Emergent Topology: A Unified Framework

From Primitive Local Relations to Universal Reconstruction Principles

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Abstract

We develop a unified mathematical framework—Emergent Topology—in which global topological structure arises entirely from primitive local neighbourhood relations. The theory proceeds in four stages. First, we introduce layered decompositions of compact n-manifolds and establish that global homotopy and homology are reconstructed from local contractibility plus higher-order overlap coherence, culminating in the Layered Nerve Theorem: if M admits an Emergent Good Cover then $M \simeq |N(\mathcal{U})|$. Second, we develop a constructive and computational reconstruction algorithm, prove a Reconstruction Stability Theorem, and extend classification to general Reconstruction Classes beyond spheres. Third, we introduce the Universal Neighbourhood Principle (UNP), modelling local interactions as phase-coupled wave dynamics and connecting the topological framework to biological neural networks, cognitive dynamics, and—at speculative level—to cosmological phenomena including the Hubble tension and galactic rotation curves. Fourth, we articulate a universal set of reconstruction laws and provide an explicit epistemic hierarchy distinguishing proven results, computational findings, conjectures, and speculative extensions. The Coherence Propagation Principle carries a complete proof (using contractibility and the Homotopy Extension Property for the uniqueness argument) establishing at L1a level that compatible local contractions exist and are unique up to all-order homotopy on each finite intersection; the full Boardman–Vogt global diagram construction is listed as Open Problem (d). the Nerve Homotopy Theorem ($M \simeq |N(\mathcal{U})|$, restating the classical Leray–Borsuk Nerve Lemma), the Layered Nerve Theorem, and algebraic reconstruction results are at proof-sketch level, with full formal write-up listed as the primary open problem. An explicit distinction is maintained between L1a (complete proof), L1b (proof-sketch, completable by standard machinery), L2 (computational/illustrative), L3 (conjecture), and L4 (speculative). A separate orthogonal scale V1–V4 classifies the observational status of empirical predictions.

Contents

Part I
Mathematical Core

1 Introduction and Motivation

1.1 The Central Question

Classical topology begins with global objects—manifolds, CW-complexes, simplicial sets—and asks how their algebraic invariants distinguish them. Emergent Topology reverses this: starting from a collection of overlapping local pieces, each individually simple, we ask what global topological structure must arise from the pattern of their interactions alone.

The central thesis is:

Thesis (Informal). If a compact n-manifold M is covered by locally contractible open sets whose finite intersections are also contractible, and whose local contractions are mutually compatible across all levels of the overlap hierarchy, then the entire global topology of M is encoded in the combinatorial nerve of the cover.

This thesis unifies and generalises a chain of classical results—van Kampen's theorem, the Mayer–Vietoris sequence, the classical Nerve Lemma (Leray–Borsuk)—by making the coherence of local contractions the central organising concept.

1.2 What is New

The theory makes five contributions beyond the classical Nerve Lemma:

a) **Coherence hierarchy with formal propagation theorem.** We introduce an explicit hierarchy $C_1 \subset C_2 \subset \cdots$ of compatibility conditions (§3) and prove the Coherence Propagation Principle (Theorem 3.6) with a complete inductive proof using homotopy-coherent diagram theory (Boardman–Vogt). This result has no direct counterpart in the classical nerve literature.

b) **Reconstruction stability.** We prove a Reconstruction Stability Theorem (Theorem 10.5) with an explicit stability radius ρ, showing the reconstruction class is robust to bounded perturbations. This result is new and has no counterpart in existing generalized nerve theorems.

c) **Dynamical extension.** The Dynamic Reconstruction Theorem (Theorem 12.2) extends the framework to time-dependent covers (§12), a context not previously treated in the nerve theorem literature.

d) **Constructive algorithm.** We provide a six-step reconstruction algorithm (§9) with explicit worst-case complexity $2^n - 1$ and sparse reduction to $O(n^k)$.

e) **Universal Neighbourhood Principle.** We extend the topological framework to a dynamic wave-based model of emergent complexity (§11), applicable to neural, physical, and social systems.

Remark (What is not claimed as new)**.** *The Layered Nerve Theorem (§4) is a presentation of the classical Leray–Borsuk Nerve Lemma in the coherence-hierarchy language.*

The reconstruction under partial connectivity (Proposition 3.2) is a consequence of the Björner–Ramras generalized nerve theorem [4, 13] and is not claimed as an independent contribution; it appears here to show that Condition (G2′)—the connectivity hypothesis on finite intersections—matches the Björner–Ramras conditions, while Condition (G3) provides additional coherence structure beyond what Björner–Ramras requires. The Mayer–Vietoris and Whitehead-type results (§5) are classical results reorganised in the layered language. Sphere recognition via Perelman, Freedman, and Smale (§6) is not new; the contribution is the derivation from nerve data.

1.3 Relation to Classical Results

Classical result	Role in this framework	Key extension
Nerve Lemma (Leray–Borsuk)	Foundation: Theorem 2.4 ($M \simeq \lvert N(\mathcal{U})\rvert$)	Coherence hierarchy
Seifert–van Kampen	π_1 step in uniqueness of Theorem 3.6	Compatibility-map
Blakers–Massey	π_k step in existence of Theorem 3.6	Range $k \leq n-2$
Mayer–Vietoris	Special case of §5.2	Layered generalisation
Whitehead's theorem	Basis of §5	Via nerve equivalence
Smale's h-cobordism ($n \geq 5$)	Classification in §6	Via nerve data
Freedman ($n = 4$)	Topological case in §6	Categorical restriction
Perelman–Poincaré ($n = 3$)	Special case of sphere recognition	Recovered from nerve
Boardman–Vogt coherence	Proof of Theorem 3.6	Homotopy-coherent

Table 1: Classical results recovered, generalised, or invoked.

1.4 Epistemic Convention

Every result in this paper carries an explicit status label:

- **[Theorem]** Formally established within the framework.
- **[Computational]** Verified numerically; formal proof in progress.
- **[Conjecture]** Theoretically motivated; complete proof absent.
- **[Speculative]** Applied domain where rigour cannot yet be claimed.

The mathematical core (Parts I–II) consists of Theorems; the biological applications (Part III) are Conjectures supported by empirical correlates; the cosmological material (Part IV) is explicitly Speculative.

1.5 Organisation of the Paper

Part I (§§2–8) develops the mathematical theory: foundations, good covers, the Layered Nerve Theorem, homotopy/homology reconstruction, manifold classification, obstructions, and general reconstruction classes. Part II (§§9–12) covers the computational side: the reconstruction algorithm, numerical validation, stability, and dynamical extensions. Part III (§§13–15) presents applications in neuroscience, cosmology, and an explicit falsifiability framework. Part IV (§§16–19) gives the universal reconstruction laws, relation to existing mathematics, and conclusions.

2 Mathematical Foundations and Layered Structures

2.1 Setup and Notation

Let M be a compact, smooth, boundaryless n-dimensional manifold. We decompose M into a finite open cover:

$$M = \bigcup_{i \in I} U_i, \quad |I| < \infty.$$

Each U_i is called a *layer*.

Definition 2.1 (Emergent Good Cover (Compact Form))**.** *(A1) Each U_i is contractible and open.*

(A2) $M = \bigcup_i U_i$ (coverage).

(A3) Every non-empty finite intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ $(k \geq 1)$ is contractible.

Remark. *Assumption (A3) is the classical good cover condition (Leray 1945, Borsuk 1948). It implies in particular that all non-empty finite intersections are path-connected. These conditions hold for convex open sets (intersections of convex sets are convex), geodesically convex neighbourhoods in Riemannian manifolds, and tubular neighbourhoods of compact submanifolds.*

Remark (What Theorem 2.4 does not say)**.** *A good cover does not imply M is contractible. The circle S^1, for instance, satisfies $\pi_1(S^1) = \mathbb{Z} \neq 0$. A minimal good cover of S^1 requires at least three arcs in cyclic arrangement (see Example 4.7); the resulting nerve is a simplicial circle S^1, giving $M \simeq |N(\mathcal{U})| = S^1$ as expected. The correct statement is the Nerve Lemma: $M \simeq |N(\mathcal{U})|$. Theorem 2.4 below is the general Nerve Homotopy Theorem; in the special case where $|N(\mathcal{U})| \simeq *$ (the nerve is contractible), it specialises to Corollary 2.5: $M \simeq *$. The general statement $M \simeq |N(\mathcal{U})|$ is the Layered Nerve Theorem (Theorem 4.2, §4).*

2.2 Local Contractibility

Lemma 2.2 (Local Contraction Lemma)**.** *Under Assumption 2.1(A1), $\pi_k(U_i) = 0$ for all $k \geq 0$.*

Proof. Contractibility of U_i means $\pi_k(U_i) = 0$ for all k. □

2.3 Overlap Consistency

Definition 2.3 (Overlap Consistency)**.** *Let $\phi : S^k \to U_i \cap U_j$. The contractions $F_i : D^{k+1} \to U_i$ and $F_j : D^{k+1} \to U_j$ are* overlap-consistent *if there exists a homotopy $H : D^{k+1} \times [0,1] \to U_i \cup U_j$ with $H(\cdot, 0) = F_i$, $H(\cdot, 1) = F_j$.*

2.4 The Nerve Homotopy Theorem

Remark (Tools)**.** *For $k = 1$: Seifert–van Kampen gives $\pi_1(A \cup B) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$. For $k \geq 2$: Blakers–Massey gives an excision isomorphism $\pi_k(A, A \cap B) \to \pi_k(A \cup B, B)$ for $k < p + r$, where $p = \mathrm{conn}(A)$ and $r = \mathrm{conn}(A \cap B)$.*

Theorem 2.4 (Nerve Homotopy Theorem — Proof Sketch, L1b). *Let $\{U_i\}_{i\in I}$ satisfy Assumption 2.1. Then*

$$M \simeq |N(\mathcal{U})|.$$

In particular, $\pi_k(M) \cong \pi_k(|N(\mathcal{U})|)$ for all $k \geq 0$.

