The Hidden-space Bridge System: A Three-axiom Foundation Connecting the Real and Imaginary Domains

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Abstract

We present a *generative-layer proposal* called HIBS (Hidden-space Bridge System): a conjecture that the complex plane $\mathbb{C}$ is not a foundational object but rather the observable slice of a hidden generative layer $\mathbb{S}$ under a projection π (§2). We introduce an abstract space $\mathbb{S}$ whose elements are called *hidden numbers* $\langle$—neither real nor imaginary. "Hidden number" is an *intermediate notation* for the generative layer. It is not a new kind of number. Under certain arithmetic operations, a hidden number is mapped by the flow operator Flow onto the real line $\mathbb{R}$ or onto the imaginary line $i\mathbb{R}$; under other operations it remains inside $\mathbb{S}$. Three axioms are postulated: (1) the existence of two maps $f\colon \mathbb{S} \to \mathbb{R}$ and $g\colon \mathbb{S} \to i\mathbb{R}$ that are non-injective (no left-inverses); (2) closure of $\mathbb{S}$ under addition and subtraction, together with the fact that multiplication and division necessarily map onto $\mathbb{R}$; and (3) the square-root operation maps onto $i\mathbb{R}$, with the positive half-axis mapping to $i\mathbb{R}^-$ and the negative half-axis to $i\mathbb{R}^+$. The main mathematical results of the paper are: (i) $\mathbb{C}$ is the projection image of $\mathbb{S}$ under π ($\pi \circ \iota = \mathrm{id}_{\mathbb{C}}$, theorem 6.5); (ii) ι is *not* a multiplicative monomorphism—the "rotation" semantics of $\mathbb{C}$ multiplication cannot pass through ι into $\mathbb{S}$ (theorem 6.2). These two results together give a *axiomatic-level account* of the structural asymmetry between the real and imaginary parts of $\mathbb{C}$.

Keywords: hidden space, hidden number, axiomatics, complex numbers, flow operator, HIBS, irreversible operations
MSC 2010 classification: Primary 12E05; Secondary 30A99

1 Introduction

The complex numbers $\mathbb{C}$ first appeared in the 16th century (Cardano, Bombelli) and have since become fundamental across analysis, algebra, and geometry. Cardano's 1545 *Ars Magna* is the first recorded use of the negative square root [3]; in 1777 Euler introduced the symbol i for $\sqrt{-1}$ together with the formula $e^{i\theta} = \cos\theta + i\sin\theta$; in 1797 Wessel, in 1806 Argand, and decisively in 1831 Gauss proposed the plane representation of $a+bi$ as the point (a,b) [18]; in the mid-19th century Cauchy, Riemann and Weierstrass elevated complex analysis into a rigorous branch of mathematics [8]. The standard construction begins with the formal definition $i^2 = -1$ and proceeds to $\mathbb{R}^2$ with componentwise addition and the obvious multiplication. This construction, while syntactically simple, does not explain the structural role of the imaginary part.

In this paper we re-examine the source of the complex plane from a more elementary perspective. We do not presuppose $\mathbb{C}$ as a foundational algebraic object; we begin from a set of three axioms on an abstract space $\mathbb{S}$ (the *hidden space*) and recover $\mathbb{C}$ as the *observable slice of* $\mathbb{S}$ *under a projection* π (§2). The construction is as follows. We introduce a space $\mathbb{S}$ whose elements, the *hidden numbers* $\langle$, are neither real nor imaginary. "Hidden number" is a *teaching notation*—a label used during the pedagogical transition to $\mathbb{C}$, not a new kind of number. Certain arithmetic operations map $\langle \in \mathbb{S}$ via the flow operator $\mathrm{Flow}(\cdot, \mathrm{op})$ onto $\mathbb{R}$ or onto $i\mathbb{R}$; other operations map onto $\mathbb{S}$ itself. The flow is treated as a primitive operator satisfying the three axioms below.

This approach offers the following:

- An axiomatic foundation for $\mathbb{C}$ that does not start from the formal definition $i^2 = -1$;

- A precise description of the asymmetry between addition/subtraction (closed in $\mathbb{S}$) and multiplication/division (mapped onto $\mathbb{R}$);
- A framework for studying nested algebraic structures via iterated flow operations.

Related work. Closest classical constructions include Hamilton's quaternions [2], Dedekind cuts [3], and the category-theoretic treatment of field extensions [4]. The methodological background is given by Hilbert's formal algebraic programme [1], the Birkhoff–Mac Lane unified-algebra perspective [5], Landau's axioms of analysis [6], and Rudin's principles of analysis [8].

2 A Structural Defect in the Standard Axiomatisations of $\mathbb{C}$

This section provides the *structural motivation* of HIBS—answering two questions: (i) why is HIBS needed, and (ii) what specific defect in the existing axiomatisations of $\mathbb{C}$ does HIBS address. Our entry point is the *complex conjugation* $\bar{z} = a - bi$.

The standard chain of field extensions. $\mathbb{C}$ is usually regarded as the terminal step of the algebraic-extension chain $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$; each step has a strict construction [3, 6]. $\mathbb{C}$ itself admits three standard axiomatisations:

- *Polynomial-ring quotient* $\mathbb{R}[x]/(x^2 + 1)$—algebraic-geometry style [7];
- *Matrix representation* $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ —linear-algebra style [7];
- *Formal symbol* $i^2 = -1$—elementary style [8].

A concrete phenomenon: the source of conjugation. Complex conjugation $\bar{z} = a - bi$ is the basic operation of $\mathbb{C}$ underlying modulus, modulus-squared, and argument. Yet the three standard paths treat $\bar{z}$ *asymmetrically*:

- Polynomial-ring path: $\bar{i} = -i$ is the *other* root of $x^2 + 1$—an *algebraic fact*, not an explanation of "why $\mathbb{C}$ naturally has two roots";
- Matrix path: $\bar{z}$ is the adjoint matrix—a *definition-for-definition translation*, not a structural account;
- Formal-symbol path: i is a formal object; conjugation $i \mapsto -i$ is *manually stipulated*, with no axiom-level source.

Common structural defect: all three paths *accept* $\mathbb{C}$ as an already-split (real-part + imaginary-part) algebraic object, and take the asymmetry between real and imaginary parts as a definition rather than as the consequence of a more elementary structure.

The one-way construction shared by all three paths. The "rotation" semantics of complex multiplication is verified directly from the expansion $(ac-bd)+(ad+bc)i$ of $z_1 \cdot z_2$. The matrix path explains this through $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \in \mathrm{SO}(2)$—but this essentially *replaces "why" with "corresponds to"*: multiplication is rotation, because it corresponds to a rotation matrix. None of the three paths *reverses the question* and asks: is the structure of $\mathbb{C}$ (real + imaginary, addition componentwise, multiplication cross-terms) *generated* by something more elementary?

HIBS conjecture (a strict statement). We propose a verifiable *Conjecture (Generative layer)*: there exists a generative layer $(\mathbb{S}, \Sigma, \oplus_{\mathbb{S}}, \otimes, \sqrt{\cdot}, f, g)$ satisfying the three axioms of §5; and a projection map $\pi : \mathbb{S} \to \mathbb{C}$ such that:

(i) $\pi \circ \iota = \mathrm{id}_{\mathbb{C}}$—$\mathbb{C}$ is the projection image of $\mathbb{S}$ (§6 gives a strict proof);

(ii) $\pi(\bar{z})$ corresponds to the *signal-reversal* operation $\langle^+ \leftrightarrow \langle^-$ in $\mathbb{S}$—an *axiom-level source of complex conjugation*: H^+-elements map to $\mathbb{R}$, reversed-signal H^--elements map to $i\mathbb{R}$; the conjugation $i \mapsto -i$ in $\mathbb{C}$ corresponds to signal reversal in $\mathbb{S}$;

(iii) π preserves addition on the image of ι but *does not* preserve multiplication (consistent with theorem 6.2).

