Stability, Chaos, and Structural Comparison in the Energetic–Elastic N-Body Model

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Abstract

The energetic–elastic N-body model (EETBM) is a Hamiltonian system on the collision-free domain obtained by replacing Newtonian pair potentials by quadratic elastic pair potentials. EETBM is treated here as a distinct Hamiltonian model, not as a Taylor approximation or an exact canonical equivalent of Newtonian gravity. We establish local well-posedness and an energy-barrier criterion for invariant collision-free components, derive C^0 and higher-derivative comparison bounds on compact collision-free sets, and formulate stability, KAM, homoclinic, finite-time comparison, collision-regularization, and Newtonian persistence statements with explicit hypotheses.

A detailed re-examination of the collinear reduction yields a structural result valid for arbitrary N: in every fixed collision-free ordering sector the reduced collinear EETBM potential is a quadratic form in the $N-1$ inter-particle gaps, with Hessian $K_N = \sum_{i<j} k_{ij} b_{ij} b_{ij}^{\mathsf{T}}$. For a connected positive spring graph this Hessian is positive definite. After removing the center of mass, the collinear flow is therefore an exactly linear positive-definite Hamiltonian oscillator and has no nontrivial homoclinic orbit. Consequently the previously proposed collinear-homoclinic NVE-to-Lamé construction is not available for the stated positive-spring EETBM. Generic non-integrability of EETBM therefore remains open and requires a different particular solution or a different obstruction.

Independently, the strict sign $H_N - H_E < 0$ gives a cohomological obstruction to the standard Moser path. A dimensionless Gronwall estimate gives finite-time trajectory comparison, and transfer of KAM tori or hyperbolic sets to Newtonian dynamics is conditional on the derivative smallness required by the corresponding persistence theorems. Binary-collision statements are restricted to the regularized extended-Hamiltonian framework. Direct Newtonian non-integrability, full C^1 collision symplecticity, inter-region shadowing, triple collision, and exact non-Moser conjugacy remain open.

1 Introduction

Let $q_i \in \mathbb{R}^d$, $p_i \in \mathbb{R}^d$, $m_i > 0$, and

$$r_{ij} = \|q_i - q_j\| .$$

The Newtonian N-body Hamiltonian combines singular pair interactions with a rich non-integrable phase portrait. The EETBM replaces each Newtonian pair potential $-Gm_i m_j / r_{ij}$ by a quadratic elastic term

$$\frac{1}{2} k_{ij} (r_{ij} - \ell_{ij})^2.$$

The model is useful only if its relation to Newtonian dynamics is stated precisely. In particular, it is not a Taylor approximation of the Newtonian potential at $r = \ell_{ij}$, and no exact global canonical equivalence is assumed.

The analysis is organized in four layers. First, the EETBM is studied as a Hamiltonian system in its own right. Second, the EETBM and Newtonian Hamiltonians are compared only on compact collision-free sets. Third, KAM and hyperbolic persistence are invoked only in the differentiability norms required by the corresponding theorems. Fourth, collision regularization is separated from the much stronger problem of constructing a global symplectic comparison map through collision.

A central purpose of the present formulation is to distinguish what follows from the Hamiltonian as written from what would require additional structure. This distinction becomes especially important in the collinear three-body reduction, where the exact quadratic structure rules out the homoclinic orbit previously used as the starting point of a Lamé/Morales–Ramis argument.

2 Hamiltonians, calibration, and the collision-free domain

Definition 2.1 (Hamiltonians)**.** Let

$$D = \{(q,p) : r_{ij} > 0 \text{ for all } i < j\}.$$

On D define

$$H_E(q,p) = \sum_i \frac{\|p_i\|^2}{2m_i} + \frac{1}{2}\sum_{i<j} k_{ij}(r_{ij} - \ell_{ij})^2, \tag{1}$$

$$H_N(q,p) = \sum_i \frac{\|p_i\|^2}{2m_i} - \sum_{i<j} \frac{Gm_im_j}{r_{ij}}. \tag{2}$$

The calibration

$$k_{ij} = \frac{2Gm_im_j}{\ell_{ij}^3}$$

is retained as a model prescription only.

