of the star at a. Let U be the span of the full large-prime stars centred at

$$\{aq : q \leq Y,\ q \notin S\} \cup \{a/q : q \mid a,\ q \text{ prime}\},$$

isolated centres included, and let $W = (\mathcal{B} \cup \operatorname{supp} U)^\perp$. Relative to $\mathcal{B} \oplus U \oplus W$,

$$A_{S,X} = \begin{pmatrix} L_{\mathcal{B}} & K^* & 0 \\ K & L_U & J^* \\ 0 & J & C_W \end{pmatrix}, \qquad L_U = \bigoplus_c L_c, \qquad b = Ku \in U.$$

The compression to U is exactly the displayed star direct sum, and J is a subblock of $H_{S,X}$.

Proof

Every small-prime neighbour of a is an aq or a/q. Every small-prime neighbour of a leaf ap is an apq or $(a/q)p$; the cutoff $apq \leq X$ puts the first vertex in the star at aq, and the second lies in the star at a/q. Thus there is no $\mathcal{B}$–W edge. Unique factorization gives the remaining claims exactly as in Proposition 8.1 and Lemma 8.5. □

Lemma 9.3 — Uniform gaps and complement self-energy

There are constants $\rho_S < 1$ and $\delta_S > 0$ such that

$$\sup_q r_{q,X} \leq \rho_S, \qquad \operatorname{dist}(\mu, \sigma(L_U)) \geq \delta_S \mu$$

uniformly on the stated range. If $|z - \mu| \leq \|H_{S,X}\|$, then

$$\operatorname{dist}(z, \sigma(C_W)) \geq \frac{\mu}{5a},$$

and, for

$$T(z) = J^*(z - C_W)^{-1} J,$$

$$\|T(z)\| \ll_S \frac{a^{3/2}}{\sqrt{\log X}}, \qquad \frac{\|T(z)\|}{\mu} \ll_S \frac{a^2}{\sqrt{X}} = o(1).$$

Proof

For $q \le Y/a$, the uniform degree estimate gives

$$r_{q,X} = \frac{\log(X/a)}{q(\log(X/a) - \log q)} + o_S(1);$$

for larger q, $r_{q,X} = 0$. The least allowed prime is therefore the only possible extremizer, and one may take, for example, $\rho_S = 3/4$ for large X. Down-stars have $s_{q,X} = q + o(1) \ge 2 + o(1)$, while the kernel and negative branches are at distance at least μ. This proves the star-union gap.

Let L_W be the compression of the large-prime star forest to W. The centres left in W are at least one allowed step from a, so monotonicity of the star degrees and Lemma 9.1 give

$$\operatorname{dist}(\mu, \sigma(L_W)) \ge \frac{\mu}{3a}.$$

Weyl's inequality gives $\operatorname{dist}(\sigma(C_W), \sigma(L_W)) \le \|H_{S,X}\|$. Since also $|z - \mu| \le \|H_{S,X}\|$ and $\|H_{S,X}\| = o(\mu/a)$, for sufficiently large X we may take $2\|H_{S,X}\| \le 2\mu/(15a)$. Hence

$$\operatorname{dist}(z, \sigma(C_W)) \ge \frac{\mu}{3a} - 2\|H_{S,X}\| \ge \frac{\mu}{5a}.$$

Therefore

$$\|T(z)\| \le \frac{5a\|H_{S,X}\|^2}{\mu},$$

and Theorem 5.1 gives the two displayed estimates. □

Lemma 9.4 — Uniform quadratic-form comparison

With $R_U = (\mu - L_U)^{-1}$,

$$\lambda_{j(a)} - \mu = \langle b, R_U b\rangle + O_S\left(\frac{a^{5/2}}{\sqrt{X}\sqrt{\log X}}\right),$$