If this conjecture holds in some concrete model of $\mathbb{S}$, then $\mathbb{C}$ is no longer "the terminal step of the algebraic-extension chain" but the observable projection of a generative layer $\mathbb{S}$—whose flow-signal mechanism gives an *axiom-level* account of $\mathbb{C}$'s structural asymmetry.

Four main contributions of the paper. Around this conjecture, the paper delivers: (1) the three axioms of the generative layer $\mathbb{S}$ (§5); (2) a concrete model $\mathbb{S} = \mathbb{R} \times \Sigma$ verifying their consistency (§4); (3) a strict proof of $\pi \circ \iota = \mathrm{id}_{\mathbb{C}}$ (theorem 6.5)—the proof of Conjecture (i); (4) verification of Conjecture (ii) on the concrete model: $\bar{z}$ corresponds to signal reversal (§6 example).

Limitation of the complex-number framework: operational vs. generative structure. The real significance of the Conjecture above is clarified by the *limitation of the complex-number framework itself.* Reviewing the historical chain of field extensions—each step was driven by an *unambiguous failure case*: $\mathbb{N}$ cannot express subtraction, hence $\mathbb{Z}$; $\mathbb{Z}$ cannot express arbitrary division, hence $\mathbb{Q}$; $\mathbb{Q}$ admits divergent Cauchy sequences, hence $\mathbb{R}$; $\mathbb{R}$ contains the unsolvable $x^2 + 1 = 0$, hence $\mathbb{C}$. Each step was *forced* by a concrete mathematical problem the prior system could not handle.

The introduction of $\mathbb{C}$ is the *terminal* step of this chain. The complex field $\mathbb{C}$ resolves algebraic closure perfectly: every non-constant complex polynomial has a root in $\mathbb{C}$ [8]. Yet this very success *obscures a long-ignored mathematical problem*—classical complex analysis *describes how complex numbers work, but does not explain why the imaginary direction appears.* Concretely:

- The theory of $\mathbb{C}$ specifies *how* complex numbers *work* (addition, multiplication, square root, conjugation, modulus, argument)—this is the *operational structure* of complex numbers;
- The theory of $\mathbb{C}$ does *not explain why complex numbers appear*: the imaginary unit i is introduced axiomatically through $i^2 = -1$—this definition determines i's algebraic properties, but does *not* explain the *existence* and *emergence mechanism* of the imaginary direction.

Specifically, three questions that complex number theory does not answer:

- Why does such an "imaginary direction" exist at all?
- Might i admit *other generative origins* (different generative layers)?
- Does the complex plane $\mathbb{C}$ itself *emerge* from a more elementary structure?

These three questions have no answer in $\mathbb{C}$'s standard axiomatisations (polynomial-ring quotient, matrix representation, formal symbol)— they are *accepted as definitions* rather than *derived and explained.*

The *limitation of the complex-number framework* can therefore be stated precisely: it studies the *operational structure* of complex numbers (how they work), but not their *generative structure* (why they appear). **HIBS studies the latter**—it does *not* replace $\mathbb{C}$ (all theorems of $\mathbb{C}$ continue to hold), but instead *questions the origin of* $\mathbb{C}$: is there a more elementary object $\mathbb{S}$ such that $\mathbb{C}$ *naturally emerges* as its observable projection?

This distinction gives HIBS a clear position in mathematics: HIBS is not "a replacement for $\mathbb{C}$" or "a new number system" but an *axiomatic study of the generative structure of complex numbers*—it studies **not "how complex numbers work" but "why complex numbers appear"**. Classical complex analysis answers the former (how $\mathbb{C}$ works); HIBS answers the latter (why $\mathbb{C}$ appears).

3 Basic Definitions

Definition 3.1 (Hidden space)**.** Denote by $\mathbb{S}$ a *space* called the *hidden space.* Its elements are not specified a priori; they are constrained only by the axioms stated below. $\mathbb{S}$ is neither a subset of $\mathbb{R}$ nor of $\mathbb{C}$: its elements are *teaching notation*, an intermediate label used during the pedagogical transition to $\mathbb{C}$.

Definition 3.2 (Hidden number)**.** An element of $\mathbb{S}$ is called a *hidden number*, denoted $\langle$. A hidden number is neither real nor imaginary: $\langle \notin \mathbb{R} \cup i\mathbb{R}$.

Definition 3.3 (Positive and negative hidden numbers)**.** $\mathbb{S}$ is partitioned into two disjoint subsets $\langle^+$ and $\langle^-$, called the *positive* and the *negative hidden numbers* respectively. By convention,

$$\langle^+ = \big\{ \langle \in \mathbb{S} : \mathrm{Flow}(\langle, \times) \subset \mathbb{R} \big\}, \qquad \langle^- = \big\{ \langle \in \mathbb{S} : \mathrm{Flow}(\langle, \sqrt{\cdot}) \subset i\mathbb{R} \big\}.$$

Definition 3.4 (Flow operator)**.** For any hidden number $\langle \in \mathbb{S}$ and any arithmetic operation $\mathrm{op} \in \{+, -, \times, \div, \sqrt{\cdot}\}$, the emphflow is a map

$$\mathrm{Flow}(\cdot, \mathrm{op}) \colon \mathbb{S} \to \{\mathbb{R},\ i\mathbb{R},\ \mathbb{S}\},$$

where the codomain element depends on op as specified in the axioms. We write $\mathrm{Flow}(\langle, \mathrm{op}) = \mathbb{R}$ to indicate that $\langle$ under op *projects onto* the real line, and similarly for $i\mathbb{R}$ and $\mathbb{S}$.

Remark 3.5. The exact partition of $\mathbb{S}$ into $\langle^+$, $\langle^-$ and the precise value of Flow need *not* be specified before the axioms; they can be derived from the axioms. The intuitive descriptions above are given first; a rigorous derivation is provided in Section 6.

Analytic foundations. The axiomatisation of arithmetic on $\mathbb{R}$ has been completed by Landau [6] and Rudin [8]; the present $\mathbb{S}$ is a second-order abstraction built on top of those: the algebraic structure of $\mathbb{R}$ enters as a sub-structure, and the flow operator is an additional signal layer wrapping it.

The projection π onto $\mathbb{C}$ (definition). In preparation for the next sections, we explicitly define the *projection map* $\pi : \mathbb{S} \to \mathbb{C}$, which sends each element of $\mathbb{S}$ to $\mathbb{C}$ according to its signal:

$$\pi(\langle) \;=\; \begin{cases} x \in \mathbb{R} \subseteq \mathbb{C} & \text{if } \mathrm{lab}(\langle) = \mathbb{R}, \\ ix \in i\mathbb{R} \subseteq \mathbb{C} & \text{if } \mathrm{lab}(\langle) = i\mathbb{R}, \\ x \in \mathbb{R} \subseteq \mathbb{C} & \text{if } \mathrm{lab}(\langle) = \mathbb{S}. \end{cases}$$

In the third case (signal $\mathbb{S}$), an element of $\mathbb{S}$ is interpreted as its real projection by the non-injective projection of (A1); the well-definedness of π is verified concretely on the labelled-pair model of §4 (see theorem 6.5 in §6). π plays the role of the *principal projection* in this paper: the complex plane $\mathbb{C}$ is the observable slice of $\mathbb{S}$ under π.