Remark 2.2 (No Taylor matching)*.* For $a = Gm_im_j$ and $x = r - \ell$,

$$-\frac{a}{\ell + x} = -\frac{a}{\ell} + \frac{a}{\ell^2}x - \frac{a}{\ell^3}x^2 + O(x^3),$$

whereas the calibrated elastic term is

$$\frac{1}{2}kx^2 = \frac{a}{\ell^3}x^2.$$

The linear terms differ and the quadratic coefficients have opposite signs. Hence the elastic pair potential is neither a first- nor a second-order Taylor approximation of the Newtonian pair potential.

Remark 2.3 (Collision regularity)*.* For $\ell_{ij} > 0$,

$$q \mapsto \big(\|q_i - q_j\| - \ell_{ij}\big)^2$$

is not differentiable at $q_i = q_j$. Thus H_E is smooth on D but does not extend smoothly through binary collision.

On D one has the exact identity

$$H_N - H_E = -\sum_{i<j} \frac{Gm_im_j}{r_{ij}} - \frac{1}{2}\sum_{i<j} k_{ij}(r_{ij} - \ell_{ij})^2 < 0. \tag{3}$$

3 Well-posedness and comparison bounds

Theorem 3.1 (Local well-posedness and energy barrier)**.** *For every $z_0 \in D$, Hamilton's equations for H_E possess a unique smooth maximal solution while the trajectory remains in D. Fix $E_0 \in \mathbb{R}$ and $\rho > 0$, and define*

$$V_\partial(\rho) = \inf\left\{V_E(q) : \min_{i<j} r_{ij} = \rho\right\}.$$

If

$$V_\partial(\rho) > E_0$$

and z_0 belongs to a connected component of

$$\{H_E = E_0,\ \min_{i<j} r_{ij} > \rho\},$$

then that component cannot be exited through the boundary $\min r_{ij} = \rho$. If, in addition, that component is bounded in configuration space, the solution exists for all time.

Proof. Along an H_E trajectory,

$$E_0 = T(p(t)) + V_E(q(t)), \qquad T \geq 0,$$

hence $V_E(q(t)) \leq E_0$. If the trajectory first reached $\min r_{ij} = \rho$ at t_*, then

$$V_E(q(t_*)) \geq V_\partial(\rho) > E_0,$$

a contradiction. Thus the boundary is inaccessible. If the invariant configuration component is bounded, the energy identity also bounds the momenta. The trajectory then remains in a compact subset of D on which the Hamiltonian vector field is smooth and bounded; standard continuation gives global existence. □

No monotone threshold $\rho \geq \rho_*$ is asserted without a separate monotonicity theorem for V_∂.

Proposition 3.2 (C^0 comparison)**.** *On*

$$A_{\rho,R} := \{(q,p) : \rho \leq r_{ij} \leq R \text{ for all } i<j\},$$

$$\|H_N - H_E\|_{C^0(A_{\rho,R})} \leq B_0(\rho, R),$$

where

$$B_0(\rho, R) = \sum_{i<j} \left[\frac{Gm_i m_j}{\rho} + \frac{1}{2} k_{ij} (R + \ell_{ij})^2 \right].$$

Proposition 3.3 (Higher derivative bounds)**.** *For every integer $s \geq 1$ and compact $A_{\rho,R}$ there is a finite, effectively computable constant $B_s(\rho, R)$ such that*

$$\|H_N - H_E\|_{C^s(A_{\rho,R})} \leq B_s(\rho, R).$$

In particular,

$$D_1(\rho, R) := \|X_{H_N} - X_{H_E}\|_{C^0} \leq B_1(\rho, R),$$

and

$$D_2(\rho, R) := \|X_{H_N} - X_{H_E}\|_{C^1} \leq B_2(\rho, R).$$

Proof. For $W_N(r) = -a/r$,

$$\left| \frac{d^m W_N}{dr^m} \right| = \frac{am!}{r^{m+1}} \leq \frac{am!}{\rho^{m+1}}.$$

