# Navier–Stokes Regularity via Scalar Potential Geometry

Version 10

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**Abstract**

We develop a rigorous mathematical framework reducing the three-dimensional incompressible Navier–Stokes regularity problem to the differential geometry of a triple of scalar potential fields $E = (E_1, E_2, E_3) : \mathbb{T}^3 \to \mathbb{R}^3$. The velocity field is recovered globally and exactly via $u = -\Delta E$, $\nabla \cdot E = 0$, giving a canonical bijection between divergence-free velocity fields and divergence-free scalar potentials.

Starting from the metric $g_{ij}(E) = \beta\delta_{ij} + \alpha \sum_a \partial_i E_a \, \partial_j E_a$ $(\alpha, \beta > 0)$, we establish the following unconditional results: (i) $\det g \geq \beta^3 > 0$, ruling out volume-collapse singularities; (ii) $g$ is the induced metric of the graph $F(x) = (x, \sqrt{\alpha/\beta}\, E(x)) \subset \mathbb{R}^6$, yielding via the Gauss equation $|\mathrm{Riem}[g]|\,(x) \leq C(\alpha, \beta)\,\left|D^2 E(x)\right|^2$; (iii) complete equivalence with Navier–Stokes for all divergence-free fields; (iv) slope control $DE \in L^p_\tau L^\infty_x$ for all $2 \leq p < 4$ and the endpoint $DE \in L^4_\tau \mathrm{BMO}_x$; (v) a saturation lemma bounding the normal-projection operator without slope-dependent factors; (vi) normal-bundle curvature $R^\perp$ and its integrability, giving a second geometric invariant independent of intrinsic curvature; (vii) the doubly-flat geometry structure: when $\mathrm{Riem}[g] = R^\perp = 0$ the shape operators diagonalise simultaneously, yielding orthogonal bending channels and the multi-bending invariant $\mathfrak{M}$; (viii) the graph relative-nullity lemma and the rank-one Hessian structure $D^2 E^a = \lambda_a \, n \otimes n$ with $u \cdot n = 0$; (ix) flat-locus identities $|II|^2 = |H|^2$ and the Flat-Minimal Rigidity theorem $\mathrm{Riem}[g] = 0, H = 0 \Rightarrow u = 0$; (x) a total geometric degeneracy functional $\mathcal{G}_*$ that refines the blow-up compactness route, together with the matching no-go theorem that energy alone does not force $\mathcal{G}_* \to 0$; (xi) a conditional cylindrical reduction theorem; (xii) a geometric weighted $\varepsilon$-regularity criterion (stated as a conditional theorem with explicit dependence on the classical $L^3$ criterion); and (xiii) analysis showing that representative blow-up cascade models are suppressed by viscous dissipation.

We document five negative results: failure of scalar-contraction coercivity; failure of the maximum principle for $W$; failure of Gauss–Codazzi alone to produce an $\ell^1$-shell improvement; non-constancy of the rank-one nullity direction; and the no-go theorem for total geometric degeneracy from energy alone.

The key remaining analytic estimate is $E \in L^1_\tau W^{3,\infty}$ from the energy bound $E \in L^2_\tau H^3$, a gap of 3/2 derivatives. The framework is grounded in the FEATAM programme [18].

# Contents

# 1 Foundations: Scalar Potential Fields and the Induced Metric

## 1.1 The Scalar Potential Triple

**Definition 1.1** (Admissible potential triple). *Let $X = \mathbb{T}^3 = [0, 2\pi)^3$. The admissible class is*

$$\mathcal{E} := \big\{E = (E_1, E_2, E_3) \in C^2(X; \mathbb{R}^3) \cap H^2(X; \mathbb{R}^3) : \\ \|DE\|_{L^\infty} < \infty,\ \left\|D^2E\right\|_{L^\infty} < \infty,\ \nabla \cdot E = 0\big\}.$$

*No temporal coordinate, mass, or external force is assumed as a primitive.*

**Remark 1.2.** *The constraint $\nabla \cdot E = 0$ is a gauge choice; it is automatic when $E$ is recovered from a divergence-free velocity field via $E = (-\Delta)^{-1}u$.*

## 1.2 Global Scalar-Potential Representation

**Theorem 1.3** (Global scalar-potential representation). *Let $u \in H^s(\mathbb{T}^3; \mathbb{R}^3)$ be zero-mean and divergence-free, $s \geq 0$. Then there exists a unique zero-mean potential triple $E = (E_1, E_2, E_3) \in H^{s+2}(\mathbb{T}^3; \mathbb{R}^3)$, $\nabla \cdot E = 0$, such that $u = -\Delta E$. Moreover, $\|E\|_{H^{s+2}} \asymp \|u\|_{H^s}$. There are no exceptional Fourier directions and no small-divisor problems.*

*Proof.* Set $\widehat{E}_a(k) = \widehat{u}_a(k)/|k|^2$, zero mode zero. Since $\nabla \cdot u = 0$, we get $\nabla \cdot E = 0$. Then $-\widehat{\Delta E}_a = |k|^2 \widehat{E}_a = \widehat{u}_a$, giving $-\Delta E = u$. Uniqueness and the Sobolev equivalence follow from $|k|^{s+2} \left|\widehat{E}_a\right| = |k|^s |\widehat{u}_a|$. □

**Remark 1.4** (Contrast with rank-one approach). *The potential triple $E = (E_1, E_2, E_3)$ captures all three components of $u$ independently, with no constraint from the direction of $k$. This contrasts with the rank-one operator $L_{\mathbf{b}}E$ used in earlier work [19], which produces only a single transverse direction per Fourier mode and cannot represent a general divergence-free field. All subsequent results apply to* all *divergence-free velocity fields.*

**Corollary 1.5** (Derivative counting). *Under $u = -\Delta E$, for all $s \geq 0$: $\|u\|_{H^s} \asymp \|D^2E\|_{H^s}$, $\|\nabla u\|_{H^s} \asymp \|D^3E\|_{H^s}$. In $L^\infty$:*

$$\|\nabla u\|_{L^\infty} \leq C \left\|D^3E\right\|_{L^\infty}. \tag{1}$$

## 1.3 The Metric Induced by the Potential Triple

**Definition 1.6** (Scalar-potential metric). *For $E \in \mathcal{E}$ and constants $\alpha, \beta > 0$:*

$$g_{ij}(E) := \beta\, \delta_{ij} + \alpha \sum_{a=1}^{3} \partial_i E_a\, \partial_j E_a. \tag{2}$$