4 Concrete Realisation, Consistency, and Independence

We first construct a model that satisfies (A1)–(A3) and thereby prove the axiom system is non-vacuous; we then exhibit three counter-models to establish that the axioms are pairwise independent. Both tasks follow directly from the operator semantics of the axioms.

Model 4.1 (Labelled-pair model)**.** Let

$$\mathbb{S} \;=\; \mathbb{R} \times \Sigma, \qquad \Sigma = \{\mathbb{S}, \mathbb{R}, i\mathbb{R}\}.$$

For $\langle_1 = (x_1, \sigma_1)$ and $\langle_2 = (x_2, \sigma_2)$ we define

$$\langle_1 + \langle_2 := (x_1 + x_2,\ \mathbb{S}), \tag{4.1}$$
$$\langle_1 - \langle_2 := (x_1 - x_2,\ \mathbb{S}), \tag{4.2}$$
$$\langle_1 \times \langle_2 := (x_1 x_2,\ \mathbb{R}), \tag{4.3}$$
$$\langle_1 \div \langle_2 := (x_1 / x_2,\ \mathbb{R}) \quad (x_2 \neq 0), \tag{4.4}$$
$$\sqrt{\langle_1} := \begin{cases} (\sqrt{x_1},\ i\mathbb{R}) & \text{if } \mathrm{lab}(\langle_1) = \langle^+, \\ (-\sqrt{-x_1},\ i\mathbb{R}) & \text{if } \mathrm{lab}(\langle_1) = \langle^-. \end{cases} \tag{4.5}$$

The projections are

$$f(\langle) = (x, \mathbb{R}) \mapsto x \in \mathbb{R}, \qquad g(\langle) = (x, i\mathbb{R}) \mapsto ix \in i\mathbb{R}.$$

Theorem 4.2 (Consistency)**.** *The labelled-pair model theorem 4.1 satisfies axioms (A1)–(A3).*

Proof. (A1): $f(\langle) = x$ depends only on the first coordinate, so $\langle_a := (3, \langle^+)$ and $\langle_b := (3, \mathbb{R})$ are two *distinct* elements of $\mathbb{S}$ ($\langle_a \neq \langle_b$) that are sent by f to the same real number $f(\langle_a) = f(\langle_b) = 3$. This directly witnesses the existential statement of (A1)—"$\exists\, \langle_a \neq \langle_b,\ f(\langle_a) = f(\langle_b)$"—so f is non-injective. The argument for g is identical. □ □

(A2a) follows from (4.1)–(4.2): the sum and difference stay in $\mathbb{S}$ (tag = $\mathbb{S}$). (A2b) follows from (4.3)–(4.4): the product and quotient have tag $\mathbb{R}$, so $\mathrm{Flow}(\langle_1 \times \langle_2, \times) = \mathbb{R}$. (A3) follows from (4.5): the square root has tag $i\mathbb{R}$.

Example 4.3 (Non-trivial flow)**.** Take $\langle_1 = (4, \langle^+)$ and $\langle_2 = (2, \mathbb{S})$.

$$\begin{aligned} \langle_1 + \langle_2 &= (6, \mathbb{S}), & \mathrm{Flow}(\langle_1 + \langle_2, +) &= \mathbb{S}, \\ \langle_1 \times \langle_2 &= (8, \mathbb{R}), & \mathrm{Flow}(\langle_1 \times \langle_2, \times) &= \mathbb{R}, \\ \sqrt{\langle_1} &= (2, i\mathbb{R}), & \mathrm{Flow}(\sqrt{\langle_1}, \sqrt{\cdot}) &= i\mathbb{R}. \end{aligned}$$

Example 4.4 (Tagged but observationally equivalent)**.** $(3, \langle^+)$ and $(3, \mathbb{R})$ are distinct hidden numbers, yet both project to the same real number 3. The flow tag records their differing *provenance* even when their *observable value* coincides.

Hidden-imaginary and hidden-real numbers. Starting from a single hidden number $\dot{\langle} = \dot{n} \in \mathbb{S}$, two kinds of derived numerical objects emerge from applying different external operators:

Definition 4.5 (Hidden-imaginary number)**.** For $\dot{\langle}_1, \dot{\langle}_2 \in \mathbb{S}$ write

$$\dot{a}\, i \;:=\; \dot{\langle}_1 \oplus_{\mathbb{S}} \dot{\langle}_2 \cdot i$$

for a *hidden-imaginary number*—the addition synthesis inside $\mathbb{S}$ of two hidden numbers with the imaginary unit i. A hidden-imaginary number carries both the $\mathbb{S}$-side information of $\dot{\langle}_1, \dot{\langle}_2$ and the i factor, so its externally observable value lands on the $i\mathbb{R}$ signal branch.

Definition 4.6 (Hidden-real number)**.** For $\dot{\langle}_1, \dot{\langle}_2 \in \mathbb{S}$ write

$$\dot{a} \;:=\; \dot{\langle}_1 \oplus_{\mathbb{S}} \dot{\langle}_2 \cdot 1$$

for a *hidden-real number*—the addition synthesis inside $\mathbb{S}$ of two hidden numbers with the real unit 1. A hidden-real number $\dot{a}$ is rendered as $\dot{a}$, i.e. a *dot above the digit*: it is not an ordinary real number (it does not participate in arithmetic on $\mathbb{R}$); it is a real number "locked into" $\mathbb{S}$, observable only through the $\mathbb{R}$ signal.

Corollary (layered formulas). For any single hidden number $\dot{\langle} = \dot{n}$, axiom (A2) (closure under $\pm$) together with axiom (A3) (square-root signal replacement) imply the following two formulas:

(a) *Hidden-imaginary formula.*

$$\dot{a}\, i + \dot{b}\, i \;=\; (a \dot{+} b)\, i, \qquad \dot{a}\, i - \dot{b}\, i \;=\; (a \dot{-} b)\, i.$$

Addition and subtraction of hidden-imaginary numbers stay on the hidden-imaginary layer.

(b) *Hidden-real formula.*

$$\dot{a} + \dot{b} \;=\; (a \dot{+} b), \qquad \dot{a} - \dot{b} \;=\; (a \dot{-} b).$$

Addition and subtraction of hidden-real numbers stay on the hidden-real layer, with the external signal landing on $\mathbb{R}$.

Proof sketch. Both (a) and (b) follow from closure of $\oplus_{\mathbb{S}}$ on $\mathbb{S}$ stated in (A2a): for any $\dot{\langle}_1, \dot{\langle}_2 \in \mathbb{S}$, $\dot{\langle}_1 \oplus_{\mathbb{S}} \dot{\langle}_2 \in \mathbb{S}$. Multiplying by the imaginary unit i or the real unit 1 changes only the signal, not the $\oplus_{\mathbb{S}}$-closure. The concrete form is realised in theorem 4.1 via (4.1) and (4.2). ☐

Remark 4.7. The notation $\dot{a} = \dot{a}$ uses the dot as a non-erasable syntactic marker of provenance: $\dot{a}$ and $a \in \mathbb{R}$ are *different objects*. Erasing the dot would conflate the hidden-real number with its real-image and lose the "this number came from $\mathbb{S}$" information.

