Prime Distribution, Relative Symmetry, and Predictive Search in Dyadic Intervals

A Unified Structural Framework

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Abstract

We present a unified framework for studying the distribution of prime numbers within dyadic intervals $[x, 2x]$. The paper has three interlocking components.

(I) Dyadic relative symmetry. We prove, unconditionally from quantitative PNT estimates (Dusart [2010]), that the normalized left–right prime imbalance in any dyad satisfies $\sigma(x) = O(1/\log x)$, and that any pre-fixed window of relative width α captures a fraction $\alpha + O(1/\log x)$ of the primes in the dyad. Explicit, computable constants are provided using effective bounds of Dusart (2010).

(II) Predictive hunt zones. We construct prime-independent predictive windows $Z(x; \alpha) \subset [x, 2x]$ defined before any primality testing, prove unconditional capture and symmetry laws with $O(1/\log x)$ relative error, and extend these to general finite window families, BV/Lipschitz weights, arithmetic progressions, short intervals, and sieve-compatible sums.

(III) Energetic formulation and the Fradkin Conjecture. We introduce an auxiliary scalar field $E(n) = \log \log n$ over $\mathbb{N}$ and study whether prime localization can be characterised via curvature conditions $|\Delta^2 E(n)| > \tau(n)$. We carefully distinguish what has been proved (asymptotic energetic symmetry, Markov semigroup generation, functional inequalities) from what remains conjectural (the energetic–prime correspondence at the individual-integer level).

The paper supersedes and consolidates a series of 47 research articles (June–August 2025). All definitions are non-circular. Part I results are fully proved with effective constants; Part III results are labelled as proved, conditional, or conjectural throughout the text. A cross-reference map (Appendix C) maps Articles 1–47 to the corresponding sections of this paper, and a falsifiability protocol is given for each empirical claim.

Proof status. Parts I and II (dyadic symmetry and predictive windows) are fully proved, with complete proofs or explicit proof sketches whose gaps are filled by standard references. Part III (energetic framework) contains a mix of proved results (Lemmata 10.3 and 10.5; the semigroup theorems of §11), conditional results stated explicitly as such (Theorems 12.5–12.7 in §12.4), and conjectures (the Fradkin Conjecture in its strongest form; the bounded-discrepancy claim of §13). A falsifiability protocol is given for each empirical claim.

Keywords: prime numbers, dyadic intervals, Prime Number Theorem, effective bounds, hunt zones, predictive windows, Fradkin Conjecture, energetic field, relative symmetry.

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Contents

1 Introduction and Research Motivation

1.1 The fundamental problem

The distribution of prime numbers is one of the oldest and most deeply studied problems in mathematics. The Prime Number Theorem (PNT) establishes

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty),$$

and its quantitative refinements (via $\mathrm{Li}(x) = \int_2^x dt/\log t$) give effective bounds on $|\pi(x) - \mathrm{Li}(x)|$. Despite this global understanding, *local* questions remain subtle: given an interval $I \subset [x, 2x]$, how many primes does I contain, and how are they distributed left and right of the dyadic midpoint?

1.2 Structural observation: dyadic rows

The starting point of this research programme is the elementary but productive observation that powers of 2 define a natural partition of $\mathbb{N}$ into *dyadic rows*

$$R_k = \{n \in \mathbb{N} : 2^{k-1} < n \leq 2^k\}, \qquad k \geq 2,$$

each of length $|R_k| = 2^{k-1}$ and midpoint

$$C_k = \frac{2^{k-1} + 2^k}{2} = \frac{3}{2} \cdot 2^{k-1}.$$

Empirical inspection of small rows reveals striking *approximate symmetry* in the distribution of primes around C_k: roughly equal numbers of primes appear in the left half $[2^{k-1}+1, C_k)$ and the right half $(C_k, 2^k]$, and the farthest primes on each side are nearly equidistant from the centre.

1.3 Goals and structure of this paper

This paper pursues three goals, which correspond to the three main parts:

(I) **Unconditional dyadic symmetry.** Prove from quantitative PNT estimates (Dusart bounds) that the left–right prime imbalance in $[x, 2x]$ is $O(1/\log x)$ in a relative sense, with explicit effective constants. This is the rigorous backbone of the empirical observation above.

(II) **Predictive hunt zones.** Construct prime-*independent* windows $Z(x; \alpha)$, defined before any primality test, that capture a predictable fraction of the primes in $[x, 2x]$, with the same $O(1/\log x)$ error control.

(III) **Energetic formulation.** Study whether an auxiliary scalar field $E(n) = \log \log n$ provides structural insight into prime localization via a curvature condition. We carefully distinguish the proved functional-analytic results from the conjectural correspondence at the integer level.

1.4 Historical note on this manuscript

The results presented here were developed over a sequence of 47 research articles written between June and August 2025. The present manuscript is a *synthesis paper*: it does not reproduce every intermediate step, but it contains every mathematically distinct result, with the final and strongest version of each result replacing earlier, weaker, or circular formulations. A complete cross-reference map (Articles 1–47 → sections of this paper) appears in Appendix C.

One important refinement is terminological. Articles 1–10 defined "hunt zones" using the nearest and farthest *prime* to the centre of each row — a definition that is circular (it presupposes knowledge of the primes) and therefore unsuitable as a predictive tool. From Article 18 onward this was replaced by windows defined purely arithmetically. The present paper adopts only the non-circular definition throughout; the historical circular definition appears only in the empirical motivation section (§9).

1.5 What is proved, what is conjectured

We use coloured boxes throughout to signal the epistemic status of each result:

Proved Result

Green box: a result with a complete proof (or a proof sketch whose gaps are explicitly noted and filled by the cited literature).

Conjecture / Requires Further Proof

Orange box: a claim that is stated as a conjecture or that requires additional work to become a theorem. The stronger *energetic* form of the Fradkin Conjecture (individual prime localisation via curvature) falls here. Note that the historical Fradkin Symmetry Conjecture ($\sigma_k \to 0$) has been proved (Theorem 10.8); its name is retained for continuity.

Proved Conditional Result (conditional on Assumption 3)

Orange box: a result that is rigorously proved *conditional on Assumption 3* (the Energy–Chebyshev correspondence, which is unproved). The proof logic is valid; the conclusion depends on the assumption.

Empirical Evidence

Blue box: an empirical observation supported by computation but not yet by a proof.

2 Dyadic Framework and Core Definitions

2.1 Dyadic intervals and prime counting

Definition 2.1 (Dyadic interval and row). *For an integer $k \geq 2$ and a real number $x \geq 2$, define the* dyadic row of level k *and the* dyadic interval at scale x*:*

$$R_k \;=\; \{n \in \mathbb{N} : 2^{k-1} < n \leq 2^k\}, \qquad [x, 2x].$$

These two frameworks are related by setting $x = 2^{k-1}$.

Definition 2.2 (Dyadic half-counts and prime counting). *Fix $x \geq 2$. Write $\pi(y) = \#\{p \leq y : p \text{ prime}\}$ and $\psi(y) = \sum_{n \leq y} \Lambda(n)$ (Chebyshev's ψ-function). Define:*

$$A(x) \;=\; \pi\left(\tfrac{3x}{2}\right) - \pi(x), \qquad B(x) \;=\; \pi(2x) - \pi\left(\tfrac{3x}{2}\right),$$

$$\Delta\pi([x, 2x]) \;=\; A(x) + B(x) \;=\; \pi(2x) - \pi(x).$$

The normalised asymmetry statistic *is*

$$\sigma(x) \;=\; \frac{|A(x) - B(x)|}{A(x) + B(x)}.$$

Definition 2.3 (Symmetry score). *For row R_k with centre $C_k = 3 \cdot 2^{k-2}$, define the left and right prime sets*

$$P_k^L \;=\; \{p \in P_k : p < C_k\}, \qquad P_k^R \;=\; \{p \in P_k : p > C_k\},$$

and the symmetry score

$$\text{Score}(k) \;=\; \left|1 - \frac{|P_k^L|}{|P_k^R|}\right| + \left|1 - \frac{|Z_1^k|}{|Z_2^k|}\right| + \left|1 - \frac{d_L(k)}{d_R(k)}\right|,$$

where $|Z_j^k|$ are the sizes of the left/right a-priori *hunt zones (see §3), and $d_L(k), d_R(k)$ are the distances from C_k to the farthest prime on each side. A score of 0 represents perfect symmetry.*

2.2 The logarithmic integral and PNT

Definition 2.4 (Li and effective PNT remainder). *Set $\text{Li}(x) = \int_2^x dt/\log t$. We say that an effective PNT remainder bound holds with parameters (A_π, B_π) if*

$$|\pi(y) - \text{Li}(y)| \;\leq\; \frac{A_\pi\, y}{(\log y)^2} \quad \text{for all } y \geq B_\pi.$$

Similarly, an effective Chebyshev bound holds with parameters (A_θ, B_θ) if

$$|\vartheta(y) - y| \;\leq\; \frac{A_\theta\, y}{\log y} \quad \text{for all } y \geq B_\theta,$$

where $\vartheta(y) = \sum_{p \leq y} \log p$.

The numerical values used throughout this paper are taken from Dusart [2010]:

$$A_u = 1.2762 \text{ (upper } \pi \text{ coeff.)}, \quad A_l = 1 \text{ (lower } \pi \text{ coeff.)}, \quad A_\theta = 1.2323, \quad X^* = 599. \tag{1}$$

These constants enter as: $\pi(x) \leq x/\log x \cdot (1 + A_u/\log x)$ and $\pi(x) \geq x/\log x \cdot (1 + A_l/\log x)$ for $x \geq 599$; and $\vartheta(x) \geq 0.9228\, x$ for $x \geq 41$ (Rosser–Schoenfeld Rosser and Schoenfeld [1962]).

2.3 Predictive windows

Definition 2.5 (Predictive hunt zone / window)**.** *Fix $\alpha \in (0,1)$. The* predictive window *of width α at scale x is*

$$Z(x;\alpha) \;=\; \Big[C(x) - \tfrac{\alpha}{2}\,x,\; C(x) + \tfrac{\alpha}{2}\,x\Big] \,\cap\, [x, 2x], \quad C(x) \;=\; \tfrac{3}{2}x.$$

This window is prime-independent*: it is defined before any primality testing and does not reference the location of any specific prime.*

Remark 2.6 (Elimination of circularity)**.** *Earlier work (Articles 1–10) used "hunt zones" defined as $[p_{\text{near}}, p_{\text{far}}]$ where p_{near} and p_{far} are the nearest and farthest primes to the centre. This is circular: the zones depend on the primes they are supposed to locate. Definition 2.5 replaces this with a purely arithmetic construction.*

2.4 Notation summary

Symbol	Meaning	First defined
R_k	Dyadic row $\{2^{k-1}+1,\dots,2^k\}$	§2
$[x,2x]$	Dyadic interval at scale x	§2
$C(x)=\frac{3x}{2}$	Midpoint of $[x,2x]$	§2
$A(x), B(x)$	Left/right prime counts	§2
$\sigma(x)$	Normalised left–right asymmetry	§2
$Z(x;\alpha)$	Predictive window of width α	Def. 2.5
$\mathrm{Li}(x)$	Logarithmic integral	§2
$A_\theta, A_\pi, \dots$	Dusart constants (1)	§2
X^*	Unified validity threshold	§6
$\sigma(x), \varepsilon(x)$	Energetic envelope rates	§12

3 From Empirical Hunt Zones to a Non-Circular Construction

3.1 Empirical motivation

Inspection of small dyadic rows reveals that primes cluster near the centre C_k and that this clustering is roughly symmetric. Table 1 (reproduced from the original computational results) shows the key statistics for $k = 3, \dots, 10$.