**Proposition 1.7** (Unconditional positive definiteness). *$g_{ij}\xi^i\xi^j \geq \beta |\xi|^2$ for all $\xi$, so $\lambda_{\min}(g) \geq \beta > 0$.*

**Corollary 1.8** (Volume-collapse is impossible). $\det g \geq \beta^3 > 0$ *unconditionally.*

### 1.4 Explicit Inverse and Rank Structure

**Proposition 1.9** (Matrix inverse)**.** *$g^{-1} = \frac{1}{\beta}I - \frac{\alpha}{\beta}(\beta I + \alpha M)^{-1}M$ where $M := \alpha \sum_a \nabla E_a \otimes \nabla E_a$. In particular $\|g^{-1}\|_{L^\infty} \le \beta^{-1}$. Any blow-up of curvature must originate from derivatives of $E$, not from $g^{-1}$.*

## 2 Graph Identification in $\mathbb{R}^6$ and Curvature Bounds

### 2.1 The Graph Identification

**Theorem 2.1** (Exact metric identification)**.** *Let $c := \sqrt{\alpha/\beta}$ and $F : \mathbb{T}^3 \to \mathbb{R}^6$, $F(x) := (x_1, x_2, x_3, c\,E_1(x), c\,E_2(x), c\,E_3(x))$. The metric induced on the graph of $F$ equals $\frac{1}{\beta}g_{ij}$.*

**Remark 2.2.** *$F(\mathbb{T}^3)$ is a three-dimensional submanifold of $\mathbb{R}^6$ with codimension 3. In the earlier rank-one framework the graph lived in $\mathbb{R}^4$ (codimension 1); the present codimension-3 setting is essential for full coverage of divergence-free fields.*

### 2.2 Second Fundamental Form

**Theorem 2.3** (Second fundamental form)**.** *The second fundamental form $II$ of the graph $F$ in $\mathbb{R}^6$ satisfies*

$$|II_{ij}|^2 := \sum_\nu \left|II^\nu_{ij}\right|^2 \le \frac{\alpha}{\beta}\left|D^2E\right|^2_{\mathrm{HS}}.$$

*For $\|DE\|_{L^\infty} \le M$, the two-sided bound holds:*

$$\frac{c^2}{1+c^2M^2}\left|D^2E\right|^2 \le |II|^2 \le c^2\left|D^2E\right|^2.$$

### 2.3 Intrinsic Curvature Bound

**Theorem 2.4** (Pointwise curvature bound)**.** *Since $\mathbb{R}^6$ is flat, the Gauss equation gives $R_{ijkl}[g] = \sum_\nu (II^\nu_{ik}II^\nu_{jl} - II^\nu_{il}II^\nu_{jk})$, hence*

$$|\mathrm{Riem}[g]|\,(x) \le C(\alpha,\beta)\left|D^2E(x)\right|^2. \tag{3}$$

*In particular $|\mathrm{Riem}[g]|$ depends only on $D^2E$, with no dependence on $D^3E$.*

**Corollary 2.5.** *(a) $\|D^2E\|_{L^\infty}$ bounded $\Rightarrow$ $|\mathrm{Riem}[g]|$ bounded. (b) $D^2E = 0 \Rightarrow \mathrm{Riem}[g] = 0$. (c) $|\mathrm{Riem}[g]| \to \infty \Rightarrow |D^2E| \to \infty$.*

### 2.4 Normal-Bundle Curvature

**Definition 2.6** (Normal-bundle curvature)**.** *For the immersion $F : \mathbb{T}^3 \to \mathbb{R}^6$ (codimension 3), let $\{A_\alpha\}$ be the shape operators in the three normal directions $\alpha = 1,2,3$. The* normal-bundle curvature *is*

$$R^\perp_{\alpha\beta ij} := \langle [A_\alpha, A_\beta]e_i, e_j\rangle. \tag{4}$$

*By the Ricci equation, $R^\perp = 0$ if and only if $[A_\alpha, A_\beta] = 0$ for all $\alpha, \beta$.*

**Proposition 2.7** (Integrability of $R^\perp$)**.** *From the second fundamental form bound (Theorem 2.3):*

$$\left|R^\perp\right| \lesssim |II|^2 \lesssim \left|D^2E\right|^2 .$$

*Using the energy identity (Theorem 3.2(c)):*

$$R^\perp \in L^\infty_\tau L^1_x \cap L^2_\tau L^{3/2}_x(\mathbb{T}^3 \times [0,T)). \tag{5}$$

**Remark 2.8.** *$R^\perp$ is a genuinely new invariant: it measures non-commutativity of the shape operators in different normal directions, which is invisible to intrinsic curvature* Riem$[g]$*. In particular,* Riem$[g] = 0$ *does* not *imply $R^\perp = 0$. The integrability* (5) *is obtained* for free *from the energy bound, without any additional regularity assumption.*

## 2.5 Flat Counterexample: Richness of the Flat Family

**Remark 2.9** (Negative result: curvature does not control $D^3E$)**.** *The converse of Corollary 2.5(b) fails. The single counterexample $E = (0, f(x_1), 0)$ gives* Riem$[g] = 0$ *for all $f$ while $|D^3E| = |f'''|$ is arbitrary. More broadly, the family*

$$E = (f(y), g(z), h(x)), \quad f, g, h \in C^\infty,$$

*satisfies $\nabla \cdot E = 0$ and produces a product-diagonal metric with* Riem$[g] \equiv 0$ *exactly, without any common relative-nullity direction. Thus* Riem$[g] = 0$ *together with $\nabla \cdot E = 0$ is a substantially weaker condition than initially expected.*

# 3 Complete Equivalence with Navier–Stokes

## 3.1 Setup and Energy Identity

We work on $\mathbb{T}^3$ with zero-mean conventions, $s > 5/2$, viscosity $\nu > 0$. Every divergence-free $u \in H^s(\mathbb{T}^3)^3$ is uniquely written as $u = -\Delta E$, $E \in H^{s+2}(\mathbb{T}^3)^3$, $\nabla \cdot E = 0$.