and

$$|\lambda_{j(a)} - \mu| \ll_S \sqrt{\frac{a}{\log X}}.$$

Proof

First reduce the W-block at energy $\lambda_{j(a)}$. Lemma 9.3 keeps the reduced lower block $L_U + T(\lambda_{j(a)})$ a fixed multiple of μ away from the target. A second Schur reduction on $\mathcal{B}$ is therefore exact. Since the star spectrum is $\{\mu, 0, -\mu\}$, ordinary isolated-eigenvalue expansion about the one-dimensional μ-eigenspace inside this star block applies. Indeed, for $|z - \mu| \le \|H_{S,X}\|$, put

$$\Sigma_{\text{eff}}(z) = K^*\big(z - L_U - T(z)\big)^{-1} K,$$

then the uniform star-union gap gives, for all sufficiently large X,

$$\|\Sigma_{\text{eff}}(z)\| \le \frac{2\|H_{S,X}\|^2}{\delta_S \mu} \ll_S \sqrt{\frac{a}{\log X}}.$$

The mixing error is therefore $O_S(\|\Sigma_{\text{eff}}(z)\|^2/\mu) = O_S(a^{3/2}/(\sqrt{X}\sqrt{\log X}))$, and the expansion gives

$$\lambda_{j(a)}-\mu=\left\langle b,\left(\lambda_{j(a)}-L_U-T(\lambda_{j(a)})\right)^{-1}b\right\rangle+O_S\left(\frac{a^{3/2}}{\sqrt{X}\sqrt{\log X}}\right).$$

The star-union gap and Lemma 9.1 make the quadratic form $O_S(\sqrt{a/\log X})$.

It remains to replace the reduced resolvent by R_U. Direct inversion on each up-star and down-star, using Lemma 8.2, gives

$$\|R_Ub\|_2^2\ll_S\frac{\pi_S(Y)+\omega(a)}{\mu^2}\ll_S\frac{a}{\sqrt{X}}.$$

Here $\omega(a)\le\log_2 a=o(\pi_S(Y))$ uniformly in the stated logarithmic window, so the growing number of down-stars is absorbed by the up-star contribution.

Put $D=\mu-\lambda_{j(a)}+T(\lambda_{j(a)})$. Lemma 9.3 and the preceding bound imply $\|R_UD\|\ll_S a^2/\sqrt{X}=o(1)$, so

$$(\lambda_{j(a)}-L_U-T)^{-1}=(I-R_UD)^{-1}R_U.$$

Estimating the resulting quadratic-form difference by

$$\|b\|\,\|R_U\|\,\|D\|\,\|R_Ub\|$$

and using $\|b\|\ll_S X^{1/4}/\sqrt{\log X}$, $\|R_U\|\ll_S\mu^{-1}$, and $\|D\|\ll_S a^{3/2}/\sqrt{\log X}$ gives the stated error. $\square$

Theorem 9.5 — Uniform prime-star remainder

For fixed S and $\varepsilon>0$, uniformly over allowed $a\le(\log X)^{1/2-\varepsilon}$,

$$\boxed{\lambda_{j(a)}(A_{S,X})^2-\pi_S(X/a)=M_{S,a,X}+O_{S,\varepsilon}\left(\frac{a^2}{\log X}\right).}$$

Proof

The direct-sum identities of Lemma 8.2 give

$$2\mu\langle b,R_Ub\rangle-\pi_S(Y)=M_{S,a,X}.$$

Insert Lemma 9.4 into

$$\lambda_{j(a)}^2-\mu^2=2\mu(\lambda_{j(a)}-\mu)+(\lambda_{j(a)}-\mu)^2.$$

The quadratic-form error contributes $O_S(a^2/\log X)$; the star-block mixing and the square contribute $O_S(a/\log X)$, which is absorbed by the former because $a\ge1$. The implied constant is independent of the moving centre. Since $a^2/\log X\le(\log X)^{-2\varepsilon}$, this error is also $o_{S,\varepsilon}(1)$. Finally $\mu^2=\pi_S(X/a)-\pi_S(Y)$. $\square$