Empirical Evidence

The data in Table 1 show a clear trend: the symmetry score decreases from 2.533 at $k = 5$ to 0.252 at $k = 10$, and the distance ratio approaches 1.000. This motivated the *Fradkin Conjecture* that symmetry continues to improve as $k \to \infty$. The unconditional theorem proved in §4 (Theorem 4.1) confirms this trend, though in a relative sense only.

Table 1: Symmetry statistics for dyadic rows $k = 3$ to $k = 10$. Score $= 0$ is perfect symmetry.

k	Range	$\lvert P_k \rvert$	Count ratio L/R	Distance ratio	Score
3	$[5, 8]$	2	1.000	1.000	0.000
4	$[9, 16]$	2	1.000	1.000	0.000
5	$[17, 32]$	5	1.500	0.800	2.533
6	$[33, 64]$	7	1.333	1.167	0.722
7	$[65, 128]$	13	0.857	0.967	0.601
8	$[129, 256]$	23	1.091	1.032	0.595
9	$[257, 512]$	47	0.880	1.024	0.642
10	$[513, 1024]$	70	1.188	1.012	0.252

3.2 The original hunt-zone construction (circular — historical only)

The original construction defined two zones:

$$Z_1^k = [p^+_{\text{near}},\ p^+_{\text{far}}], \quad Z_2^k = [p^-_{\text{far}},\ p^-_{\text{near}}],$$

where $p^+_{\text{near}}, p^+_{\text{far}}$ are the nearest and farthest primes to the right of C_k, and $p^-_{\text{near}}, p^-_{\text{far}}$ those to the left. This construction is explicitly **circular**: it uses the primes in R_k to define the search zones for primes in R_k. It cannot serve as a predictive tool.

3.3 The corrected construction

The predictive window $Z(x; \alpha)$ of Definition 2.5 replaces the above. Its key properties are:

- It is defined before any primality test.
- Its size is $|Z(x; \alpha)| = \alpha x$ (up to rounding).
- By Theorem 5.1, it captures a fraction $\alpha + O(1/\log x)$ of the primes in $[x, 2x]$.

3.4 Calibration

By Theorem 5.1, *any* fixed $\alpha \in (0, 1)$ gives a well-defined unconditional capture law with $O(1/\log x)$ relative error; the choice of α does not affect the asymptotic prime density within the window. The optimal α is therefore a computational tuning parameter that depends on the application.

Empirical Evidence

In finite-sample tests on rows $k \leq 31$, windows of width $\alpha \approx 0.02$–0.04 relative to the dyad length yielded a favourable balance between search area and prime count. This is a finite-sample empirical observation only; it does not reflect an asymptotic property of the method, and no claim of prime-density enrichment is made (see Remark 8.1).

Empirical Evidence

Computational tests (Article 10 of the original series) used $v13$ (symmetric Hunt Zone sampling) on the row $k = 31$ (integers of approximately 10 digits): 500 candidates tested, 21 prime hits, hit ratio 4.2%, search area $\approx 20\%$. Compared with uniform random sampling (500 candidates, 23 hits, 4.6%, 100% search area). The window covers 20% of the dyad while yielding 91% of the primes found by uniform sampling—demonstrating spatial focus without any change in prime density (Remark 8.1).

4 Prime Distribution and Relative Symmetry in Dyadic Intervals

4.1 Main symmetry theorem

Proved Result

Theorem 4.1 (Dyadic relative symmetry). (a) From quantitative PNT estimates (Dusart bounds):

$$\sigma(x) \;=\; \frac{|A(x) - B(x)|}{\pi(2x) - \pi(x)} \;=\; O\!\left(\frac{1}{\log x}\right), \qquad x \to \infty.$$

(b) From the de la Vallée Poussin zero-free region ($|\pi(y) - \mathrm{Li}(y)| = O(y\, e^{-c\sqrt{\log y}})$):

$$A(x) - B(x) \;=\; \frac{\log(32/27)\, x}{\log^2 x} + O\!\left(\frac{x}{\log^3 x}\right),$$

so $\sigma(x) \sim \log(32/27)/\log x$ *as* $x \to \infty$, *with* $\log(32/27) \approx 0.1699$.

Proof sketch. Part (a). The proof uses the Dusart upper and lower π-bounds of Definition 6.1 directly, without passing through a bound on $|\pi - \mathrm{Li}\,|$. The complete argument is in Appendix A.6: one bounds $2\pi_U(3x/2) - \pi_L(x) - \pi_L(2x)$ from above and uses the ϑ-based denominator lower bound (6), yielding $\sigma(x) = O(1/\log x)$. The precise constant is $\widehat{K}_1 = 1.20$ (Theorem 6.2).

Part (b). The leading term $\log(32/27)\, x/\log^2 x$ is a property of Li alone, obtained from the expansion $\mathrm{Li}(cx) = cx(\log^{-1} x + (1 - \log c)\log^{-2} x + O(\log^{-3} x))$ at $c \in \{1, 3/2, 2\}$. Transferring this to π requires $|\pi(y) - \mathrm{Li}(y)| = o(y/\log^2 y)$, which follows from the zero-free region estimate and not from the Dusart bounds alone. □

4.2 General partitions

Proved Result

Theorem 4.2 (Proportional symmetry for arbitrary splits). *Fix $\lambda \in (0,1)$ and set $x_\lambda = (1+\lambda)x$. Then*

$$\frac{\pi(x_\lambda) - \pi(x) \;-\; \lambda(\pi(2x) - \pi(x))}{\pi(2x) - \pi(x)} \;=\; O\!\left(\frac{1}{\log x}\right),$$

uniformly for λ in compact subsets of $(0,1)$. Every subinterval of $[x, 2x]$ captures its length-proportional share of primes up to a vanishing relative error.

Proof sketch. Substitute Li for π and expand $\mathrm{Li}((1+\lambda)x)$ around $\log x$. The proportional term $\lambda(\mathrm{Li}(2x) - \mathrm{Li}(x))$ cancels the $x/\log x$ contribution; the residual is $O(x/\log^2 x)$, giving $O(1/\log x)$ after normalisation. □

4.3 Sliding windows and local dyadicity

Proved Result

Corollary 4.3 (Moving relative symmetry). *Let $h = \theta x$ with $\theta \in (0, 1/2]$. Consider adjacent windows $[x, x+h]$ and $[x+h, x+2h] \subset [x, 2x]$. Then*

$$\frac{|\pi(x+h) - \pi(x) \;-\; (\pi(x+2h) - \pi(x+h))|}{\pi(x+2h) - \pi(x)} \;=\; O\!\left(\frac{1}{\log x}\right),$$

uniformly for θ in compact subsets of $(0, 1/2]$.

4.4 Conditional sharpening under RH

Remark 4.4 (Sharp asymptotic constant — de la Vallée Poussin). *The expansion $A(x) - B(x) = \log(32/27)\, x/\log^2 x + O(x/\log^3 x)$ holds at the level of the logarithmic integral* Li. *Transferring the leading term to π — establishing $\sigma(x) \sim \log(32/27)/\log x$ — requires $|\pi(y) - \mathrm{Li}(y)| = o(y/\log^2 y)$. This is* stronger *than the Dusart effective bounds (which give $|\pi(y) - \mathrm{Li}(y)| = O(y/\log^2 y)$ but not $o(y/\log^2 y)$) and follows from the de la Vallée Poussin zero-free region estimate $|\pi(y) - \mathrm{Li}(y)| = O(y\, e^{-c\sqrt{\log y}}) = o(y/\log^A y)$ for any $A > 0$. The Dusart effective bounds used in Theorem 6.2 are sufficient for the $O(1/\log x)$ claim; the sharp leading constant $\log(32/27)/\log x$ is a separate (classical, unconditional) result.*

Remark 4.5 (Improvement under RH). *Under the Riemann Hypothesis, $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x)$, and the argument above yields $\sigma(x) = O(\sqrt{\log^2 x/x})$, a much faster rate. This is a* conditional *result (conditional on RH); all other results in this paper are unconditional.*

5 Predictive Window and Prime-Capture Laws

5.1 Single-window capture

Proved Result

Theorem 5.1 (Predictive capture — unconditional). *For any fixed $\alpha \in (0,1)$, the predictive window $Z(x;\alpha)$ (Definition 2.5) satisfies*

$$\frac{\pi(Z(x;\alpha))}{\pi(2x)-\pi(x)} = \alpha + O\left(\frac{1}{\log x}\right), \qquad x \to \infty.$$

Proof sketch. $|Z(x;\alpha)| = \alpha x$, so $\mathrm{Li}(Z(x;\alpha)) = \alpha x/\log x + O(x/\log^2 x)$. The denominator $\pi(2x)-\pi(x) = x/\log x + O(x/\log^2 x)$. Dividing yields $\alpha + O(1/\log x)$. □

5.2 Multiple windows and unions

Proved Result

Theorem 5.2 (Simultaneous capture for finite families). *Let $\{Z(x;\alpha_j)\}_{j=1}^m$ be a pre-fixed finite family of predictive windows with widths α_j. For all $x \geq X^*$ and all j simultaneously,*

$$\left|\frac{\pi(Z(x;\alpha_j))}{\pi(2x)-\pi(x)} - \alpha_j\right| \leq \frac{\widehat{K}_2}{\log x},$$

where $\widehat{K}_2 = 3.10$ (see §6). The bound holds simultaneously for all j and is independent of m.

5.3 Non-symmetric partitions and perturbation stability

Proved Result

Corollary 5.3 (Non-symmetric partition). *Let $\{W_j\}_{j=1}^m$ be any pre-fixed partition of $[x,2x]$ with proportional widths α_j summing to 1. For all $x \geq X^*$ and all j,*

$$\left|\frac{\pi(W_j)}{\pi(2x)-\pi(x)} - \alpha_j\right| \leq \frac{\widehat{K}_2}{\log x}.$$

Proved Result

Theorem 5.4 (Perturbation stability). *Let Z' be a δ-perturbation of $Z(x;\alpha)$ (boundaries shifted by $\leq \delta x$, width changed by $\leq \delta$). Then*

$$\left|\frac{\pi(Z')}{\pi(2x)-\pi(x)} - \frac{\pi(Z(x;\alpha))}{\pi(2x)-\pi(x)}\right| \leq \delta + \frac{2\widehat{K}_2}{\log x}.$$

6 Effective Bounds and Finite-x Theory

6.1 Explicit constants from Dusart (2010)

All asymptotic statements in §§4–5 can be made *effective*: valid for all $x \geq X_{\min}$ with a computable $X_{\min}$ and explicit constants. We use the Dusart [2010] bounds (1) to derive the following.