Connection to quantum mechanics. The labelled-pair model $\mathbb{S} = \mathbb{R} \times \Sigma$ is structurally isomorphic to the "mod-squared plus phase" decomposition of quantum states—Σ marking whether the state lives on $\mathbb{S}$, $\mathbb{R}$ or $i\mathbb{R}$, in analogy with "observed / unobserved". Arguments for the necessity of complex numbers in quantum computing appear in [19]; the geometry of imaginary coordinates in quantum state space is studied in [17].

4.1 Independence of the axioms

To prove that axioms (A1)–(A3) are pairwise independent, we use theorem 4.1 as the *full model* M and modify it within the **same signature** $\left(\mathbb{S}, \Sigma, \oplus_{\mathbb{S}}, \otimes, \sqrt{\cdot}, f, g\right)$ to construct three counter-models, each violating exactly one axiom. The full model M satisfies all three axioms (by theorem 4.2); the three counter-models M_1, M_2, M_3 each violate a different one, so no two of the axioms imply the third.

- **Counter-model M_1 (failing (A1)).** Keep $\mathbb{S} = \mathbb{R} \times \Sigma$ and the definitions of $\oplus_{\mathbb{S}}, \otimes, \sqrt{\cdot}$ unchanged, but **collapse** Σ to a singleton $\Sigma_1 := \{*\}$—every element's tag becomes the same label. The projections f_1, g_1 are still well-defined, but now f_1 and g_1 are bijective (signals cannot be distinguished), so **(A1) fails**. (A2) and (A3) hold in the same form as in the full model: closure under $\pm$, product onto $\mathbb{R}$, and square root onto $i\mathbb{R}$ remain true once all tags are identified to $*$.

- **Counter-model M_2 (failing (A2b)).** Keep $\mathbb{S} = \mathbb{R} \times \Sigma$, f, g, $\oplus_{\mathbb{S}}$ and $\sqrt{\cdot}$ unchanged, but **redefine multiplication** as $\otimes_2 : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$, $\otimes_2(\langle_1, \langle_2) := (x_1 x_2, \mathbb{S})$— the product's tag stays $\mathbb{S}$ instead of being replaced by $\mathbb{R}$. **(A2b) fails**. (A1) still holds (f is still non-injective); (A3) still holds (square root's tag is independently set to $i\mathbb{R}$, regardless of the product's tag behaviour).
- **Counter-model M_3 (failing (A3)).** Keep $\mathbb{S} = \mathbb{R} \times \Sigma$, f, g, $\oplus_{\mathbb{S}}$ and $\otimes$ unchanged, but **redefine the square root** as $\sqrt{\langle_1} := (\sqrt{|x_1|}, \mathbb{S})$—the square root's tag stays $\mathbb{S}$ instead of being replaced by $i\mathbb{R}$. **(A3) fails**. (A1) and (A2) are unaffected.

All three counter-models are obtained from the full model M by a local modification within a single signature, so they are structurally identical to M except at the rule of the violated axiom. This guarantees the model-theoretic strictness of the independence proof.

Corollary 4.8. *No two of (A1)–(A3) imply the third.* □

5 Axioms

We postulate three axioms for HIBS. Before stating them, we make two structural preconditions explicit—they form the background against which Axiom (A3) below becomes meaningful.

Precondition 1: three spaces are inequivalent. HIBS involves three spaces—the hidden space $\mathbb{S}$, the real line $\mathbb{R}$, and the imaginary line $i\mathbb{R}$—and they are structurally distinct:

- $\mathbb{S}$ is a wider container than $\mathbb{R}$; its elements $\langle = (x, \sigma)$ can retain their signal σ under arithmetic, so $\mathbb{S} \ncong \mathbb{R}$ as algebraic objects in HIBS;
- $i\mathbb{R}$ is algebraically isomorphic to $\mathbb{R}$ (via $ix \mapsto x$), but in HIBS it is *designated* as the direction orthogonal to $\mathbb{R}$, and is not identified with $\mathbb{R}$;
- their observable projections are independent: the map $f\colon \mathbb{S} \to \mathbb{R}$ and the map $g\colon \mathbb{S} \to i\mathbb{R}$ are two separate channels.

This inequivalence is precisely what makes Axiom (A3) possible. If the three spaces could be merged, the "signal destination" of a square-root operation would have no independent meaning.

Precondition 2: zero is the symmetric anchor of square and square-root. The zero element $\mathbf{0} = (0, \mathbb{S})$ of $\mathbb{S}$ forms a symmetric pair under the two seemingly dual operations—$\mathrm{sq}(\cdot) = (\cdot)^2$ and $\sqrt{\cdot}$:

- *Direction 1.* For any $\langle \in \mathbb{S}$, $\mathrm{sq}(\langle) := \langle \otimes \langle$ squares it;
- *Direction 2.* For any element of $\langle^+$ or $\langle^-$, $\sqrt{\langle}$ takes its square root.

The two operations meet at $\langle = \mathbf{0}$: $\mathbf{0}^2 = \mathbf{0}$, $\sqrt{\mathbf{0}} = \mathbf{0}$, and $\mathbf{0} \in \mathbb{R} \cap i\mathbb{R}$. Zero sits on both the real and imaginary axes, acting as a *bridgehead* of the two operations inside $\mathbb{S}$. This symmetry is what makes Axiom (A3) formally consistent: although the square-root result lands in $i\mathbb{R}$, its square returns inside $\mathbb{S}$ and does not break the axiom system.

Axiom 5.1 ((A1) Existence of one-way flow)**.** There exist two maps $f\colon \mathbb{S} \to \mathbb{R}$ and $g\colon \mathbb{S} \to i\mathbb{R}$ such that both are **non-injective**:

$$\exists\, \langle_a, \langle_b \in \mathbb{S},\ \langle_a \neq \langle_b \ \wedge\ f(\langle_a) = f(\langle_b),$$
$$\exists\, \langle_c, \langle_d \in \mathbb{S},\ \langle_c \neq \langle_d \ \wedge\ g(\langle_c) = g(\langle_d).$$

Equivalently, neither f nor g admits a *left inverse*: there is no map $r\colon \mathbb{R} \to \mathbb{S}$ such that $r \circ f = \mathrm{id}_{\mathbb{S}}$, and similarly for g. That is, there are one-way "downward" passages from $\mathbb{S}$ into the real line and into the imaginary line, respectively, along which information cannot be recovered.

Axiom 5.2 ((A2) Closure under $\pm$; mapping under $\times, \div$)**.** For any $\langle_1, \langle_2 \in \mathbb{S}$:

- *Closure under* $\pm$: the sum and difference satisfy $\langle_1 + \langle_2 \in \mathbb{S}$ and $\langle_1 - \langle_2 \in \mathbb{S}$; moreover $\mathrm{Flow}(\langle_1 \pm \langle_2, \pm) = \mathbb{S}$, i.e. the result is mapped onto $\mathbb{S}$ (no projection onto $\mathbb{R}$ or $i\mathbb{R}$ occurs).
- *Mapping under* $\times, \div$: for the product and quotient we have $\mathrm{Flow}(\langle_1 \times \langle_2, \times) = \mathbb{R}$ and $\mathrm{Flow}(\langle_1 \div \langle_2, \div) = \mathbb{R}$; that is, multiplication and division necessarily project the result onto the real line.

Axiom 5.3 ((A3) Square root maps onto the imaginary axis)**.** Let $\sqrt{\cdot}$ denote the square-root operation on $\mathbb{S}$. Then:

$$\mathrm{Flow}(\langle, \sqrt{\cdot}) \in i\mathbb{R} \quad \text{for all} \quad \langle \in \mathbb{S},$$

and moreover $\mathrm{Flow}(\langle, \sqrt{\cdot}) \in i\mathbb{R}^-$ when $\langle \in \langle^+$, and $\mathrm{Flow}(\langle, \sqrt{\cdot}) \in i\mathbb{R}^+$ when $\langle \in \langle^-$.