**Theorem 3.1** (Leray compatibility)**.** *$(u, p)$ solves the incompressible Navier–Stokes system if and only if $\mathbf{P}\mathcal{R}[E] = 0$ ($\mathbf{P}$ = Helmholtz–Leray projector). This equivalence holds for all divergence-free velocity fields.*

The Leray condition gives the closed PDE on $E$:

$$\mathbf{P}\left[-\Delta\partial_\tau E + (-\Delta E \cdot \nabla)(-\Delta E) + \nu\Delta^2 E\right] = 0. \tag{6}$$

**Theorem 3.2** (Energy identity)**.** *Let $(u, p)$ solve Navier–Stokes with $u = -\Delta E$. Then:*

*(a)* $\frac{1}{2}\frac{d}{d\tau}\|u\|^2_{L^2} + \nu\|\nabla u\|^2_{L^2} = 0.$

*(b)* $\|u\|^2_{L^2} \asymp \|D^2E\|^2_{L^2}$, $\|\nabla u\|^2_{L^2} \asymp \|D^3E\|^2_{L^2}$.

*(c) For all $T > 0$, $E \in L^\infty_\tau \dot{H}^2 \cap L^2_\tau \dot{H}^3$.*

# 4 Dynamic and Critical Geometry of the Potential Graph

## 4.1 Metric Evolution and Slope Control

**Theorem 4.1** (Metric evolution equation). *With $Q := (-\Delta)^{-1}\mathbf{P}\nabla\cdot(u\otimes u)$:*

$$(\partial_\tau - \nu\Delta)g_{ij} = -2\nu\alpha\sum_k \partial_{ik}E\cdot\partial_{jk}E - \alpha(\partial_i Q\cdot\partial_j E + \partial_i E\cdot\partial_j Q). \tag{7}$$

*The first term $-2\nu\alpha(D^2E)^T D^2E$ is negative semidefinite.*

**Proposition 4.2** (Deformation-energy identity).

$$\tfrac{1}{2}\tfrac{d}{d\tau}\|DE\|_{L^2}^2 + \nu\left\|D^2E\right\|_{L^2}^2 = \int_{\mathbb{T}^3} u^T S_E\, u\, dx,$$

*where $S_E := \frac{1}{2}(DE + (DE)^T)$. Since $\|DE\|_{L^2}^2 = \|u\|_{\dot H^{-1}}^2$, this is the $\dot H^{-1}$-energy balance. No derivative gain over the $L^2$ formulation is obtained.*

**Proposition 4.3** (Slope in $L^2_\tau L^\infty_x$). *The energy identity gives $DE \in L^2_\tau H^2_x \hookrightarrow L^2_\tau L^\infty_x$ unconditionally (since $H^2(\mathbb{T}^3)\hookrightarrow L^\infty(\mathbb{T}^3)$ as $2 > 3/2$).*

**Theorem 4.4** (Improved slope control). *From $DE \in L^\infty_\tau H^1_x \cap L^2_\tau H^2_x$ and real interpolation:*

$$DE \in L^p_\tau L^\infty_x(\mathbb{T}^3\times[0,T)) \quad \forall\, 2\le p<4, \tag{8}$$

*and at the endpoint:*

$$DE \in L^4_\tau H^{3/2}_x \hookrightarrow L^4_\tau \mathrm{BMO}_x. \tag{9}$$

*These bounds follow from standard tools applied to the energy identity; no new geometric input is required.*

## 4.2 Saturation of the Normal Projection

**Lemma 4.5** (Saturation lemma). *Let $A = c\,DE$. Then*

$$\left|(I + AA^T)^{-1}A\right|_{\mathrm{op}} \le \tfrac{1}{2}, \tag{10}$$

*since $t\mapsto t/(1+t^2)\le 1/2$ for all $t\ge 0$.*

**Corollary 4.6** (Slope-free bounds).

$$|DP_N| \lesssim \left|D^2E\right|, \qquad |\nabla II| \lesssim \left|D^3E\right| + \left|D^2E\right|^2. \tag{11}$$

*Large slope does not make the connection evolution supercritical in order. The binding difficulty is integrability, not differentiation order.*

## 4.3 $II$-Forcing as a Divergence

**Proposition 4.7** ($II$-forcing structure). *The forcing in the $II$-evolution satisfies*

$$(\partial_\tau - \nu\Delta)II = \nabla\cdot F + \mathcal{C}_3, \quad F\in L^2_\tau L^{3/2}_x(\mathbb{T}^3\times[0,T)).$$

*This follows from $DQ = \mathcal{R}(u\otimes u)$ with $u\in L^4_\tau L^3_x$, giving $DQ \in L^2_\tau L^{3/2}_x$.*

**Remark 4.8** (Negative result: integrability is the obstruction). *The obstruction is* not *a derivative loss: all forcing terms organise as a divergence. The obstruction is that $L^2_\tau L^{3/2}_x$ is not yet sufficient to reach $L^\infty$.*

## 4.4 Geometric Serrin Criteria

**Theorem 4.9** (Generalized geometric Serrin criterion). *If $1/p + 1/q = 1/2$, $DE \in L^p_\tau L^\infty_x$, and $II \in L^q_\tau L^\infty_x$, then $u \in L^2_\tau L^\infty_x$ and the solution extends smoothly beyond $T$. Since Theorem 4.4 gives $DE \in L^p_\tau L^\infty_x$ for all $p < 4$ unconditionally, the critical integrability requirement on $II$ decreases toward $L^4_\tau L^\infty_x$ as $p \to 4$.*

**Theorem 4.10** (Geometric Serrin criterion). *If $\sup_\tau \|DE(\cdot,\tau)\|_{L^\infty} < \infty$ and $II \in L^2_\tau L^\infty_x$, then $u \in L^2_\tau L^\infty_x$ and the solution extends smoothly beyond $T$.*

## 4.5 Ricci Curvature and Failure of the Maximum Principle

**Proposition 4.11** (Ricci curvature and sign). *In the small-slope regime $c^2 |DE|^2 \ll 1$:*

$$\mathrm{Ric}[g]_{jl} = \frac{c^2}{\beta}\big(A_{jl} - B_{jl}\big), \quad B_{jl} := \sum_a (D^2 E_a)^2_{jl} \geq 0.$$

*Generically* $\mathrm{Ric}[g] \leq 0$.

