The Riemann Zeta Function in Hidden-Number Coordinates : From Classical Inputs to the Critical-Circle Reduction

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Abstract: We give a logically ordered account of the hidden-number coordinate description of the Riemann zeta function. The hidden-number space used here was introduced and developed by the present author in earlier work (Liu and Xu, 2026); the present paper applies that prior coordinate framework to the zero set of ζ. The argument starts with classical analytic facts: the functional equation, the zero-free half-plane, and Hardy's theorem on zeros on the critical line. The change of variables $w = e^s$ then translates reflection into the inversion $w \mapsto e/w$ and the critical line into the unique inversion-fixed circle $|w| = \sqrt{e}$. From these facts we derive the annular location and reflection pairing of nontrivial zeros, and Hardy's theorem transports directly to the statement that infinitely many lifted zeros lie over this circle. The energy and frequency calculations are included as exact coordinate diagnostics; they do not by themselves create zeros or prove a density law. The Riemann hypothesis is consequently isolated as the remaining assertion that every nontrivial reflection pair is self-paired. Lean identifiers are retained in the source for traceability but are suppressed in the typeset article.

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1. Question, inputs, and logical order

The Riemann hypothesis concerns the nontrivial zeros of $\zeta(s)$. It asserts

$$0 < \Re s < 1, \quad \zeta(s) = 0 \quad \Longrightarrow \quad \Re s = \tfrac{1}{2}.$$

The purpose of the present paper is not to replace the classical analytic theory with a list of formal names. It is to make the implication structure visible. We will use three kinds of statements :

Classical inputs. The functional equation, the standard zero-free regions, and Hardy's theorem are taken as known analytic results.

Exact deductions. Coordinate changes, modulus identities, reflection pairing, and the equivalence between the critical-line and critical-circle descriptions are proved directly.

Open statements. The exclusion of non-self-paired zeros in the critical annulus is exactly the Riemann hypothesis. Numerical density and energy heuristics are not used to hide this gap.

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Bibliographic context. The foundational memoir is Riemann's original work (Riemann, 1859). The zero distribution and its relation to prime counting are treated in classical work by von Mangoldt, Hadamard, and de la Vallée Poussin (von Mangoldt, 1905; Hadamard, 1893; de la Vallée Poussin, 1899); elementary equivalent formulations are discussed by von Koch, Schoenfeld, and Lagarias (von Koch, 1901; Schoenfeld, 1976; Lagarias, 2002). Hardy's theorem gives the critical-line existence input used below (Hardy, 1914), with later quantitative developments by Hardy–Littlewood, Selberg, Levinson, Heath-Brown, and Conrey (Hardy and Littlewood, 1921; Selberg, 1942; Levinson, 1974; Heath-Brown, 1979; Conrey, 1989). Pair-correlation and spectral perspectives are represented by Montgomery, Connes, and Bombieri–Hejhal (Montgomery, 1973, 1977; Connes, 1999; Bombieri and Hejhal, 1995); large finite-height computations are reported by Platt and Trudgian (Platt and Trudgian, 2021). Standard accounts and surveys include Edwards, Titchmarsh, Iwaniec–Kowalski, Iwaniec, and Conrey (Edwards, 1974; Titchmarsh, 1986; Iwaniec and Kowalski, 2004; Iwaniec, 2014; Conrey, 2003).

The rest of the paper follows this order : first the coordinate dictionary, then the deductions about the zero set, then the direct transport of Hardy's theorem, and finally the auxiliary energy/frequency diagnostics and the remaining open problem.

This order is intentionally asymmetric. The s-plane is the place where the analytic facts about ζ are stated, whereas the w-plane is the place where the reflection geometry becomes transparent. We therefore do not begin by asking the coordinates to prove that a zero exists. We first ask what the change of coordinates preserves, then apply it to facts already known about the zero set, and only afterwards discuss the information that finite sums and numerical experiments can provide. This separation prevents a geometric picture from being mistaken for an analytic zero-counting argument.