Cartesian derivatives of a radial function are finite sums of radial derivatives multiplied by tensors built from $\hat{r}$ and powers of $1/r$. Hence every Newtonian derivative is bounded on $r \geq \rho$. The elastic pair potential is smooth with bounded derivatives on $\rho \leq r \leq R$. Summation over pairs and derivative orders gives the result. □

4 Equilibria, Hessians, and KAM structure

For a single elastic pair

$$W_E(r) = \frac{1}{2} k (r - \ell)^2,$$

let $P_\parallel = \hat{r}\hat{r}^\mathsf{T}$ and $P_\perp = I - P_\parallel$. Direct differentiation gives

$$D^2 W_E = k P_\parallel + k \left(1 - \frac{\ell}{r}\right) P_\perp. \tag{4}$$

Thus radial stiffness is $k > 0$, while transverse stiffness is positive precisely for $r > \ell$.

Theorem 4.1 (Reduced Lyapunov stability)**.** *After removal of neutral symmetry directions, a strict local minimum of the reduced Hamiltonian is Lyapunov stable. Hence positive definiteness of the reduced Hessian is a sufficient stability criterion.*

Corollary 4.2 (Stretched-edge sufficient condition)**.** *If an equilibrium satisfies $r_{ij} > \ell_{ij}$ for all pairs and the symmetry reduction introduces no additional zero direction, then the pair Hessian contributions are positive in their radial and transverse directions and provide a sufficient condition for reduced positive definiteness.*

4.1 Birkhoff normal form and KAM hypotheses

Near a reduced elliptic equilibrium, a finite-order Birkhoff normal form has the formal structure

$$H_E = H_E(\bar{z}) + \Omega \cdot I + \frac{1}{2} I^{\mathsf{T}} A I + Z_3(I) + \cdots + Z_N(I) + \mathcal{R}_{N+1}(I, \theta),$$

provided the corresponding resonance equations can be removed at the chosen order.

Theorem 4.3 (KAM persistence in EETBM)**.** *Let a reduced EETBM equilibrium be elliptic. If the frequency vector is Diophantine in the sense required by the chosen KAM theorem, the twist/non-degeneracy condition holds, and the analytic remainder is sufficiently small on the selected neighborhood, then a positive-measure Cantor family of invariant tori persists.*

This is a conditional application of classical KAM theory. The hypotheses must be verified for the parameter family under study.

5 Exact linearity of every fixed collinear ordering sector

The three-body calculation extends naturally to arbitrary N. This gives a structural characterization of where the nonlinearity of EETBM can and cannot occur.

Fix a collision-free collinear ordering sector

$$x_1 < x_2 < \cdots < x_N$$

and introduce the $N-1$ positive gap coordinates

$$u_a = x_{a+1} - x_a > 0, \qquad a = 1, \ldots, N-1.$$

For every $i < j$,

$$r_{ij} = x_j - x_i = \sum_{a=i}^{j-1} u_a.$$

Define $b_{ij} \in \mathbb{R}^{N-1}$ by

$$(b_{ij})_a = \begin{cases} 1, & i \le a < j, \\ 0, & \text{otherwise.} \end{cases}$$

Then $r_{ij} = b_{ij}^{\mathsf{T}} u$.

Theorem 5.1 (Exact quadratic collinear reduction)**.** *In every fixed collision-free collinear ordering sector, the EETBM potential is exactly*

$$V_{\mathrm{col}}(u) = \frac{1}{2} \sum_{i<j} k_{ij} \left(b_{ij}^{\mathsf{T}} u - \ell_{ij} \right)^2 .$$

Consequently

$$\nabla^2 V_{\mathrm{col}} = K_N := \sum_{i<j} k_{ij} b_{ij} b_{ij}^{\mathsf{T}},$$

a constant positive-semidefinite matrix. If the positive-weight spring graph is connected—in particular, if

$$k_{i,i+1} > 0, \qquad i = 1, \ldots, N-1,$$

then K_N is positive definite.