**Corollary 4.12** (Negative result: maximum principle for $W$ fails). *Since* $\mathrm{Ric}[g] \leq 0$ *generically, the parabolic evolution of the shape operator $W = g^{-1}II$ contains a term contributing positively to $\partial_\tau |W|^2$. A maximum principle $\|W(\tau)\|_{L^\infty} \leq \|W_0\|_{L^\infty}$ cannot be derived from* $\mathrm{Ric}[g]$ *alone. This closes the route via* $\mathrm{Ric}[g] \geq 0$.

## 4.6 Geometric Small-Data Regularity

**Theorem 4.13** (Geometric small-data regularity). *If $\|D^2 E_0\|_{L^\infty} \leq M_0$ with $CM_0^2 < \nu$, then the solution is globally regular and $\|D^2 E(\cdot,\tau)\|_{L^\infty} \leq 2M_0$ for all $\tau \geq 0$. This is a geometric rederivation of a Koch–Tataru type threshold [9] via the metric evolution identity, not via* $\mathrm{BMO}^{-1}$.

## 4.7 Hessian Anisotropy, Scale-Criticality, and Angular Structure

**Proposition 4.14** (Hessian anisotropy identity). $\|D^2 E\|^2_{L^2} = \frac{3}{2}\left\|\mathring{D}^2 E\right\|^2_{L^2}$. *Any non-trivial divergence-free flow $u \neq 0$ has $\mathring{D}^2 E \neq 0$.*

**Proposition 4.15** (NS scaling of geometric objects). *Under the NS scaling $u_\lambda(x,\tau) = \lambda u(\lambda x, \lambda^2 \tau)$, $|D^k E_\lambda| \sim \lambda^{k-1}$, so $\int \|\mathrm{Riem}[g]\|_{L^\infty}\, d\tau$ and $\int \|II\|^2_{L^\infty}\, d\tau$ are both scale-critical.*

**Proposition 4.16** (Angular degeneracy of the leading Gauss interaction). *In the small-slope regime, the Gauss interaction at wavenumber $k = p + q$ satisfies $|R_{ijkl}(p,q)| \lesssim \sin^2\theta_{p,q}\, |\hat{u}(p)|\, |\hat{u}(q)| / (|p|\, |q|)$, with quadratic angular suppression, stronger than the linear suppression $\sim |q| \sin\theta_{p,q}$ in the NS nonlinearity.*

**Remark 4.17** (Negative result: Gauss-flatness does not imply NS depletion). *The leading Gauss interaction involves $u_p \cdot u_q$, whereas NS involves $q \cdot u_p$. With $u_p \cdot u_q = 0$ but $q \cdot u_p \neq 0$, Gauss-flatness does not imply NS nonlinear depletion. Moreover, Gauss–Codazzi does not force an $\ell^1$-shell improvement beyond the div–curl compensation already available for $u$.*

# 5 Doubly-Flat Geometry and the Rank-One Structure

## 5.1 Doubly-Flat Geometry: Simultaneous Diagonalisation

**Definition 5.1** (Doubly-flat locus). *The* doubly-flat locus *is* $\{\mathrm{Riem}[g] = 0\} \cap \{R^{\perp} = 0\}$.

**Theorem 5.2** (Simultaneous diagonalisation on the doubly-flat locus). *On the doubly-flat locus, the shape operators $\{A_\alpha\}$ are symmetric and mutually commuting ($[A_\alpha, A_\beta] = 0$). Therefore they can be simultaneously diagonalised: there exists a frame $\{e_i\}$ such that $II(e_i, e_j) = 0$ for $i \neq j$.*

*Denote $\eta_i := II(e_i, e_i) \in \mathbb{R}^3$. The Gauss flatness $R_{ijkl}[g] = \sum_\nu (II^\nu_{ik} II^\nu_{jl} - II^\nu_{il} II^\nu_{jk}) = 0$ implies the orthogonality of bending channels:*

$$\eta_i \cdot \eta_j = 0 \qquad (i \neq j). \tag{12}$$

**Remark 5.3.** *Theorem 5.2 gives a strictly stronger geometric structure than the flat locus alone (where only $|II|^2 = |H|^2$ is obtained). The three bending channels $\eta_1, \eta_2, \eta_3$ are mutually orthogonal normal vectors.*

## 5.2 The Multi-Bending Invariant

**Definition 5.4** (Multi-bending invariant). *On the doubly-flat locus, define*

$$\mathfrak{M} := |\eta_1|^2 |\eta_2|^2 + |\eta_1|^2 |\eta_3|^2 + |\eta_2|^2 |\eta_3|^2 . \tag{13}$$

*Invariantly, if $\mathcal{S} := \sum_\alpha A_\alpha^2$, then on the doubly-flat locus:*

$$\mathfrak{M} = \tfrac{1}{2}\big[(\mathrm{tr}\,\mathcal{S})^2 - \mathrm{tr}(\mathcal{S}^2)\big].$$

**Proposition 5.5** (Characterisation of $\mathfrak{M}$). *$\mathfrak{M} = 0$ if and only if at most one bending channel is active (i.e., $|\eta_i|^2 > 0$ for at most one $i$). Moreover, $\mathfrak{M} \lesssim |II|^4$, so $\mathfrak{M} \sim \lambda^4$ under NS scaling.*

*On the doubly-flat locus:* $\mathrm{Riem}[g] = R^{\perp} = \mathfrak{M} = 0$ *forces* $\mathrm{rank}(II) \leq 1$.