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2. The coordinate dictionary

Origin of the space and contribution of this paper. The hidden-number space is not an auxiliary notation introduced only for this manuscript. It is the coordinate framework proposed and developed by the present author in earlier work (Liu and Xu, 2026). Its basic idea is to retain the radial coordinate and the lifted angular coordinate at the same time : projection to $w \neq 0$ forgets the sheet, whereas the logarithmic cover remembers it. This distinction is what allows vertical translation in the s-plane to be tracked as motion between lifts over the same point of the punctured w-plane. The contribution here is therefore more specific : we use that prior construction to organize the zero-set argument for ζ, derive the reflection/inversion dictionary, and separate exact coordinate deductions from classical analytic inputs and the remaining open claim.

Write $s=\sigma+it$ and introduce

$$\Phi(s)=e^{s}=w=e^{\sigma}e^{it}.$$

The radius records the real part, while the argument records the imaginary part only modulo 2π. Thus the natural domain is a logarithmic cover : a point of the cover remembers both $w\neq 0$ and a sheet index. This is a change of coordinates, not an additional assumption about ζ.

Definition 1 (hidden-number coordinates)**.** *The hidden-number coordinate of $s=\sigma+it$ is the pair (w,k) with $w=e^{s}$ and k recording the chosen lift of t modulo 2π. In particular,*

$$|w|=e^{\sigma},\qquad \log|w|=\sigma.$$

More concretely, choose an angle $\theta\in[0,2\pi)$ and a sheet index $k\in\mathbb{Z}$. A point on the logarithmic cover can then be written as

$$w=re^{i\theta},\qquad s=\log r+i(\theta+2\pi k),\qquad r>0.$$

The projection forgets k, since all these lifts have the same image w. The cover does not identify the corresponding values of s : changing k changes the height by 2π and records a different point upstairs. This is the precise sense in which the hidden-number space keeps information that is invisible in the punctured w-plane. It is also why the construction does not replace the zeta function by a new function. We still evaluate ζ at the lifted point s; only its radial and angular data are being stored in a different form.

For the shifted Dirichlet terms, the same split reads

$$(n+1)^{-s}=(n+1)^{-\sigma}\exp\bigl(-it\log(n+1)\bigr).$$

The first factor is radial data and the second is lifted angular data. This identity is elementary, but it explains why the later energy and frequency calculations belong naturally after the geometric reduction : they inspect the two coordinates separately without yet making a claim about analytic continuation.

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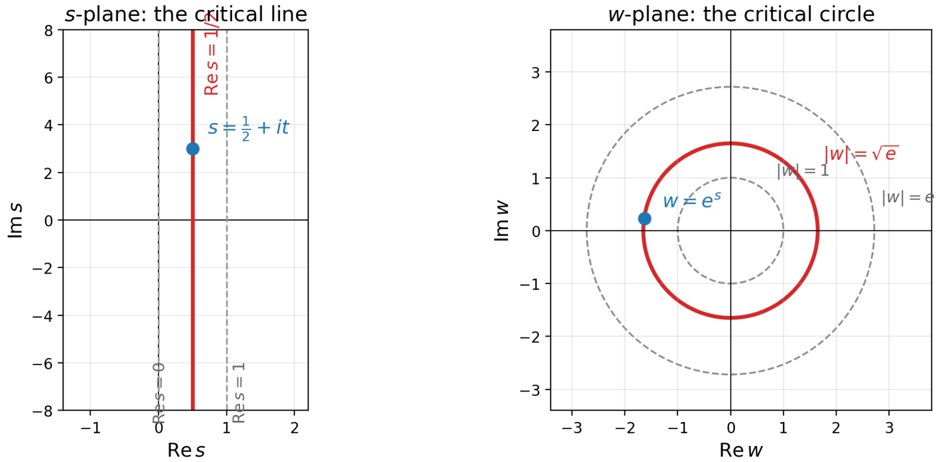

FIGURE 1. *The coordinate projection $w = e^s$. Vertical lines $\Re s = \sigma$ become circles $|w| = e^{\sigma}$; the critical line becomes $|w| = \sqrt{e}$.*

reduction : they inspect the two coordinates separately without yet making a claim about analytic continuation.