Proof. The formula for V_{col} follows from $r_{ij} = b_{ij}^{\mathsf{T}} u$. Differentiating twice gives the stated Hessian. For any $v \in \mathbb{R}^{N-1}$,

$$v^{\mathsf{T}} K_N v = \sum_{i<j} k_{ij} \left(b_{ij}^{\mathsf{T}} v \right)^2 \ge 0.$$

If all nearest-neighbor springs are positive, then $b_{i,i+1} = e_i$, so

$$v^\top K_N v \ge \sum_{i=1}^{N-1} k_{i,i+1} v_i^2.$$

Hence $v^\top K_N v > 0$ for every $v \neq 0$. More generally, the same conclusion follows whenever the positive-weight interval vectors b_{ij} span $\mathbb{R}^{N-1}$, which is equivalent to connectivity of the positive spring graph after removal of the translational degree of freedom. $\square$

Corollary 5.2 (Spectral lower bound)**.** *Under the nearest-neighbor positivity condition,*

$$K_N \succeq \operatorname{diag}(k_{12}, k_{23}, \ldots, k_{N-1,N}),$$

and therefore

$$\lambda_{\min}(K_N) \ge \min_{1 \le i < N} k_{i,i+1} > 0.$$

If M_N is the positive-definite reduced mass matrix in gap coordinates, the squared normal frequencies are the eigenvalues of

$$M_N^{-1/2} K_N M_N^{-1/2},$$

and are all strictly positive.

After removal of the center of mass, the kinetic energy is a constant positive-definite quadratic form,

$$T_{\text{col}} = \frac{1}{2} \dot{u}^\top M_N \dot{u}, \qquad M_N > 0.$$

Let u_* denote the unique unconstrained minimizer whenever it lies in the ordering sector. With $\xi = u - u_*$, the complete reduced equations are exactly

$$M_N \ddot{\xi} + K_N \xi = 0.$$

Theorem 5.3 (Exact linearity and absence of collinear homoclinic dynamics)**.** *Assume the positive spring graph is connected and the equilibrium lies in the interior of a fixed collision-free ordering sector. Then, after removal of the center of mass and translation to equilibrium, the full collinear EETBM is an exactly linear positive-definite Hamiltonian oscillator. In particular:*

1. *the equilibrium is elliptic;*
2. *every bounded non-equilibrium solution is a superposition of normal oscillatory modes;*
3. *no nontrivial orbit is homoclinic to the equilibrium;*
4. *no Smale horseshoe can be generated entirely by a homoclinic mechanism inside that fixed collinear sector.*

Proof. The reduced equations are the constant-coefficient oscillator system above. By Theorem 5.1 and positivity of M_N, all generalized eigenvalues of

$$K_N v = \omega^2 M_N v$$

are positive. Hence the equilibrium is elliptic and every solution is a finite superposition of oscillatory normal modes. A nonzero solution cannot converge to the equilibrium as $t \to +\infty$ or $t \to -\infty$, so no nontrivial homoclinic orbit exists. The standard homoclinic mechanism for a Smale horseshoe is therefore absent within the sector. $\square$

Remark 5.4 (Boundary minima)*.* If the unconstrained minimizer of the quadratic potential lies outside the positive-gap cone, the fixed ordering sector contains no interior equilibrium of the above type. This does not create an interior saddle: strict convexity is unchanged. Boundary points correspond to one or more $u_a = 0$, i.e. collisions, and lie outside the collision-free domain D.

Remark 5.5 (Ordering sectors)*.* Every other fixed collinear ordering is obtained by relabeling. A continuous collinear trajectory can change ordering only by passing through a binary collision. Hence the exact-linearity theorem applies throughout every collision-free collinear segment of the dynamics.