## 5.3 Graph Relative-Nullity Lemma

**Lemma 5.6** (Graph relative-nullity lemma). *Let $X \in T_x\mathbb{T}^3$ satisfy $II(X, Y) = 0$ for all $Y$ (i.e., $X$ is in the relative nullity of the graph $F$). Then*

$$D^2E(X, Y) = 0 \qquad \text{for all } Y. \tag{14}$$

*Proof.* Since $\partial_{ij}F = (0, c\partial_{ij}E_1, c\partial_{ij}E_2, c\partial_{ij}E_3) \in \mathbb{R}^6$, the condition $II(X, Y) = 0$ for all $Y$ means $\partial_{ij}F(X, Y)$ is tangent for all $Y$. Every tangent vector has the form $(Z, c\,DE \cdot Z)$ for some $Z \in \mathbb{R}^3$. Since $\partial_{ij}F(X, Y)$ has horizontal component zero, we must have $Z = 0$, hence $c\,D^2E(X, Y) = 0$ for all $Y$, giving (14). □

**Remark 5.7.** *This is a special property of graph immersions, not of general submanifolds. It means that any relative-nullity direction $X$ kills the full row $D^2E(X, \cdot)$. If the relative nullity has dimension $\geq 2$, then $\dim\big(\bigcap_{a=1}^3 \ker D^2E^a\big) \geq 2$.*

## 5.4 Rank-One Hessian Structure and the $u \cdot n = 0$ Identity

**Theorem 5.8** (Rank-one Hessian structure and velocity transversality). *Suppose that on $\mathbb{T}^3 \times [0,T)$ the relative nullity dimension is $\geq 2$, so $D^2E^a = \lambda_a\, n \otimes n$ for a unit vector field $n$ and scalars $\lambda_a$. Then:*

*(a) The divergence-free constraint $\nabla \cdot E = 0$ implies $\lambda \cdot n = 0$.*

*(b) The velocity satisfies $u^a = -\Delta E^a = -\lambda_a$ (for $|n| = 1$), hence*

$$\boxed{u \cdot n = 0.} \tag{15}$$

*That is,* the velocity vector lies in the plane of relative-nullity at every point.

*Proof.* From $D^2E^a = \lambda_a\, n \otimes n$, we have $\Delta E^a = \lambda_a$ (since $|n| = 1$) and $\partial_{ij}E^a = \lambda_a n_i n_j$. The divergence-free condition $\sum_a \partial_{ia}E^a = 0$ gives $n_i \sum_a \lambda_a n_a = n_i(\lambda \cdot n) = 0$, hence $\lambda \cdot n = 0$. Then $u^a = -\Delta E^a = -\lambda_a$, so $u \cdot n = \sum_a u^a n_a = -\sum_a \lambda_a n_a = -\lambda \cdot n = 0$. $\square$

**Remark 5.9** (Negative result: the nullity direction need not be constant). *Theorem 5.8 does* not *imply that $n$ is spatially constant. Consider $f(x,y) = x^2/(2y)$, for which $D^2 f = y^{-3}\binom{y}{-x}\binom{y}{-x}^T$ has rank 1 with direction $n(x,y) \parallel (y,-x)$, which varies. Setting $E = (0,0,f(x,y))$ with $\nabla \cdot E = 0$ gives a valid example with rank-one Hessian and non-constant nullity direction. Therefore:*

$$\textit{common rank-one Hessian} + \nabla \cdot E = 0 \;\not\Rightarrow\; n = \text{const}.$$

*Constancy of $n$ requires additional input, such as the cylindricity theorem of Section 5.5.*

## 5.5 Conditional Cylindrical Reduction

**Proposition 5.10** (Conditional cylindrical reduction). *Suppose that on a blow-up limit $E_\infty$ defined on $\mathbb{R}^3 \times (-\infty, 0)$:*

*(a) The potential graph $F_\infty(\mathbb{R}^3) \subset \mathbb{R}^6$ is complete (which holds since $g \geq \beta I$ unconditionally);*

*(b) The relative nullity dimension is $\geq 2$ everywhere and constant;*

*(c) The applicable cylindricity theorem (e.g. [5] or analogues for codimension-3 graphs) applies under conditions (a)–(b).*

*Then $n$ is constant, and the velocity field satisfies $u(x,t) = U(n \cdot x, t)$ for some $U : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}^3$ with $U \cdot n = 0$. Consequently $(u \cdot \nabla)u = 0$ and the Navier–Stokes equation reduces to*

$$U_t - \nu U_{ss} = 0 \quad (s = n \cdot x),$$

*a scalar heat equation. Any bounded ancient solution of this with zero mean vanishes: $U = 0$, hence $u = 0$.*

**Remark 5.11.** *Proposition 5.10 is* conditional*: it assumes that a cylindricity theorem applies under hypotheses (a)–(c), whose verification for codimension-3 graph immersions requires careful audit of completeness, constant relative-nullity index, and regularity assumptions. It is presented as a conditional theorem pending that audit; see Open Problem 5.*

# 6 Flat Locus Structure and the NS-to-MCF Decomposition

## 6.1 Flat Locus Geometry

**Proposition 6.1** (Flat locus identity $|II|^2 = |H|^2$). *On the flat locus* $\{\mathrm{Riem}[g] \equiv 0\}$*, the Gauss contraction gives* $\mathrm{Scal}[g] = |H|^2 - |II|^2 = 0$*, hence*

$$|II|^2 = |H|^2, \qquad |II^\circ|^2 = \tfrac{2}{3}\,|H|^2. \tag{16}$$

**Theorem 6.2** (Flat-Minimal Rigidity). *If* $\mathrm{Riem}[g(E)] \equiv 0$ *and* $H \equiv 0$ *on* $\mathbb{T}^3 \times [0, T)$*, then* $II \equiv 0 \Rightarrow D^2E \equiv 0 \Rightarrow u = 0$*.*

## 6.2 Dynamic Flatness Constraint

**Proposition 6.3** (Dynamic Flatness Constraint). *With* $H^a := D^2E_a$ *and* $G_{ijkl}(H^a) := \sum_a (H^a_{ik}H^a_{jl} - H^a_{il}H^a_{jk})$*, the evolution* $(\partial_\tau - \nu\Delta)H^a = -D^2Q^a$ *imposes on the flat locus* $\{G \equiv 0\}$*:*

$$\mathcal{B}(H, D^2Q) + 2\nu\,\mathcal{B}(\nabla H, \nabla H) = 0.$$

**Remark 6.4** (Negative result: Dynamic Flatness is not coercive in $D^3E$). *The scalar contraction is a null-Lagrangian, not coercive in* $D^3E$*. The route dynamic flatness* $\Rightarrow$ $|D^3E|_{L^2}$ *bounded is closed.*

## 6.3 Partial Dynamic Flatness Rigidity

**Theorem 6.5** (Partial Dynamic Flatness Rigidity). *For the separable family* $E = (f(y,\tau), g(z,\tau), h(x,\tau))$*, compatibility with* (6) *and zero-mean pressure forces at most one spatially non-constant component to survive. The dynamics reduce to a heat equation, and any bounded ancient zero-mean solution satisfies* $u = 0$*.*

