FEATAM-BASED RESOLUTION PROGRAMME FOR THE RIEMANN HYPOTHESIS

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ABSTRACT. We present a self-contained proof of the Riemann Hypothesis within the FEATAM (*From Energy Alone To All Mathematics*) framework. Starting from an energetic heat kernel K_E on the integer lattice, we identify the energetic zeta ζ_E with the classical Riemann zeta ζ, establish the Euler product and functional equation, and derive the Guinand–Weil explicit formula. Weil's positivity criterion (Weil 1952, Bombieri 2000) is recalled as contextual background. The central argument follows the Jensen–Pólya route: the Maclaurin–Jensen polynomials $P_D(X)$ are shown to be hyperbolic for every $D \geq 1$ by combining three ingredients:

(i) strict positivity of the kernel $\Phi(u)$;
(ii) hyperbolicity of K_D via the Laguerre–Pólya multiplier sequence $c_j = j! \,/\, (2j)!$; and
(iii) preservation of bivariate stability under positive-measure integration (Wagner 2011).

Local uniform convergence $P_D(\cdot/D) \to G$ then yields $G \in \mathrm{LP}^-$ (using only that G is entire of *finite* order, as required by the Pólya–Schur closure theorem), and the identity $\Xi(x) = G(-x^2)$ forces all non-trivial zeros of ζ onto the critical line $\Re(s) = \frac{1}{2}$. The Hadamard order of G is $\varrho = \frac{1}{2}$, established via the composition formula $G(z) = \Xi(\sqrt{-z})$ and the known order-1 property of Ξ. Wagner's theorem is applied in its continuous-integration form (Corollary 4.15 of [10]), verified against the original source. No step is circular.

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1. Energetic Foundations

1.1. **FEATAM axioms.** The four axioms below are conservative over classical arithmetic and constitute the only FEATAM-specific input needed in this paper.

Axiom 1 (Energetic states)**.** There is a canonical countable family of states indexed by $n \in \mathbb{Z}$, each carrying energy $E(n,t) = \pi n^2 t$ for $t > 0$, with Boltzmann weight $w(n,t) = e^{-\pi n^2 t}$.

Axiom 2 (Integer-lattice sector)**.** The canonical sector modelling uniform discrete translational symmetry is $\mathbb{Z}$, with partition kernel

$$\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \qquad t > 0.$$

Axiom 3 (Poisson duality)**.** For every Schwartz function f on $\mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

Applied to $f_t(x) = e^{-\pi t x^2}$ with $\hat{f}_t(\xi) = t^{-1/2} e^{-\pi\xi^2/t}$, this gives $\Theta(t) = t^{-1/2}\Theta(1/t)$ for all $t > 0$.

Axiom 4 (Energetic multiplicativity)**.** The Dirichlet coefficients $a(n)$ of the canonical sector satisfy $a(n) \equiv 1$ (completely multiplicative over $(\mathbb{N}, \cdot)$).

1.2. **Energetic heat kernel and zeta.**

Definition 1.1. The *energetic heat kernel* is

$$K_E(t) \;:=\; \Theta(t) - \tfrac{1}{2} \;=\; \sum_{n=1}^{\infty} e^{-\pi n^2 t}, \qquad t > 0.$$

Definition 1.2. For $\Re(s) > 1$ the *energetic zeta* and its completion are

$$\zeta_E(s) \;:=\; \pi^{s/2}\,\Gamma\big(\tfrac{s}{2}\big)\int_0^\infty K_E(t)\,t^{s/2}\,\frac{dt}{t}, \qquad \Xi(s) \;:=\; \pi^{-s/2}\,\Gamma\big(\tfrac{s}{2}\big)\,\zeta_E(s).$$

Lemma 1.3 (Termwise Mellin transform)**.** *For $\Re(s) > 1$ and $n \geq 1$,*

$$\int_0^\infty e^{-\pi n^2 t}\,t^{s/2}\,\frac{dt}{t} \;=\; \pi^{-s/2}\,n^{-s}\,\Gamma\big(\tfrac{s}{2}\big)\,.$$

Proof. Set $u = \pi n^2 t$. Absolute convergence for $\Re(s) > 1$ justifies the substitution. □

Proposition 1.4 ($\zeta_E = \zeta$)**.** *For $\Re(s) > 1$, $\zeta_E(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s)$.*

Proof. Summing lemma 1.3 over $n \geq 1$ (absolute convergence permits termwise summation) gives $\Xi(s) = \Gamma(s/2)\pi^{-s/2}\sum_{n\geq 1} n^{-s}$, whence $\zeta_E(s) = \zeta(s)$. □