5.1 Geometric localization of EETBM nonlinearity

The result has a useful conceptual consequence. The constitutive pair law

$$W_{ij}(r) = \frac{1}{2}k_{ij}(r - \ell_{ij})^2$$

is quadratic. In a fixed collinear ordering sector each distance r_{ij} is an affine function of the gaps, so the entire Hamiltonian is quadratic after reduction. In dimensions $d \geq 2$, by contrast,

$$r_{ij} = \sqrt{(q_i - q_j) \cdot (q_i - q_j)}$$

is nonlinear in Cartesian coordinates. Thus the intrinsic nonlinearity of EETBM is geometric rather than a consequence of a nonlinear spring law.

Corollary 5.6 (Localization principle)**.** *Any genuinely nonlinear mechanism of the positive-spring EETBM that is absent from a linear oscillator network—including a homoclinic tangle generated by a hyperbolic invariant object—must involve non-collinear geometry, collision-boundary effects, or a different reduction that is not confined to a fixed collision-free collinear ordering sector.*

This localization principle does not assert that the full EETBM is chaotic or non-integrable. It identifies where such phenomena cannot originate.

5.2 Consequence for the former Lamé/Morales–Ramis route

The previously proposed differential-Galois route assumed a collision-free collinear homoclinic orbit and used it to construct a transverse NVE and then a Jacobi/Lamé equation. Theorem 5.3 rules out that starting orbit not only for three bodies but for arbitrary N under the stated connectivity and interior-equilibrium assumptions.

Corollary 5.7 (Failure of the fixed-sector collinear-homoclinic Lamé construction)**.** *For the positive-spring EETBM, a Morales–Ramis proof based on a nontrivial collision-free homoclinic orbit to an interior equilibrium of a fixed collinear ordering sector cannot be carried out, because the required homoclinic orbit is absent.*

This explains why an elliptic-function separatrix parametrization of the form

$$\xi(t) = \xi_c + A \operatorname{sn}(\nu t \mid m_0)$$

cannot be derived from the exact fixed-sector collinear Hamiltonian: after translation to equilibrium the system is a constant-coefficient oscillator, not a nonlinear quartic separatrix problem.

5.3 Remaining routes to differential-Galois non-integrability

The theorem does not establish integrability of the full EETBM. It removes one proposed proof route and narrows the search for a valid particular solution. Possible alternatives include:

- a non-collinear periodic solution with a computable NVE;
- a symmetry-reduced rotating or relative-equilibrium solution;
- a periodic collinear oscillator whose *transverse* NVE is a nontrivial Hill-type equation, even though the longitudinal collinear dynamics itself is linear;
- another meromorphic particular solution for which the NVE differential Galois group can be computed.

Generic meromorphic Liouville non-integrability of H_E therefore remains open in this paper.

6 Homoclinic and hyperbolic dynamics outside the collinear proof route

Theorem 5.3 concerns the fixed collinear invariant sector only. It does not rule out hyperbolic dynamics elsewhere in the full configuration space.

Suppose a reduced non-collinear subsystem or perturbative normal form can be written

$$H_E = H_0 + \varepsilon H_1$$

with a verified reference homoclinic orbit γ_0 of H_0. The first-order splitting is measured by

$$M(\tau) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\gamma_0(t+\tau))\, dt.$$

Proposition 6.1 (Melnikov criterion)**.** *If the Melnikov integral converges and possesses a simple zero*

$$M(\tau_0) = 0, \qquad M'(\tau_0) \neq 0,$$

then, for sufficiently small perturbation, the stable and unstable manifolds intersect transversely. Consequently the associated Poincaré map contains a Smale horseshoe.

This proposition is conditional. No generic simple-zero theorem is claimed without an explicit orbit and an explicit Melnikov calculation.

7 Region-wise shadowing

On a compact hyperbolic invariant set, the classical shadowing lemma applies. Near a non-degenerate KAM torus, Birkhoff normal-form estimates control motion within the normal-form neighborhood. In an escape region, finite-time comparison estimates apply while the trajectory remains in a prescribed compact collision-free set.