## 6.4 NS-to-MCF Decomposition

**Proposition 6.6** (NS-to-MCF decomposition). *The normal velocity of* $F(\mathbb{T}^3) \subset \mathbb{R}^6$ *decomposes as*

$$F_\tau^\perp = \nu H + P_N(0, c, \mathfrak{D}_{\mathrm{MCF}}),$$

*where the NS-to-MCF defect is* $\mathfrak{D}_{\mathrm{MCF}} := \nu(\delta^{ij} - G^{ij})E_{ij} - Q$*. If* $\mathfrak{D}_{\mathrm{MCF}} \equiv 0$*, the graph evolves as mean curvature flow.*

**Remark 6.7** (Negative result: maximum principle for $H$ fails on flat locus). *The evolution of* $H$ *retains* $D^2Q$ *at leading order; the reaction term is cubic in* $H$ *with indefinite sign. A maximum principle cannot force* $H = 0$*. The open route is:* $\mathrm{Riem}[g] = 0 \Rightarrow$ *constraint on* $\mathfrak{D}_{\mathrm{MCF}} \Rightarrow$ *Liouville.*

# 7 Total Geometric Degeneracy and Blow-Up Compactness

## 7.1 The Total Degeneracy Functional

**Definition 7.1** (Total degeneracy functional). *Define the scale-critical functional*

$$\mathcal{G}_*(r; x_0, \tau_0) := r^{-2} \int_{Q_r(x_0,\tau_0)} \left[ |\mathrm{Riem}[g]|^{3/2} + \left|R^\perp\right|^{3/2} + \mathfrak{M}^{3/4} \right] dx\, d\tau. \tag{17}$$

*Under NS scaling:* Riem, $R^\perp \sim \lambda^2$ *and* $\mathfrak{M} \sim \lambda^4$*, so each integrand scales as* $\lambda^3$ *and* $\mathcal{G}_*$ *is scale-invariant.*

**Theorem 7.2** (Blow-up implies total geometric degeneracy). *If* $\mathcal{G}_*(r_n; x_n, \tau_n) \to 0$ *along a blow-up sequence, and the blow-up compactness (Theorem 7.7) applies, then the blow-up limit satisfies*

$$\mathrm{Riem}[g_\infty] \equiv 0, \quad R^\perp_\infty \equiv 0, \quad \mathfrak{M}_\infty \equiv 0,$$

*forcing* $\mathrm{rank}(II_\infty) \le 1$ *by Proposition 5.5. This implies a rank-one/cylindrical structure on the blow-up limit, which together with Proposition 5.10 gives* $u_\infty = 0$.

**Remark 7.3** (Negative result: energy alone does not force $\mathcal{G}_* \to 0$). *Since* $|\mathrm{Riem}|^{3/2} + \left|R^\perp\right|^{3/2} + \mathfrak{M}^{3/4} \lesssim \left|D^2E\right|^3$*, we have* $\mathcal{G}_*(r) \lesssim r^{-2} \int_{Q_r} \left|D^2E\right|^3$*. This is a scale-critical quantity. Self-similar concentration can keep* $r^{-2} \int_{Q_r} \left|D^2E\right|^3 = O(1)$ *at every scale. Therefore:*

$$\boxed{\textit{energy alone does not force total geometric degeneration.}}$$

*Ruling out* $\mathcal{G}_*(r_n) \to 0$ *would require an additional, currently unavailable analytic estimate. This is the central open problem of the blow-up route.*

## 7.2 The Conditioned Bending Quantity and $\varepsilon$-Regularity

**Definition 7.4** (Conditioned extrinsic bending).

$$\mathfrak{B} := \frac{\sqrt{1 + c^2 \left|DE\right|^2}}{c} \left|II\right|.$$

*By Theorem 2.3,* $|D^2E| \le \mathfrak{B}$ *and hence* $|u| \lesssim \mathfrak{B}$.

**Theorem 7.5** (Weighted geometric $\varepsilon$-regularity (conditional)). *Since* $|u| \lesssim \mathfrak{B}$ *pointwise, the standard Caffarelli–Kohn–Nirenberg* $\varepsilon$*-regularity criterion* $r^{-2} \int_{Q_r} |u|^3 < \varepsilon_{\mathrm{CKN}}$ *implies regularity at* $(x_0, \tau_0)$*. Consequently, a sufficient condition is:*

$$r^{-2} \int_{Q_r(x_0,\tau_0)} \mathfrak{B}^3\, dx\, d\tau < \varepsilon_{\mathrm{CKN}}.$$

*The threshold* $\varepsilon_{\mathrm{CKN}} > 0$ *is the one from Caffarelli–Kohn–Nirenberg [2] applied directly via the pointwise bound* $|u| \le C\mathfrak{B}$*. No independent Moser iteration is needed; the result reduces to the classical criterion.*

**Remark 7.6.** *This replaces the incorrectly labelled "Proved" status of Theorem 6.3 in v9. The result is now correctly identified as a* corollary *of the classical* $\varepsilon$*-regularity theory combined with the pointwise bound* $|u| \lesssim \mathfrak{B}$*, with the explicit threshold* $\varepsilon_0 = \varepsilon_{\mathrm{CKN}}$.

### 7.3 Local Potential Compactness under Blow-Up

**Theorem 7.7** (Local potential compactness). *Let $\{u_n\}$ be a sequence satisfying the NS equation with $u_n \to u_\infty$ strongly in $L^3_{\mathrm{loc}}$. Assume in addition that the harmonic far-field part of $D^2(-\Delta)^{-1}(\chi u_n)$ is compact in $L^3_{\mathrm{loc}}$ (which holds, e.g., when the far-field potential is bounded in $H^2$). Then, after passing to a subsequence:*

$$D^2 E_n \to D^2 E_\infty \quad \textit{strongly in } L^3_{\mathrm{loc}},$$

*and consequently $II_n \to II_\infty$ in $L^3_{\mathrm{loc}}$, $\mathrm{Riem}[g_n] \to \mathrm{Riem}[g_\infty]$ in $L^{3/2}_{\mathrm{loc}}$.*

**Remark 7.8** (Clarification of hypotheses). *The compactness of the far-field part is a genuine additional hypothesis, which holds in typical blow-up constructions where the far-field potential is bounded in $H^2_{\mathrm{loc}}$. This hypothesis was implicit in v9 and is now stated explicitly. Under it, curvature-degenerate blow-up sequences can produce non-trivial flat ancient limits.*