Definition 6.1 (Program constants). *The following constants are derived in Appendix A (Theorem A.6) directly from the Dusart Dusart [2010] explicit π-bounds:*

$$\frac{x}{\log x}\left(1+\frac{1}{\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1+\frac{1.2762}{\log x}\right), \qquad x \geq 599.$$

These bounds are used directly (without passing through $|\pi - \operatorname{Li}|$):

$$\begin{aligned}
\widehat{K}_1 &= 1.20 \quad \textit{(valid for all } x \geq 599\textit{)},\\
\widehat{K}_2 &= 3.10 \quad \textit{(valid for all } x \geq 599,\ 0<\alpha<1\textit{)},\\
X^* &= 599.
\end{aligned}$$

For a band $B_\Lambda(x) = [x/\Lambda, x]$ with $\Lambda \geq 2$:

$$c_\Lambda = \frac{\ln \Lambda + 2}{2}, \qquad \widehat{K}_i^{(\Lambda)} = c_\Lambda \widehat{K}_i \quad (i = 1, 2).$$

Proved Result

Theorem 6.2 (Explicit dyadic bound). *Under the Dusart bounds (1), for all $x \geq 599$:*

$$\sigma(x) \leq \frac{\widehat{K}_1}{\log x} = \frac{1.20}{\log x}, \tag{2}$$

$$\left|\frac{\pi(Z(x;\alpha))}{\pi(2x)-\pi(x)} - \alpha\right| \leq \frac{\widehat{K}_2}{\log x} = \frac{3.10}{\log x}, \qquad 0<\alpha<1. \tag{3}$$

Both bounds hold simultaneously for all pre-fixed finite families. The derivation is in Appendix A.

Proof. See Appendix A.6, which applies the Dusart upper and lower π-bounds directly (without passing through $|\pi - \operatorname{Li}|$) to bound $|A(x) - B(x)|$ and uses the Chebyshev ϑ-bound for the denominator. □

6.2 Finite-ε thresholds

Proved Result

Theorem 6.3 (ε-level guarantees). *Given $\varepsilon \in (0,1)$ and $\Lambda \geq 2$, define the scheme* *maximum* $\widehat{K}^{(\Lambda)} = \max\{\widehat{K}_1^{(\Lambda)}, \widehat{K}_2^{(\Lambda)}\} = \widehat{K}_2^{(\Lambda)}$ *(since* $K_2 > K_1$*), and set*

$$x_{\min}(\varepsilon) \;=\; \max\left\{X^*,\, e^2\Lambda,\, \exp\left(\frac{\widehat{K}_2^{(\Lambda)}}{\varepsilon}\right)\right\}.$$

For every $x \geq x_{\min}(\varepsilon)$*, every pre-fixed finite family of windows* $\{W_j\}$ *on* $B_\Lambda(x)$*, and every offset* $\rho \in (0,1)$*:*

$$\max_j \left|\frac{\pi(W_j)}{\pi(B_\Lambda(x))} - \alpha_j\right| \;\leq\; \varepsilon, \qquad \sigma^{(\Lambda,\rho)}(x) \;\leq\; \varepsilon.$$

Both bounds hold simultaneously because $\widehat{K}_2^{(\Lambda)} \geq \widehat{K}_1^{(\Lambda)}$.

6.3 Numerical table of thresholds

Table 2: Validity thresholds $x_{\min}(\varepsilon)$ for $\Lambda \in \{2,4\}$, driven by the capture constant $\widehat{K}_2 = 3.10$ (which exceeds $\widehat{K}_1 = 1.20$).

Λ	ε	$\widehat{K}_2^{(\Lambda)}$	$x_{\min}(\varepsilon)$
2	0.10	4.1744	$\approx 10^{18.1}$
2	0.05	4.1744	$\approx 10^{36.3}$
4	0.10	5.2488	$\approx 10^{22.8}$
4	0.05	5.2488	$\approx 10^{45.6}$

Remark 6.4 (Magnitude of thresholds)**.** *The large values in Table 2 reflect the conservatism of unconditional PNT-level bounds. They should not be read as saying that the symmetry is absent at smaller scales — indeed, Table 1 shows excellent symmetry already at* $k = 10$ *(*$x \approx 10^3$*). Rather, these thresholds guarantee* uniform *bounds valid across all finite families and offsets, using elementary effective PNT inputs. Stronger explicit bounds (e.g., zero-free region improvements) would lower* $x_{\min}$ *substantially.*

6.4 Replication protocol

The following non-circular protocol can be executed by any independent team:

R1. **Freeze constants.** Fix $(A_\theta, B_\theta) = (1.2323, 2)$, $A_u = 1.2762$, $A_l = 1$, $A_\theta = 1.2323$, $X^* = 599$ from Dusart [2010] and Rosser and Schoenfeld [1962].

R2. **Choose scheme.** Fix $\Lambda \geq 2$; compute c_Λ and $\widehat{K}_i^{(\Lambda)}$.

R3. **Pick target.** Choose $\varepsilon \in (0,1)$; compute $x_{\min}(\varepsilon)$.

R4. **Freeze families.** Pre-specify windows $\{W_j\}$, offsets ρ, before any primality work.

R5. **Count.** Evaluate π on $B_\Lambda(x)$ without changing frozen choices.

R6. **Evaluate.** Verify Theorem 6.2; archive outputs with SHA-256 hashes.

7 Generalized Window Theory

7.1 General bands and scheme invariance

Proved Result

Theorem 7.1 (Scheme invariance). *For any fixed $\Lambda \geq 2$ and the band $B_\Lambda(x) = [x/\Lambda, x]$, the capture and imbalance bounds of Theorem 6.2 hold with constants $\widehat{K}_i^{(\Lambda)} = c_\Lambda \widehat{K}_i$, $c_\Lambda = (\ln \Lambda + 2)/2$, and validity threshold $x \geq \max\{X^*, e^2\Lambda\}$.*

Proof sketch. The key step is the inequality $1/\log(x/\Lambda) \leq c_\Lambda/\log x$ for $x \geq e^2\Lambda$ (Appendix A, Lemma A.2). All other steps follow from the $\Lambda = 2$ argument with the band $[x/\Lambda, x]$ replacing $[x, 2x]$. □

7.2 BV and Lipschitz weighted windows

Definition 7.2 (Weighted capture). *For a weight $w : B_\Lambda(x) \to [0, 1]$ with total variation* $\mathrm{TV}(w)$, *define the normalised expected mass $\alpha_w = \frac{1}{\ln x - \ln(x/\Lambda)} \int_{x/\Lambda}^{x} \frac{w(t)}{\ln t}\, dt$ and the weighted capture $C^{(\Lambda)}(x; w) = \sum_{p \in B_\Lambda(x)} w(p)/\#\{p \in B_\Lambda(x)\}$.*

Proved Result

Theorem 7.3 (Weighted capture). *For any pre-fixed $w \in \mathrm{BV}(B_\Lambda(x))$ with $0 \leq w \leq 1$ and finite complexity $\mathrm{Comp}(w) = \min\{\mathrm{TV}(w), L \cdot x(1 - 1/\Lambda)\}$:*

$$\left| C^{(\Lambda)}(x; w) - \alpha_w \right| \leq \frac{\widehat{K}_2^{(\Lambda)}}{\log x}\Big(1 + \mathrm{Comp}(w)\Big).$$

For finite families $\{w_j\}$, the bound holds simultaneously with $\max_j \mathrm{Comp}(w_j)$.

7.3 Arithmetic progressions and short intervals

The capture law of Theorem 5.1 extends to individual residue classes via Dirichlet characters.

Proved Result

Theorem 7.4 (Capture in arithmetic progressions — PNT version). *Fix modulus $q \geq 2$, residue a with $\gcd(a, q) = 1$, and $\alpha \in (0, 1)$ fixed (independent of x). For any pre-fixed window $W = [x + h_1, x + h_2] \subset [x, 2x]$ of relative width $\alpha = (h_2 - h_1)/x$,*

$$\frac{\#\{p \in W : p \equiv a \pmod q\}}{\#\{p \in [x, 2x] : p \equiv a \pmod q\}} = \alpha + O_q\!\left(\frac{1}{\log x}\right),$$

where the implicit constant depends on q but not on W. If $\alpha = \alpha(x) \to 0$, the $O(1/\log x)$ error bound holds only when $\alpha(x) \gg e^{-c\sqrt{\log x}}$ for some $c > 0$. The proof uses PNT in arithmetic progressions directly; see Appendix A.

Remark 7.5 (Dependence on q and full proof)**.** *The O_q constant grows with q through the zero-free region constant $c(q)$ in de la Vallée Poussin's theorem for Dirichlet L-functions, and through $1/\phi(q)$ in the prime count normalisation. For fixed q and $x \to \infty$ this is absorbed in $O(1/\log x)$. The complete proof is in Appendix A, Theorem A.5. For unions of intervals the error adds linearly; for short intervals $[x, x + H]$ with $H \to \infty$ and $H \gg x\, e^{-c\sqrt{\log x}}$ for a suitable absolute constant $c > 0$ (e.g. $H \geq x^{2/3+\varepsilon}$ unconditionally suffices), the same argument gives an $O(1/\log x)$ relative error for the capture fraction on the short interval. For $H = o(x)$ without a growth condition on $H/x\, e^{-c\sqrt{\log x}}$, the relative error is not controlled by $O(1/\log x)$ and the statement should not be applied.*

Remark 7.6 (Conditional energetic extensions)**.** *The results of this section extend to the energetic localization indicator $1_{\mathrm{loc},\tau}$ of §12* conditionally on Assumption 3 *(the Energy–Chebyshev correspondence). Those conditional versions — for arithmetic progressions, short intervals, bilinear sums, and sieve-compatible sums — are stated in §12.4.*

8 Computational Prime-Search Framework

8.1 Algorithm overview

The predictive hunt-zone method yields the following prime-search algorithm:

Step 1. Freeze. Choose $\Lambda \geq 2$, width parameter α, and scale x. Compute $Z(x; \alpha)$ from Definition 2.5 without any reference to primes.

Step 2. Construct candidates. List all integers in $Z(x; \alpha)$; optionally apply modular pre-filtering (e.g., exclude multiples of 2,3,5).

Step 3. Test primality. Apply a deterministic or probabilistic primality test (Miller–Rabin, BPSW, ECPP) to each candidate.

Step 4. Evaluate. Compare the capture fraction $\pi(Z(x; \alpha))/(\pi(2x) - \pi(x))$ against the theoretical prediction $\alpha + O(1/\log x)$.

8.2 Efficiency analysis

Theorem 5.1 gives the baseline efficiency: a window of width α contains a fraction α of the primes and has αx candidates, so the prime density within the window equals the ambient prime density $\approx 1/\log x$ — no better.

The efficiency advantage comes from *reducing the search space*: a window of width α reduces the candidate set by a factor α. By Theorem 5.1, it also captures a fraction α of the primes; the prime *density* within the window (primes per candidate) therefore equals the ambient density $\approx 1/\log x$ — no enrichment. The saving is purely in the size of the candidate set that must be tested.

If the application requires finding primes in a bounded sub-region of $[x, 2x]$, a window of width α covering that region reduces the number of primality tests by a factor α while retaining all primes in the region. This is the sense in which the method is efficient.

Remark 8.1 (No prime-density enrichment)**.** *The prime density inside $Z(x; \alpha)$ equals that of $[x, 2x]$, namely $\approx 1/\log x$. The window does not concentrate primes; it localises the search. In particular, the expected number of primality tests to find the* first *prime in*

$Z(x;\alpha)$ is $\approx \log x$, the same as in $[x, 2x]$ itself. The saving arises only when the application is to find primes within a prescribed region*, not to minimise the cost per prime found.*

The more significant saving occurs in *targeted search* (e.g., for primes within a pre-specified sub-region of $[x, 2x]$ of application-specific interest), where the window restricts testing to that region without reference to any enrichment of prime density.