The functional equation reflects s to $1-s$. Under Φ this becomes

$$\Phi(1-s) = e^{1-s} = \frac{e}{e^s} = \frac{e}{w}.$$

We therefore obtain the following exact dictionary :

$$s \longmapsto 1-s \qquad \Longleftrightarrow \qquad w \longmapsto \iota(w) := \frac{e}{w}.$$

3. The candidate circle forced by reflection

We now derive the geometric candidate before discussing any zero. For real t,

$$\left|e^{1/2+it}\right| = e^{1/2} = \sqrt{e}.$$

Theorem 2 (the critical line maps to a circle)**.** *The image of $\Re s = 1/2$ under $w = e^s$ is exactly $|w| = \sqrt{e}$.*

The inversion also determines this radius independently. If $|w| = \rho$, then $|\iota(w)| = e/\rho$. The circle is preserved precisely when $e/\rho = \rho$.

Proposition 3 (unique inversion-fixed circle)**.** *Among circles centered at the origin, the only circle preserved by $\iota(w) = e/w$ is $|w| = \sqrt{e}$. Its logarithmic radius is $\log\sqrt{e} = 1/2$.*

There are two different meanings of “fixed” here. Setwise, the inversion preserves the circle of radius ρ when it sends that circle back to the same circle. Pointwise, a particular point must satisfy $e/w = w$. The first condition selects the radius $\sqrt{e}$; the second selects only the two points $w = \pm\sqrt{e}$. Thus the critical circle is the unique geometric candidate for self-reflection, but it is not a claim that every point on the circle is fixed, nor that every point on it is a zero.

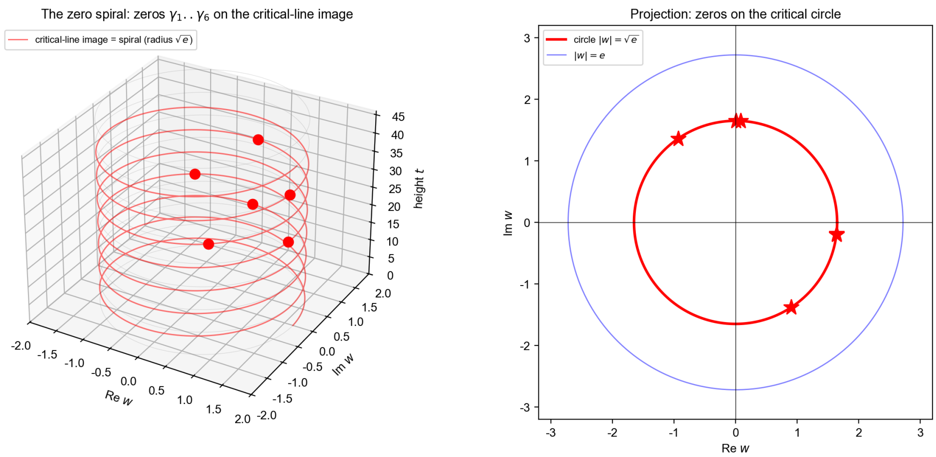

FIGURE 2. *A static view of the logarithmic cover. The critical line lifts to a helix of radius $\sqrt{e}$ and projects to the critical circle.*

Remark (geometry is not zero existence)**.** *The proposition identifies the only possible self-reflection circle. It does not say that ζ vanishes at any point of the circle. The existence of zeros is a separate analytic question and will be supplied below by Hardy's classical theorem.*

4. What the classical zero information implies

We next use known information about ζ, in the direction needed for the Riemann hypothesis. The Euler product gives the standard zero-free region $\Re s \geq 1$; the functional equation and the known trivial zeros handle the opposite side. Consequently every nontrivial zero lies in the critical strip :

$$\zeta(s) = 0 \text{ and } s \text{ nontrivial} \quad \Longrightarrow \quad 0 < \Re s < 1.$$

Applying $|w| = e^{\Re s}$ gives the coordinate version

$$1 < |w| < e.$$

The strictness of this annular statement matters. The outer boundary $|w| = e$ corresponds to $\Re s = 1$, where the Euler product gives zero-freeness in the usual formulation. The inner boundary $|w| = 1$ corresponds to $\Re s = 0$; after the trivial zeros and the exceptional points in the functional equation are separated, the nontrivial zeros are confined to the open strip. The coordinate change therefore transports a classical strip statement into an open annulus, but it does not sharpen the classical zero-free information by itself.