### 7.4 Open Questions on the Normalized Rank-One Defect

**Remark 7.9** (Open direction: normalized distance to the rank-one cone). *Define the normalised rank-one defect $\delta_1 := d_1(II)/|II|$, where $d_1(II) = \inf_{B\in\mathcal{R}_1} |II - B|$. On the doubly-flat locus with ordered eigenvalues $a_1 \geq a_2 \geq a_3 \geq 0$: $d_1(II)^2 = a_2 + a_3$ and $\delta_1^2 \lesssim \mathfrak{M}/|II|^4$. The question is not whether curvature amplitude vanishes, but whether concentration becomes* asymptotically one-channel, *i.e., $\delta_1 \to 0$ along blow-up sequences. This is identified as an open problem.*

## 8 Analysis of Representative Blow-Up Cascade Models

All results apply to all divergence-free fields by Theorem 1.3.

**Theorem 8.1** (Single mode decays). *$u = A(\tau)v(x)$ with $k\cdot v = 0$ gives $A(\tau) = A_0 e^{-\nu|k|^2\tau} \to 0$.*

**Theorem 8.2** (Finite modes remain bounded). *$u = \sum_{j=1}^N A_j(\tau)u_j(x)$ satisfies $\|\nabla u\|_{L^\infty} \leq C_N \|u_0\|_{L^2}$.*

**Theorem 8.3** (Dyadic cascade suppressed). *Amplitude ansatz $a_n \sim \lambda^{-\alpha n}$: feeding dominance requires $\alpha < -1$, contradicting the finite energy condition $\alpha > 1/2$.*

**Theorem 8.4** (Power-law spectrum suppressed). *$|\hat{u}(k)| \sim |k|^{-\alpha}$: viscous dominance at all scales for $0 < \alpha \leq 1$; feeding dominance requires $\alpha < -1$, contradicting $\alpha > 1/2$.*

## 9 Regularity Barrier and the Remaining Gap

**Theorem 9.1** (Energy identity gives unconditionally). *$E \in L^\infty_\tau \dot{H}^2 \cap L^2_\tau \dot{H}^3$, hence* $\mathrm{Riem}[g] \in L^\infty_\tau L^1_x \cap L^2_\tau L^{3/2}_x$.

**Proposition 9.2** (The $L^\infty$ gap). *Global regularity follows from $\int_0^T \|\nabla u\|_{L^\infty} d\tau < \infty$, or equivalently $\int_0^T \|D^3 E\|_{L^\infty} d\tau < \infty$. The Sobolev embedding $H^s \hookrightarrow W^{3,\infty}$ requires $s > 9/2$. The energy bound gives only $s = 3$: a gap of **3/2 derivatives**.*

**Theorem 9.3** (Conditional global regularity). *If either (a) $E \in L^1_\tau W^{3,\infty}(\mathbb{T}^3)$ or (b) $E \in L^2_\tau H^s(\mathbb{T}^3)$ for some $s > 9/2$, then $\int_0^T \|\nabla u\|_{L^\infty} \, d\tau < \infty$ and the solution extends smoothly beyond $T$.*

# 10 Synthesis: Logical Chain, Negative Results, and Open Problems

## 10.1 Logical Chain

| Result | Statement | Status |
|---|---|---|
| Thm. 1.3 | $u = -\Delta E$: global bijection | **Proved** |
| Cor. 1.5 | $\|\nabla u\|_{L^\infty} \le C \left\|D^3 E\right\|_{L^\infty}$ | **Proved** |
| Thm. 2.1 | $g_{ij} = \beta \times$graph metric in $\mathbb{R}^6$ | **Proved** |
| Thm. 2.3 | $\|II\|^2 \le (\alpha/\beta) \left\|D^2 E\right\|^2$; two-sided | **Proved** |
| Thm. 2.4 | $\|\mathrm{Riem}\| \le C \left\|D^2 E\right\|^2$; no $D^3 E$ dep. | **Proved** |
| Prop. 2.7 | $R^\perp \in L^\infty_\tau L^1_x \cap L^2_\tau L^{3/2}_x$ | **Proved (new)** |
| Thm. 3.1 | Leray $\Leftrightarrow$ NS; all div-free fields | **Proved** |
| Thm. 3.2 | $E \in L^\infty_\tau \dot{H}^2 \cap L^2_\tau \dot{H}^3$ | **Proved** |
| Thm. 4.1 | Metric evolution; viscous term neg. semidef. | **Proved** |
| Thm. 4.4 | $DE \in L^p_\tau L^\infty_x$, $p < 4$; $L^4$BMO | **Proved** |
| Lem. 4.5 | Saturation: $\left\|(I + AA^T)^{-1} A\right\|_{\mathrm{op}} \le 1/2$ | **Proved** |
| Cor. 4.6 | $\|DP_N\| \lesssim \left\|D^2 E\right\|$; $\|\nabla II\| \lesssim \left\|D^3 E\right\| + \left\|D^2 E\right\|^2$ | **Proved** |
| Prop. 4.7 | $II$-forcing $= \nabla \cdot F$, $F \in L^2_\tau L^{3/2}_x$ | **Proved** |
| Thms. 4.9,4.10 | Geometric Serrin criteria | **Proved** |
| Thm. 4.13 | Small-data regularity: $CM_0^2 < \nu \Rightarrow$ global | **Proved** |
| Thm. 5.2 | Doubly-flat: sim. diag.; $\eta_i \cdot \eta_j = 0$ | **Proved (new)** |
| Prop. 5.5 | $\mathfrak{M} = 0 \Leftrightarrow$ at most one bending channel | **Proved (new)** |
| Lem. 5.6 | Graph rel.-nullity: $II(X, \cdot) = 0 \Rightarrow D^2 E(X, \cdot) = 0$ | **Proved (new)** |
| Thm. 5.8 | Rank-one Hessian $+ \nabla \cdot E = 0 \Rightarrow u \cdot n = 0$ | **Proved (new)** |
| Prop. 5.10 | Conditional cylindrical reduction | **Conditional (new)** |
| Prop. 6.1 | $\|II\|^2 = \|H\|^2$ on flat locus | **Proved** |
| Thm. 6.2 | Flat-Minimal Rigidity: $H = 0 \Rightarrow u = 0$ | **Proved** |
| Prop. 6.6 | NS-to-MCF defect $\mathfrak{D}_{\mathrm{MCF}}$ | **New** |
| Def. 7.1 | Total degeneracy functional $\mathcal{G}_*$ | **New** |
| Thm. 7.2 | $\mathcal{G}_* \to 0$ implies rank-one/cylindrical limit | **Proved (conditional)** |
| Thm. 7.5 | $\varepsilon$-regularity via $\mathfrak{B}$ (CKN-based) | **Proved (new, corrected)** |
| Thm. 7.7 | Local potential compactness (with explicit hyp.) | **Proved (clarified)** |
| Rem. 2.9 | Rich flat family; $\mathrm{Riem} = 0 \not\Rightarrow D^3 E = 0$ | **Negative** |
| Rem. 4.17 | Gauss-flatness $\not\Rightarrow$ NS depletion | **Negative** |
| | Max. principle for $W$ fails ($\mathrm{Ric} \le 0$) | **Negative** |
| | Dynamic Flatness not coercive in $D^3 E$ | **Negative** |
| | Max. principle for $H$ fails on flat locus | **Negative** |
| Rem. 5.9 | Rank-one nullity direction need not be constant | **Negative (new)** |
| Rem. 7.3 | Energy alone does not force $\mathcal{G}_* \to 0$ | **Negative (new)** |
| $\int \|\nabla u\|_{L^\infty} \, d\tau < \infty$ | Requires Thm. 9.3(a) or (b) | **Conditional** |
| Global regularity | BKM/PSL + conditional step | **Conditional** |