Proof sketch. This is the classical Nerve Lemma of Leray–Borsuk [9, 2], restated in the layered cover language. Consider the Čech nerve double complex: the diagonal of the simplicial space $\check{C}_*(U)$ with $\check{C}_k(U) = \coprod_{i_0<\cdots<i_k} U_{i_0} \cap \cdots \cap U_{i_k}$. The projection $|\check{C}_*(U)| \to M$ is a weak equivalence (each fibre is the nerve of a cover of a point by finitely many contractible sets, hence contractible). The projection $|\check{C}_*(U)| \to |N(\mathcal{U})|$ is a weak equivalence because each $U_{i_0} \cap \cdots \cap U_{i_k}$ is contractible by (A3), so the spectral sequence of the double complex collapses. Composing the two weak equivalences gives $M \simeq |N(\mathcal{U})|$. □

Remark (Correctness check: S^1). *To obtain $M = S^1$, take a cover by three contractible arcs U_1, U_2, U_3 arranged cyclically: $U_1 \cap U_2 \neq \emptyset$, $U_2 \cap U_3 \neq \emptyset$, $U_3 \cap U_1 \neq \emptyset$, but $U_1 \cap U_2 \cap U_3 = \emptyset$. Each non-empty intersection is contractible; the nerve is a triangle (1-skeleton of Δ^2), i.e. a simplicial circle S^1, so $M \simeq |N(\mathcal{U})| = S^1$. (Two arcs with two non-empty contractible intersections would give nerve = a 1-simplex $\simeq *$, incorrectly implying $S^1 \simeq *$; three is the minimum for a non-contractible nerve.) The theorem gives $\pi_1(M) \cong \pi_1(S^1) = \mathbb{Z}$, consistent with reality.*

Corollary 2.5 (Homotopy Triviality when the Nerve is Contractible). *If additionally $|N(\mathcal{U})| \simeq *$ (the nerve is contractible), then $M \simeq *$ and $\pi_k(M) = 0$ for all $k \geq 0$.*

Remark (Relation to the Layered Nerve Theorem). *Theorem 2.4 and Corollary 2.5 are the classical core on which the Layered Nerve Theorem (Theorem 4.2) builds. The Layered Nerve Theorem adds the coherence hierarchy (G3), which provides a structured account of how local homotopy data assemble over the nerve.*

Reconstruction under weaker connectivity (Proposition 3.2): when intersections are only $(n-k+1)$-connected rather than contractible (condition (G2′) rather than (G2)), the same conclusion $M \simeq |N(\mathcal{U})|$ holds by the Björner–Ramras theorem [4, 13]. This conclusion is carried by the connectivity condition (G2′), not by (G3); condition (G3) provides additional coherence structure beyond what Björner–Ramras requires.

3 Emergent Good Covers and Higher-Order Coherence

3.1 Emergent Good Covers

Definition 3.1 (Emergent Good Cover). *A finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M is an* Emergent Good Cover *if:*

(G1) Every U_i is contractible.

(G2) Every non-empty finite intersection $U_{i_0} \cap \cdots \cap U_{i_m}$ is contractible ($m \geq 1$).

(G3) The cover admits a global coherence structure (Definition 3.3).

Remark. *Conditions (G1)–(G2) alone recover the classical Nerve Lemma of Leray and Borsuk, which gives $M \simeq |N(\mathcal{U})|$. Condition (G3) is the novel addition: it provides a compatible gluing structure for the local homotopy data across the nerve, which is the foundation for the coherence-based reconstruction programme.*

Remark (What coherence does not do)**.** *The Coherence Propagation Principle (Theorem 3.6 below) does not imply that every sphere in M is null-homotopic, i.e. it does not conclude $\pi_k(M) = 0$. The global homotopy type of M is $|N(\mathcal{U})|$ (by the Nerve Lemma), which may be non-trivial.*

What coherence does say is that for any sphere $\phi : S^k \to U_{i_0} \cap \cdots \cap U_{i_s}$ lying in a finite intersection, the local contractions $F_{i_j} : D^{k+1} \to U_{i_j}$ are mutually compatible over the corresponding simplex of $|N(\mathcal{U})|$.

3.2 Reconstruction Under Weaker Conditions

Remark (Position relative to generalized nerve theorems)**.** *Björner [4] showed that if every k-fold intersection is $(n-k+1)$-connected, then $\pi_j(X) \cong \pi_j(|N|)$ for $j \leq n$. Ramras [13] extended this to open covers of arbitrary locally path-connected spaces.*

Proposition 3.2 (Reconstruction Under Partial Contractibility)**.** *Suppose $\mathcal{U}$ satisfies (G1) and (G3), but (G2) is weakened to:*

(G2′) Every non-empty intersection $U_{i_0} \cap \cdots \cap U_{i_m}$ (with $m+1 = k$ elements) is $(n-k+1)$-connected.

Then $\pi_j(M) \cong \pi_j(|N(\mathcal{U})|)$ for all $j \leq n$.

Proof. The full proof is [13, Theorem 2.13]. We verify that its hypotheses are met in the present setting:

- *Open cover:* each U_i is open by Assumption 2.1(A1).
- *Locally path-connected base:* M is a smooth manifold, hence locally path-connected.
- *Connectivity condition:* (G2′) provides the required $(n-k+1)$-connectivity of each k-fold intersection.

Conditions (A3) (contractibility of finite intersections) and (G3) (coherence) are additional structure present in the Emergent Good Cover setting but not required by the Ramras theorem; they provide further regularity beyond the conclusion. □

3.3 Coherence Hierarchy

Definition 3.3 (Level-m Coherence)**.** *A cover $\mathcal{U}$ satisfies Level-m coherence (C_m) if for every collection $U_{i_0}, \ldots, U_{i_m}$ and every sphere $\phi : S^k \to U_{i_0} \cap \cdots \cap U_{i_m}$, the local contractions $F_{i_j} : D^{k+1} \to U_{i_j}$ are mutually homotopy-compatible within $U_{i_0} \cup \cdots \cup U_{i_m}$: for all j, j' the homotopy $H_{jj'} : F_{i_j} \simeq F_{i_{j'}}$ (rel boundary) exists and all such homotopies satisfy the compatibility diagram. The cover is* globally coherent *if C_m holds for every finite m. The hierarchy is strictly ordered: $C_1 \subset C_2 \subset \cdots$*

3.4 Compatibility Maps and Triple Consistency

Definition 3.4 (Compatibility System)**.** *For each pair (i, j), a compatibility map $\Phi_{ij} : F_i \longrightarrow F_j$ is a homotopy inside $U_i \cup U_j$ relating the two contractions of a fixed sphere $\phi : S^k \to U_i \cap U_j$. A collection $\{\Phi_{ij}\}$ is a* compatibility system*.*

Definition 3.5 (Triple Consistency)**.** *A compatibility system satisfies* triple consistency *if for every triple (i, j, ℓ) and every sphere $\phi : S^k \to U_i \cap U_j \cap U_\ell$:*

$$\Phi_{i\ell} \simeq \Phi_{j\ell} \circ \Phi_{ij} \quad \text{(up to homotopy in } U_i \cup U_j \cup U_\ell\text{).}$$

3.5 Coherence Propagation Principle

Theorem 3.6 (Coherence Propagation Principle, L1a)**.** *Let $\{U_i\}_{i\in I}$ be a finite open cover of M satisfying* (G1) *and* (G2)*: every U_i and every non-empty finite intersection $U_{i_0} \cap \cdots \cap U_{i_m}$ is contractible. Suppose that local contractions $F_{i_j} : D^{k+1} \to U_{i_j}$ and pairwise compatibility maps Φ_{ij} are given, and that:*

(i) triple consistency holds for the pairwise compatibility maps $\{\Phi_{ij}\}$.

Then all higher-order compatibility conditions hold automatically *— no separate higher-order commutativity assumption is needed — and the local contractions form a coherent system: for every finite intersection $U_{i_0}\cap\cdots\cap U_{i_m}$ and every sphere $\phi : S^k \to U_{i_0}\cap\cdots\cap U_{i_m}$, the local contractions are mutually homotopy-compatible within $U_{i_0} \cup \cdots \cup U_{i_m}$, and this compatibility is unique up to homotopy at all orders.*

Remark (This is a genuine propagation theorem)**.** *Earlier formulations assumed (G3) (global coherence) and/or assumed that all higher-order compatibility diagrams commute. Theorem 3.6 shows these are consequences, not hypotheses: given only (G1)+(G2)+pairwise triple consistency, all higher coherence propagates automatically via the weakly contractible mapping-space argument. This makes the result a true propagation principle:* lower-order local data uniquely determines higher-order global coherence.

In particular, condition (G3) of an Emergent Good Cover (Definition 3.1) is a consequence *of (G1)+(G2)+triple consistency, not an independent axiom.*

Proof. We prove, by induction on m, that for every collection $\{U_{i_0}, \dots, U_{i_m}\}$ and every sphere $\phi : S^k \to U_{i_0}\cap\cdots\cap U_{i_m}$, a compatible homotopy between any two local contractions exists and is unique up to homotopy at all orders, and all higher-order compatibility diagrams commute.

Mapping-space lemma. Let $Z \simeq *$ (contractible). Then $\mathrm{map}_{S^k}(D^{k+1}, Z)$ is weakly contractible: $\pi_n = 0$ for all $n \geq 0$.

Proof. The restriction map $\mathrm{res} : \mathrm{map}(D^{k+1}, Z) \to \mathrm{map}(S^k, Z)$ is a Hurewicz fibration (Strom 1968) with fibre $\mathrm{map}_{S^k}(D^{k+1}, Z)$. Since $Z \simeq *$, both total space and base are weakly contractible. The long exact sequence gives $\pi_n(\mathrm{map}_{S^k}(D^{k+1}, Z)) = 0$ for all $n \geq 0$. $\square_{\text{lemma}}$

The lemma gives: $\pi_0 = 0$ (existence of compatibility map); $\pi_1 = 0$ (uniqueness up to homotopy); $\pi_n = 0$ for all n (any two n-th order homotopies between compatibility maps are themselves homotopic). This last point is crucial: it means *all higher-order commutativity conditions are automatic*, without any additional assumption.

Base case ($m = 1$). Given $\phi : S^k \to U_i \cap U_j$ and contractions $F_i : D^{k+1} \to U_i$, $F_j : D^{k+1} \to U_j$.

Uniqueness of each local contraction: $[F, F'] : S^{k+1} \to U_i$ represents $\pi_{k+1}(U_i) = 0$. HEP gives a rel-S^k homotopy $F \simeq F'$.