8.3 Comparison with classical methods

Table 3: Comparison of prime-search methods.

Method	Type	Efficiency	Scalability	Struct.
Trial division	Det.	Very low	Poor	No
Sieve of Er.	Det.	High/n small	Mem.-ltd	No
Miller–Rabin	Prob.	Moderate	Very good	No
GPU sieving	Par.	High/hw	Excellent	No
ECPP	Det.	Very high/n	Conditional	No
Pred. windows	Hyb.	Baseline+sp.	High	Yes

The predictive window method does not replace any existing method; it acts as a *pre-filter* that reduces the search space before handing off to Miller–Rabin or ECPP. It does not certify primality.

8.4 Stepwise computation and symmetry-based inference

A further computational consequence of the symmetry law is a stopping criterion. Given primes found in $Z(x;\alpha) \cap [x, C(x)]$ (left half), the bound $\sigma(x) = O(1/\log x)$ predicts that approximately the same *count* of primes lies in $Z(x;\alpha) \cap [C(x), 2x]$ (right half). This global count symmetry yields:

- a stopping criterion for the right-half search when its prime count matches the left-half count to within $O(1/\log x)$ tolerance;
- a prior on the expected right-half count for resource allocation.

Remark 8.2 (Count symmetry is not pointwise)**.** *The symmetry $|P_L| \approx |P_R|$ is a* global count *property. It does not imply that if $p \in P_L$ with $p = C - d$, then $C + d$ is prime. Primes are not mirror images of each other; the distribution is symmetric on average, not pointwise. Any "mirror-pair" prioritization strategy is a heuristic that requires independent empirical validation and is not justified by Theorem 4.1.*

The formal stepwise efficiency analysis appeared in Article 9 of the original series. The claimed $\approx 52\%$ saving is a heuristic estimate applicable in specific scenarios; the rigorous content of Theorem 4.1 is only the $O(1/\log x)$ count balance.

9 Empirical Results and Computational Evidence

9.1 Symmetry convergence

Table 1 (p. 9) showed the historical symmetry score (Definition 2.3) decreasing from 2.533 at $k = 5$ to 0.252 at $k = 10$. Extending the computation to $k = 11, 12$:

Table 4: Historical symmetry score Score(k) and count imbalance $\sigma_k = ||P_k^L| - |P_k^R||/|P_k|$ for $k = 5$ to $k = 12$. Only σ_k is controlled by Theorem 4.1.

k	Score(k)	σ_k
5	2.533	0.200
6	0.722	0.143
7	0.601	0.077
8	0.595	0.043
9	0.642	0.064
10	0.252	0.086
11	0.212	—
12	0.183	—

Empirical Evidence

The count imbalance $\sigma_k = ||P_k^L| - |P_k^R||/|P_k|$ (right column) is the quantity controlled by Theorem 4.1: $\sigma_k = O(1/((k-1)\log 2))$. The historical composite Score(k) (Definition 2.3, left column) incorporates additional statistics (hunt-zone sizes, distance ratios) not directly implied by Theorem 4.1; it is presented here for historical continuity only. For $k = 5$–10, σ_k is computed from the exact prime counts in Table 6; for $k = 11, 12$ only the composite Score is available from Article 17.

9.2 Predictive model for hunt-zone size

Empirical Evidence

From rows $k = 3$ to $k = 10$, the relative hunt-zone size (total width of the two a-priori zones as a fraction of row length) is approximated by

$$P_k \approx 0.96 + \frac{0.2}{k}.$$

This is a curve-fitted formula from $k = 3$–10; it should not be extrapolated without validation at larger k. It is *not* a theorem.

9.3 Computational benchmarks at $k = 31$

The following data were obtained from a primality-testing experiment on the dyadic row with $k = 31$ (integers of approximately 10 digits):

Table 5: Benchmarks at $k = 31$. v12: uniform random; v13: predictive window.

Method	Candidates	Prime hits	Hit ratio	Search area
v12 (uniform)	500	23	4.6%	100%
v13 (window, $\alpha \approx 0.2$)	500	21	4.2%	≈20%

The raw prime count is similar (21 vs. 23) and the hit ratio (4.2% vs. 4.6%) is comparable, confirming that prime density inside the window equals the ambient density $\approx 1/\log x$. The v13 window covers only 20% of the dyad while capturing 91% of the primes found by uniform sampling. Since both methods tested 500 candidates, there is no reduction in primality tests per prime: both require $\approx \log x$ tests (Remark 8.1). The benefit of the window is *spatial focus*: the search is restricted to a pre-defined 20% of the dyad with predictable prime capture.

9.4 Falsifiability protocol for empirical claims

Each empirical claim in this section is testable as follows:

1. Reproduce $\pi(2^k)$ for $k = 3, \ldots, 12$ using any certified sieve (values are deterministic and well-known).
2. Compute $\sigma_k = ||P_k^L| - |P_k^R||/|P_k|$ from the exact prime counts (Definition 10.7 and Theorem 10.8; note this is the count-imbalance σ_k, *not* the composite Score(k) of Definition 2.3).
3. Verify that $\sigma_k < \widehat{K}_1/((k-1)\log 2)$ for $k \geq 11$ (Theorem 6.2 with $x = 2^{k-1}$, so $\log x = (k-1)\log 2$; $k \geq 11$ ensures $x = 2^{k-1} \geq 1024 \geq 599$).
4. For the predictive model P_k: sieve rows $k = 13, \ldots, 20$ and check whether the formula remains accurate within ± 0.05.

10 Energetic Formulation and the Fradkin Conjecture

10.1 Motivation

Parts I and II of this paper are unconditional and complete. Part III introduces an auxiliary perspective: viewing prime distribution through the lens of a scalar field $E : \mathbb{N} \to \mathbb{R}$. This perspective is *motivated* by analogy with physical potential theory and yields a rich functional-analytic framework (§11), but the direct connection to prime localization at the integer level remains conjectural.

Remark 10.1 (Important caveat)**.** *The reader should be aware that Article 18 of the original series contains the following self-corrective remark (Proposition 7.1):*

> *"Any rule of the form 'declare n interesting if $|\Delta^2 E(n)| > \tau(n)$' yields a set whose density in $[x, 2x]$ must go to 0 . . . Thus curvature-only triggers cannot (and should not be claimed to) capture 'almost all' primes."*

This is a correct mathematical statement (verified below) and it limits the scope of the energetic-localization programme. We present that programme honestly, with all claims marked as proved or conjectural.

10.2 The energetic field

Definition 10.2 (Energetic field and curvature)**.** *Define $E : \mathbb{N}_{\geq 3} \to \mathbb{R}^+$ by $E(n) = \log\log n$. The discrete gradient and curvature are*

$$\nabla E(n) = E(n+1) - E(n),$$
$$\Delta^2 E(n) = E(n+1) - 2E(n) + E(n-1).$$

Proved Result

Lemma 10.3 (Curvature magnitude)**.** *For all $n \geq 3$:*

$$\Delta^2 E(n) \;=\; O\!\left(\frac{1}{n^2 \log n}\right).$$

In particular, for any threshold $\tau(n) = c/\log n$ with $c > 0$, the inequality $|\Delta^2 E(n)| > \tau(n)$ fails *for all sufficiently large n.*

Proof. Since $E(x) = \log\log x$, we have $E'(x) = 1/(x \log x)$ and $E''(x) = -1/(x^2 \log x) - 1/(x^2 \log^2 x) = O(1/(x^2 \log x))$. The discrete second difference satisfies $\Delta^2 E(n) = E'(n+1) - E'(n) + O(E'''(n)) = E''(\xi)$ for some $\xi \in [n-1, n+1]$ (by the mean value theorem applied to E'), so $|\Delta^2 E(n)| \leq \sup_{\xi\in[n-1,n+1]} |E''(\xi)| = O(1/(n^2 \log n))$.

Since $|\Delta^2 E(n)| = O(1/(n^2 \log n))$ and $\tau(n) = c/\log n$, we get $|\Delta^2 E(n)|/\tau(n) = O(1/n^2) \to 0$. $\square$

Remark 10.4 (Consequence for energetic localization)**.** *Lemma 10.3 shows that the energetic curvature condition $|\Delta^2 E(n)| > c/\log n$ is* not *a viable prime-detection rule for large n: the curvature is too small relative to the threshold. The localization programme (Articles 11–16, 44–47) is therefore not a deterministic characterisation of primes in the sense originally hoped.*

10.3 Asymptotic energetic symmetry

Despite Lemma 10.3, the energetic field possesses genuine structural properties that are relevant to Part I.

Proved Result

Lemma 10.5 (Asymptotic energetic symmetry — Lemma 1, Art. 11)**.** *Define $\varphi_k(x) = 2C_k - x$ (mirror map about C_k). For every $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $k > K$ and all $x \in R_k$:*

$$|E(x) - E(\varphi_k(x))| < \varepsilon.$$

Proof. Since $\varphi_k(x) = 2C_k - x$ and $x \in R_k = (2^{k-1}, 2^k]$, the distance satisfies $|\varphi_k(x) - x| = 2|C_k - x| \leq |R_k| = 2^{k-1}$ (the maximum occurs when x is at an endpoint of R_k). Hence $|E(x) - E(\varphi_k(x))| \leq \sup_{t\in R_k} |E'(t)| \cdot 2^{k-1} \leq \frac{1}{2^{k-1}\log(2^{k-1})} \cdot 2^{k-1} = \frac{1}{(k-1)\log 2} \to 0$. $\square$

Remark 10.6. *Lemma 10.5 is a correct and non-trivial structural fact about E. However, it does not imply that* primes *are symmetrically distributed, because (a) Lemma 10.3 shows the curvature condition does not localise primes, and (b) the symmetry of E is a consequence of its smoothness, not of any arithmetic property of $\mathbb{P}$. The observed prime symmetry (Part I) is explained by PNT, not by the energetic field.*

10.4 The Fradkin Conjecture

Proved Result

Definition 10.7 (Fradkin Symmetry Conjecture (historical; proved below as Theorem 10.8))**.** *The symmetrical prime distribution phenomenon observed in dyadic rows — characterised by stable and symmetric hunt zones around well-defined axes — persists infinitely across all dyadic rows. More precisely,*

$$\lim_{k\to\infty} \sigma_k = 0, \quad \textit{where} \quad \sigma_k = \frac{||P_k^L| - |P_k^R||}{|P_k|}.$$

*This is a **proved theorem** (Theorem 10.8 below); the name "Fradkin Conjecture" is preserved for historical continuity. The stronger energetic characterisation of individual primes remains open.*

Proved Result

Theorem 10.8 (Fradkin Conjecture — proved)**.** *$\sigma_k \to 0$ as $k \to \infty$, and more precisely $\sigma_k = O(1/((k-1)\log 2))$.*

Proof. This is a direct consequence of Theorem 4.1(a) with $x = 2^{k-1}$, so $\log x = (k-1)\log 2$. □

Remark 10.9 (What the Fradkin Conjecture means)**.** *The Fradkin Conjecture, as just stated, is* proved *as a consequence of PNT. Its significance is that the original empirical observation was correct. The stronger version — that primes are individually localised by an energetic curvature rule — remains unproved and is contradicted by Lemma 10.3.*

11 Analytic and Functional-Analytic Framework

11.1 Overview

Articles 31–36 of the original series developed a rigorous functional-analytic framework around the energetic potential $\Phi = \mathcal{T}[E]$. We summarise the main results here; all proofs are in the cited literature.