No theorem is asserted that glues these regimes into a single global mixed-phase-space shadowing theorem. Inter-region transition control remains open.

8 Newtonian comparison and cohomological obstruction

Theorem 8.1 (Perturbative Hamiltonian bound)**.** *Let $\phi : A_{\rho,R} \to D$ be symplectic with compact image and*

$$\|\phi - \mathrm{id}\|_{C^1} \leq \varepsilon.$$

Then

$$\|H_N \circ \phi - H_E\|_{C^0} \leq B_0(\rho, R) + \varepsilon K_N,$$

where

$$K_N = \sup_{\phi(A_{\rho,R})} \|\nabla H_N\|.$$

Proposition 8.2 (Obstruction to the standard Moser path)**.** *Let*

$$H_t = (1-t)H_E + tH_N.$$

A necessary condition for a smooth single-valued solution of the cohomological equation

$$X_{H_t} G_t = -(H_N - H_E)$$

on a domain containing a closed orbit γ is

$$\oint_\gamma (H_N - H_E)\, dt = 0.$$

By (3)*, this condition fails for every nontrivial closed orbit in D.*

Proof. Integrating $X_{H_t}G_t = dG_t/dt$ over a closed orbit gives zero. The right-hand side has strictly positive integral after the minus sign because $H_N - H_E < 0$ pointwise. Hence the equation cannot admit a single-valued solution on a domain containing that closed orbit. □

Thus the standard Moser path cannot yield a global exact symplectomorphism satisfying

$$H_N \circ T = H_E$$

on such a domain. This is an obstruction to that construction, not a proof that every possible symplectic conjugacy is impossible.

8.1 Dimensionless finite-time comparison

Choose scales $L_0, M_0, T_0 > 0$ and define

$$\widetilde{q} = q/L_0, \qquad \widetilde{p} = p/(M_0 L_0/T_0), \qquad \widetilde{t} = t/T_0.$$

On a compact dimensionless collision-free set $\widetilde{A}$, define

$$\delta_1 = \sup_{\widetilde{A}} \left\| \widetilde{X}_N - \widetilde{X}_E \right\|, \qquad \Lambda = \sup_{\widetilde{A}} \left\| D\widetilde{X}_N \right\|.$$

Theorem 8.3 (Dimensionless Gronwall bound)**.** *For solutions with the same initial point that remain in $\widetilde{A}$,*

$$\left\| \widetilde{z}_N(\widetilde{t}) - \widetilde{z}_E(\widetilde{t}) \right\| \le \frac{\delta_1}{\Lambda} \left(e^{\Lambda \widetilde{t}} - 1 \right),$$

with the limiting value $\delta_1 \widetilde{t}$ if $\Lambda = 0$.

Proof. Let

$$f(\widetilde{t}) = \| \widetilde{z}_N - \widetilde{z}_E \|.$$

Then

$$\begin{aligned} f' &\le \left\| \widetilde{X}_N(\widetilde{z}_N) - \widetilde{X}_N(\widetilde{z}_E) \right\| + \left\| \widetilde{X}_N(\widetilde{z}_E) - \widetilde{X}_E(\widetilde{z}_E) \right\| \\ &\le \Lambda f + \delta_1. \end{aligned}$$

Gronwall's inequality gives the estimate. □

9 Binary-collision regularization

For a binary collision introduce

$$r = s^2, \qquad dt = s^2\, d\tau.$$

On a fixed Newtonian energy E, use the extended Hamiltonian

$$K_N = s^2(H_N - E).$$

The singular Newtonian contribution

$$-\frac{a}{r} = -\frac{a}{s^2}$$

becomes the finite term $-a$ in K_N.

Proposition 9.1 (Regularized boundedness)**.** *In the binary regularized chart, the singular Newtonian pair contribution to K_N extends continuously across $s = 0$. The remaining terms are bounded on compact chart domains away from simultaneous higher-order collisions.*

This boundedness result is weaker than a full canonical collision theorem. A complete C^1 symplectic extension would require a canonical momentum transformation of Levi–Civita or equivalent type, verification of the regularized symplectic form, non-degeneracy at the collision stratum, and compatible overlap maps. These steps remain open, as does triple collision.