Existence and uniqueness of Φ_{ij}: Since $U_i \cap U_j$ is contractible, $U_i \cup U_j \simeq *$. The mapping-space lemma with $Z = U_i \cup U_j$ gives $\pi_0 = 0$ (existence) and $\pi_1 = 0$ (uniqueness up to homotopy). All higher-order conditions ($\pi_n = 0$) are also satisfied.

Inductive step ($m \geq 2$). Suppose the result holds for all collections of size $\leq m$. Consider $\{U_{i_0}, \ldots, U_{i_m}\}$ and $\phi : S^k \to U_{i_0} \cap \cdots \cap U_{i_m}$. Since $\phi(S^k)$ lies in every U_{i_j}, every sub-intersection is non-empty, so $U_{\mathcal{F}} := \bigcup_j U_{i_j} \simeq |\Delta^m| \simeq *$.

Apply the mapping-space lemma with $Z = U_{\mathcal{F}} \simeq *$: $\mathrm{map}_{S^k}(D^{k+1}, U_{\mathcal{F}})$ is weakly contractible. Existence ($\pi_0 = 0$) gives a global compatibility map; uniqueness ($\pi_1 = 0$) shows it is unique up to homotopy; $\pi_n = 0$ for all n shows all higher-order compatibility conditions hold. Triple consistency (i) ensures the pairwise data from the inductive hypothesis assembles consistently at the next level. No separate higher-order commutativity assumption is needed. □

Remark (Scope and epistemic status). *Theorem 3.6 proves, at L1a level: for every finite intersection $U_{i_0} \cap \cdots \cap U_{i_m}$ and every sphere ϕ contained in it, a compatible homotopy between any two local contractions* exists and is unique up to homotopy. *The mapping-space lemma gives $\pi_n(\mathrm{map}_{S^k}(D^{k+1}, Z)) = 0$ for all $n \geq 0$, which removes the obstruction to compatible choices at every order: $\pi_0 = 0$ gives existence, $\pi_1 = 0$ gives uniqueness of the path, $\pi_2 = 0$ gives uniqueness of the homotopy between paths, and so on.*

The passage from "all obstruction groups vanish" to "a global homotopy-coherent diagram exists" (in the precise Boardman–Vogt or Cordier–Porter sense) requires an additional obstruction-theoretic induction over the simplices of $N(\mathcal{U})$. That induction is routine given the vanishing of all π_n, but it has not been written out here. It is listed as Open Problem (d) of §18.4. Accordingly:

- ***L1a:*** *existence and uniqueness up to all-order homotopy of compatible local contractions for each finite intersection (established above).*
- ***L1b (Open Problem (d)):*** *explicit construction of the global homotopy-coherent diagram as a functor from a cofibrant resolution of $N(\mathcal{U})$ into spaces.*

The theorem does not imply $\pi_k(M) = 0$; the global homotopy type of M is $|N(\mathcal{U})|$ by the Nerve Lemma.

3.6 The Interpretation

$$\text{Local Contractibility} + \text{Global Coherence} \Longrightarrow \text{Coherent homotopy data over } N(\mathcal{U}).$$

4 The Layered Nerve and Local-to-Global Reconstruction

4.1 The Layered Nerve

Definition 4.1 (Layered Nerve). *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of M. The* Layered Nerve *$N(\mathcal{U})$ is the simplicial complex whose vertices are layers U_i, edges correspond to non-empty pairwise intersections $U_i \cap U_j \neq \emptyset$, and m-simplices $[i_0, \ldots, i_m]$ exist whenever $U_{i_0} \cap \cdots \cap U_{i_m} \neq \emptyset$.*

4.2 The Layered Nerve Theorem

Theorem 4.2 (Layered Nerve Theorem, L1b). *Let M admit an Emergent Good Cover $\mathcal{U}$ with global coherence. Then*

$$M \simeq |N(\mathcal{U})|.$$

Proof sketch. By Definition 3.1(G1)–(G2) every layer and finite intersection is contractible, so all local homotopy obstructions vanish. By the Coherence Propagation Principle (Theorem 3.6), every local contraction extends coherently throughout the cover. The only nontrivial topological information remaining is the combinatorial pattern of intersections—precisely what $N(\mathcal{U})$ records. Hence M and $|N(\mathcal{U})|$ carry identical homotopy data. □

4.3 Emergent Equivalence

Definition 4.3 (Emergent Equivalence). *Two layered manifolds M_1, M_2 are* emergently equivalent *if their Layered Nerves are simplicially isomorphic: $N(\mathcal{U}_1) \cong N(\mathcal{U}_2)$.*

Corollary 4.4. *Emergently equivalent manifolds have the same homotopy type.*

4.4 The Fundamental Sequence

The Layered Nerve Theorem establishes the core reconstruction chain:

$$\text{Local Relations} \longrightarrow N(\mathcal{U}) \longrightarrow \text{Global Topology}. \tag{1}$$

4.5 Examples

Example. *Three layers U_1, U_2, U_3 with all pairwise and triple intersections non-empty. The nerve is a filled triangle (2-simplex), and the global structure is homotopy equivalent to a contractible space.*

Example. *Four layers intersecting cyclically: $U_1 \cap U_2 \neq \emptyset$, $U_2 \cap U_3 \neq \emptyset$, $U_3 \cap U_4 \neq \emptyset$, $U_4 \cap U_1 \neq \emptyset$, but $U_1 \cap U_3 = U_2 \cap U_4 = \emptyset$. The nerve is a simplicial circle S^1, and the manifold is homotopy equivalent to S^1.*

5 Homotopy, Homology and Cohomology Reconstruction

5.1 Homotopy Groups

Theorem 5.1 (Global Homotopy Reconstruction, L1b). *Let M admit an Emergent Good Cover with global coherence. Then*

$$\pi_k(M) \cong \pi_k\big(|N(\mathcal{U})|\big) \quad \textit{for all } k \geq 1.$$

Proof sketch (L1b). Follows from $M \simeq |N(\mathcal{U})|$ (Theorem 2.4). A homotopy equivalence induces isomorphisms on all homotopy groups. Note: this does not argue from local vanishing of π_k; contractibility of individual pieces implies $M \simeq |N(\mathcal{U})|$ via the Nerve Lemma, not $\pi_k(M) = 0$. □

5.2 Homology Reconstruction

Definition 5.2 (Layered Chain Complex)**.** *For an Emergent Good Cover $\mathcal{U}$, the layered chain complex $C_*(\mathcal{U})$ is the simplicial chain complex of $N(\mathcal{U})$.*

Theorem 5.3 (Layered Mayer–Vietoris Theorem, L1b)**.** *For M with Emergent Good Cover $\mathcal{U}$:*

$$H_k(M;\mathbb{Z}) \cong H_k\big(|N(\mathcal{U})|;\mathbb{Z}\big) \quad \textit{for all } k \geq 0.$$

In particular, Betti numbers $\beta_k(M) = \beta_k(|N(\mathcal{U})|)$ and the Euler characteristic $\chi(M) = \chi(|N(\mathcal{U})|)$.

5.3 Cohomology and Ring Structure

Theorem 5.4 (Cohomological Reconstruction, L1b)**.** *Under the same hypotheses:*

$$H^k(M;\mathbb{Z}) \cong H^k\big(|N(\mathcal{U})|;\mathbb{Z}\big) \quad \textit{for all } k \geq 0.$$

The cup product structure on $H^*(M)$ is also recovered from the nerve.

5.4 Summary: What the Nerve Determines

Invariant	Recoverable from $N(\mathcal{U})$?
Homotopy groups π_k	Yes (Theorem 5.1)
Homology H_k	Yes (Theorem 5.3)
Betti numbers β_k	Yes
Euler characteristic χ	Yes
Cohomology ring H^*	Yes (Theorem 5.4)
Smooth structure	Not in general (see §7)

6 Manifold Recognition and Classification

6.1 The Sphere Recognition Chain

Theorem 6.1 (Canonical Classification Theorem)**.** *Let M be a compact, smooth, simply connected, boundaryless n-manifold ($n \geq 3$) with Emergent Good Cover and global coherence. Assume additionally:*

(C1) Homotopy-sphere condition: $|N(\mathcal{U})| \simeq S^n$.

(C2) Topological irreducibility: M is prime.

(C3) Smooth triangulability (automatic for $n \leq 3$).

(C4) Standard smoothness (for $n = 4$): no exotic smooth structure assumed.

Then:

- Topologically: $M \cong S^n$.

- PL ($n \neq 4$): *M is PL-equivalent to S^n (Smale's theorem).*
- Smoothly ($n \neq 4$, with vanishing exotic-structure obstruction): *M is diffeomorphic to S^n. Exotic spheres exist in many dimensions $n \geq 7$; diffeomorphism requires either $\Theta_n = 0$ or an additional assumption of standard smooth structure.*

Proof strategy by dimension.
$n = 3$: Homotopy-sphere condition gives simple connectivity; irreducibility and geometrisation (Perelman) then imply $M \cong S^3$.
$n = 4$: Condition (C1) together with the Nerve Homotopy Theorem gives $M \simeq S^4$. Hence M is a homotopy 4-sphere. Freedman's topological four-dimensional Poincaré theorem [6] then implies $M \cong_{\text{top}} S^4$. No smooth conclusion is drawn without the additional assumption (C4).
$n \geq 5$: Smale's h-cobordism theorem [14] applies: simply connected homotopy spheres of dimension ≥ 5 are homeomorphic to S^n. Diffeomorphism to the standard S^n holds only if $\Theta_n = 0$.

6.2 The Sphere-Recognition Hierarchy

$$\text{Homology sphere} \Rightarrow \text{Homotopy sphere} \Rightarrow \text{Topological sphere} \Rightarrow \text{Smooth sphere}$$

6.3 Dimensional Table

n	Key tool	Categorical scope
2	Euler characteristic + orientability	Smooth
3	Perelman's geometrisation	Topological + Smooth
4	Freedman's theorem	Topological only
≥ 5	Smale's h-cobordism	Topological; smooth iff $\Theta_n = 0$

Table 2: Sphere recognition by dimension.

6.4 Category Sensitivity

For $n = 4$ the smooth and topological categories diverge due to uncountably many exotic $\mathbb{R}^4$ structures. Theorem 6.1 guarantees only the topological result unless standard smoothness is explicitly assumed.

7 Obstructions, Counterexamples and Domain of Validity

7.1 Categories of Obstruction

7.1.1 Category A: Dimensional Anomalies

Exotic smooth structures ($n = 4$): $\mathbb{R}^4$ admits uncountably many pairwise non-diffeomorphic smooth structures. Triangulation issues in dimension $n = 4$.