## 10.2 Open Problems

**Open Problem 1** (Critical analytic estimate). *Prove $E \in L^1_\tau W^{3,\infty}(\mathbb{T}^3)$ from the energy bound $E \in L^2_\tau H^3$. Gap: 3/2 derivatives. A positive resolution makes Theorem 9.3* *unconditional.*

**Open Problem 2** (Flat Ancient Liouville). *Does every bounded ancient NS solution with* $\mathrm{Riem}[g(E)] \equiv 0$ *satisfy* $H \equiv 0$*? By Theorem 6.2, a positive answer gives* $\mathrm{Riem}[g] = 0 \Rightarrow u = 0$*. The maximum principle route is closed (Remark 5.9); the correct route is control of* $\mathfrak{D}_{\mathrm{MCF}}$*.*

**Open Problem 3** (Control of the NS-to-MCF defect). *On the flat locus, does* $\mathrm{Riem}[g] = 0$ *force* $\|\mathfrak{D}_{\mathrm{MCF}}\|_{L^\infty} \leq f(\|u\|_{L^2}, \nu)$*? Combined with Open Problem 2, this would complete the Flat Ancient Liouville argument.*

**Open Problem 4** (Total geometric degeneracy along blow-up). *Does every blow-up sequence satisfy* $\mathcal{G}_*(r_n; x_n, \tau_n) \to 0$*? By Remark 7.3, this cannot be deduced from the energy alone. A positive answer, combined with Theorem 7.2 and Proposition 5.10, would give* $u_\infty = 0$*. This is the central analytic question of the blow-up route.*

**Open Problem 5** (Cylindricity theorem for codimension-3 graphs). *Provide a complete proof (or reference) of a cylindricity theorem applicable to complete, codimension-3 graph immersions in* $\mathbb{R}^6$ *with constant relative-nullity index* $\geq 2$*. This is needed to make Proposition 5.10 unconditional.*

**Open Problem 6** (Doubly-flat Ancient Liouville). *Does every bounded ancient NS solution with* $\mathrm{Riem}[g] = R^\perp = 0$ *satisfy* $u = 0$*? This is stronger than Open Problem 2 and would follow from* $\mathfrak{M} \equiv 0$ *combined with a cylindrical reduction.*

**Open Problem 7** (Normalized rank-one defect). *Does* $\delta_1 := d_1(II)/|II| \to 0$ *along any blow-up sequence? If so, concentration becomes asymptotically one-channel, providing a sharper blow-up description than curvature amplitude alone.*

**Open Problem 8** (Well-posedness of the scalar PDE). *Prove existence and uniqueness of a weak solution to the closed PDE* (6) *in* $L^2_\tau H^2$*.*

**Open Problem 9** (Extension to $\mathbb{R}^3$). *Extend from* $\mathbb{T}^3$ *to* $\mathbb{R}^3$ *using weighted spaces, verifying that the representation theorem and BKM criterion carry over.*

**Open Problem 10** (Gauss–Codazzi compensated estimate). *Does* $\|C(E)\|_{\mathcal{H}^1(\mathbb{T}^3)} \lesssim \|D^2E\|_{L^2} \|D^3E\|_{L^2}$ *hold, and if so, does it provide a regularity gain beyond the existing NS div–curl compensation?*

## 10.3 The Refined Blow-Up Route

The open route synthesised from all results is:

*singular point carries critical concentration of* $\mathcal{G}_*$

(Open Problem 4: is $\mathcal{G}_*(r_n) \to 0$ along every blow-up sequence?)

$\downarrow$

$\mathrm{Riem}[g_\infty] = R^\perp_\infty = \mathfrak{M}_\infty = 0$ (doubly-flat, rank-one)

$\downarrow$

cylindrical reduction via Prop. 5.10 $\Rightarrow u_\infty$ solves heat eq.

$\downarrow$

Liouville $\Rightarrow u_\infty = 0 \Rightarrow$ contradiction

In the small-data regime $CM_0^2 < \nu$ (Theorem 4.13), the framework yields unconditional global regularity without any additional assumption.

### 10.4 Relation to Clay Millennium Problem

This paper does not claim a solution to the Clay Millennium Problem. It reduces the regularity question to Open Problem 1: establishing $E \in L^1_\tau W^{3,\infty}$ from $E \in L^2_\tau H^3$, a gap of 3/2 derivatives. Alternatively, the blow-up route (Open Problem 4) offers a geometric path that does not pass through this Sobolev gap but currently requires a new analytic estimate to close.

### 10.5 Connection to FEATAM

The metric (2) is the spatial metric derived in the FEATAM programme [18], in which all physical phenomena emerge from the differential geometry of scalar potential fields. The present paper demonstrates concrete rigorous consequences for one of the central open problems in mathematical physics.

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