Calculation procedure for the square root in (A3). We spell out the full computation that justifies the asymmetric treatment of $\langle^+$ and $\langle^-$ in Axiom (A3).

For any non-zero hidden number $\langle$ represented (via the embedding $\iota_{\mathbb{R}}$) by a complex number $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi]$, the principal-value square root is

$$\sqrt{z} \;=\; \sqrt{r}\, e^{i\theta/2}, \qquad \theta/2 \in (-\pi/2, \pi/2].$$

Step 1 (non-negative real part). Since $\theta/2 \in (-\pi/2, \pi/2]$, the principal value $\sqrt{z}$ always has non-negative real part and lies in the right half-plane or the upper half of the imaginary axis. **The principal-value square root never enters the lower half of the imaginary axis.**

Step 2 (entering the lower half via minus sign). The value $-\sqrt{z}$ has argument $\theta/2 + \pi \in (\pi/2, 3\pi/2]$; its real part is $-\sqrt{r}\cos(\theta/2) \leq 0$, so $-\sqrt{z}$ lies in the lower half of the imaginary axis (between $\pi/2$ and $3\pi/2$).

Step 3 (correspondence with (A3)). Combining the two steps:

- For $\langle^+$ (corresponding to $z = re^{i\cdot 0}$, $\theta = 0$), $\sqrt{z} = \sqrt{r}$ still lies in $\mathbb{R}$; reaching $i\mathbb{R}$ requires a leading minus sign $-\sqrt{\langle}$. This matches the first clause of (A3): "$-\sqrt{\langle^+} \in i\mathbb{R}$".
- For $\langle^-$ (corresponding to $z = re^{i\pi}$, $\theta = \pi$), $\sqrt{z} = \sqrt{r}e^{i\pi/2} = i\sqrt{r}$ *already* lies in $i\mathbb{R}$; no leading minus sign is needed. This matches the second clause of (A3): "$\sqrt{\langle^-} \in i\mathbb{R}$".

The asymmetric treatment of $\langle^+$ and $\langle^-$ in (A3) is therefore not an arbitrary stipulation: it is the geometric constraint of the principal-value square root. $\langle^+$ requires a leading minus sign to reach the lower half of the imaginary axis; $\langle^-$ already lands in the upper half. See [8, 6, 18].

Relation to the axiomatic-algebraic tradition. The form of axioms (A1)–(A3) follows Hilbert's axiomatic programme [1]. At the operator-algebra level the flow tag is analogous to a spectral measure in a Banach algebra; see [9, 10]. In the categorical setting, the distinct "signal branches" of $\mathbb{S}$ can be viewed as a hyperpolyadic structure [13]. The independence proof follows the standard method for Hilbert-style axiom systems, cf. [11]; the axiomatic dependencies in quaternionic pedagogy are discussed in [12].

6 Consequences and Basic Properties

This section gives the full proof of the section's main result. The result has two parts: (1) the embedding map $\iota : \mathbb{C} \to \mathbb{S}$ is an additive monomorphism (theorem 6.1); (2) ι is *not* a multiplicative homomorphism (theorem 6.2). The two together yield the algebraic characterisation that $\mathbb{S}$ is an *additive pre-space* of $\mathbb{C}$ (theorem 6.3).

Theorem 6.1 ($\mathbb{C}$ appears as an additive sub-structure of $\mathbb{S}$)**.** *Define the embedding map $\iota : \mathbb{C} \to \mathbb{S}$ by*

$$\iota(a + bi) \;=\; \iota_{\mathbb{R}}(a) \oplus_{\mathbb{S}} \dot{b}\, i,$$

where $\iota_{\mathbb{R}} : \mathbb{R} \to \mathbb{S}$ is the standard embedding $\iota_{\mathbb{R}}(a) = (a, \mathbb{R})$ and $\dot{b}\, i$ is the hidden-imaginary number of b (Definition 4.5). Then ι is an additive monomorphism *from $(\mathbb{C}, +)$ to $(\mathbb{S}, \oplus_{\mathbb{S}})$.*

Proof. Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ in $\mathbb{C}$. By definition,

$$\iota(z_1) = \iota_{\mathbb{R}}(a_1) \oplus_{\mathbb{S}} \dot{b_1}\, i, \qquad \iota(z_2) = \iota_{\mathbb{R}}(a_2) \oplus_{\mathbb{S}} \dot{b_2}\, i.$$

By the definition of $\oplus_{\mathbb{S}}$ ((4.1)), addition inside $\mathbb{S}$ is componentwise, hence

$$\begin{aligned}
\iota(z_1) \oplus_{\mathbb{S}} \iota(z_2) &= \big((a_1, \mathbb{R}) \oplus_{\mathbb{S}} (a_2, \mathbb{R})\big) \oplus_{\mathbb{S}} \big(\dot{b_1}\, i \oplus_{\mathbb{S}} \dot{b_2}\, i\big) \\
&= \big((a_1 + a_2, \mathbb{S})\big) \oplus_{\mathbb{S}} \big((b_1 \dot{+} b_2)\, i\big) \\
&= \iota_{\mathbb{R}}(a_1 + a_2) \oplus_{\mathbb{S}} (b_1 \dot{+} b_2)\, i \\
&= \iota\big((a_1 + a_2) + (b_1 + b_2)\, i\big) \\
&= \iota(z_1 + z_2).
\end{aligned}$$

Thus $\iota(z_1) \oplus_{\mathbb{S}} \iota(z_2) = \iota(z_1 + z_2)$, so ι is an additive homomorphism. Injectivity of ι follows directly from the injectivity of $\iota_{\mathbb{R}}$ on $\mathbb{R}$. □ □

Theorem 6.2 (ι is not a multiplicative homomorphism)**.** *In the labelled-pair model theorem 4.1, the map $\iota : \mathbb{C} \to \mathbb{S}$ preserves addition (theorem 6.1) but* ***does not preserve multiplication****: there exist $z_1, z_2 \in \mathbb{C}$ such that*

$$\iota(z_1 \cdot z_2) \;\neq\; \iota(z_1) \otimes \iota(z_2)$$

(the two sides do not lie in the same signal branch of $\mathbb{S}$).

Proof. Take $z_1 = a + bi, z_2 = c + di \in \mathbb{C}$ with $a, c \in \mathbb{R}$, $b, d \in \mathbb{R}$, and the imaginary part of the product *non-zero*: $ad + bc \neq 0$ (e.g. $z_1 = 1 + i, z_2 = i$, giving $z_1 \cdot z_2 = -1 + i$ with imaginary part $1 \neq 0$).

Left-hand side. The standard expansion of complex multiplication is

$$z_1 \cdot z_2 \;=\; (ac - bd) + (ad + bc)\, i,$$

whose imaginary part is $ad + bc$. By the definition of ι:

$$\begin{aligned} \iota(z_1 \cdot z_2) &= \iota_{\mathbb{R}}(ac - bd) \oplus_{\mathbb{S}} (ad \dot{+} bc)\, i \\ &= \big((ac - bd,\ \mathbb{R})\big) \oplus_{\mathbb{S}} \big((ad \dot{+} bc)\, i\big) \\ &= \big((ac - bd,\ \mathbb{R}),\ (ad \dot{+} bc)\, i\big). \end{aligned}$$

In the labelled-pair model, $\oplus_{\mathbb{S}}$ is componentwise addition by (4.1); the first component is $\iota_{\mathbb{R}}(ac - bd) \oplus_{\mathbb{S}} (ad \dot{+} bc)\, i$ in $\mathbb{S}$, but $(ad \dot{+} bc)\, i$ has signal $i\mathbb{R}$—it is **not an element of** $\mathbb{R}$.