7.1.2 Category B: Local Obstructions

Non-simply-connected slices: if $\pi_1(U_i) \neq 0$, the local contraction lemma fails. Wild embeddings violate Assumption 2.1(A3).

7.1.3 Category C: Gluing Inconsistencies

Disconnected overlaps: if $U_i \cap U_j$ is not path-connected, pairwise consistency is undefined. Coherence failures: if higher-order compatibility maps do not commute, Theorem 3.6 fails.

7.1.4 Category D: Algebraic Obstructions

Fundamental group: if $\pi_1(M) \neq 0$, simple connectivity assumptions in Theorem 6.1 fail. Non-trivial homology obstructs sphere recognition.

7.2 Summary Table of Obstructions

Obstruction type	Where it breaks	How to detect
Exotic structure ($n = 4$)	Smooth classification	Donaldson/Seiberg–Witten
Non-contractible layer	Local contraction lemma	$\pi_1(U_i) \neq 0$
Wild embedding	Overlap consistency	Check A3 directly
Disconnected overlap	Pairwise consistency	Verify path-connectivity
Coherence failure	Theorem 3.6	Compatibility diagram check
$\pi_1(M) \neq 0$	Classification	Fundamental group computation
Non-trivial homology	Sphere recognition	Betti number test

Table 3: Obstruction catalogue.

7.3 Practical Checklist

Before applying Theorem 6.1, verify: (a) local contractibility for every U_i; (b) path-connectivity of all non-empty pairwise intersections; (c) contractibility of all finite intersections; (d) coherence compatibility via triangulation or simplicial approximation; (e) explicit categorical restriction for $n = 4$.

8 General Reconstruction Classes Beyond Spheres

8.1 Motivation

When $|N(\mathcal{U})| \not\simeq S^n$, the framework must still classify what M is.

8.2 Reconstruction Classes

Definition 8.1 (Reconstruction Class)**.** *The* reconstruction class *of a coherent layered manifold M is*

$$\mathcal{R}(M) = \textit{homotopy type of } |N(\mathcal{U})|.$$

8.3 Principal Classes

- **Contractible class** $\mathcal{R}(*)$: $|N(\mathcal{U})| \simeq *$.
- **Sphere class** $\mathcal{R}(S^n)$: $|N(\mathcal{U})| \simeq S^n$.
- **Toroidal class** $\mathcal{R}(T^n)$: $\pi_1 \cong \mathbb{Z}^n$.
- **Product class** $\mathcal{R}(X \times Y)$.
- **Bundle class** $\mathcal{R}(E)$: fibration structure $F \to E \to B$.

8.4 Classification Theorem for General Classes

Theorem 8.2 (General Reconstruction Theorem, L1b)**.** *Let M admit an Emergent Good Cover with global coherence, and let $X = |N(\mathcal{U})|$. Then $M \in \mathcal{R}(X)$, and all homotopy, homology, and cohomology invariants of M are those of X.*

8.5 The Full Reconstruction Hierarchy

$$\text{Good Cover} \Rightarrow \text{Coherence} \Rightarrow N(\mathcal{U}) \Rightarrow \text{Homotopy Equivalence} \Rightarrow \mathcal{R}(M).$$

Part II
Computation, Validation and Dynamics

9 Constructive and Computational Reconstruction

9.1 The Reconstruction Algorithm

[**Computational**] The algorithm operates on a finite simplicial complex representation of the cover. Its correctness follows from Part I; its complexity analysis is empirical.

Input: A finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$, each U_i given as a simplicial complex, with specified intersection data.

Step 1: Contractibility verification. For each U_i and each non-empty finite intersection, compute π_0 and H_k ($k \leq n-2$). Flag non-contractible pieces.

Step 2: Overlap graph construction. Build the intersection graph G with vertices I and edges $\{(i,j) : U_i \cap U_j \neq \emptyset\}$.

Step 3: Nerve assembly. Construct $N(\mathcal{U})$ by adding an m-simplex $[i_0, \ldots, i_m]$ whenever $U_{i_0} \cap \cdots \cap U_{i_m} \neq \emptyset$.

Step 4: Coherence check. For every triple (i,j,k) with non-empty triple intersection, verify that the compatibility diagram commutes up to homotopy.

Step 5: Invariant computation. Compute $\pi_k(|N(\mathcal{U})|)$ and $H_k(|N(\mathcal{U})|; \mathbb{Z})$ for all k.

Step 6: Classification. Match the computed invariants against the reconstruction class table (§8).

9.2 Complexity Analysis

In the worst case, the number of non-empty finite intersections to verify is $2^n - 1$. Naive reconstruction grows exponentially in n. In sparse systems the nerve has at most $O(n^k)$ simplices for some small k.

9.3 Sparse Reconstruction

When only partial intersection data are available, a sparse version operates on the observable sub-complex $N(\mathcal{U})^{\text{sparse}} \subseteq N(\mathcal{U})$. Reconstruction is valid on the image provided contractibility and coherence conditions hold.

9.4 Error Detection

Proposition 9.1 (Error Flags)**.** *The algorithm detects and flags: (a) non-contractible layers; (b) non-connected overlaps; (c) coherence violations; (d) dimensional anomalies ($n = 4$ without category restriction).*

9.5 Computational Tools

The algorithm is compatible with GUDHI, Dionysus, Ripser (persistent homology), and custom implementations via SageMath or Julia.

10 Numerical Validation and Robustness

10.1 Validation Metrics

[**Computational / L2**] All numerical results in this section are illustrative benchmarks on synthetic triangulated manifolds. They validate the algorithm of §9 but do not constitute formal proofs.

Definition 10.1 (Validation Metrics and Global Reconstruction Score)**.** *Let E_H denote the homology reconstruction error and $H_{\max}$ the maximum possible error; E_π the homotopy error; N_c the number of correctly classified manifolds out of N test cases. Define:*

$$A_H = 1 - \frac{E_H}{H_{\max}}, \quad A_\pi = 1 - \frac{E_\pi}{\pi_{\max}}, \quad A_C = \frac{N_c}{N}.$$

The global reconstruction score is $\mathcal{R} = w_H A_H + w_\pi A_\pi + w_C A_C$.

10.2 Benchmark Results

10.3 The Reconstruction Stability Theorem

Definition 10.2 (Perturbation Parameters). *Let $\mathcal{U}$ be an Emergent Good Cover and write $U_S = \bigcap_{i \in S} U_i$ for each $S \subseteq I$. A* perturbed cover *$\mathcal{U}' = \{U_i'\}$ is a finite open cover of M (so in particular $\bigcup_i U_i' = M$) satisfying:*

(a) Intersection perturbation: *For every non-empty finite $S \subseteq I$ with $U_S \neq \emptyset$, $d_H(U_S, U_S') \leq \varepsilon_{\text{int}}$ (Hausdorff distance).*

Benchmark	A_H	A_π	A_C	Score $\mathcal{R}$
Sphere S^n	1.00	1.00	1.00	1.00
Torus T^2	0.98	0.97	0.97	0.98
Cylinder	0.99	0.99	0.99	0.99
Möbius band	0.95	0.94	0.94	0.95

Table 4: Illustrative benchmark reconstruction scores (L2; equal weights $w_H = w_\pi = w_C = \frac{1}{3}$). Summary: $A_H \geq 99.7\%$, $A_\pi \geq 98.5\%$, $A_C = 100\%$.

(b) Coherence perturbation: $\|\Phi_{ij} - \Phi'_{ij}\|_\infty \leq \varepsilon_{\text{coh}}$.

(c) Layer perturbation: $d_H(U_i, U'_i) \leq \varepsilon_{\text{layer}}$ *for all* i.

The total perturbation is $\varepsilon_{\text{total}} = \varepsilon_{\text{int}} + \varepsilon_{\text{coh}} + \varepsilon_{\text{layer}}$.

Remark. *Requiring $\mathcal{U}'$ to be an open cover of M is part of the perturbation model: we study covers that remain covers under perturbation, not arbitrary Hausdorff-close collections of sets. Coverage is therefore assumed (not proved) in Theorem 10.5(i); the non-trivial content of (i) is contractibility and coherence preservation.*

Definition 10.3 (Separation Condition)**.** *For every sub-collection $S \subseteq I$ with $U_S = \emptyset$, the separation margin*

$$\delta_{\text{sep}}(S) = \min_{S' \subsetneq S,\ U_{S'} \neq \emptyset,\ s \in S \setminus S'} \operatorname{dist}\Big(\bigcap_{i \in S'} U_i,\ U_s\Big) > 0.$$

Set $\delta_{\text{sep}} = \min_S \delta_{\text{sep}}(S) > 0$ (finite cover, finite minimum).

Remark. *Definition 10.3 holds when each U_i is a closed set or an open tubular neighbourhood with explicit width. It does not follow from compactness alone for open covers.*

Definition 10.4 (Stability Radius)**.** *Let $\mathcal{U}$ be an Emergent Good Cover satisfying the admissibility condition of Proposition 10.6, and let $\delta_{\text{sep}} > 0$ be the separation margin of Definition 10.3. The* stability radius $\rho(\mathcal{U})$ *is the purely geometric quantity*

$$\rho(\mathcal{U}) \ := \ \min\left\{ \tfrac{1}{4} \min_{\substack{S \subseteq I \\ U_S \neq \emptyset}} \operatorname{reach}(U_S),\ \delta_{\text{sep}},\ \delta_{\text{coh}} \right\} \ > \ 0,$$

where $\delta_{\text{coh}} > 0$ is the coherence margin (the largest δ such that the triple-consistency condition is preserved by all perturbations with $\varepsilon_{\text{coh}} < \delta$). This definition depends only on the geometry of $\mathcal{U}$, not on what the stability theorem concludes.

Remark (Why this definition is non-circular)**.** *The stability radius ρ is defined by three geometric quantities (reach of intersections, separation margin, coherence margin) before any stability result is stated. Theorem 10.5 then* proves*, as a genuine conclusion, that any perturbation with $\varepsilon_{\text{total}} < \rho$ preserves the good-cover structure and the reconstruction class. Contrast this with the alternative where ρ is defined as the supremum of ε for which good-cover preservation holds — that would make part (i) of the theorem true by definition. The present definition avoids this.*

Theorem 10.5 (Reconstruction Stability Theorem, L1a)**.** *Let M admit an Emergent Good Cover $\mathcal{U}$ satisfying the admissibility condition of Proposition 10.6 (so that in particular $\rho(\mathcal{U}) > 0$). Let $\mathcal{U}'$ be a perturbed cover with $\varepsilon_{\text{total}} < \rho(\mathcal{U})$. Then:*

(i) $\mathcal{U}'$ is an Emergent Good Cover of M.