Theorem 4 (reflection pairing)**.** *If $0 < \Re s < 1$ and $\zeta(s) = 0$, then $\zeta(1-s) = 0$, and conversely. In hidden-number coordinates the pair is $(w, e/w)$.*

This is a direct consequence of the functional equation after checking that its factors do not vanish in the open critical strip. It is a symmetry of the zero set, not yet a statement that the two members of a pair coincide.

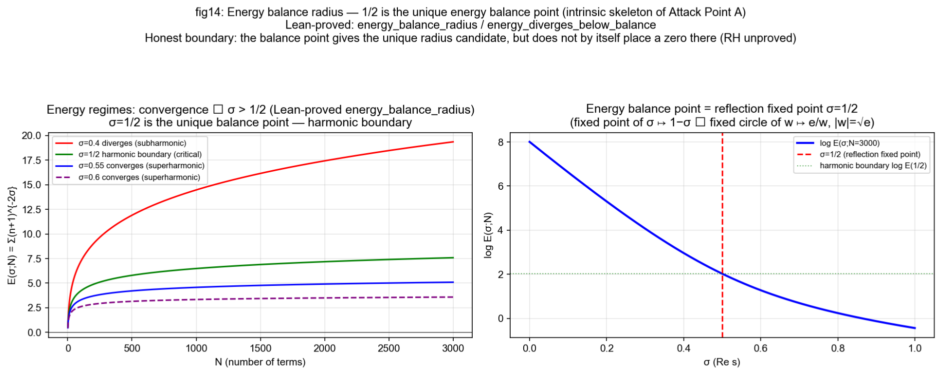

FIGURE 3. *The geometric and analytic scales in the coordinate picture. The reflection-fixed radius is $\sqrt{e}$, while the zero-free boundary $\Re s = 1$ corresponds to $|w| = e$.*

Geometrically, a zero away from the candidate circle comes with a distinct partner on the opposite side of the circle. The map $w \mapsto e/w$ exchanges radii ρ and e/ρ and preserves their product e. Only when $\rho = \sqrt{e}$ can the two radii agree. The analytic question left by the functional equation is therefore not whether reflection exists, but whether the zero set contains any genuinely non-self-paired orbit.

Theorem 5 (self-pairs and the critical circle)**.** *For $w \neq 0$,*

$$\frac{e}{w} = w \iff w^2 = e \iff w = \pm\sqrt{e}.$$

Thus a self-paired zero, if present, lies on $|w| = \sqrt{e}$.

The preceding deductions give the exact reduction :

Corollary 6 (the remaining statement is RH)**.** *For nontrivial zeros, the Riemann hypothesis is equivalent to the assertion that every reflection pair in $1 < |w| < e$ is self-paired. Equivalently, there are no non-self-paired zeros away from $|w| = \sqrt{e}$.*

This is a reduction rather than a solution. If one starts with the Riemann hypothesis, every nontrivial zero has $\Re s = 1/2$ and hence lies on the circle. Conversely, if every reflection orbit in the annulus collapses to a self-pair, the algebra above forces $w^2 = e$, and the corresponding lift has $\Re s = 1/2$. The two directions use different ingredients : the first is the definition of the hypothesis in the original coordinate, while the second uses the reflection pairing and the self-pair calculation.

5. Abundance on the circle : the known theorem transported

The preceding sections identify and constrain the location. They do not produce a zero. For existence and abundance we now invoke the classical theorem of Hardy : infinitely many nontrivial zeros of ζ lie on the line $\Re s = 1/2$ (Hardy, 1914).