10 Conditional transfer of invariant structures to Newtonian dynamics

Corollary 10.1 (Conditional KAM transfer)**.** *Let $\mathcal{T}_E$ be a non-degenerate Diophantine KAM torus of H_E contained in $A_{\rho,R}$, and let r be the differentiability order required by the selected KAM theorem. If*

$$B_r(\rho, R) < \varepsilon_{\mathrm{KAM}}(\mathcal{T}_E),$$

then H_N possesses a nearby invariant KAM torus.

Corollary 10.2 (Conditional hyperbolic transfer)**.** *Let Λ_E be a compact locally maximal hyperbolic invariant set of H_E contained in $A_{\rho,R}$. If*

$$D_2(\rho, R) = \|X_{H_N} - X_{H_E}\|_{C^1(A_{\rho,R})} < \varepsilon_{\mathrm{hyp}}(\Lambda_E),$$

then structural stability yields a nearby hyperbolic invariant set for H_N conjugate to Λ_E.

Direct non-integrability of H_N is not inferred from these comparison results.

11 Energetic Effective Cycle and numerical illustrations

Let S_Δ be a symplectic integrator and let Π_E be an explicitly specified energy-anchoring operation. Define

$$\Psi_\Delta = \Pi_E \circ S_\Delta.$$

Theorem 11.1 (Energy-anchored error)**.** *If Π_E is an exact projection onto $\{H_E = E_0\}$, then*

$$H_E(\Psi_\Delta^m(x_0)) = E_0$$

after every projection. If the numerical projection instead has uniform local energy residual $O(\Delta^{k+1})$ on the relevant compact set, then for fixed $T = m\Delta$ the accumulated energy residual is $O(T\Delta^k)$.

Energy control alone does not imply trajectory control in a chaotic region.

The numerical values below are retained only as illustrative data from the source study.

Table 1: Illustrative seven-day EEC comparison.

Δt	Steps	$\lvert\Delta E/E\rvert$ (EEC)	$\lvert\Delta E/E\rvert$ (RK4)
1 h	168	$\sim 10^{-13}$	$\sim 10^{-12}$
6 h	28	$\sim 10^{-10}$	$\sim 10^{-6}$
12 h	14	$\sim 10^{-9}$	$\sim 10^{-3}$
24 h	7	$\sim 10^{-13}$	$> 10^{-1}$

Table 2: Illustrative benchmark values.

System	Method	Reported performance
Globular ($N = 10^5$)	KS	$\lvert\Delta E/E\rvert < 10^{-10}$; $1\times$
Globular ($N = 10^5$)	Hybrid	$\lvert\Delta E/E\rvert < 10^{-9}$; $8\times$
Core collapse	KS	$\lvert\Delta E/E\rvert < 10^{-10}$; $1\times$
Core collapse	Hybrid	$\lvert\Delta E/E\rvert < 10^{-9}$; $12\times$
Planetary triple	WHFast	$\sim 10^{-9}$; $5.3\times$
Planetary triple	Hybrid	$\sim 10^{-11}$; $7\times$

12 Status of principal results

Table 3: Illustrative weak-scaling data.

Cores	N	Time (h)	Efficiency
64	6.4×10^5	0.8	100%
256	2.6×10^6	0.9	97%
1024	1.0×10^7	1.0	91%
4096	4.1×10^7	1.2	83%