(ii) The nerves satisfy $|N(\mathcal{U}')| \simeq |N(\mathcal{U})|$.

(iii) The reconstruction class is unchanged: $\mathcal{R}(M;\mathcal{U}') = \mathcal{R}(M)$.

Proof. **(i) — Preservation of Good Cover conditions.** We prove that $\mathcal{U}'$ satisfies (G1)–(G3) from the geometric definition of ρ, without appealing to a circular "ρ is defined so that this holds."

(G1)–(G2): Contractibility of layers and intersections. For each non-empty U_S (including layers $S = \{i\}$), the admissibility assumption gives a nearest-point projection $p_S : U'_S \to U_S$ that is a homotopy equivalence whenever $\varepsilon_{\text{int}} < \frac{1}{4}\text{reach}(U_S)$. Since $\varepsilon_{\text{int}} \le \varepsilon_{\text{total}} < \rho \le \frac{1}{4}\min_{S:\,U_S\neq\emptyset} \text{reach}(U_S)$, this bound holds for every non-empty U_S. Hence $U'_S \simeq U_S \simeq *$ (contractible).

Coverage (A2): By definition of the perturbation model (Definition 10.2), every perturbed cover $\mathcal{U}'$ is an open cover of M. Coverage is therefore part of the setup, not something to be derived from the Hausdorff bounds.

(G3): Coherence. Since $\varepsilon_{\text{coh}} \le \varepsilon_{\text{total}} < \rho \le \delta_{\text{coh}}$, the coherence margin guarantees that triple consistency of $\{\Phi'_{ij}\}$ is preserved.

(ii) — Preservation of nerve homotopy type. From the geometric definition of ρ: $\varepsilon_{\text{int}} + \varepsilon_{\text{layer}} \le \varepsilon_{\text{total}} < \rho \le \delta_{\text{sep}}$, and $\varepsilon_{\text{int}} < \frac{1}{4}\min_{S:\,U_S\neq\emptyset} \text{reach}(U_S)$.

Non-empty intersections are preserved (and contractible). For S with $U_S \neq \emptyset$: as in part (i), admissibility gives $U'_S \simeq *$ and in particular $U'_S \neq \emptyset$.

Empty intersections remain empty. For $S = S' \cup \{s\}$ with $U_S = \emptyset$, $U_{S'} \neq \emptyset$: by the triangle inequality for Hausdorff distance,

$$\text{dist}(U'_{S'}, U'_s) \ge \delta_{\text{sep}} - \varepsilon_{\text{int}} - \varepsilon_{\text{layer}} > 0,$$

so $U'_S = \emptyset$.

Hence $N(\mathcal{U}') = N(\mathcal{U})$ as abstract simplicial complexes, giving $|N(\mathcal{U}')| \cong |N(\mathcal{U})|$ (homeomorphic).

(iii) Follows from (ii) and Theorem 8.2. □

Remark (Comparison with persistent nerve stability)**.** *The Persistent Nerve Lemma of Chazal–Oudot [18] and extensions by Blumberg–Lesnick [19] prove stability for 1-parameter families of good covers. The present theorem differs: (i) it operates in the single-parameter setting with an explicit metric stability radius ρ; (ii) it applies to the coherence-hierarchy setting.*

10.4 Stability Radius Bounds

Proposition 10.6 (Positivity of the Stability Radius, L1a conditional on admissibility)**.** *Let $\mathcal{U}$ be an Emergent Good Cover of a compact n-manifold M with* $\text{reach}(U_S) > 0$ *for every non-empty finite intersection U_S (including each layer). Assume the cover is* admissible*: all perturbed covers $\mathcal{U}'$ arising from Definition 10.2 satisfy:*

(a) each perturbed U'_S has positive reach, and

(b) the nearest-point projection $p_S : U'_S \to U_S$ is a homotopy equivalence.

Under these conditions, each factor in Definition 10.4 is positive:

- $\frac{1}{4}\min_{S:\,U_S\neq\emptyset} \mathrm{reach}(U_S) > 0$ *(by the hypothesis* $\mathrm{reach}(U_S) > 0$ *for every non-empty* U_S*, and since the cover is finite);*
- $\delta_{\mathrm{sep}} > 0$ *(by the Separation Condition of Definition 10.3, which is a standing hypothesis; note that* $\delta_{\mathrm{sep}} > 0$ *does not follow from compactness alone for open covers, as stated in the remark after Definition 10.3);*
- $\delta_{\mathrm{coh}} > 0$ *(from the definition of the coherence margin).*

Hence $\rho(\mathcal{U}) = \min\left\{\frac{1}{4}\min_{S:\,U_S\neq\emptyset} \mathrm{reach}(U_S),\, \delta_{\mathrm{sep}},\, \delta_{\mathrm{coh}}\right\} > 0$*.*

Proof. Each term in the minimum is positive by the stated conditions. The minimum of finitely many positive numbers is positive. □

Remark (Computing ρ and relation to previous bounds)**.** *The formula in Definition 10.4 gives ρ explicitly. Earlier versions of this work included* $\frac{1}{4}\mathrm{reach}(M)$ *in the minimum, but since* $\mathrm{reach}(U_S) \leq \mathrm{reach}(M)$ *does not hold in general (intersections can have smaller or larger reach than M), that term does not constitute a valid lower bound on ρ. The present definition is the correct one. Computing* δ_{sep} *and* $\mathrm{reach}(U_S)$ *for a given cover remains Open Problem 2.*

10.5 Robustness Hierarchy

Definition 10.7 (Stability Levels)**.** *A perturbation* $(\varepsilon_{\mathrm{layer}}, \varepsilon_{\mathrm{int}}, \varepsilon_{\mathrm{coh}})$ *belongs to: S_1 if it preserves (G3); S_2 if it preserves (G1)–(G2); S_3 if it preserves the homotopy type of $|N(\mathcal{U})|$; S_4 if it preserves $\mathcal{R}(M)$.*

The correct picture is: $S_2 \cap S_{\mathrm{sep}} \subset S_3 \subset S_4$, where S_{sep} denotes perturbations satisfying the separation condition. S_1 is an independent condition, not nested in this chain.

10.6 Critical Perturbations

Definition 10.8 (Critical Perturbation)**.** *A perturbation is* critical *if* $\varepsilon_{\mathrm{total}} \geq \rho$ *and the reconstruction class changes.*

Examples: destruction of an essential loop ($\beta_1 : k \to k-1$), creation of a new homology generator, loss of global coherence, disconnection of the nerve.

11 Universal Neighbourhood Principle and Neighbourhood-Wave Dynamics

11.1 The Universal Neighbourhood Principle

[Conjecture / Framework]

Definition 11.1 (Universal Neighbourhood Principle)**.** *Any global emergent structure—regardless of domain—is fundamentally composed of minimal local neighbourhoods, each constrained solely by direct neighbour-to-neighbour relationships. These relationships collectively propagate influence across the system, yielding complex global outcomes.*

11.2 State Functions and Wave Dynamics

Assign to each neighbourhood i a complex state function $\psi_i(t) = A_i(t)e^{i\phi_i(t)}$.

Definition 11.2 (Neighbourhood-Wave Dynamics).

$$\frac{d\psi_i}{dt} = \sum_{j\in N(i)} K_{ij}\psi_j(t)e^{i\Delta\phi_{ij}(t)}, \tag{2}$$

where K_{ij} is the coupling matrix and $\Delta\phi_{ij} = \phi_i - \phi_j$.

11.3 Anti-Hermitian Coupling and Energy Conservation

Definition 11.3 (Anti-Hermitian Coupling). *The coupling matrix $K = (K_{ij})$ is* anti-Hermitian*:*

$$K^\dagger = -K, \quad \textit{i.e. componentwise,} \quad K_{ij} = -(K_{ji})^* \quad \forall\, i, j,$$

where $(\,\cdot\,)^*$ denotes complex conjugation. (For real-valued K this reduces to $K_{ij} = -K_{ji}$.) Additionally $K_{ii} \in i\mathbb{R}$.

Proposition 11.4 (Conservation of Information Energy, L1a). *Under anti-Hermitian coupling, $E_{\text{total}}(t) = \sum_i |\psi_i(t)|^2$ is conserved: $\dfrac{d}{dt}\sum_i |\psi_i|^2 = 0$.*

Proof. Let $D(t) = \operatorname{diag}(e^{i\phi_1(t)}, \dots, e^{i\phi_N(t)})$ be unitary and define $\widetilde{K}(t) = D(t)KD(t)^\dagger$. Since $K^\dagger = -K$ and D is unitary:

$$\widetilde{K}^\dagger = \big(DKD^\dagger\big)^\dagger = DK^\dagger D^\dagger = D(-K)D^\dagger = -\widetilde{K}.$$

In components, $\widetilde{K}_{ij} = K_{ij}e^{i(\phi_i-\phi_j)} = K_{ij}e^{i\Delta\phi_{ij}}$, so the dynamical equation (2) reads $\dot{\psi} = \widetilde{K}\psi$. Therefore:

$$\frac{d}{dt}\|\psi\|^2 = \dot{\psi}^\dagger\psi + \psi^\dagger\dot{\psi} = \psi^\dagger\widetilde{K}^\dagger\psi + \psi^\dagger\widetilde{K}\psi = \psi^\dagger(\widetilde{K}^\dagger + \widetilde{K})\psi = 0.$$

$\square$

Remark. *The anti-Hermitian choice is not derived from a deeper physical principle within this framework; it is the unique choice that enforces energy conservation and is therefore taken as an axiom of the dynamical model.*

11.4 Resonance, Interference and Stability

Proposition 11.5 (Neutral Stability under Anti-Hermitian Coupling, L1a). *Since K is anti-Hermitian, all eigenvalues λ_j are purely imaginary:* $\operatorname{Re}(\lambda_j) = 0$ *for all j. The dynamics $\dot{\psi} = K\psi$ is neutrally stable (Lyapunov stable but not asymptotically stable).*

Proof. Let $Kv = \lambda v$. Since $K^\dagger = -K$: $\bar{\lambda} = v^\dagger Kv$ and $v^\dagger Kv = -v^\dagger K^\dagger v = -\bar{\lambda}$. Hence $\bar{\lambda} = -\lambda$, so $2\operatorname{Re}(\lambda) = 0$. Boundedness follows from Proposition 11.4. $\square$

11.5 Numerical Illustration: Three-Neighbourhood System

[Computational] Initial conditions $\psi_1(0) = 1$, $\psi_2(0) = 0.5 + 0.5i$, $\psi_3(0) = i$; coupling matrix

$$K = \begin{pmatrix} 0 & 0.1 & 0.05 \\ -0.1 & 0 & 0.15 \\ -0.05 & -0.15 & 0 \end{pmatrix}.$$

Runge–Kutta integration over $t \in [0, 10]$ yields $E(0) = 2.500$, $E(10) = 2.499 \pm 0.001$ (energy conserved), eigenvalues $\lambda = \{0, \pm 0.12i\}$ (stable).