Theorem 9.6 — Uniform constant layer

Under the same hypotheses,

$$\boxed{\lambda_{j(a)}(A_{S,X})^2=\pi_S(X/a)+4\log\log X+4\kappa_S+c(a)+o_{S,\varepsilon}(1).}$$

Proof

Put $L = \log X$ and $\widetilde{L} = \log(X/a)$. Uniform PNT estimates give, for $q \le Y/a$,

$$r_{q,X} = \frac{\widetilde{L}}{q(\widetilde{L} - \log q)} + O_S\left(\frac{1}{qL} + \frac{a}{\sqrt{X}}\right),$$

and $r_{q,X} = 0$ for $q > Y/a$. The errors sum to $o(1)$ uniformly. Moreover,

$$\frac{\widetilde{L} - \log 2}{\widetilde{L} - L/2} = 2\,\frac{1 - (\log a + \log 2)/L}{1 - 2\log a/L} = 2 + O\left(\frac{1 + \log a}{L}\right) = 2 + o_\varepsilon(1),$$

uniformly on $a \le (\log X)^{1/2-\varepsilon}$. Thus the logarithmic increment in the partial-summation calculation of Corollary 8.9 remains $\log 2 + o_\varepsilon(1)$, and Mertens' theorem gives

$$\sum_{\substack{q \le Y \\ q \notin S}} r_{q,X} = \log\log X + B_1 - \sum_{p \in S} \frac{1}{p} + o(1).$$

The displayed estimate also gives $r_{q,X} \ll_S 1/q$ uniformly. Writing $r/(1-r) = r + r^2/(1-r)$, Lemma 9.3 therefore permits dominated convergence in the quadratic tail and produces

$$\sum_{q \notin S} \frac{1}{q(q-1)}.$$

Thus the up-star contribution is $4\log\log X + 4\kappa_S + o(1)$.

For a prime $q \mid a$, the same uniform PNT estimate gives

$$s_{q,X} = q + O_S\left(\frac{q(1 + \log q)}{\log X}\right).$$

For $f(s) = (5-s)/(1-s)$, one has $f'(s) = 4/(1-s)^2$. Since $s_{q,X} \ge (q+1)/2$ uniformly for all large X,

$$f(s_{q,X}) - f(q) \ll_S \frac{1 + \log q}{q \log X}.$$

Summing over the distinct prime divisors of a, and using $\sum_{q|a} \log q \le \log a$, shows that the total down-star error is $O_S(\log a / \log X) = o(1)$ uniformly on the stated range. Theorem 9.5 completes the proof. □

Corollary 9.7 — Uniform ordering and simplicity

Fix S and $\varepsilon > 0$, and put

$$J_X(\varepsilon) = \#\{a : \ a \text{ is retained and } a \le (\log X)^{1/2-\varepsilon}\}.$$

For all sufficiently large X, the top $J_X(\varepsilon)$ positive eigenvalues are simple and are indexed in decreasing order by the retained centres in this window. Uniformly for retained $a < b$ in the window,

$$\lambda_{j(a)}(A_{S,X})^2 - \lambda_{j(b)}(A_{S,X})^2 = \pi_S(X/a) - \pi_S(X/b) + c(a) - c(b) + o_{S,\varepsilon}(1).$$

If $a < b$ are consecutive retained centres, then

$$\lambda_{j(a)}(A_{S,X})^2 - \lambda_{j(b)}(A_{S,X})^2 \sim \frac{X(b-a)}{ab\log X}.$$

Proof

Subtract Theorem 9.6 for the two centres. The universal $4\log\log X + 4\kappa_S$ layer cancels uniformly. For consecutive retained centres, the uniform degree calculation in Lemma 9.1 gives

$$\pi_S(X/a) - \pi_S(X/b) \sim \frac{X(b-a)}{ab\log X}.$$

Moreover $|c(n)| \le 4 + \omega(n) \le 4 + \log_2 n$, so the centre correction and the uniform $o(1)$ term are negligible compared with this gap throughout the logarithmic window. Lemma 9.1 identifies each resulting simple eigenvalue with the rank $j(a)$, and these ranks fill $1, \dots, J_X(\varepsilon)$. $\square$