Result	Status
Local well-posedness on D	Proved
Energy-barrier criterion $V_\partial(\rho) > E_0$	Proved
Global existence on bounded invariant collision-free component	Proved
C^0 and higher-derivative comparison bounds	Proved
Pair Hessian formula and reduced stability criterion	Proved
KAM tori in EETBM	Conditional on standard KAM hypotheses
Arbitrary-N fixed-sector collinear potential is quadratic	Proved
Positive connected spring graph gives $K_N > 0$	Proved
Spectral lower bound for collinear stiffness	Proved
Arbitrary-N fixed-sector collinear dynamics is exactly linear	Proved
Nontrivial collision-free fixed-sector collinear homoclinic orbit	Ruled out
Former collinear-homoclinic NVE-to-Lamé route	Not applicable
Generic EETBM non-integrability	Open
Melnikov horseshoe elsewhere in phase space	Conditional on verified orbit and simple zero
Cohomological obstruction to standard Moser path	Proved
Exact global conjugacy by other methods	Open
Dimensionless finite-time Gronwall comparison	Proved on compact set
Binary regularized boundedness	Proved
Full C^1 binary-collision symplectic extension	Open
KAM transfer to H_N	Conditional on C^r smallness
Hyperbolic transfer to H_N	Conditional on C^1 vector-field smallness
Direct non-integrability of H_N	Open
Region-wise shadowing	Standard/conditional by region
Inter-region shadowing	Open
Triple collision	Open
EEC energy control	Conditional on projection implementation
Numerical tables	Illustrative

13 Open problems

The most important unresolved mathematical directions are now sharply localized.

First, a valid non-integrability proof for H_E requires a different explicit non-equilibrium solution than the nonexistent collinear homoclinic orbit. A promising route is to derive the NVE along a non-collinear periodic or rotating solution and determine its differential Galois group.

Second, exact global symplectic equivalence between H_E and H_N remains open beyond the proved obstruction to the standard Moser path. Symplectic volume and action invariants should be tested before attempting a construction.

Third, direct Newtonian non-integrability must be established from the Newtonian equations themselves rather than transferred from EETBM.

Fourth, a full binary-collision theorem requires completion of the canonical regularization and chart-overlap analysis, while triple collision requires a separate multi-scale blow-up.

Finally, global shadowing across KAM, hyperbolic, and escape regions requires explicit transition maps and uniform control of accumulated errors.

14 Conclusion

The EETBM provides a mathematically well-defined Hamiltonian model on the collision-free domain, with rigorous energy-barrier and comparison estimates and a clear perturbative relationship to Newtonian dynamics on compact sets. The pair Hessian and KAM framework give controlled local stability statements, while the Moser-path obstruction shows that exact Hamiltonian matching is not obtained by the standard cohomological deformation.

The detailed collinear analysis produces a structural theorem for arbitrary N. In every fixed collision-free ordering sector, all pair distances are affine functions of the gap variables and the reduced EETBM potential is exactly quadratic. For a connected positive spring graph its Hessian is positive definite, so the reduced collinear dynamics is an exactly linear oscillator network and admits no nontrivial homoclinic orbit. The former homoclinic-to-Lamé non-integrability route is consequently unavailable for the Hamiltonian as stated. This does not establish integrability; it identifies the correct status of the problem and narrows the search for a valid Morales–Ramis obstruction to other particular solutions.

The remaining claims are separated into proved, conditional, and open statements. This separation yields a more defensible platform for further work than a stronger but unsupported global equivalence or non-integrability claim.

References

[1] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier–Villars, 1892.

[2] D. G. Saari, *Collisions, Rings, and Other Newtonian N-Body Problems*, AMS, 2005.

[3] J. J. Morales-Ruiz and J.-P. Ramis, Methods Appl. Anal. 8, 33 (2001).

[4] K. F. Sundman, Acta Math. 36, 105 (1913).

[5] P. Kustaanheimo and E. Stiefel, J. Reine Angew. Math. 218, 204 (1965).

[6] R. McGehee, Invent. Math. 27, 191 (1974).

[7] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer, 1989.

[8] J. Moser, Trans. Amer. Math. Soc. 120, 286 (1965).

[9] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd ed., Springer, 2003.

[10] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.

[11] J. Kovacic, J. Symbolic Comput. 2, 3 (1986).

[12] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration*, 2nd ed., Springer, 2006.

[13] M. F. Singer, *Galois Theory of Linear Differential Equations*, Springer, 2003.