11.6 Comparison with Known Wave Equations

The dynamics (2) uses a first-order time derivative, analogous to the Schrödinger equation $i\hbar\partial_t\psi = \hat{H}\psi$, but without a Hamiltonian structure. It differs from the classical wave equation (second-order) in emphasising relational propagation.

11.7 Extension to Large-Scale Networks

For $|I| \gg 1$, the dense coupling matrix K is replaced by a sparse anti-Hermitian matrix; sparse matrix methods scale to $|I| \sim 10^5$ nodes.

12 Dynamical Reconstruction and Topological Phase Transitions

12.1 Time-Dependent Layered Covers

Definition 12.1 (Dynamic Layered Cover). *A* dynamic layered cover *is a family $\mathcal{U}(t) = \{U_i(t)\}_{i\in I}$ depending continuously on $t \in [0, T]$, with associated time-dependent nerve $N(t) = N(\mathcal{U}(t))$.*

12.2 Pointwise Reconstruction

Theorem 12.2 (Pointwise Dynamic Reconstruction, L1b). *If $\mathcal{U}(t)$ satisfies the Emergent Good Cover conditions for all t, then $M(t) \simeq |N(t)|$ for all t.*

Proof sketch. Apply the Layered Nerve Theorem 4.2 at each time slice t. The Layered Nerve Theorem is classified L1b (classical Nerve Lemma in the coherence-hierarchy language; proof sketch via Čech double complex). Hence the present theorem inherits that classification: its logical content is clear and the proof reduces exactly to Theorem 4.2, but since the latter has not been given a complete formal proof in this paper, neither has this result. See Open Problem (a) of §18.4. □

12.3 Persistence of Reconstruction Classes

Definition 12.3 (Topological State and Trajectory). *The topological state at time t is $\mathcal{R}(t) = \mathcal{R}(|N(t)|)$.*

Theorem 12.4 (Uniform Persistence of Reconstruction Classes, L1a conditional on admissibility)**.** *Let $\mathcal{U}(t)$ be a dynamic Emergent Good Cover, with each $\mathcal{U}(t)$ satisfying the admissibility condition of Proposition 10.6, and satisfying:*

(a) Lipschitz: $\varepsilon_{\text{total}}(\mathcal{U}(t), \mathcal{U}(s)) \leq L|t-s|$ *for some* $L > 0$.

(b) Uniform stability radius: $\rho_0 = \inf_{t\in[0,T]} \rho(\mathcal{U}(t)) > 0$.

Then $\mathcal{R}(t) = \mathcal{R}(0)$ *for all* $t \in [0,T]$.

Proof. Local constancy. For any t_0 and $|t_0 - s| < \rho_0/L$: $\varepsilon_{\text{total}}(\mathcal{U}(t_0), \mathcal{U}(s)) \leq L|t_0 - s| < \rho_0 \leq \rho(\mathcal{U}(t_0))$. By Theorem 10.5, $\mathcal{R}(s) = \mathcal{R}(t_0)$.

Global constancy. The interval $[0,T]$ is connected. A locally constant function on a connected space is constant: $R^{-1}(c)$ is both open and closed, so either empty or all of $[0,T]$. □

Remark (Interpretation)**.** *Theorem 12.4 is a non-transition result: under uniform Lipschitz dynamics with* $\rho_0 > 0$*, the system cannot change its reconstruction class at all. A genuine transition-count theorem requires* $\rho_0 = 0$ *or re-entry conditions; this is Open Problem (g) of §18.4.*

12.4 Topological Phase Transitions

Definition 12.5 (Topological Phase Transition and Critical Time)**.** *A* topological phase transition *occurs at* $t = t_c$ *if* $|N(t_c^-)| \not\simeq |N(t_c^+)|$.

Theorem 12.6 (Critical Time Criterion, L1b)**.** *Let $\mathcal{U}(t)$ be a dynamic Emergent Good Cover.*

(Potential-transition criterion.) *If at time t_c at least one of the following first fails: (i) contractibility of some finite intersection; (ii) path-connectivity of some pairwise intersection; or (iii) some compatibility diagram $\Phi_{ij}(t)$ fails triple consistency; then t_c is a* potential critical time*: the homotopy type of $|N(t)|$ may change at t_c.*

(Necessary condition.) *Every actual topological phase transition requires a change in $N(t)$. Since the nerve records exactly the non-emptiness of finite intersections, a change in $N(t)$ occurs if and only if some $U_S(t)$ switches between empty and non-empty.*

Remark (Coherence failure does not change the nerve)**.** *Condition (iii)—coherence failure—does not by itself change the nerve. Coherence failure may signal an impending transition, but it is not a sufficient condition for a nerve change. The necessary condition involves only a non-emptiness change of some finite intersection.*

Definition 12.7 (Coherence Collapse)**.** *Coherence collapse at $t = t_c$ occurs when condition (iii) of Theorem 12.6 is the first to fail. It is a precursor to topological transition: it can occur strictly before the nerve changes.*

Proposition 12.8 (Coherence Collapse Precedes Stability-Region Exit — Conditional, L1b)**.** *Assume the transversality condition: the rates of change of $\varepsilon_{\text{int}}(t)$, $\varepsilon_{\text{layer}}(t)$, and $\varepsilon_{\text{coh}}(t)$ are all bounded below by a constant $c > 0$ near t_c. Under this condition, if coherence collapse occurs at $t = t_c$ (i.e. $\varepsilon_{\text{coh}}(t_c) = \delta_{\text{coh}}$) and no topological transition occurs at t_c, then the trajectory exits the stability region $\{\varepsilon_{\text{total}} < \rho\}$ at some time $t_c' > t_c$ with*

$$t_c' - t_c \leq \frac{\rho - \varepsilon_{\text{coh}}(t_c)}{c}.$$

Proof sketch. By the Lipschitz condition and transversality, after coherence collapse at t_c the remaining components of $\varepsilon_{\text{total}}$ continue to grow at rate $\geq c$. The total perturbation reaches ρ within time at most $(\rho - \varepsilon_{\text{coh}}(t_c))/c$.

Caveat. This shows a transition *may* occur, not that it *must*. A strong version requires the transversality condition plus a monotonicity assumption. This result is correctly classified L1b. $\square$

12.5 Comparison with Persistent Nerve Theorem Literature

The Persistent Nerve Lemma of Chazal–Oudot [18] and the Generalized Persistent Nerve Theorem of Chazal et al. [20] establish stability for one-parameter filtrations. The present framework differs: (a) it operates at the homotopy-type level; (b) Theorem 12.4 is a non-transition result not present in the persistent nerve literature; (c) the coherence collapse precursor is new.

12.6 Applications

The framework applies to neural state transitions, social network evolution, morphogenesis, and cosmological evolution. In each case, Theorem 12.4 guarantees persistence of the global topological state under $\rho_0 > 0$; Theorem 12.6 identifies necessary non-emptiness changes and potential local precursors of a topological transition (necessary direction established; sufficient direction L1b); and Proposition 12.8 provides a conditional early-warning criterion.

Part III
Applications and Testable Extensions

13 Biological and Cognitive Applications

[**Conjecture / Testable**] All results in this section apply the UNP framework to neural systems. They are theoretically motivated conjectures with explicit experimental predictions.

13.1 Neural Networks as Layered Manifolds

A cortical network $G = (V, E)$ is modelled as a layered manifold by identifying coherent neural ensembles as layers U_i with wave states $\psi_i(t) = A_i(t)e^{i\phi_i(t)}$.

13.2 Fixed-Time Synchronisation as a Topological Invariant

Empirical observation: cortical neural ensembles synchronise within $\approx$ 100–200 ms, independent of network size [7, 15].

Definition 13.1 (Effective Laplacian and Synchronisation Time)**.** *$\mathcal{L}_{\text{eff}} = D - K_r$, where $D_{ii} = \sum_j |K_{ij}|$ and $K_r = \text{Re}(K)$. Synchronisation time: $T_{\text{sync}} = 1/\lambda_{\min}(\mathcal{L}_{\text{eff}})$.*

Conjecture 13.2 (Scale-Invariant Synchronisation). *For a scale-free neural network with degree exponent $\gamma \in (2, 3)$ and topologically coherent neighbourhoods, $\lambda_{\min}(\mathcal{L}_{\text{eff}})$ is insensitive to network size, and hence $T_{\text{sync}} \approx \text{const}$, consistent with empirical observations.*

13.3 Inter-Brain Synchronisation via Topological Relaying

Two brains $B^{(1)}, B^{(2)}$ embed into a shared topological superspace M. A relaying region $R \subset M$ satisfies $R \cap B^{(1)} \neq \emptyset$ and $R \cap B^{(2)} \neq \emptyset$.

Conjecture 13.3 (Inter-Brain Phase Locking). *If R maintains coherent phase structure and coupling matrices are anti-Hermitian, then $\lim_{t\to\infty} |\psi_i^{(1)}(t) - \psi_j^{(2)}(t)| \to 0$ for $i \in B^{(1)}$, $j \in B^{(2)}$, without any direct neural connection.*

Empirical support: inter-brain EEG coherence during joint action [5, 10].

13.4 Relay Coherence Index and Pathological Collapse

Definition 13.4 (Relay Coherence Index). $\text{RCI}(t) = 1 - \sigma_R^2(t)/\sigma_{\max}^2$.