11.2 Weighted Hilbert space and operator

Fix $\beta > 0$ (dimensionless gauge) and set $w(x) = e^{-\beta\Phi(x)/c^2}$, $d\mu = w\,dx$ on $\mathbb{R}^d$. Assume $w \in A_2$ (this holds when Φ has bounded mean oscillation; see Article 31).

Proved Result

Theorem 11.1 (Semigroup generation — Art. 31, Thm. 5.1)**.** *Under uniform ellipticity of a, BMO regularity of Φ, and $w \in A_2$, the closure L of the divergence-form operator $L_0 f = \nabla\cdot(a\nabla f) - b\cdot\nabla f$ ($b = \gamma\nabla\Phi$) is m-dissipative on $H = L^2(\mathbb{R}^d, \mu)$ and generates a contraction C_0-semigroup $(T(t))_{t\geq 0}$.*

Proved Result

Theorem 11.2 (Markov property and L^p-contractivity — Art. 33). *Under the sector condition and Dirichlet property of the closed form, $(T(t))$ is positivity-preserving, sub-Markovian, and contractive on $L^p(\mu)$ for all $p \in [1, \infty]$.*

Proved Result

Theorem 11.3 (Poincaré and log-Sobolev inequalities — Art. 34). *If the curvature-dimension bound $\Gamma_2(f) \geq \rho\,\Gamma(f)$ holds for some $\rho > 0$ (Bakry–Émery), then:*

$$\begin{aligned} \int |f|^2 \, d\mu \ &\leq \ \frac{1}{\rho} \int \Gamma(f) \, d\mu \quad (\textit{Poincaré, for } \bar{f} = 0), \\ \mathrm{Ent}_\mu(f^2) \ &\leq \ \frac{2}{\rho} \int \Gamma(f) \, d\mu \quad (\textit{log-Sobolev}). \end{aligned}$$

The semigroup is hypercontractive with schedule determined by ρ.

Proved Result

Theorem 11.4 (Holomorphic extension and H^∞ calculus — Art. 35). *Under the sector bound, $-L$ is m-sectorial with angle $\vartheta \in (0, \pi/2]$; $(T(t))$ extends to a bounded holomorphic semigroup on $\Sigma_{\pi/2-\vartheta}$. Under additional geometric hypotheses (doubling, Poincaré on $(\mathbb{R}^d, d_a, \mu)$), L admits a bounded $H^\infty(\Sigma_\varphi)$ functional calculus for $\varphi > \vartheta$.*

Proved Result

Theorem 11.5 (Kato square root — Art. 35, Thm. 5.3). *Under the AKM hypotheses,* $\mathrm{Dom}(L^{1/2}) = H^1(\mu)$ *and* $\|L^{1/2} f\|_2 \simeq \|\nabla f\|_{L^2(\mu, a)}$.

11.3 Quantum CPTP dynamics

Article 36 showed that, under additional hypotheses (a dense $*$-subalgebra, convergent Lindblad channels, and a quantum detailed-balance condition), the classical semigroup lifts to a CPTP quantum dynamical semigroup satisfying entropy decay. This is a conditional result depending on assumptions that are not verified for $E(n) = \log \log n$ in particular.

Conjecture / Requires Further Proof

The CPTP lifting (Article 36) is a rigorous conditional result. Whether the specific energetic field $E(n) = \log \log n$ satisfies the required algebraic hypotheses is an open question.

11.4 Relevance to prime distribution

The functional-analytic framework is structurally sound and independent of the prime-localization conjecture. Its connection to primes rests on the (unproved) energetic–prime correspondence of §12. We present it here as a self-contained mathematical contribution that stands on its own merits.

12 Energetic Localization and the Prime-Set Correspondence

12.1 Admissible thresholds and localization indicators

Definition 12.1 (Admissible threshold)**.** *A function $\tau : \mathbb{N}_{\geq 2} \to (0, \infty)$ is* admissible *if:*

(T1) Slow variation: $|\tau(n+1) - \tau(n)| \leq \eta(n)\tau(n)$ *with* $\eta(n) \to 0$ *and* $\sum_{n \geq 2} \eta(n)/n < \infty$*;*

(T2) Scale compatibility: $\tau_\lambda(n) := \tau(n)\Phi_\lambda(n)/\Phi(n)$ *satisfies* $K_1\tau(n) \leq \tau_\lambda(n) \leq K_2\tau(n)$ *for constants* $K_1, K_2 > 0$*;*

(T3) Asymptotic adequacy: *the calibration $C(\tau(n))$ is eventually bounded away from 0 and ∞ on prime arguments.*

The localization indicator is $1_{\mathrm{loc},\tau}(n) = \mathbf{1}\{\Phi(n) \geq \tau(n)\}$*.*

12.2 Threshold-independent convergence

Proved Conditional Result (conditional on Assumption 3)

Theorem 12.2 (Threshold independence — Art. 37, Thm. 6)**.** *Under Assumptions 1–3 (stability, dyadic symmetry, energy–Chebyshev correspondence), for any admissible τ and all $x \geq 2$:*

$$\sum_{n \leq x} \Big(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)\Big) \;=\; O(R_\pi(x)), \quad R_\pi(x) := \sigma_{\mathrm{env}}(x) + \varepsilon(x)\, x,$$

with constants depending only on the structural data (C_1, C_2, A, B), uniformly in τ.

Remark 12.3 (Status of Assumptions 1–3)**.** *Assumptions 1 (envelope stability) and 2 (dyadic symmetry) follow from the PNT-level results of Part I and the functional-analytic properties of E.* ***Assumption 3*** *(Energy–Chebyshev correspondence) is the critical one: it asserts that $\sum_{n \leq x} C(\Phi(n)) - \psi(x) = O(\sigma_{\mathrm{env}}(x))$ for a monotone calibration C. This is not proved from first principles; it is assumed as a structural hypothesis. Lemma 10.3 shows that the naive curvature rule $|\Delta^2 E(n)| > \tau(n)$ does not supply this correspondence for large n.*

12.3 Quantitative rates and weighted stability

Proved Conditional Result (conditional on Assumption 3)

Theorem 12.4 (Quantitative convergence — Art. 38)**.** *For any BV weight $w : [0,1] \to \mathbb{R}$ and admissible τ:*

$$\sum_{n \leq x} w\!\left(\tfrac{n}{x}\right) \Big(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)\Big) \;=\; O\Big((\|w\|_\infty + \mathrm{TV}(w))\, R_\pi(x)\Big),$$

uniformly in τ.

12.4 Conditional extensions to APs, short intervals, and sieves

Under Assumptions 1–3 and for any admissible τ, the Theorem 12.2 extends to more structured settings. All results here are *conditional on Assumption 3.*

Proved Conditional Result (conditional on Assumption 3)

Theorem 12.5 (AP extension — conditional on Assumption 3). *Fix modulus $q \geq 2$, residue a with $\gcd(a,q) = 1$. Then*

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Big(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)\Big) \;=\; O\Big(R_\pi(x)\Big),$$

uniformly in τ, with constants depending on (C_1, C_2, A, B) and q.

Proved Conditional Result (conditional on Assumption 3)

Theorem 12.6 (Short-interval extension — conditional on Assumption 3). *Let $H = H(x)$ with $H \to \infty$, $H = o(x)$, and $K \in \mathrm{BV}[0,1]$. Then*

$$\sum_{x<n\leq x+H} K\Big(\tfrac{n-x}{H}\Big) \Big(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)\Big) \;=\; O\Big((\|K\|_\infty + \mathrm{TV}(K))R_\pi(x+H)\Big) + O(H^{2/3}).$$

Proved Conditional Result (conditional on Assumption 3)

Theorem 12.7 (Sieve-compatible stability — conditional on Assumption 3). *Let $(\lambda_d)_{1\leq d\leq D}$ be a Selberg-type weight with squarefree support, ℓ^1-bound $\|\lambda\|_1^{[D]}$, and admissible projector remainder $E_{\mathrm{proj}}(x)$. Then*

$$\sum_{n\leq x} S_{P,D}(\lambda; n)\Big(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)\Big) \;=\; O\Big(\|\lambda\|_1^{[D]} R_\pi(x) + E_{\mathrm{proj}}(x)\Big),$$

uniformly in admissible τ.

Remark 12.8 (Bilinear and convolution sums). *Analogous results hold for bilinear sums $\sum_{uv\leq x} a(u)b(v)d_{uv}(\tau)$ (Art. 42) and Dirichlet convolution sums (Art. 43), under the same Assumption 3 and ℓ^1-bounds on the coefficient sequences. The key device is the weighted Abel summation lemma (Lemma A.3) applied to the partial sums $D(x;\tau)$ with BV weights derived from the coefficient sequences.*

Remark 12.9 (Variance and L^2 control). *Art. 41 of the original series established L^2 (second-moment) control:*

$$\sum_{n\leq x} \Big(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)\Big)^2 \;=\; O\Big(R_\pi(x)\Big),$$

leading to density-one convergence: $1_{\mathrm{loc},\tau}(n) = 1_{\mathbb{P}}(n)$ for a set of integers of natural density 1, under summability $\sum_k R_\pi(x_k) < \infty$ along a sequence (x_k). This is conditional on Assumption 3.

13 Strong Energetic Localization Claims and Their Status

Articles 44–47 of the original series put forward stronger claims about the energetic framework: that the discrepancy $|P_k \triangle Q_k|$ is *bounded* (not just of vanishing density), and that primes are *deterministically* characterised by an energetic curvature condition. We assess these claims carefully.

13.1 The bounded-discrepancy claim (Art. 44)

Conjecture / Requires Further Proof

Claim (Art. 44, Thm. 1, stated as theorem): There exists $M > 0$ such that $|P_k \triangle Q_k| \leq M$ for all sufficiently large k.
Status: Not proved; argument contains a logical gap.
The argument in Art. 44 proceeds: (1) $|P_k \triangle Q_k|/|R_k| \to 0$; (2) therefore $|P_k \triangle Q_k|$ is bounded. Step (2) does not follow from step (1). If $|R_k| = 2^{k-1}$ and the relative error is $O(1/((k-1)\log 2))$, then $|P_k \triangle Q_k|$ could grow like $2^{k-1}/(k \log 2) \to \infty$. A bounded discrepancy would require a *much* stronger statement, not available from the curvature analysis.
Moreover, Lemma 10.3 shows $|\Delta^2 E(n)| \ll \tau(n)$ for large n, so Q_k (the energetic localization set) is eventually *empty*, making $|P_k \triangle Q_k| = |P_k| \to \infty$.

13.2 The structural prime-emergence theorem (Art. 45)

Conjecture / Requires Further Proof

Claim (Art. 45, Lemma 2): Prime numbers generate localized energetic curvature deviations exceeding the threshold $\tau(n)$.
Status: Contradicted by Lemma 10.3.
Lemma 10.3 shows $|\Delta^2 E(n)| = O(1/(n^2 \log n))$, while $\tau(n) = c/\log n$. For large n, $|\Delta^2 E(n)|/\tau(n) \to 0$, so the inequality $|\Delta^2 E(n)| > \tau(n)$ fails for all large n, whether n is prime or composite. The claim in Art. 45 is therefore false for large n.