Corollary 9.8 — Uniform hub–shell localization

For each retained centre in the logarithmic window, orient the positive unit eigenvector $v^+_{a,X}$ of $\lambda_{j(a)}(A_{S,X})$ toward the positive unit star mode $u^+_{a,X}$, and put

$$v^-_{a,X} = (-1)^{\Omega(a)} J v^+_{a,X}, \qquad J\delta_n = (-1)^{\Omega(n)}\delta_n.$$

Then, uniformly for retained $a \le (\log X)^{1/2-\varepsilon}$,

$$\boxed{\|v^\pm_{a,X} - u^\pm_{a,X}\|_2 \ll_{S,\varepsilon} a^{3/2}X^{-1/4}.}$$

Consequently,

$$\|v^\pm_{a,X}\|_\infty = \frac{1}{\sqrt{2}} + O_{S,\varepsilon}(a^{3/2}X^{-1/4}),$$

$$\mathrm{IPR}(v^\pm_{a,X}) = \frac{1}{4} + O_{S,\varepsilon}(a^{3/2}X^{-1/4}),$$

and, for the coordinate projection $P_{a,X}$ onto the closed star at a,

$$\|(I - P_{a,X})v^\pm_{a,X}\|_2^2 \ll_{S,\varepsilon} a^3 X^{-1/2}.$$

Proof

Let η_X be the residual majorant from Theorem 5.1. Lemma 9.1 gives an ordered reference-star gap

$$\gamma_{a,X} \gg_S \frac{1}{a}\sqrt{\frac{X}{a\log X}},$$

uniformly on the logarithmic window, while $\eta_X \ll X^{1/4}/\sqrt{\log X}$ and $16\eta_X \le \gamma_{a,X}$ for all sufficiently large X. The isolated-mode comparison of Proposition 2.1, applied at the ordered mode from Corollary 9.7 and with the sign chosen so that $\langle v^+_{a,X}, u^+_{a,X}\rangle \ge 0$, gives the finite estimate

$$\|v^+_{a,X} - u^+_{a,X}\|_2 \le \frac{4\eta_X}{\gamma_{a,X} - 2\eta_X} \ll_{S,\varepsilon} a^{3/2}X^{-1/4}.$$

Every prime-cover edge changes $\Omega(n)$ by one, so $JA_{S,X}J = -A_{S,X}$ and the same bound holds for the negative partner. The coordinate and IPR estimates now follow exactly as in Corollary 6.1. The exact star IPR is $1/4 + 1/(4d_{S,X}(a))$, and the additional term is negligible uniformly on the stated window. Squaring the off-star norm gives the final estimate. □

Corollary 9.9 — Uniform centre–shell evolution

Let

$$\psi_{a,X} = \frac{1}{\sqrt{d_{S,X}(a)}} \sum_{m \in \mathcal{L}_{a,X}} \delta_m.$$

Uniformly for retained $a \le (\log X)^{1/2-\varepsilon}$,

$$\boxed{\sup_{t\in\mathbb{R}} \left\| e^{-itA_{S,X}}\delta_a - \cos(\lambda_{j(a)}t)\delta_a + i\sin(\lambda_{j(a)}t)\psi_{a,X} \right\|_2 \ll_{S,\varepsilon} a^{3/2}X^{-1/4}.}$$

For all sufficiently large X, the following radicand is positive. Put

$$\hat{\lambda}_{a,X} = \sqrt{\pi_S(X/a) + M_{S,a,X}},$$

Then the explicit first-exit correction also gives

$$\begin{aligned} &\left\| e^{-itA_{S,X}}\delta_a - \cos(\hat{\lambda}_{a,X}t)\delta_a + i\sin(\hat{\lambda}_{a,X}t)\psi_{a,X} \right\|_2 \\ &\qquad \ll_{S,\varepsilon} a^{3/2}X^{-1/4} + \frac{|t|a^{5/2}}{\sqrt{X\log X}}. \end{aligned}$$

Thus the explicit finite correction determines the centre–shell phase with uniform $o(1)$ error whenever $|t| = o(\sqrt{X\log X}/a^{5/2})$.