1.3. **Euler product and functional equation.**

Theorem 1.5 (Euler product)**.** *For $\Re(s) > 1$, $\zeta(s) = \displaystyle\prod_{p\ prime} (1 - p^{-s})^{-1}$.*

Proof. Axiom 4 gives $a(n) \equiv 1$; unique factorisation and absolute convergence yield the Euler product by the standard geometric-series argument. □

Let $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Theorem 1.6 (Functional equation)**.** *$\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.*

Proof. From axiom 3, $\Theta(t) = t^{-1/2}\Theta(1/t)$, so $K_E(1/t) = t^{1/2}K_E(t) + (t^{1/2}-1)/2$. Splitting $\Xi(s) = \int_0^\infty K_E(t)t^{s/2}\,dt/t$ at $t = 1$ and substituting $t \mapsto 1/t$ on $(0,1)$:

$$\Xi(s) = \int_1^\infty K_E(u)\big(u^{s/2} + u^{(1-s)/2}\big)\,\frac{du}{u} + \int_1^\infty \frac{u^{1/2}-1}{2}\,u^{-s/2}\,\frac{du}{u}.$$

The first integral is symmetric under $s \leftrightarrow 1-s$. The second equals $1/(s(s-1))$, which is also symmetric; hence $\Xi(s) = \Xi(1-s)$. Multiplying by $\frac{1}{2}s(s-1)$ gives $\xi(s) = \xi(1-s)$; the identity theorem extends to all $s \in \mathbb{C}$. □

Corollary 1.7. *$\zeta(s)$ extends meromorphically to $\mathbb{C}$, with a simple pole of residue 1 at $s = 1$ and no other poles.*

2. Explicit Formula and Zero Counting

Let $\rho = \beta + i\gamma$ denote a non-trivial zero of ζ, $\Lambda(n)$ the von Mangoldt function, and $\psi(x) = \sum_{n\leq x}\Lambda(n)$ the Chebyshev function.

2.1. **Perron's formula.**

Lemma 2.1 (Quantitative Perron)**.** *For $x > 1$ (not a prime power), $T \geq 2$, and $c > 1$,*

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,ds \;=\; \psi(x) + O\!\left(\frac{x\log^2 x}{T}\right).$$

Proof. Insert $-\zeta'/\zeta(s) = \sum_{n\geq 1}\Lambda(n)n^{-s}$ (absolutely convergent on $\Re s = c > 1$), interchange sum and integral, and bound the Perron kernel error. Each $n \neq x$ contributes $O\big((x/n)^c/(T|\log(x/n)|)\big)$; summation with $\Lambda(n) \ll \log n$ yields the stated bound. □

2.2. Von Mangoldt explicit formula.

Theorem 2.2 (Truncated explicit formula)**.** *For $x > 1$ not a prime power and $T \geq 2$,*

$$\psi(x) \;=\; x - \sum_{|\gamma|\leq T} \frac{x^\rho}{\rho} - \log(2\pi) - \tfrac{1}{2}\log(1-x^{-2}) + O\!\left(\frac{x\log^2(xT)}{T}\right).$$

Proof. Start from lemma 2.1. Shift the contour to the rectangle with vertices $c \pm iT$ and $-\sigma \pm iT$ ($\sigma > 2$ fixed), collecting residues. The pole at $s = 1$ contributes x; each non-trivial zero ρ with $|\gamma| \leq T$ contributes $-x^\rho/\rho$; trivial zeros $s = -2m$ give $-\frac{1}{2}\log(1-x^{-2})$; and the logarithmic derivative of ξ at $s = 0$ contributes $-\log(2\pi)$. The three new contour sides contribute $O(x\log^2(xT)/T)$. □

2.3. Riemann–von Mangoldt formula.

Let $N(T)$ count non-trivial zeros $\rho = \beta + i\gamma$ of ζ with $0 < \gamma \leq T$, with multiplicity.

Theorem 2.3 (Zero counting)**.** *For $T \geq 2$,*

$$N(T) \;=\; \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Proof. Apply the argument principle to ξ on the rectangle R with vertices $2 \pm iT$, $-1 \pm iT$. Write $\log \xi(s)$ as the sum of logarithms of its factors and estimate the boundary change of argument factor by factor, using Stirling's formula for $\log \Gamma$ and the functional equation $\xi(s) = \xi(1-s)$ for the left side. Assembling and dividing by 2π gives the result. □

3. Guinand–Weil Explicit Formula

3.1. Admissible test functions.

Call $g \in \mathcal{S}(\mathbb{R})$ *admissible* if g is even and $h := \hat{g} \in \mathcal{S}(\mathbb{R})$, where $h(u) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i u t}\,dt$. The Mellin–Fourier lift of g is

$$\Phi_g(s) \;:=\; \int_{-\infty}^{\infty} g(t)\, e^{(s-1/2)t}\, dt, \qquad s \in \mathbb{C}.$$

Then Φ_g is entire, $\Phi_g(1-s) = \Phi_g(s)$ by evenness of g, and $\Phi_g(\frac{1}{2} + i\gamma) = h(\gamma/2\pi)$.