13.3 The completion theorem (Art. 46)

Conjecture / Requires Further Proof

Claim (Art. 46, Thm. 1): $P_k = Q_k \setminus F$ for a finite exceptional set F and all large k.
Status: Does not follow; depends on the two claims above.
Since both Art. 44 (bounded discrepancy) and Art. 45 (curvature $>$ threshold) are unproved/false, the completion theorem has no valid proof.

13.4 What can be salvaged

The following weaker statement *is* true:

Proved Result

Theorem 13.1 (Relative symmetry as proved version of the conjecture)**.** *For the count-imbalance statistic σ_k of Definition 10.7 and Theorem 10.8,*

$$\sigma_k = \frac{||P_k^L| - |P_k^R||}{|P_k|} \to 0 \quad (k \to \infty),$$

with rate $O(1/((k-1)\log 2))$. This is the quantitative, proved version of the Fradkin Conjecture, following from Theorem 4.1. Note that σ_k is not *the composite* Score(k) *of Definition 2.3; the latter is not directly controlled by Theorem 4.1.*

13.5 Recommended reformulation

The energetic framework suggests the following well-posed open problem:

Conjecture / Requires Further Proof

Open Problem (Energetic–Prime Correspondence): Does there exist a threshold sequence $(\tau(n))$ and a scalar field $\Phi : \mathbb{N} \to \mathbb{R}$ such that the localization set $Q = \{n : \Phi(n) \geq \tau(n)\}$ satisfies $|Q \triangle \mathbb{P}|/x \to 0$ with an explicit rate superior to $O(1/\log x)$?
If $\Phi(n) = \log\log n$, Lemma 10.3 shows that the curvature-based rule fails. A direct potential rule $\Phi(n) > \tau(n)$ might fare better for a suitable choice of Φ.

14 Relationship to Classical Prime-Search Methods

The predictive window framework is complementary to, not a replacement for, existing prime-search methods.

14.1 Sieve of Eratosthenes and GPU sieving

Classical sieves produce all primes in a range by elimination. GPU sieving achieves very high throughput. The predictive window method reduces the search space before testing; it does not eliminate composites. The two are compatible: apply the sieve within $Z(x;\alpha)$ rather than $[x, 2x]$.

14.2 Miller–Rabin and BPSW

Probabilistic tests certify individual candidates as probable primes. The predictive window provides a prioritised candidate list; Miller–Rabin then certifies each candidate. Using a window $Z(x;\alpha)$ as the candidate queue reduces the number of Miller–Rabin calls by a factor α, while also reducing the expected prime count by the same factor α; the expected number of tests *per prime found* remains $\approx \log x$. The saving is therefore in total computational cost when the application requires covering a specific sub-region, not in per-prime cost. (See Remark 8.1 in §8.)

14.3 ECPP

Elliptic Curve Primality Proving gives deterministic certificates but is slow per candidate. Predictive windows reduce the candidate queue fed to ECPP, potentially significant for large primes.

14.4 What this paper does not claim

- The method does not improve the *density* of primes within the window: by Theorem 5.1, the prime density in $Z(x;\alpha)$ equals the ambient density $\approx 1/\log x$.
- It does not provide a primality certificate.
- It does not improve on PNT for counting primes in intervals.
- The energetic curvature rule does not deterministically identify individual primes (Lemma 10.3, §13).

15 Limitations, Open Problems, and Falsifiability

15.1 Limitations of Part I (dyadic symmetry)

- The $O(1/\log x)$ rate of $\sigma(x)$ follows from quantitative PNT estimates (specifically the Dusart bounds used in Theorem 6.2) and is tight in the sense that the leading term cannot be improved by a PNT-level argument alone. The Li model has a genuine leading term of size $\log(32/27)\,x/\log^2 x$ in the numerator $|A(x) - B(x)|$ (Theorem 4.1).
- Establishing the sharp asymptotic $\sigma(x) \sim \log(32/27)/\log x$ requires transferring the Li-model leading term to π. This transfer needs $|\pi(y) - \mathrm{Li}(y)| = o(y/\log^2 y)$, which follows from the de la Vallée Poussin zero-free region (giving $|\pi(y) - \mathrm{Li}(y)| = O(y\,e^{-c\sqrt{\log y}})$) but not from the Dusart bounds alone (which give only $O(y/\log^2 y)$). The sharp asymptotic is therefore an additional unconditional result obtained using the classical de la Vallée Poussin zero-free-region theorem, not a consequence of the Dusart bounds used in Theorem 6.2.
- The validity thresholds $x_{\min}(\varepsilon)$ are large (Table 2) due to the conservatism of the Dusart/ϑ-function constants. In practice, the symmetry holds at much smaller scales; stronger effective PNT bounds would reduce $x_{\min}$ substantially.

15.2 Limitations of Part II (predictive windows)

- The capture law gives the *same* prime density as the ambient dyad. The window does not “enrich” the prime density; it merely localises the search.
- Narrowing α increases efficiency (fewer candidates) but linearly reduces the expected prime count. The optimal α depends on the application.

15.3 Open problems

1. **Explicit PNT constants.** Improve the Dusart 2010 bounds to reduce $x_{\min}(\varepsilon)$ to computationally accessible ranges.

2. **Energetic–prime correspondence.** Find Φ and τ such that $1_{\text{loc},\tau}$ asymptotically identifies primes with super-PNT accuracy.

3. **Functional-analytic connection.** Prove or disprove the hypothesis that the semigroup $(T(t))$ of §11 carries arithmetic information about $\mathbb{P}$.

4. **Post-quantum key generation.** Investigate integration of predictive windows with post-quantum cryptographic prime generation pipelines.

15.4 Falsifiability

1. **Symmetry claim.** Falsified if, for some k, the exact count $\sigma_k > \widehat{K}_1/((k-1)\log 2)$ for some $k \geq 11$ with certified sieve (using $x = 2^{k-1}$ so $\log x = (k-1)\log 2$, valid for $k \geq 11$ to ensure $x \geq 599$).

2. **Capture law.** Theorem 6.2 is falsified if, for some $x \geq 599$ and $\alpha \in (0,1)$, the empirical capture fraction deviates from α by more than $\widehat{K}_2/\log x = 3.10/\log x$. (A factor-of-3 tolerance would be an *empirical validation criterion*, not a falsification of the theorem.)

3. **Predictive model P_k.** Falsified if $|P_k - (0.96 + 0.2/k)| > 0.05$ for $k = 13, \ldots, 20$.

4. **Energetic localization.** The claim that $|\Delta^2 E(n)| > c/\log n$ for prime n is already falsified by Lemma 10.3 for all large n.

16 Conclusions

16.1 What has been proved

1. **(Unconditional, Part I)** The relative left–right imbalance in any dyadic interval $[x, 2x]$ satisfies $\sigma(x) = O(1/\log x)$ from quantitative PNT estimates. The sharp leading constant $\log(32/27)$ (i.e. $\sigma(x) \sim \log(32/27)/\log x$) is an additional result requiring the de la Vallée Poussin zero-free region estimate (Theorem 4.1(b) and Remark 4.4). Any pre-fixed window of width α captures a fraction $\alpha + O(1/\log x)$ of the dyadic primes. Both results hold for any finite family of windows, BV-weighted windows, arithmetic progressions (fixed α), and short intervals (with the H condition of §7).

2. **(Effective, Part I)** With Dusart 2010 constants, the bounds are: $\sigma(x) \leq 1.20/\log x$ and capture error $\leq 3.10/\log x$, valid for $x \geq 599$ (Theorem 6.2).

3. **(Functional-analytic, Part III)** The operator L with drift $b = \gamma\nabla\Phi$ and weight $w = e^{-\beta\Phi/c^2} \in A_2$ generates a contraction C_0-semigroup on $L^2(\mu)$ that is Markov, positivity- preserving, L^p-contractive, hypercontractive (under Bakry–Émery), holomorphic, and admits an H^∞ calculus and a Kato square root.

4. **(Proved version of Fradkin Conjecture)** $\sigma_k \to 0$ with rate $O(1/((k-1)\log 2))$.

16.2 What is conjectured or open

1. The energetic curvature rule $|\Delta^2 E(n)| > \tau(n)$ does *not* deterministically characterise primes for large n (Lemma 10.3).
2. The energy–Chebyshev correspondence (Assumption 3 of §12) is unproved.
3. The bounded-discrepancy claims of Articles 44–47 contain logical gaps and are not established.
4. The Fradkin Conjecture in its original, strongest form (an energetic structural characterisation of individual primes) remains open.

16.3 Contribution of this work

This paper establishes a rigorous, non-circular, falsifiable framework for studying prime distribution in dyadic intervals. Its principal contribution is the *predictive window* construction together with unconditional capture and symmetry laws with effective constants. The energetic perspective provides a rich auxiliary structure whose full arithmetic implications remain to be determined.

A Mathematical Proofs and Technical Lemmas

A.1 Li-half asymmetry bound

Lemma A.1 (Li asymmetry). *For $x \geq e$:*

$$|\Delta_1(x) - \Delta_2(x)| \;=\; \left|\mathrm{Li}(\tfrac{3x}{2}) - \mathrm{Li}(x) - \left(\mathrm{Li}(2x) - \mathrm{Li}(\tfrac{3x}{2})\right)\right| \;\leq\; \frac{x}{4\log^2 x}.$$

The absolute constant $C_1 = 1/4$ in $|\Delta_1 - \Delta_2| \leq C_1 x/\log^2 x$ is explicit, matching the bound $x/(4\log^2 x)$ proved below.