Right-hand side. By (A2b), multiplication on $\mathbb{S}$ is $\otimes : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ with $\otimes(\langle_1, \langle_2) = (x_1 x_2,\ \mathbb{R})$, where $\langle_i = (x_i, \sigma_i)$. Concretely:

$$\begin{aligned} \iota(z_1) \otimes \iota(z_2) &= \iota_{\mathbb{R}}(a) \otimes \dot{b}\, i \otimes \iota_{\mathbb{R}}(c) \otimes \dot{d}\, i \\ &= (a \cdot b \cdot c \cdot d,\ \mathbb{R}) \in \mathbb{R}. \end{aligned}$$

This is a real number in $\mathbb{R}$, viewed as the degenerate element $(abcd, \mathbb{R}) \in \mathbb{S}$; its signal is **strictly** $\mathbb{R}$, with no $i\mathbb{R}$ component.

Contradiction. By the definition of ι, the left-hand side $\iota(z_1 \cdot z_2)$ has first coordinate $ac - bd$ and second coordinate $(ad \dot{+} bc)\, i$—the latter only degenerates to $\iota_{\mathbb{R}}(0)$ (with signal $\mathbb{R}$) when $ad + bc = 0$. By the assumption $ad + bc \neq 0$, the imaginary part of $\iota(z_1 \cdot z_2)$ has signal $i\mathbb{R}$. The right-hand side $\iota(z_1) \otimes \iota(z_2)$ is strictly in $\mathbb{R}$ (signal $\mathbb{R}$), so the two sides are never in the same signal branch of $\mathbb{S}$. Hence $\iota(z_1 \cdot z_2) \neq \iota(z_1) \otimes \iota(z_2)$. □ □

Corollary 6.3 ($\mathbb{S}$ is an additive pre-space of $\mathbb{C}$)**.** *By theorem 6.1 and theorem 6.2:*

- *The* additive *structure of $\mathbb{C}$ appears as a sub-structure inside $\mathbb{S}$ (via the additive monomorphism ι);*
- *The* multiplicative *structure of $\mathbb{C}$ cannot be embedded in $\mathbb{S}$ (ι is not a multiplicative homomorphism).*

Hence $\mathbb{S}$ is strictly an additive pre-space *of $\mathbb{C}$, not a subring. This asymmetry is a direct corollary of (A2)—it explains at the axiomatic level why HIBS is not "another extension of $\mathbb{C}$", but a larger container that admits $\mathbb{C}$ as a* partial *sub-structure.*

Remark 6.4. The central conclusion of §5—that $\mathbb{C}$ appears as an *additive sub-structure* (not a ring sub-structure) inside $\mathbb{S}$—has its existential basis in the construction of theorem 4.1 (the projections f and g have images filling $\mathbb{R}$ and $i\mathbb{R}$ respectively). ι is an additive monomorphism but *not* a multiplicative monomorphism; this asymmetry is a direct algebraic consequence of (A2).

Theorem 6.5 (Projection factorisation: $\mathbb{C}$ is the projection image of $\mathbb{S}$)**.** *Let $\pi : \mathbb{S} \to \mathbb{C}$ be the projection map defined at the end of §6. In the labelled-pair model theorem 4.1,*

$$\pi \circ \iota \;=\; \mathrm{id}_{\mathbb{C}}.$$

That is, every complex number z is uniquely recovered from its image $\iota(z)$ in $\mathbb{S}$ via π; $\mathbb{C}$ is, strictly speaking, the observable slice *of $\mathbb{S}$ under the projection π.*

Proof. Let $z = a + bi \in \mathbb{C}$. By the definition of ι (theorem 6.1),

$$\iota(z) \;=\; \iota_{\mathbb{R}}(a) \oplus_{\mathbb{S}} \dot{b}\,i \;=\; \big((a,\mathbb{R})\big) \oplus_{\mathbb{S}} \big(\dot{b}\,i\big),$$

where the first coordinate $(a, \mathbb{R})$ has signal $\mathbb{R}$ and the second coordinate $\dot{b}\,i$ has signal $i\mathbb{R}$ (by the definition of $\dot{b}\,i$ in §4). By the definition of π (end of §6):

- First coordinate $(a, \mathbb{R})$: lab $= \mathbb{R}$, so π extracts the real part, giving a;
- Second coordinate $\dot{b}\,i$: lab $= i\mathbb{R}$, so π extracts the imaginary part, giving ib.

After composition via $\oplus_{\mathbb{S}}$ (which is componentwise addition by (4.1)), π sends the whole element to $a + ib = z$. Hence $\pi(\iota(z)) = z$, i.e. $\pi \circ \iota = \mathrm{id}_{\mathbb{C}}$. ☐ ☐

Remark 6.6. **$\mathbb{S}$ as the generative layer of $\mathbb{C}$.** theorem 6.5 is the strict proof of Conjecture (i) of §2: $\mathbb{C}$ is the observable slice of $\mathbb{S}$ under π. Together with theorem 6.2 (ι does not preserve multiplication), the paper gives a complete picture of the relation between $\mathbb{C}$ and $\mathbb{S}$—$\mathbb{C}$ is the *addition-invertible, multiplication-non-invertible* projection image of $\mathbb{S}$: the complex-conjugation operation $\bar{z} = a - bi$ in $\mathbb{C}$ corresponds to the signal reversal $\langle^+ \leftrightarrow \langle^-$ in $\mathbb{S}$ (Conjecture (ii) of §2); while the "rotation" semantics of $\mathbb{C}$ multiplication cannot be recovered inside $\mathbb{S}$ (theorem 6.2). Together, these two results give an *axiom-level account* of the structural asymmetry of $\mathbb{C}$ between its real and imaginary parts.

Theorem 6.7 (Emergence: $\mathbb{C}$ emerges from axioms (A1)–(A3))**.** *The three axioms (A1)–(A3)* do not explicitly contain a definition of $\mathbb{C}$. *They involve only* $\mathbb{S}, \Sigma, \oplus_{\mathbb{S}}, \otimes, \sqrt{\cdot}, f, g$*; the complex field* $\mathbb{C}$*, the complex-field operations, and complex conjugation* do not appear at the axiom level. *Yet from (A1)–(A3) the complex plane* $\mathbb{C}$ naturally emerges *as a two-dimensional composite structure:*

(i) (A1) supplies two non-injective projections*:* $f : \mathbb{S} \to \mathbb{R}$ *and* $g : \mathbb{S} \to i\mathbb{R}$*—giving the two* observable directions $\mathbb{R}$ *and* $i\mathbb{R}$*;*

(ii) (A2) supplies internal addition*:* $\oplus_{\mathbb{S}}$ *is closed inside* $\mathbb{S}$*, allowing two hidden numbers to be* composed into *a new hidden number— the structural basis of "two-dimensional composition";*

(iii) (A3) supplies imaginary-direction tagging*: the square-root operation forces the signal onto* $i\mathbb{R}$*,* explicitly distinguishing *the "real direction* $\mathbb{R}$*" from the "imaginary direction* $i\mathbb{R}$*" inside* $\mathbb{S}$*;*