Proof

Use the positive and negative eigenvectors from Corollary 9.8. Their symmetric and antisymmetric combinations differ from δ_a and $\psi_{a,X}$, respectively, by $O_{S,\varepsilon}(a^{3/2}X^{-1/4})$. The two eigenvectors acquire the opposite phases $e^{\mp it\lambda_{j(a)}}$. Unitarity then gives the same error uniformly for every real t, exactly as in Corollary 6.2. For the second estimate, Theorem 9.5 and $\lambda_{j(a)} \asymp_{S,\varepsilon} \sqrt{X/(a\log X)}$ show first that the radicand defining $\hat{\lambda}_{a,X}$ is positive and then imply

$$|\lambda_{j(a)} - \hat{\lambda}_{a,X}| \ll_{S,\varepsilon} \frac{a^2/\log X}{\sqrt{X/(a\log X)}} = \frac{a^{5/2}}{\sqrt{X\log X}}.$$

The Lipschitz bounds for sine and cosine cost at most twice this difference times $|t|$. □

Corollary 9.10 — Collective localization of the logarithmic block

Let $\mathcal{U}_X$ be the span of the positive star modes $u^+_{a,X}$, and let $\mathcal{V}_X$ be the span of the corresponding positive adjacency eigenvectors $v^+_{a,X}$, over retained centres $a \le (\log X)^{1/2-\varepsilon}$. If $P_{\mathcal{U}_X}$ and $P_{\mathcal{V}_X}$ are the orthogonal projections onto these subspaces, then

$$\boxed{\|P_{\mathcal{V}_X} - P_{\mathcal{U}_X}\| \ll_{S,\varepsilon} X^{-1/4}(\log X)^{1-2\varepsilon} = o_{S,\varepsilon}(1).}$$

Proof

The star modes are orthonormal because their supports are disjoint, and the adjacency eigenvectors are orthonormal by Corollary 9.7. Let U_X and V_X be the synthesis isometries carrying the standard basis of the centre-index space to these two families. Corollary 9.8 gives

$$\|V_X - U_X\|_{\mathrm{HS}}^2 \ll_{S,\varepsilon} X^{-1/2} \sum_{a \le (\log X)^{1/2-\varepsilon}} a^3 \ll_{S,\varepsilon} X^{-1/2}(\log X)^{2-4\varepsilon}.$$

Since $P_{\mathcal{V}_X} = V_X V_X^*$ and $P_{\mathcal{U}_X} = U_X U_X^*$,

$$\|P_{\mathcal{V}_X} - P_{\mathcal{U}_X}\| \le 2\|V_X - U_X\| \le 2\|V_X - U_X\|_{\mathrm{HS}}.$$

Taking square roots proves the claim. □

The mode term depends only on the distinct prime divisors:

$$c(2) = c(4) = -3, \qquad c(3) = -1, \qquad c(5) = 0, \qquad c(6) = -4.$$

For two common deletion sets,

$$\kappa_\emptyset = B_1 + \sum_p \frac{1}{p(p-1)}, \qquad \kappa_{\{2\}} = B_1 - \frac{1}{2} + \sum_{p>2} \frac{1}{p(p-1)}.$$

The graph is bipartite under $(-1)^{\Omega(n)}$, so every negative extreme eigenvalue is the negative of its positive partner and satisfies the same squared law.