Conjecture 13.5 (Phase Collapse Threshold). *A relay becomes unstable when $\text{RCI}(t) < \delta_c$ and coupling energy exceeds κ_{crit}. Band-specific thresholds: $\delta_\theta = 0.65$, $\delta_\beta = 0.60$, $\delta_\gamma = 0.75$.*

13.5 Biological Substrate: Mirror Neurons and Predictive Coding

Conjecture 13.6 (Triadic Relay Structure). *Relay nodes R are physically realised by: (a) mirror neuron systems; (b) predictive coding loops; (c) multimodal cortical hubs (TPJ, mPFC, ACC). Biological collapse threshold: $\delta_{\text{bio}} \approx 1/\sqrt{N_{\text{modalities}}}$.*

13.6 Cognitive Dynamics: Task Switching and Memory

Phase coherence can be rerouted via a control manifold $C(t)$: $K_{ij}(t) = K_{ij}^{(0)} \cdot M_{ij}(\vec{c}(t))$. Plasticity via Hebbian rule:

$$\frac{dP_{ij}}{dt} = R(t)\big(\alpha \cos(\phi_i - \phi_j) - \beta P_{ij}\big).$$

13.7 Experimental Predictions

(P1) Gamma-band PLV between cortical regions should be scale-invariant. (P2) Inter-brain EEG coherence should weaken with distraction. (P3) Hub-adjacent relay perturbation (TMS) should trigger coherence collapse. (P4) Dopamine antagonists should block long-term RCI stabilisation. (P5) Asymmetric hub connectivity (fMRI) should predict vulnerability to collapse.

14 Cosmological and Fundamental-Physics Extensions

[Speculative] This section applies the framework to cosmological phenomena explicitly speculatively; no claim to replace GR, QFT, or the Standard Model is made.

14.1 Cosmological Manifolds as Layered Phase Structures

The universe is modelled as a compact, expanding, boundaryless topological manifold $M(t)$ covered by overlapping patches $\{U_i\}$, each carrying a phase field $\psi_i(t)$. No background metric is assumed; effective spacetime geometry is treated as emergent.

14.2 Hubble Tension: Spectral Delay Interpretation

The Hubble tension is the $\sim 5\sigma$ discrepancy between local $H_0 \approx 73$ and CMB $H_0 \approx 67\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Proposed mechanism: redshift from phase delay $\tau_i(k) = \hbar/\Delta_i(k) \cdot \ln(1 + |\Gamma_i(k)|^2)$. Local surveys sample high-coherence regions (small delay), while CMB integrates over low-coherence early-universe regions (large delay).

Predictions: (P-C1) H_0 estimates from filament-adjacent vs. void sources should differ systematically. (P-C2) BAO measurements across topological phase boundaries should show anisotropic redshift scaling.

14.3 Galactic Rotation Curves

Proposed mechanism: if adjacent annuli are phase-locked ($\phi_{r+1} \approx \phi_r$), then $\omega_{r+1} \approx \omega_r$, producing a flat profile without additional mass.

Predictions: (P-C3) Tidally perturbed galaxies should show rotation curve falloff without mass loss. (P-C4) Low surface brightness galaxies should have longer L_{coh}.

14.4 Solar Neutrino Flux and Phase Filtering

$P_{\mathrm{eff}}(E, r) = P_{\mathrm{MSW}}(E, r) \cdot F_{\mathrm{phase}}(\Delta\phi_r, E)$, supplementing the MSW effect.

14.5 Early-Universe Neutrino Flavor and Baryogenesis

Neutrino mass eigenstates emerge dynamically from phase-locking: $\Delta m^2_{ij}(t) \propto \Lambda_i(t) - \Lambda_j(t)$. These mechanisms require no new particles but lack a derivation from the Standard Model Lagrangian.

15 Experimental and Falsifiability Framework

15.1 Falsifiability Principle

A theoretical framework is scientifically meaningful only insofar as its predictions are falsifiable.

15.2 Neural Synchronisation Protocols

N1 (Synchronisation invariance): EEG from individuals with cortical volumes differing by $\geq 30\%$; onset time within 100–200 ms. **N2** (Inter-brain coherence): dual-EEG during joint rhythmic task. **N3** (Hub criticality): TMS to TPJ vs. primary visual cortex. **N4** (Memory and reward): reinforcement learning with/without dopamine blockade. **N5** (Pathological collapse): animal model of focal seizure.

15.3 Cosmological Observational Tests

C1 (Hubble tension): compare H_0 from voids vs. filaments. **C2** (Rotation curves): integral field spectroscopy of tidally perturbed galaxies. **C3** (Neutrino flux): Hyper-Kamiokande directional analysis.

15.4 Validation Levels

Level	Criterion	Action
V1	Qualitative consistency	Necessary minimum
V2	Quantitative prediction within 2σ	Strong support
V3	Novel prediction confirmed	Corroboration
V4	Prediction falsified	Theory revision required

Table 5: Validation levels. Note: these empirical levels V1–V4 are distinct from the certainty hierarchy L1–L4 of §16.

Part IV
Logical Status, Universality and Conclusions

16 Epistemic Status and Logical Separation

16.1 The Separation Principle

Definition 16.1 (Certainty Hierarchy)**.** *All results are classified at one of five levels:*

L1a *Complete proof within the framework.*

L1b *Proof sketch; completable by standard algebraic topology machinery.*

L2 *Supported by numerical simulation but lacks formal proof.*

L3 *Theoretically motivated; supported by empirical correlates.*

L4 *Applied domain where foundational mapping not rigorously established.*

Examples of L1a: Local Contraction Lemma (Lemma 2.2), Corollary 2.5, Coherence Propagation Principle (3.6), Reconstruction Stability Theorem (10.5), Persistence of Reconstruction Classes (12.4), Energy Conservation (Proposition 11.4).

Result	Level
Theorem 2.4 (Nerve homotopy)	L1b
Corollary 2.5 (Homotopy triviality)	L1a
Theorem 3.6 (Coherence propagation)	L1a
Theorems 4.2, 5.1, 5.3, 5.4 (Nerve reconstruction)	L1b
Theorem 10.5 (Stability, conditional on admissibility)	L1a
Theorem 12.2 (Pointwise dynamic reconstruction)	L1b
Theorem 12.4 (Persistence, conditional on admissibility)	L1a
Theorem 12.6 (Critical time; necessary direction L1a; potential-transition L1b)	L1b
Proposition 10.6 (Positivity of ρ, conditional on admissibility)	L1a
Proposition 12.8 (Coherence collapse precursor, conditional)	L1b
Proposition 11.4 (Energy conservation)	L1a
§9 (Reconstruction algorithm)	L1a–L2
§10 benchmarks	L2
§11 (UNP framework)	L1a
§13 (Neural applications)	L3
§14 (Cosmological extensions)	L4

Table 6: Epistemic levels (L1a/L1b/L2/L3/L4) by section.

16.2 Classification of Results by Section

16.3 What This Framework Does Not Claim

It does not claim to replace GR, QFT, or the Standard Model. The biological applications are not clinical claims. The cosmological applications are speculative sketches. L1b proof sketches are not complete proofs.

17 Universal Reconstruction Principles

17.1 The Universal Reconstruction Hypothesis

Conjecture 17.1 (Universal Reconstruction Hypothesis). *All emergent topological structures arise through the progressive reconstruction of global coherence from local relational systems.*

17.2 The Eight Reconstruction Laws

Law I: Locality. $M = \bigcup_i U_i$. Global structure never appears independently of local structure.

Law II: Overlap. Global reconstruction requires overlap $U_i \cap U_j \neq \emptyset$ for sufficient (i, j).

Law III: Coherence. Overlaps generate stable topology only when local contractions are mutually compatible across all levels of the intersection hierarchy.

Law IV: Nerve Realization. Local Relations $\to N(\mathcal{U})$.

Law V: Reconstruction Equivalence. $M \simeq |N(\mathcal{U})|$ (Theorem 4.2).

Law VI: Stability. Topological structures persist whenever perturbations remain below ρ (Theorem 10.5).

Law VII: Phase Transition. A change in reconstruction class occurs precisely when the homotopy type of $N(\mathcal{U})$ changes.

Law VIII: Emergent Hierarchy. Local Layers $\to$ Intersections $\to$ Coherence $\to$ $N(\mathcal{U})$ $\to$ Homotopy $\to$ Classification $\to$ Stability $\to \cdots$

17.3 The Universal Reconstruction Equation

$\mathcal{R} = F(L, O, C)$, where L = local structure, O = overlap structure, C = coherence, $\mathcal{R}$ = reconstruction class.

17.4 Philosophical Import

Traditional topology begins with global objects. Emergent Topology begins with local relations: global structure is reconstructed, not fundamental.

18 Discussion, Limitations and Relation to Existing Mathematics

18.1 What Is Genuinely New

a) **Coherence Propagation Principle (L1a/L1b):** The hierarchy $C_1 \subset C_2 \subset \cdots$ and Theorem 3.6 establish, at L1a level, that for every finite intersection compatible local contractions exist and are unique up to all-order homotopy (see Remark 3.5). The passage to a global homotopy-coherent diagram in the Boardman–Vogt sense is L1b (Open Problem (d)): the vanishing of all obstruction groups is shown, but the formal functor construction over $N(\mathcal{U})$ has not been written out.

b) **Reconstruction Stability Theorem (L1a):** Theorem 10.5 derives invariance of $\mathcal{R}(M)$ from the geometric stability radius ρ of Definition 10.4. Proposition 10.6 proves $\rho > 0$ under admissibility.

c) **Dynamical Extension (L1a/L1b):** Pointwise Dynamic Reconstruction (Theorem 12.2, L1b — inherits from Layered Nerve Theorem); Persistence Theorem (Theorem 12.4, L1a conditional on admissibility); Potential-Transition Criterion (Theorem 12.6, L1b); Coherence Collapse Precursor (Proposition 12.8, L1b).

d) **Universal Neighbourhood Principle (L3–L4):** The UNP and wave-dynamics model provide a formalism for applying topological emergence to non-manifold contexts.

18.2 What Is a Reformulation, Application, or Known Result

Layered Nerve Theorem (Theorem 4.2): a presentation of the classical Leray–Borsuk Nerve Lemma. Reconstruction under partial connectivity (Proposition 3.2): consequence of Björner–Ramras. Homotopy reconstruction (Theorem 5.1): classical once nerve equivalence is established. Sphere recognition (§6): Perelman, Freedman, and Smale are classical.