(iv) From (i)–(iii) one constructs the embedding $\iota : \mathbb{C} \to \mathbb{S}$ *via* $\iota(a + bi) = \iota_{\mathbb{R}}(a) \oplus_{\mathbb{S}} \dot{b}\,i$*— the complex number* $a + bi$ *is* uniquely composed *into a hidden number with two signal components ("real part" and "imaginary part").*

Key point*: the* entire two-dimensional structure of $\mathbb{C}$ *(real part, imaginary part, addition, complex conjugation, modulus)* emerges *from the construction process of (A1)–(A3)—*$\mathbb{C}$ *is* not *an axiom input, but an* axiom output*.*

The emergence is irreversible*: (A1) explicitly states that* f *and* g *are non-injective (see* **??***); distinct elements of* $\mathbb{S}$ *can project to the same* $\mathbb{R}$ *or* $i\mathbb{R}$ *element. Thus from* $\mathbb{C}$ *(a composite of* $\mathbb{R}$ *and* $i\mathbb{R}$*) one* cannot uniquely recover *the generative layer* $\mathbb{S}$*—the emergence is* irreversible at the axiom level. *This, together with theorem 6.5 (*π *is a left inverse of* ι *but not a right inverse),* jointly guarantees *the "generative" semantics of HIBS:* $\mathbb{C}$ *emerges from* $\mathbb{S}$*, but* $\mathbb{C}$ *does not* fully determine $\mathbb{S}$ *in return.*

Proof. **Axiom audit**: inspecting the statements of (A1)–(A3) (§5): (A1) involves $\mathbb{S}, \mathbb{R}, i\mathbb{R}, f, g$, and does *not* contain $\mathbb{C}$; (A2) involves $\mathbb{S}, \oplus_{\mathbb{S}}, \otimes, \Sigma$, and does *not* contain $\mathbb{C}$; (A3) involves $\mathbb{S}, \Sigma, \sqrt{\cdot}, \langle^+, \langle^-$, and does *not* contain $\mathbb{C}$. Thus $\mathbb{C}$ is *not* an axiom input.

Construction audit: on the labelled-pair model $\mathbb{S} = \mathbb{R} \times \Sigma$ of theorem 4.1,

- By (A1), $f(x, \sigma) = x$ gives the $\mathbb{R}$-projection; $g(x, \sigma) = ix$ (when lab $= i\mathbb{R}$) gives the $i\mathbb{R}$-projection;
- By (A2), $\oplus_{\mathbb{S}}$ gives internal addition inside $\mathbb{S}$ ((4.1)), allowing the cross-signal composition $(a, \mathbb{R}) \oplus_{\mathbb{S}} (b, i\mathbb{R})$;
- By (A3), the square-root tag $\langle^+, \langle^-$ corresponds to $i\mathbb{R}$ ((4.5)), explicitly distinguishing the real and imaginary directions.

From these three steps one explicitly defines the embedding $\iota : \mathbb{C} \to \mathbb{S}$ (theorem 6.1), sending $a+bi \in \mathbb{C}$ to $(a, \mathbb{R}) \oplus_{\mathbb{S}} (b, i\mathbb{R})$. The two-dimensional structure of $\mathbb{C}$ (real + imaginary) is *emergent* from the *definition process* of ι, *not* pre-assumed as axiom input.

Irreversibility: by **??**, both f and g are non-injective—there exist $\langle_a \neq \langle_b$ with $f(\langle_a) = f(\langle_b)$. Thus the image $\iota(\mathbb{C}) \subseteq \mathbb{S}$ does *not* cover all of $\mathbb{S}$ (of the $3 \times 3 = 9$ signal combinations in the labelled-pair model, ι covers only 1); hence $\mathbb{S}$ is *strictly larger* than $\iota(\mathbb{C})$, and from $\mathbb{C}$ one *cannot uniquely recover* $\mathbb{S}$. ☐ ☐

Remark 6.8. theorem 6.7 is the *core mathematical result* of HIBS: it shows that HIBS is not "a replacement for $\mathbb{C}$" ($\mathbb{C}$ still exists inside the image of ι), but a *axiomatic emergence theory of* $\mathbb{C}$—the existence of $\mathbb{C}$ is a *consequence* of (A1)–(A3), not an *input*. This distinction separates HIBS from all *definitional* constructions of $\mathbb{C}$ (polynomial-ring quotient, matrix representation, formal symbol)—the latter three *presuppose* the existence of $\mathbb{C}$; HIBS *derives* the existence of $\mathbb{C}$.

Related operator-algebraic results. The embedding $\iota : \mathbb{C} \to \mathbb{S}$ has natural analogues in quaternionic and Clifford settings; see [18] for the differential-geometric viewpoint, [9, 10] for the quaternionic spectral theory, and [14] for a recent alternative extension of complex numbers. Closely related Hilbert-space phenomena—such as sums of closed subspaces [15] and the wandering-subspace property [16]—show that the structural asymmetries of HIBS are not isolated: similar irreversibilities appear in operator theory. See also [20] for complex-valued functions in plane differential geometry.

Proposition 6.9 (Layering of reversibility)**.** *The operations of HIBS fall into two layers by reversibility:*

1. Reversible layer*: addition and subtraction.* $\langle_1 + \langle_2 = \langle_3$ *and* $\langle_3 - \langle_2 = \langle_1$ *are mutual inverses, all carried out inside* $\mathbb{S}$*.*
2. Irreversible layer*: multiplication, division and square root. Each of these operations necessarily triggers a flow:* $\mathrm{Flow}(\cdot, \times) \colon \mathbb{S} \to \mathbb{R}$*,* $\mathrm{Flow}(\cdot, \div) \colon \mathbb{S} \to \mathbb{R}$*, and* $\mathrm{Flow}(\cdot, \sqrt{\cdot}) \colon \mathbb{S} \to i\mathbb{R}$*. Because these maps are not injective, the forward direction does not admit a unique inverse back to* $\mathbb{S}$*.*

The lack of a unique inverse for the irreversible layer follows directly from Axioms 2 and 3.

Corollary 6.10 (Asymmetry of factorisation)**.** *From Proposition 4.2, multiplication is reversible while factorisation is not; the arithmetic intuition that "factorisation is harder than multiplication" therefore has axiomatic status in HIBS.*

7 Relation to Existing Work

Core relation. The central relation of this paper is that $\mathbb{S}$ is an *additive pre-space* of $\mathbb{C}$: the additive structure of $\mathbb{C}$ appears as a sub-structure inside $\mathbb{S}$, but its multiplicative structure cannot be embedded because (A2b) sends every internal multiplication to a real number. This asymmetry is established in Theorem 5.1 and is the core corollary of (A2). More precisely: every element of $\mathbb{C}$ can be obtained by applying the flow operator Flow to some hidden number in $\mathbb{S}$; but $\mathbb{S}$ also contains elements that are not fully mapped to $\mathbb{C}$ (they participate in operations *inside* $\mathbb{S}$ and are observable only via the $i\mathbb{R}$ signal branch). Hence $\mathbb{S} \supsetneq \mathbb{C}$: $\mathbb{C}$ is the visible slice of $\mathbb{S}$ on the real plus imaginary axes, but $\mathbb{S}$ is strictly larger.

With this clarification in place, we compare HIBS with four classical frameworks.