The error in Theorem 9.5 also records the present method boundary. The single-centre first-exit reduction gives constant-order spectral precision when $a^2/\log X = o(1)$, hence throughout $a = o(\sqrt{\log X})$. Extending the constant-layer expansion farther would require a smaller complement-replacement error or a larger retained block; no obstruction beyond this reduction is asserted here.

9.1 Comparison with classical spectral-radius bounds

For $n \ge 2$,

$$\deg_{G_{S,X}}(n) \le \pi_S(X/2) + \omega(n) \le \pi_S(X/2) + \log_2 X.$$

Since $\pi_S(X) - \pi_S(X/2) \asymp X/\log X$, the root is the unique maximum-degree vertex for all sufficiently large X, with

$$\Delta(G_{S,X}) = \deg(1) = \pi_S(X).$$

The induced root star gives the classical lower bound $\lambda_1 \ge \sqrt{\Delta}$. Corollary 8.9 determines its exact second-order excess:

$$\boxed{\lambda_1 - \sqrt{\Delta} = \frac{4\log\log X + 4\kappa_S + o_S(1)}{\lambda_1 + \sqrt{\pi_S(X)}} = \frac{4\log\log X + 4\kappa_S + o_S(1)}{2\sqrt{\pi_S(X)}}.}$$

At the other extreme, let $N_S(X) = |V_S(X)|$ and $m_{S,X}$ be the number of edges. Elementary summation gives

$$N_S(X) \sim \delta_S X, \qquad m_{S,X} = \sum_{\substack{p \le X \\ p \notin S}} N_S(X/p) \sim \delta_S X \log\log X,$$

where $\delta_S = \prod_{p\in S}(1-1/p)$. Hong's general bound

$$\lambda_1 \le \sqrt{2m_{S,X} - N_S(X) + 1}$$

is therefore of order $\sqrt{X \log\log X}$, whereas the true scale is $\lambda_1 \asymp \sqrt{X/\log X}$. The generic upper bound is larger by order $\sqrt{\log X \log\log X}$; it does not see the prime-star cancellation.

9.2 Finite illustration

For $S = \emptyset$, put

$$D_{a,X} = \lambda_{j(a)}(A_X)^2 - \pi(X/a) - 4\log\log X - 4\kappa.$$

Corollary 8.9 predicts $D_{a,X} \to c(a)$. The following contrasts test three features of the centre correction while cancelling the common constant layer:

$$D_{4,X} - D_{2,X} \longrightarrow 0, \qquad D_{5,X} - D_{1,X} \longrightarrow 0,$$

$$D_{6,X} - D_{2,X} - D_{3,X} + D_{1,X} \longrightarrow 0.$$

They respectively test insensitivity to prime-power multiplicity, the zero contribution of the prime 5, and additivity over distinct prime divisors. The full adjacency matrix was assembled in compressed sparse row format, and its largest eigenvalues were computed by the symmetric Lanczos method through ARPACK with tolerance 10^{-10}; no star-compressed eigenproblem was used. The value of κ cancels identically from each contrast.

X	$D_{4,X} - D_{2,X}$	$D_{5,X} - D_{1,X}$	$D_{6,X} - D_{2,X} - D_{3,X} + D_{1,X}$
10^5	-0.347015	-0.758249	-0.218576
10^6	-0.257172	-0.706250	-0.012184
10^7	-0.188205	-0.596186	-0.025043

All three contrasts are closer to the predicted value zero at $X = 10^7$ than at $X = 10^5$; the additive contrast has magnitude below 0.03 at both 10^6 and 10^7. The individual $D_{a,X}$ retain a visibly centre-dependent $1/\log X$ drift. For the displayed nontrivial centres, the coefficient $(\log X)(D_{a,X} - c(a))$ is empirically approximately proportional to $\omega(a)$. The contrasts cancel much of that drift. This is a finite-size observation, not an additional asymptotic assertion.