18.3 Limitations

Proof status. Results are classified L1a or L1b throughout. The anti-Hermitian condition on K is an axiom. The cosmological applications lack derivation from the Einstein field equations or Standard Model. Benchmark figures are illustrative (L2).

18.4 Open Mathematical Problems

a) Complete formal proofs of all L1b results.

b) Sharp stability radius: compute $\rho(\mathcal{U})$ exactly.

c) Extension to infinite covers of non-compact manifolds.

d) Categorical (higher-category / HoTT) formulation.

e) Formal verification in Lean 4 or Coq/HoTT.

f) Reproducible benchmark suite with public code.

g) Transition-count theorem under piecewise-stable or re-entry hypotheses.

h) Critical time iff: full biconditional characterisation of Theorem 12.6.

i) Forced transition: strong form of Proposition 12.8 without transversality assumption.

19 Conclusions and Future Directions

19.1 Summary of Results at L1a (Complete Proofs)

- **Local Contraction Lemma (2.2):** contractibility implies $\pi_k(U_i) = 0$.
- **Homotopy Triviality when Nerve is Contractible (Corollary 2.5):** $|N| \simeq * \Rightarrow M \simeq *$.
- **Coherence Propagation Principle (Theorem 3.6):** for every finite intersection, compatible local contractions exist and are unique up to all-order homotopy (L1a). The explicit Boardman–Vogt global diagram construction is L1b (Open Problem (d)).
- **Reconstruction Under Weaker Conditions (Proposition 3.2):** corollary of Björner–Ramras.
- **Reconstruction Stability Theorem (Theorem 10.5):** $\varepsilon_{\text{total}} < \rho \Rightarrow \mathcal{R}(M)$ unchanged. The stability radius ρ is the explicit geometric quantity of Definition 10.4; Proposition 10.6 proves $\rho > 0$ under admissibility.
- **Persistence of Reconstruction Classes (Theorem 12.4):** under uniform $\rho_0 > 0$ and admissibility, $\mathcal{R}(t) = \mathcal{R}(0)$ for all $t \in [0, T]$. *(L1a conditional on admissibility assumption of Proposition 10.6.)*
- **Energy Conservation (Proposition 11.4):** anti-Hermitian coupling conserves total information energy.

19.2 Summary of Results at L1b (Proof-Sketch or Conditional)

- Nerve Homotopy Theorem (Theorem 2.4): classical Leray–Borsuk; proof sketch via Čech double complex.
- Layered Nerve Theorem (Theorem 4.2), Global Homotopy Reconstruction (Theorem 5.1), Layered Mayer–Vietoris (Theorem 5.3), Cohomological Reconstruction (Theorem 5.4).
- Pointwise Dynamic Reconstruction (Theorem 12.2): reduces to Layered Nerve Theorem (L1b) applied pointwise.
- Stability Radius Positivity (Proposition 10.6): now L1a conditional on admissibility; admissibility hypothesis not verified in general (see Open Problem 2).
- Full Critical Time "iff" characterisation.
- Coherence Collapse Precursor (Proposition 12.8): conditional on transversality.

19.3 Summary of Open Problems

a) Formal proofs of all L1b results.

b) Stability radius computation.

c) Non-compact case.

d) Categorical formulation.

e) Formal verification.

f) Reproducible benchmarks.

g) Neural predictions (V2–V3).

h) Cosmological predictions.

19.4 The Unified Picture

The unifying thread is the chain:

$$\text{Local Relations} \longrightarrow N(\mathcal{U}) \longrightarrow \text{Global Topology} \longrightarrow \mathcal{R}(M).$$

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A Boundary Conditions, Energy Couplings, and Inter-Body Interactions

[**Conjecture / Framework**] All material in this appendix extends the UNP wave dynamics of §11 to physical media and multi-body systems.

A.1 Boundary Conditions for Open and Closed Systems

Closed (Reflective) System: No energy escapes; $E_{\text{total}} = \text{const}$ (Proposition 11.4) holds exactly.

Open (Absorbing) System: Energy may dissipate: $\psi_i(t) \leftarrow \psi_i(t) + \xi_{\text{ext}}(t)$, where $\xi_{\text{ext}}(t)$ encodes external influence. Modified energy balance: $dE_{\text{total}}/dt = \sum_{i\in\partial\Omega} \text{Re}(\bar{\psi}_i(t)\xi_{\text{ext}}(t))$.

A.2 Interactions with Physical Media and Energy Forms

The general interaction equation for node i in medium $\mathcal{M}$:

$$\frac{d\psi_i}{dt} = \sum_j K_{ij}\psi_j e^{i\Delta\phi_{ij}} + \sum_\alpha \gamma_{i\alpha} E_\alpha(t).$$

Medium / Energy	Governing equation	E_α	Coupling γ
Air (damping)	$\dot{\psi}_i = \sum_j K_{ij}\psi_j - \mu\psi_i$	—	$\mu > 0$
Water (fluid)	$\dot{\psi}_i = \sum_j K_{ij}\psi_j - (\mu_f + \beta_b)\psi_i$	—	μ_f, β_b
Vacuum (radiative)	$\dot{\psi}_i = \sum_j K_{ij}\psi_j + \kappa_r E_0 e^{i\omega_r t}$	$E_0 e^{i\omega_r t}$	κ_r
Electromagnetic	Coupled to Maxwell	E-field	σ
Acoustic	$\partial^2 p/\partial t^2 = c_s^2\nabla^2 p + \gamma_a\dot{\psi}_i$	Pressure p	γ_a
Thermal	$\partial T/\partial t = \alpha_T\nabla^2 T + \gamma_T\lvert\psi_i\rvert^2$	Temperature T	γ_T
Mechanical	$\rho\ddot{u} = (\lambda + \mu_L)\nabla(\nabla\cdot u) + \mu_L\nabla^2 u + \gamma_m\psi_i$	Displacement u	γ_m

Table 7: Neighbourhood-wave coupling to physical energy forms.

A.3 Interactions Between Distinct Topological Bodies

When two distinct topological bodies $B^{(1)}$ and $B^{(2)}$ interact:

$$\frac{d\psi_i^{(1)}}{dt} = \sum_{j\in B^{(1)}} K_{ij}^{(1)}\psi_j^{(1)} + \sum_{k\in B^{(2)}} C_{ik}^{(12)}\psi_k^{(2)}, \tag{1}$$

$$\frac{d\psi_k^{(2)}}{dt} = \sum_{l\in B^{(2)}} K_{kl}^{(2)}\psi_l^{(2)} + \sum_{i\in B^{(1)}} C_{ki}^{(21)}\psi_i^{(1)}, \tag{2}$$

where $(C^{(12)})^\dagger = -C^{(21)}$ to conserve total energy $E^{(1)} + E^{(2)} + E^{(12)}$.

Physical realisations: Gravitational: $C_{ik}^{(12)} = -Gm_im_k/|x_i - x_k|^2$. Electromagnetic: $C_{ik}^{(12)} = k_e q_i q_k/|x_i - x_k|^2$. Mechanical: $C_{ik}^{(12)} = \kappa_{ik}$.

In the limit where $B^{(1)}, \dots, B^{(N)}$ exhaust all physical entities: $K_{\text{univ}} = K_{\text{grav}} + K_{\text{EM}} + K_{\text{strong}} + K_{\text{weak}}$, giving $dE_{\text{univ}}/dt = 0$. This is a formal sketch; its relation to the Standard Model Lagrangian is not established.

B Article-to-Section Correspondence

The following table maps every article of the original 55-article series to the section(s) of this unified paper. No article has been silently discarded.

Article(s)	Title / Content	Section here
1	Layered decomposition, local rules, Mayer–Vietoris sketch	§§1,2
2	Local contraction of embedded spheres	§2
3	Global homotopy triviality via layered patching	§§3,5
4	Canonical identification with S^n	§6
5	Exceptional cases and dimensional pathologies	§7
6	Applications, extensions, future directions	§§19,18
7	Explicit examples and simulations	§§4,9
8	Comparative analysis, classical connections	§§1,18
9	Universal Neighbourhood Principle	§11
10	Wave dynamics, resonance, interference	§11
11	Conservation laws, stability, physical definitions	§§11,A.1
12	Anti-Hermitian coupling, numerical validation	§§11,A.1
13	Numerical results, empirical validation (EEG)	§§10,13
14	Large-scale networks, scalability	§11
15	Future directions (UNP series)	§19
16	Challenges, limitations, mitigation	§18
17	Summary and integration (UNP series)	§§17,19
18	Body–environment interactions (air/water/vacuum)	§A.2
19	Energy forms (EM/acoustic/thermal/mechanical)	§A.2
20	Inter-body interactions	§A.3
21	Universe as complete topological system	§A.3
22	Theoretical implications, limitations, future research	§18
23	Fixed-time neural synchronisation	§13
24	Topological relaying and inter-brain synchronisation	§13
25	Phase collapse and pathological propagation	§13
26	Mirror neurons and multimodal synchronisers	§13
27	Simulated topological robustness	§13
28	Phase reconfiguration, task switching, attention	§13
29	Topological memory formation	§13
30	General framework for topological phase dynamics	§14
31	Emergent spectral delay, Hubble tension	§14
32	Non-local angular momentum, galactic rotation curves	§14
33	Phase-filtered neutrino transitions, solar flux	§14
34	Early-universe neutrino phase transitions	§14
35	Topological phase coherence, baryogenesis	§14
36	Emergent Good Covers, finite intersection theory	§3
37	Higher-order overlap consistency, coherence hierarchies	§3
38	Layered Nerve Theorem, global reconstruction	§4
39	Whitehead-type reconstruction, homotopy recovery	§5
40	Homological and cohomological reconstruction	§5
41	Sphere recognition from nerve data	§6
42	Obstruction theory, non-recognisable manifolds	§7

Article(s)	Title / Content	Section here
43	Constructive reconstruction algorithm	§9
44	Computational benchmarks and validation	§10
45	Epistemic status, logical separation principle	§16
46	Experimental falsifiability framework	§15
47	Nerve equivalence theorem (formal)	§4
48	Algebraic topology reconstruction (formal)	§5
49	Cohomological reconstruction (formal)	§5
50	Classification hierarchy	§6
51	Smooth vs. topological sphere recognition	§6
52	Generalised reconstruction classes	§8
53	Reconstruction stability, perturbation theory	§10
54	Dynamic topological stability	§12
55	Universal laws and synthesis	§§17,19