Theorem 7 (Hardy's theorem in hidden-number coordinates)**.** *There are infinitely many lifted zeros of ζ over the critical circle $|w| = \sqrt{e}$.*

Démonstration. Let Z be the infinite set of zeros on the critical line supplied by Hardy's theorem. The logarithmic cover remembers the lifted height, and Theorem 2 sends every element of Z to a lift over the circle $|w| = \sqrt{e}$. Hence the set of lifted circle zeros is infinite. $\square$

This is a transport theorem, not a new proof of Hardy's result. It is the correct logical role of the hidden-number coordinates : they change the geometric language of a known abundance theorem while preserving its content.

The word "lifted" is important in this statement. The exponential map forgets heights modulo 2π, while the hidden-number space retains the sheet on which a height occurs. Consequently the transport preserves the information needed to regard Hardy's infinitely many zeros as infinitely many objects upstairs, even when their projections are described only by the single radius $\sqrt{e}$. No new zero is manufactured by the projection ; the existence and infinitude enter entirely through Hardy's theorem.

6. Energy and frequency diagnostics

The coordinate system also exposes two useful diagnostics. They explain why $1/2$ is a natural scale and how height enters the finite approximants, but they must not be promoted to zero-existence arguments.

For the Dirichlet terms,

$$(n+1)^{-s} = (n+1)^{-\sigma} \exp\bigl(-it\log(n+1)\bigr), \qquad \bigl|(n+1)^{-s}\bigr| = (n+1)^{-\sigma}.$$

On the critical line the squared lengths are $(n+1)^{-1}$, so the diagonal energy has the harmonic scale

$$\sum_{n\le N} \bigl|(n+1)^{-1/2-it}\bigr|^2 = \sum_{n\le N} \frac{1}{n+1} \sim \log N.$$

This is an exact calculation about terms and finite sums. It is not an estimate for the analytically continued value of ζ in the critical strip.

This distinction is the main reason for placing the diagnostics after the zero-set argument. In the half-plane of absolute convergence, a Dirichlet series and its partial sums can be compared by standard limiting arguments. In the critical strip, however, the analytically continued function is the object whose zeros are being studied. A finite vector sum may cancel while the corresponding value of the continuation is not zero, and an unbounded diagonal energy may coexist with substantial cross-term cancellation. The calculation identifies a scale ; it does not supply the missing analytic bridge.

The consecutive logarithmic frequencies satisfy

$$\Delta\mathrm{freq}(n) = \log\frac{n+2}{n+1}, \qquad \Delta\mathrm{freq}(n) \longrightarrow 0, \qquad \Delta\mathrm{freq}(n+1) \le \Delta\mathrm{freq}(n).$$

For an alternating approximation the phase difference is

$$\Delta\theta_n = \pi - t\,\Delta\mathrm{freq}(n).$$

When $t\log 2 > \pi$, this monotone quantity changes sign once ; the crossing index is of order t/π. These identities describe the finite frequency skeleton and are useful for organizing numerical experiments.

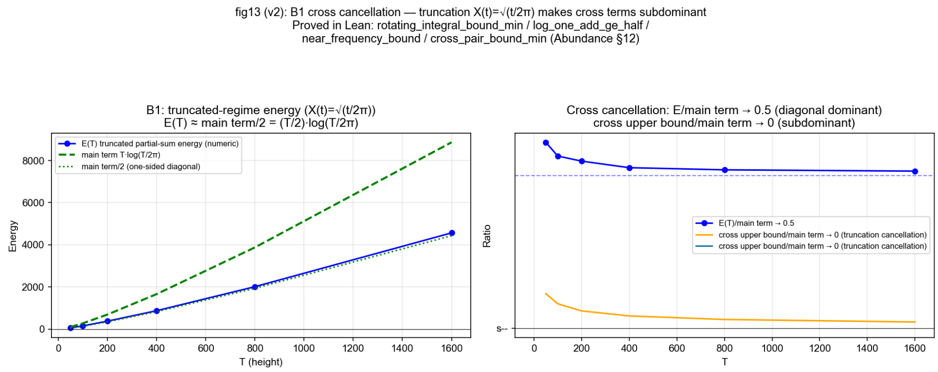

FIGURE 4. *A numerical energy budget for truncated sums. The plot illustrates the harmonic scale ; it is not used as a substitute for a zero-counting theorem.*