- *ZFC / set theory*: HIBS is not a set-theoretic axiomatic system; it introduces operator axioms on a space $\mathbb{S}$ whose concrete construction is left unspecified, and is therefore consistent with ZFC; see [11] for categorical large-cardinal perspectives on axiomatic independence.
- *Axiomatisations of the complex numbers*: classical axiomatisations (e.g. Landau, Benedetto) start from $\mathbb{R}^2$; HIBS starts from the flow on $\mathbb{S}$, and $\mathbb{C}$ is recovered by projecting the $i\mathbb{R}$ signal branch onto the real axis. The Hilbert-style approach to algebraic axiomatics is surveyed in [1]; modern operator-algebraic analogues include [10, 9], while [14] proposes a recent alternative extension of $\mathbb{C}$. *Key difference*: Hilbert's programme aims to axiomatise *existing* mathematics, whereas HIBS introduces auxiliary notation *before* the existing axiomatisations of $\mathbb{C}$—as a teaching bridge, not a replacement.
- *Hamilton's quaternions*: Hamilton's quaternions [2] form a four-dimensional real algebra with non-commutative multiplication. *Key difference*: quaternions extend $\mathbb{C}$ *outwards* ($\mathbb{C} \hookrightarrow \mathbb{H}_{\mathrm{Ham}}$), whereas HIBS extends $\mathbb{C}$ *inwards* ($\mathbb{S} \twoheadrightarrow \mathbb{C}$). The two are not competing extensions but extensions in opposite directions.
- *Dedekind cuts*: Dedekind cuts [3] construct $\mathbb{R}$ from $\mathbb{Q}$ via lower-closed subsets. HIBS reuses the idea of a "tag" recording provenance but reverses the direction: $\mathbb{S}$ is built *downwards* from a richer structure (both $\mathbb{R}$ and $i\mathbb{R}$ are its projections), not *upwards* by partition of a larger set.

- *Field extensions in category theory*: in the Mac Lane–Birkhoff framework [5], $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$. HIBS gives the opposite hierarchy $\mathbb{S} \xrightarrow{\text{flow}} \mathbb{C} \xhookrightarrow{\text{embed}} \mathbb{R}^2$: HIBS is not built from $\mathbb{R}$ up to $\mathbb{C}$, but from $\mathbb{S}$ down to $\mathbb{C}$.
- *AI "latent spaces"*: the two notions are unrelated. An AI latent space is a parameter embedding space (a high-dimensional $\mathbb{R}^n$ in which neural network weights live); the HIBS hidden space is a *bridge connecting the real and the imaginary*, not a parameter embedding space.
- *Complex-plane geometry and analytic functions*: Wessel, Argand and Gauss identified $a+bi$ with the point (a, b) in the plane [18]; complex addition follows the parallelogram law, complex multiplication combines dilation with rotation. Within the labelled-pair model of HIBS these geometric behaviours are reproduced inside $\mathbb{S}$ via the embedding $\iota_{\mathbb{R}} : \mathbb{R} \to \mathbb{S}$. See [20] for a modern treatment of complex-valued functions in plane differential geometry.

Complex numbers and hidden numbers. A complex number $z = a + bi$ corresponds inside $\mathbb{S}$ to the composition $\dot{\langle} = \dot{a} \oplus_{\mathbb{S}} \dot{b}\, i$, where $\dot{a}$ and $\dot{b}\, i$ are the hidden-real and hidden-imaginary parts of z. This correspondence avoids the error of viewing z as a single element of $\mathbb{S}$: z is *not* a hidden number; it is the synthesis of two hidden numbers.

Application interfaces. *Removed.* Earlier drafts of this section listed three potential application interfaces (quantum computing, signal processing, and Calabi–Yau geometry). These were speculative connections: the references they cite do not engage with the HIBS axioms, and including them diluted the algebraic focus of the paper. We retain only the structural comparison with quantum mechanics in **??** above.

8 Conclusion

This paper presents a *generative-layer proposal* called HIBS (Hidden-space Bridge System): a conjecture that the complex plane $\mathbb{C}$ is not a foundational object but rather the observable slice of a hidden generative layer $\mathbb{S}$ under a projection π (§2). HIBS is **not a new kind of number, a new mathematical structure, nor a closed theory**; nor does it replace any theorem of $\mathbb{C}$—all theorems of $\mathbb{C}$ continue to hold, but the *structural source* of $\mathbb{C}$ (the asymmetry between its real and imaginary parts) is given an *axiom-level account* by the flow-signal mechanism of $\mathbb{S}$ (theorem 6.5). The three axioms are:

- **(A1) Existence of one-way flow**—two non-invertible projections from $\mathbb{S}$ to $\mathbb{R}$ and to $i\mathbb{R}$;
- **(A2) Closure under $\pm$; signal replacement under $\times, \div$**—$\oplus_{\mathbb{S}}$ is closed inside $\mathbb{S}$, while $\otimes$ immediately replaces the signal with $\mathbb{R}$;
- **(A3) Square-root signal replacement**—the square root of an element of $\langle^+$ flows onto $i\mathbb{R}^-$, while that of $\langle^-$ flows onto $i\mathbb{R}^+$.

Main results established.

1. **Consistency.** The axiom system (A1)–(A3) is non-empty: the labelled-pair model $\mathbb{S} = \mathbb{R} \times \Sigma$, with $\Sigma = \{\mathbb{S}, \mathbb{R}, i\mathbb{R}\}$, satisfies all three axioms (theorem 4.1).
2. **Independence.** (A1), (A2), (A3) are pairwise independent: the three counter-models M_1, M_2, M_3 each violate one axiom while preserving the other two.
3. **$\mathbb{C}$ as a sub-structure of $\mathbb{S}$.** The embedding $\iota(a + bi) = \iota_{\mathbb{R}}(a) \oplus_{\mathbb{S}} \dot{b}\, i$ is a monomorphism from $(\mathbb{C}, +)$ into $(\mathbb{S}, \oplus_{\mathbb{S}})$; ι is *not* a multiplicative monomorphism, because $\otimes$ once executed flows out of $\mathbb{S}$ into $\mathbb{R}$ (theorem 6.2). This asymmetry is the algebraic justification for the claim in theorem 6.3 that $\mathbb{S}$ is a *pre-space* of $\mathbb{C}$, not a sub-structure.
4. **Hidden-imaginary and hidden-real formulas.** From the $\oplus_{\mathbb{S}}$-closure of (A2a), the layered identities $\dot{a}\, i \pm \dot{b}\, i = (a \,\dot{\pm}\, b)\, i$ and $\dot{a} \pm \dot{b} = (a \,\dot{\pm}\, b)$ are derived.
5. **Geometric origin of (A3).** The principal value $\sqrt{re^{i\theta}} = \sqrt{r}\, e^{i\theta/2}$ with $\theta/2 \in (-\pi/2, \pi/2]$ never enters the lower half of the imaginary axis; a leading minus sign rotates the argument by π, entering $(\pi/2, 3\pi/2]$. This geometric fact directly yields the asymmetric square-root behaviour of $\langle^+$ and $\langle^-$.

Position of the system. The relation between HIBS and the existing mathematical structures can be summarised in one line:

$$\mathbb{S} \xrightarrow{\text{Flow}} \mathbb{C} \xhookrightarrow{\iota} \mathbb{R}^2,$$

where Flow is an irreversible many-to-one map (consistent with (A1)) and ι is a monomorphism that is not multiplicative (consistent with theorem 6.1 and theorem 6.2). HIBS is not a generalisation of $\mathbb{R}$, $\mathbb{C}$ or the quaternions; it is their *common pre-structure*—axiomatically "prior" to them, and recovered via Flow and ι.

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