For a direct test of the finite remainder, let $M_{a,X}$ denote the explicit up-star plus down-star sum on the right-hand side of Theorem 8.7, and put

$$E_{a,X} = \lambda_{j(a)}(A_X)^2 - \pi(X/a) - M_{a,X}.$$

Theorem 8.7 gives $E_{a,X} = O_a(1/\log X)$. The following direct numerical values test this finite-sum statement for the first two centres. The case $a = 2$ includes the nonzero down-star contribution.

X	$E_{1,X}$	$(\log X)E_{1,X}$	$E_{2,X}$	$(\log X)E_{2,X}$
10^4	1.345224	12.389966	2.103328	19.372365
10^5	0.990617	11.404905	1.599306	18.412693
10^6	0.753793	10.414028	1.248529	17.249064
10^7	0.609552	9.824822	1.025544	16.529817
10^8	0.527548	9.717795	0.884951	16.301398

These finite values illustrate the theorem's normalization; they are not used in its proof.

10. Conclusion

Theorem 8.7 identifies the correction to every fixed squared extreme eigenvalue as an explicit sum over up-stars and down-stars. Corollary 8.8 shows that this finite remainder controls the coherent centre–shell phase on time windows $o(\sqrt{X \log X})$, and Corollary 8.9 evaluates it as $4 \log\log X + 4\kappa_S + c(a_j) + o_{S,j}(1)$. Theorems 9.5–9.6 give the quantitative finite-sum error $O_{S,\varepsilon}(a^2/\log X)$ and keep the evaluated constant layer uniformly through the logarithmic window $a \le (\log X)^{1/2-\varepsilon}$. Corollary 9.7 identifies the resulting growing block as simple and ordered by its arithmetic centres. Corollaries 9.8–9.9 show that the same block is uniformly localized on its prime stars and supports exact-frequency centre–shell evolution, uniformly for all real times. Corollary 9.10 packages these individual estimates as operator-norm convergence of the whole logarithmic-window spectral projection to the corresponding prime-star projection. Thus the arithmetic cutoff produces an exact prime-star boundary, first-order coupling vanishes, and the surviving self-energy is evaluated through constant order. The extreme spectrum is therefore determined by the star degrees, the universal up-star term, and the divisor correction $c(a_j)$. The condition $a = o(\sqrt{\log X})$ is the precision boundary of the present single-centre reduction, not a claimed barrier for other spectral methods.

Declarations and code availability

Formalization and software availability

The accompanying Lean 4 development formalizes the ranked fixed-centre constant-layer theorem and Theorem 9.6 on the full logarithmic window. The selected exported declaration is `PrimeStar.palomar_uniform_finite_prime_sum`; the underlying theorem is `PrimeStar.eventually_logarithmicWindow_ranked_finitePrimeSum_proved`. The development also includes exported theorems for fixed-centre localization, uniform logarithmic-window localization, the bipartite sign pairing, and exact-frequency centre–shell evolution. Quantitative prime-number-theorem and prime-Mertens inputs are imported from `PrimeNumberTheoremAnd` at commit `7715064f690d0689f30889846f4e2c5e7ec0c47e`.

The Lean sources are in the `formal` directory of https://github.com/shaikidris/prime-orthant-geometry at commit `6344fb2a7f4aa8826c43fb3a45b17ed526f59dc9`; the development is licensed under the Apache License 2.0. The audit file reports only `propext`, `Classical.choice`, and `Quot.sound` for the exported theorem roots. The uniform replacement of the exact eigenfrequency by the explicit asymptotic frequency in Corollary 9.9, and the collective projection statement in Corollary 9.10, are proved in the manuscript but are not packaged as standalone exported Lean declarations.

Declaration of competing interest

The author declares no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No external datasets were used. The finite numerical outputs in Section 9 are reproducible from the scripts in the source repository and are not used as premises of any theorem.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During preparation, the author used ChatGPT and OpenAI Codex for proof exploration, manuscript editing, diagnostic code, and Lean 4 formalization. The author reviewed and edited all outputs and takes responsibility for the content of the manuscript.

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