Proof. Let $f(t) = 1/\log t$ and $a = 3x/2$. Then

$$\Delta_1 - \Delta_2 = \int_0^{x/2} \left(f(a-u) - f(a+u)\right) du = -\int_0^{x/2} 2u\, f'(\xi_u)\, du$$

for some $\xi_u \in [a-u, a+u]$ by the MVT. Since $|f'(t)| = 1/(t\log^2 t) \leq 1/(x\log^2 x)$ for $t \in [x, 2x]$:

$$|\Delta_1 - \Delta_2| \leq \int_0^{x/2} 2u \cdot \frac{1}{x\log^2 x}\, du = \frac{(x/2)^2}{x\log^2 x} = \frac{x}{4\log^2 x}.$$

□

A.2 Log-ratio bound for general bands

Lemma A.2. *For $x \geq e^2\Lambda$ and $\Lambda \geq 2$:*

$$\frac{1}{\log(x/\Lambda)} \;\leq\; \frac{c_\Lambda}{\log x}, \quad c_\Lambda = \frac{\ln\Lambda + 2}{2}.$$

Proof. $1/\log(x/\Lambda) = 1/(\log x - \ln\Lambda)$. Since $x \geq e^2\Lambda$, $\log x \geq \ln\Lambda + 2$, so $1 - \ln\Lambda/\log x \geq 2/(\ln\Lambda + 2)$, giving $\log(x/\Lambda) \geq 2\log x/(\ln\Lambda + 2)$ and the claim. □

A.3 Abel summation by parts

Lemma A.3 (Abel summation by parts — weighted form)**.** *Let* $(a_n)_{n=1}^{\infty}$ *be a real sequence,* $A(t) = \sum_{n\le t} a_n$*, and* $w : \{u+1,\dots,v\} \to \mathbb{R}$ *a weight with total variation* $\mathrm{TV}(w) = \sum_{n=u+1}^{v-1} |w(n+1) - w(n)|$*. Then*

$$\left|\sum_{n=u+1}^{v} w(n)\, a_n\right| \;\le\; \Big(\|w\|_\infty + \mathrm{TV}(w)\Big) \cdot \sup_{u<t\le v} |A(t) - A(u)|.$$

Proof. By Abel summation by parts:

$$\sum_{n=u+1}^{v} w(n)\, a_n = w(v)\Big(A(v) - A(u)\Big) - \sum_{n=u+1}^{v-1} \Big(A(n) - A(u)\Big)\Big(w(n+1) - w(n)\Big).$$

Taking absolute values and bounding each factor:

$$\Big|\cdots\Big| \le \|w\|_\infty \sup |A(t) - A(u)| + \mathrm{TV}(w) \cdot \sup |A(t) - A(u)|.$$

□

Remark A.4 (Correction to earlier version)**.** *An unweighted statement* $\sum |a_n| \le 2\sup |A(t){-}A(u)|$ *that appeared in an earlier version of this paper is* false*: take* $a_n = (-1)^n$*, where* $\sup |A(t){-}A(u)| = O(1)$ *but* $\sum |a_n| = v - u \to \infty$*. The weighted form above is the correct statement and is all that is needed in the proofs of Theorems 7.3, 12.2, 12.5, 12.6, and 12.7.*

A.4 Denominator positivity

Lemma A.5. *For* $\Lambda \ge 2$ *and* $x \ge \max\{599, e^2\Lambda\}$*, the denominator* $\pi(x) - \pi(x/\Lambda) \ge 1$ *and the normalised capture functional is well-defined.*

Proof. By Bertrand's postulate, for every real $y \ge 1$ there exists a prime p with $y < p \le 2y$. For $\Lambda \ge 2$, $[x/2, x] \subseteq [x/\Lambda, x]$, so it suffices to find a prime in $(x/2, x]$. Applying Bertrand with $y = x/2$ gives a prime p with $x/2 < p \le x$, hence $\pi(x) - \pi(x/\Lambda) \ge \pi(x) - \pi(x/2) \ge 1$. □

A.5 Proof of AP capture (Theorem 7.4)

We give a direct proof from PNT in arithmetic progressions, avoiding the Fejér-BV construction.

Proof of Theorem 7.4. Fix $q \ge 2$ and $\gcd(a, q) = 1$. Write $\pi(y; q, a) = \#\{p \le y : p \equiv a \pmod q\}$. By PNT in arithmetic progressions (Dirichlet's theorem with quantitative error):

$$\pi(y; q, a) \;=\; \frac{\mathrm{Li}(y)}{\phi(q)} + E(y; q, a),$$

where $E(y; q, a) = O(ye^{-c\sqrt{\log y}})$ unconditionally (de la Vallée Poussin zero-free region), with $c > 0$ depending only on q.

For a pre-fixed window $W = [x + h_1, x + h_2] \subset [x, 2x]$ of width $|W| = (h_2 - h_1)$, let $\alpha = (h_2 - h_1)/x$ be its relative width. Then:

$$\begin{aligned}\#\{p \in W : p \equiv a \pmod q\} &= \pi(x + h_2; q, a) - \pi(x + h_1; q, a)\\ &= \frac{\mathrm{Li}(x + h_2) - \mathrm{Li}(x + h_1)}{\phi(q)} + O\Big(xe^{-c\sqrt{\log x}}\Big).\end{aligned}$$

Since $\mathrm{Li}(x + h_2) - \mathrm{Li}(x + h_1) = \alpha x/\log x + O(\alpha x/\log^2 x)$,

$$\frac{\#\{p \in W : p \equiv a \pmod q\}}{\pi(2x; q, a) - \pi(x; q, a)} = \frac{\alpha x/(\phi(q)\log x) + O(\alpha x/\log^2 x)}{x/(\phi(q)\log x) + O(x/\log^2 x)} = \alpha + O_q\left(\frac{1}{\log x}\right).$$

The error is $O_q(1/\log x)$ because the exponential PNT error is $o(1/\log x)$ for all x and the $O(\cdot/\log^2 x)$ terms contribute $O(1/\log x)$ after division. The implicit constant depends on q through $\phi(q)$ and the zero-free region constant $c(q)$. □

Remark A.6 (Extension to general windows)**.** *Theorem 7.4 as stated covers interval windows $[x + h_1, x + h_2]$. For unions of intervals, the same argument applies to each interval and the errors add. The key point is that the proof uses only PNT in APs and is fully self-contained; no Fejér kernel construction is required.*

A.6 Complete derivation of $\widehat{K}_1 = 1.20$ and $\widehat{K}_2 = 3.10$

Proof of Theorem 6.2 (complete and self-contained). **The Dusart bounds used.** We use the Dusart Dusart [2010] explicit π-bounds (valid for $x \geq 599$):

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{A_u}{\log x}\right), \quad A_u = 1.2762, \tag{4}$$

$$\pi(x) \geq \frac{x}{\log x}\left(1 + \frac{A_l}{\log x}\right), \quad A_l = 1. \tag{5}$$

We do not use a bound on $|\pi(y) - \mathrm{Li}(y)|$. All estimates use (4)–(5) directly.

Step 1: Bounding $|A(x) - B(x)|$ using upper/lower π-bounds. Write $A(x) = \pi(3x/2) - \pi(x)$ and $B(x) = \pi(2x) - \pi(3x/2)$. To bound $A(x) - B(x)$ from above:

$$A(x) - B(x) \leq \pi_U(3x/2) - \pi_L(x) - \Big[\pi_L(2x) - \pi_U(3x/2)\Big] = 2\pi_U(3x/2) - \pi_L(x) - \pi_L(2x),$$

where $\pi_U(y)$ and $\pi_L(y)$ denote the upper and lower bounds (4)–(5). This is the logically correct combination: the upper bound for $\pi(3x/2)$ (large) and the lower bounds for $\pi(x)$ and $\pi(2x)$ (subtracted).

Expanding with $f_u(c) = cx/(\log x + \log c)(1 + A_u/(\log x + \log c))$ and $f_l(c) = cx/(\log x + \log c)(1 + A_l/(\log x + \log c))$:

$$2\pi_U(3x/2) - \pi_L(x) - \pi_L(2x) = \frac{x}{\log^2 x}\Big[3A_u - 3A_l - 3\log(3/2) + 2\log 2 + O(1/\log x)\Big].$$

Numerically: $3A_u - 3A_l - 3\log(3/2) + 2\log 2 = 3(1.2762) - 3(1) - 3(0.4055) + 2(0.6931) = 3.8286 - 3 - 1.2164 + 1.3863 = 0.9985$ (where $\log(3/2) \approx 0.4055$). This asymptotic expression motivates the quartic proof in Step 3, which does not rely on it.

Step 2: Denominator lower bound.

$$\pi(2x) - \pi(x) \;\geq\; \frac{0.8456\,x}{\log(2x)} \quad (x \geq 41), \tag{6}$$

where $0.8456 = 0.9228 \cdot 2 - 1$ uses $\vartheta(2x) \geq 0.9228 \cdot 2x$ and $\vartheta(x) \leq x$ (Rosser and Schoenfeld Rosser and Schoenfeld [1962]).

Step 3: Assembling $\widehat{K_1}$ via polynomial positivity. Set $t = \log x$, $u = t - \log 599 \geq 0$. Define

$$\begin{aligned} H_+(t) &= 2\pi_U(3x/2)/x - \pi_L(x)/x - \pi_L(2x)/x, \\ H_-(t) &= \pi_U(2x)/x + \pi_U(x)/x - 2\pi_L(3x/2)/x. \end{aligned}$$

Using $A - B \leq xH_+(t)$ and $B - A \leq xH_-(t)$ gives $|A - B| \leq x \max(H_+, H_-)$. The inequality $\sigma(x) < 1.20/\log x$ is equivalent to

$$t(t + \log 2)\, H_\pm(t) < 1.20 \times 0.8456 \quad (t \geq \log 599).$$

Substituting the explicit Dusart formulas for π_U, π_L, clearing positive denominators, and changing variable to $u = t - \log 599 \geq 0$, both inequalities reduce to quartic polynomials $P_\pm(u)$. Using rational enclosures $\log 599 \in [6.3952, 6.3953]$, $\log 2 \in [0.6931, 0.6932]$, $\log(3/2) \in [0.4054, 0.4055]$ (each with 4-digit lower and upper bounds), the coefficients of $P_+(u)$ and $P_-(u)$ satisfy:

$$\begin{aligned} P_+(u) &\geq 0.0162\,u^4 + 0.5470\,u^3 + 7.7721\,u^2 + 50.067\,u + 118.36 \;>\; 0, \\ P_-(u) &\geq 0.3560\,u^4 + 9.824\,u^3 + 100.35\,u^2 + 449.96\,u + 747.04 \;>\; 0. \end{aligned}$$

Since all coefficients are strictly positive and $u \geq 0$, both polynomials are positive everywhere, proving

$$\sigma(x) \;<\; \frac{1.20}{\log x} \quad \text{for all } x \geq 599.$$

Capture bound $\widehat{K_2} = 3.10$ — fully analytic proof. Write $\pi(y) = M(y) + e(y)$ where $M(y) = y/\log y \cdot (1 + 1/\log y)$ is the Dusart lower bound and $0 \leq e(y) \leq 0.2762\, y/\log^2 y$ (since $A_u - A_l = 0.2762$). Split the capture deviation $D = \pi(Z) - \alpha(\pi(2x) - \pi(x)) = D_M + D_e$.

Model term D_M. Since $M'(y) = 1/\log y - 2/\log^3 y$ is monotone for $y \geq 599$, $|D_M| \leq \alpha x[M'(x) - M'(2x)]$. Using $|M''(y)| \leq 1/(y \log^2 y)$:

$$M'(x) - M'(2x) \leq \int_x^{2x} \frac{dy}{y \log^2 y} < \frac{\log 2}{\log^2 x}.$$

Hence $|D_M| \leq \log(2)\, x/\log^2 x$.

Error term D_e. The four error terms satisfy $|D_e| \leq e(y_2) + e(y_1) + \alpha e(2x) + \alpha e(x)$. Since $y_1, y_2 \in [x, 2x]$ and $e(y) \leq 0.2762y/\log^2 x$:

$$\begin{aligned} e(y_1) + e(y_2) \;&\leq\; 0.2762 \cdot 3x/\log^2 x, \\ \alpha[e(2x) + e(x)] \;&\leq\; 0.2762 \cdot 3x/\log^2 x. \end{aligned}$$

Hence $|D_e| \leq 6(0.2762)x/\log^2 x$.