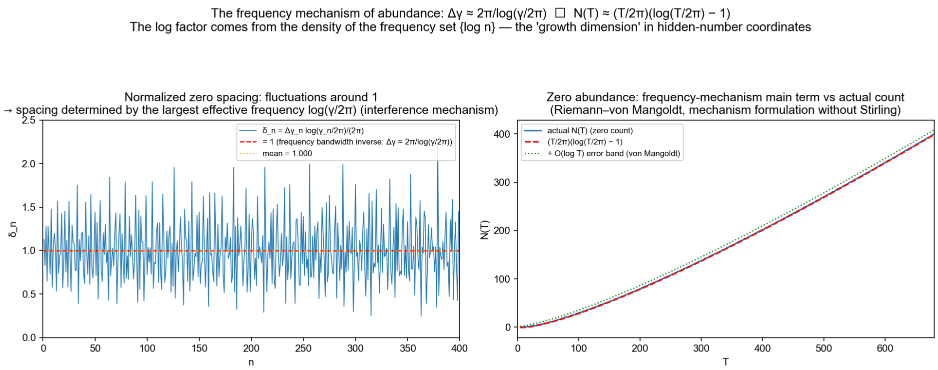

FIGURE 5. *Numerical spacing and frequency diagnostics. Such comparisons can suggest a density law but do not prove one.*

The frequency statement has a similarly limited but useful role. It says that the logarithmic frequencies become locally closer as the index grows, so a high-height experiment contains long stretches in which adjacent phase increments are small. That observation can suggest where partial sums may show coherent motion or a fold. It does not say that such motion reaches the origin, and even reaching the origin for a truncation would still require a tail estimate before it could be identified with a zero of ζ.

Remark (why the diagnostics do not close RH)**.** *In the critical strip the Dirichlet series is not an interchangeable name for its analytic continuation. A finite partial sum can display alignment without being a zero of ζ, and a harmonic energy divergence does not give a zero-counting estimate. The missing analytic bridge would require boundary estimates and an argument-principle or zero-counting theorem.*

7. Formalization and the exact boundary of the argument

The Lean development records many of the algebraic and order-theoretic components used above : the coordinate identities, the inversion map, the reflection involution, the self-pair algebra, the harmonic and frequency lemmas, and the elementary zero-free estimate in its stated domain. The source retains the corresponding identifiers such as , , and ; their absence from the typeset output keeps the mathematical prose readable.

The formal names should consequently be read as a traceability layer, not as a substitute for hypotheses. A theorem about e^s proves a coordinate identity ; it does not import Hardy's theorem. A predicate describing an unbounded zero family records the shape of an analytic input ; it does not provide an instance of that input. Keeping these levels separate is especially important for the final RH reduction, where the remaining statement is genuinely open.

The logical boundary is now explicit :

1. The change of variables derives the critical circle and explains why its logarithmic radius is $1/2$.
2. Classical zero-free information derives the critical annulus, and the functional equation derives reflection pairs.
3. Hardy's known theorem derives infinitely many circle zeros by direct transport.
4. The Riemann hypothesis is the still-open exclusion of non-self-paired zeros in the annulus. Neither the coordinate change nor the diagnostic calculations prove that exclusion.

Any future attempt to replace Hardy's input with an internal proof must supply the missing analytic bridge : a zero-counting or boundary-winding estimate for the analytically continued ζ. Naming that requirement is part of the argument, not an omission in its presentation.

8. Conclusion

The hidden-number coordinate $w = e^s$ reorganizes the known theory into a short deduction chain. Reflection becomes $w \mapsto e/w$; its unique fixed circle is $|w| = \sqrt{e}$; standard zero-free information places nontrivial zeros in $1 < |w| < e$; the functional equation pairs them ; and Hardy's theorem gives infinitely many lifted zeros over the fixed circle. The remaining off-circle exclusion is exactly the Riemann hypothesis. Energy and frequency structures clarify the coordinate geometry and guide computation, but they remain supporting diagnostics rather than unproved steps in the main deduction.

All formalized components were developed in Lean (de Moura et al., 2015) with Mathlib (The mathlib Community, 2020).

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