Combining.

$$|D| \le |D_M| + |D_e| \le (\log 2 + 6 \times 0.2762)\frac{x}{\log^2 x} = 2.35035\,\frac{x}{\log^2 x}.$$

Dividing by denominator (6) and multiplying through:

$$\left|\frac{\pi(Z)}{\pi(2x)-\pi(x)} - \alpha\right| \le \frac{2.35035\,\log(2x)}{0.8456\,\log^2 x} = \frac{2.35035\,(1+\log 2/\log x)}{0.8456\,\log x}.$$

Since $(1+\log 2/\log x)$ is decreasing in x, the right side is maximised at $x = 599$: $(1 + \log 2/\log 599) < 1.109$, giving

$$\left|\frac{\pi(Z)}{\pi(2x)-\pi(x)} - \alpha\right| \le \frac{2.35035 \times 1.109}{0.8456\,\log x} \approx \frac{3.081}{\log x} < \frac{3.10}{\log x} \quad (x \ge 599,\ 0 < \alpha < 1).$$

This bound is independent of α and requires no assumption on the location of the supremum in α. Setting $\widehat{K_2} = 3.10$ completes the proof. □

Remark A.7 (Correction history). *The constants $\widehat{K_1} = 1.20$ and $\widehat{K_2} = 3.10$ are derived here from the Dusart Dusart [2010] π-bounds used directly, without passing through $|\pi - \mathrm{Li}\,|$. Versions through v9 of this paper incorrectly applied the coefficient 1.2762 as a bound for $|\pi - \mathrm{Li}\,|$ — not supported by the cited theorem. The constant sequence has been:* $7.48/3.74 \to 7.91/4.08 \to 8.7/4.7 \to 8.8/9.0 \to 10.4/10.6 \to 1.20/1.00 \to 1.20/3.10$.

A.7 Energy–Chebyshev correspondence: status and examples

Assumption 3 in §12 is the weakest point of the energetic correspondence programme. We document what can and cannot currently be proved.

Remark A.8 (What is known about Assumption 3). *For the specific choice $E(n) = \log\log n$ and the curvature rule $\Phi(n) = |\Delta^2 E(n)|$, Lemma 10.3 shows $|\Delta^2 E(n)| \ll 1/(n^2 \log n)$ for all n, while $\tau(n) = c/\log n$. Thus $1_{\mathrm{loc},\tau}(n) \equiv 0$ for all large n, and the correspondence $\sum_{n\le x}(1_{\mathrm{loc},\tau}(n) - 1_{\mathbb{P}}(n)) = O(R_\pi(x))$ trivially fails: the left side equals $-\pi(x) \to -\infty$.*

Assumption 3 can hold for other choices of Φ, for instance:

- *$\Phi(n) = \log n$, $\tau(n) = \log(2)/2 + \varepsilon$: then $1_{\mathrm{loc},\tau}(n) \equiv 1$ and the sum is $-\pi(x) \to -\infty$ again. One needs $\tau(n)$ to be calibrated so that $|\{\Phi(n) \ge \tau(n)\}| \approx \pi(x)$, which is non-trivial.*
- *For $\Phi(n) = -\Lambda(n)$ (negative von Mangoldt), the set $\{\Phi(n) \ge \tau\}$ for $\tau < 0$ captures primes and prime powers. This choice makes Assumption 3 trivially true but also trivially circular.*

Finding a non-circular Φ satisfying Assumption 3 with rate $R_\pi(x) = o(\pi(x))$ is the central open problem of the energetic programme.

B Algorithms, Data, and Reproducibility Protocol

B.1 Segmented sieve of Eratosthenes

The following pseudocode generates $\pi(y)$ for all $y \leq 2X_{\max}$:

```
Input: X_max, block size B = 2^20
base = primes up to floor(sqrt(2*X_max))   # small sieve
Pi[0..2*X_max] = 0
count = 0
for L in {2, 2+B, 2+2B, ...}:
    R = min(L+B-1, 2*X_max)
    mark[L..R] = True                      # initialise block
    for p in base:
        m = ceil(L/p)*p
        if m == p: m += p
        for t in {m, m+p, ..., R}:
            mark[t] = False
    for y in {L..R}:
        if mark[y] and y >= 2:
            count += 1
        Pi[y] = count
return Pi
```

Correctness checks: (i) Monotonicity: $\Pi[y+1] - \Pi[y] \in \{0, 1\}$. (ii) Spot-check: deterministic Miller–Rabin with bases $\{2, 3, 5, 7, 11, 13, 17\}$ on a 10% stratified sample. (iii) Conservation: block count matches mark array.

B.2 Li evaluation

$\mathrm{Li}(x)$ is evaluated by 15-point Gauss–Kronrod quadrature on $[2, x]$ with absolute tolerance 10^{-12} and relative tolerance 10^{-12}. The integrand $f(t) = 1/\log t$ is regular on $[2, x]$ ($t = 2$ is not a singularity of Li).

B.3 Data schema

The output dataset `dyads.csv` has one row per x with columns:

Column	Content
`x`	Scale
`pi_x, pi_3x2, pi_2x`	Prime counts
`d_pi_total, d_pi_left, d_pi_right`	Half-counts
`sigma`	Normalised asymmetry $\sigma(x)$
`li_x, li_3x2, li_2x`	Li values
`d_pi_zone_alpha_%`	Capture for each α

Integrity: SHA-256 hashes for all data files in `checksums.txt`.

B.4 Reproducibility checklist

1. Fix the Dusart constants $A_u = 1.2762$, $A_l = 1$, $A_\theta = 1.2323$, $X^* = 599$ from Dusart [2010] and Rosser and Schoenfeld [1962].
2. Compute $X^* = 599$ and $x_{\min}(\varepsilon)$ via Definition 6.1.
3. Run sieve to produce $\Pi[y]$ up to $2X_{\max}$.
4. Build `dyads.csv` from Π.
5. Evaluate Li with stated tolerances.
6. Execute evaluation protocol and record pass/fail.
7. Generate hashes and publish data bundle.

B.5 Consolidated empirical data (rows $k = 3$ to $k = 12$)

The following table consolidates all empirical results from Articles 1–10 and 17 of the original series. Every entry can be reproduced from a certified prime sieve.

Table 6: Complete empirical data for dyadic rows $k = 3$ to $k = 12$. Distances d_L, d_R are from C_k to the farthest prime on each side.

k	$\lvert R_k\rvert$	$\lvert P_k\rvert$	$\lvert P_k^L\rvert$	$\lvert P_k^R\rvert$	d_L	d_R	Score
3	4	2	1	1	1.5	0.5	0.000
4	8	2	1	1	1.5	0.5	0.000
5	16	5	3	2	7.5	7.5	2.533
6	32	7	4	3	11.5	9.5	0.722
7	64	13	6	7	29.5	30.5	0.601
8	128	23	12	11	61.5	58.5	0.595
9	256	47	22	25	127.5	124.5	0.642
10	512	70	38	32	247.5	228.5	0.252
11	1024	–	–	–	–	–	0.212
12	2048	–	–	–	–	–	0.183

Key observations:

- Distance ratios d_L/d_R converge to 1.000 from below (rows 3–7) and from above (rows 8–10), consistent with the $O(1/\log k)$ symmetry law.
- The score < 0.5 for all $k \geq 10$, confirming the threshold $k_0 = 11$ identified in Art. 17.
- Rows $k = 3, 4$ achieve perfect score 0.000 trivially (only one prime per side); these should not be extrapolated.

B.6 Predictive model calibration data

The fitted formula $P_k \approx 0.96 + 0.2/k$ was derived from rows $k = 3$–10 only and is not a theorem. The large residuals at $k = 5$–7 suggest the fit is poor for small k. Validation at $k = 13$–20 is needed before the formula can be used as a predictive tool.

Table 7: Relative hunt-zone size P_k (circular definition, historical only) vs. the fitted model $0.96 + 0.2/k$.

k	P_k (empirical)	$0.96 + 0.2/k$	Residual
3	1.000	1.027	−0.027
4	1.000	1.010	−0.010
5	0.938	1.000	−0.062
6	0.625	0.993	−0.368
7	0.672	0.989	−0.317
8	0.898	0.985	−0.087
9	0.938	0.982	−0.044
10	0.953	0.980	−0.027

C Cross-Reference Map: Articles 1–47 to This Paper

The following table maps each article in the original series to the corresponding section(s) of this unified paper. Articles whose content has been superseded by later corrections are marked with †.

#	Original title (abbreviated)	This paper	Notes
1	Symmetry and Hunt Zones (base-2)	§9, §3	Empirical motivation
2	Statistical Analysis and Validation	§4, §9	
3	Optimal Calibration	§3	
4	Comparative Analysis	§14	
5	Computational Implementations	§8, App. B	
6	Predictive Model for Hunt Zone Size	§9	Empirical only
7	The Fradkin Conjecture (symmetry table)	§10, Tab. 1	
8	Stepwise Computation (Art. 8)	§8	
9	Formal Derivation of Efficiency Gains	§8	
10	Scientific Comparison (v13)	§14	
11	Energetic Model for Exponential Rows	§10, §11	Lemma 10.5
12	Prime Localization via Energy Gradient	§10†	See Lemma 10.3
13	Proof of Fradkin Conjecture (energetic)	§13†	Superseded by Thm. 4.1
14	Theoretical Justification of Energetic Framework	§10	
15	Formal Induction Proof	§13†	Gap in induction step
16	Formal Error Estimation	§13†	False positive analysis unreliable
17	Establishing Minimal Threshold k_0	§9	Empirical only
18	Predictive Hunting Zones Independent of Primes	§3, §5, §2	Key paper; corrects circularity
19	Relative Symmetry of Primes in Dyadic Intervals	§4	
20	Direct Error Bounds for Dyadic Asymmetry	§6, App. A	
21	Standardized Notation	§2	
22	Standardization (self-contained)	§2, §6	
23	Open Data and Replication Protocol	App. B	
24	Effective Constants and Explicit $x_{\min}$	§6	
25	Uniform Window Laws (multi-window)	§7	
26	Scale Transfer and Scheme Invariance	§7	
27	Finite Minimax Thresholds (ε-level)	§6, Tab. 2	
28	Weighted Capture Laws (BV/Lipschitz)	§7	
29	Fully Numeric Instantiation	§6	
30	Unified Consolidation (Arts. 24–29)	§6, §7	
31	Axioms and Well-Posedness (Φ framework)	§11	
32	Calibration-Free Structural Constants	§11	

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33	Positivity and Markov Regularity	§11	
34	Functional Inequalities and Hypercontractivity	§11	
35	Sectoriality, Analyticity, H^∞ Calculus	§11	
36	Energetic Forms to CPTP Semigroups	§11	Conditional
37	Energetic Localization and the Prime Set	§12	
38	Quantitative Convergence of Energetic Localization	§12	
39	Energetic Localization in Arithmetic Progressions	§7	
40	Energetic Localization in Short Intervals	§7	
41	Variance and Almost-Everywhere Convergence	§7	
42	Bilinear and Convolution Stability	§7	
43	Sieve-Compatible Stability	§7	
44	Deterministic Collapse of Localization Error	§13†	Logical gap
45	Structural Theorem for Prime Emergence	§13†	Contradicted by Lem. 10.3
46	Completion Theorem (Fradkin Structure)	§13†	Depends on Arts. 44–45
47	Structural Stability of Energetic Localization	§13†	Repeats Art. 44

† = superseded or contains logical gaps; the correct version or assessment appears in the section indicated.

References

Pierre Dusart. Estimates of some functions over primes without R.H. *arXiv preprint*, 2010. arXiv:1002.0442.

J. Barkley Rosser and Lowell Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois Journal of Mathematics*, 6:64–94, 1962.