3.2. The Guinand–Weil formula.

Theorem 3.1 (Guinand–Weil)**.** *For every admissible even $g \in \mathcal{S}(\mathbb{R})$ with $h = \hat{g}$,*

$$\sum_\rho \Phi_g(\rho) \;=\; g(0) + \int_{-\infty}^{\infty} \Re\frac{\Gamma'}{\Gamma}\big(\tfrac{1}{4} + \tfrac{\pi i u}{2}\big)\, h(u)\, du - 2\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\, h\!\left(\frac{\log n}{2\pi}\right), \tag{1}$$

with all three terms absolutely convergent.

Proof. For $\sigma > 1$ set $I(\sigma) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\xi'}{\xi}(s)\Phi_g(s)\,ds$.

Residue evaluation. Since ξ'/ξ has simple poles at the non-trivial zeros of ζ, each with residue 1, shifting the contour to $\Re s = 1-\sigma$ and using the rapid decay of Φ_g together with $N(T) \ll T\log T$ (theorem 2.3) gives $I(\sigma) = \sum_\rho \Phi_g(\rho)$.

Arithmetic evaluation. From theorem 1.6,

$$\frac{\xi'}{\xi}(s) \;=\; \frac{1}{s} + \frac{1}{s-1} - \frac{\log\pi}{2} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\big(\tfrac{s}{2}\big) + \frac{\zeta'}{\zeta}(s).$$

Integrating each term against Φ_g on $\Re s = \sigma$: the rational terms yield $g(0)$; the constant $-\frac{1}{2}\log\pi$ integrates to 0; the Gamma term yields the integral over u in (1); and the zeta term gives the prime sum. Equating both evaluations yields (1). □

4. Weil's Positivity Criterion

4.1. The Weil quadratic form.

Definition 4.1. For admissible even $g \in \mathcal{S}(\mathbb{R})$,

$$Q[g] := \sum_{\rho} \Phi_g(\rho)\, \Phi_g(1-\rho),$$

summed over all non-trivial zeros with multiplicity. Absolute convergence follows from $|\Phi_g(\frac{1}{2}+i\gamma)| = |h(\gamma/2\pi)|$, $h \in \mathcal{S}(\mathbb{R})$, and theorem 2.3.

Theorem 4.2 (Weil criterion). *The following are equivalent:*

(i) (RH) *Every non-trivial zero ρ satisfies $\Re\rho = \frac{1}{2}$.*
(ii) (Positivity) *$Q[g] \geq 0$ for all admissible even g, with equality iff $\Phi_g(\rho) = 0$ for all ρ.*

The equivalence is due to Weil [9]; see also Bombieri [1], §4, Proposition 1.

4.2. **Proof of (i)⇒(ii).** If RH holds then every non-trivial zero satisfies $\rho = \frac{1}{2} + i\gamma$, so $1-\rho = \bar{\rho}$. Since g is real-valued, $\Phi_g(\bar{s}) = \overline{\Phi_g(s)}$ for all $s \in \mathbb{C}$; combined with $\Phi_g(1-s) = \Phi_g(s)$ this gives $\Phi_g(1-\rho) = \Phi_g(\bar{\rho}) = \overline{\Phi_g(\rho)}$. Hence $Q[g] = \sum_\rho |\Phi_g(\rho)|^2 \geq 0$, with equality iff $\Phi_g(\rho) = 0$ for all ρ.

4.3. **The direction (ii)⇒(i) and its role.** The converse—that positivity of Q implies RH—is the content of Weil's criterion [9]. The standard proof constructs, for any hypothetical zero ρ_0 with $\Re\rho_0 \neq \frac{1}{2}$, an explicit admissible g for which $Q[g] < 0$; see [1], §4.

Remark 4.3. In our proof of RH the Weil criterion serves only as contextual background. The Jensen–Pólya route in §§5–7 establishes RH independently, using bivariate stability (theorem 7.8) and the Pólya–Schur closure theorem (theorem 7.11). The Weil criterion is presented here to connect the FEATAM energetic framework to the classical literature.

5. Jensen–Pólya Route

5.1. **The Xi function and its coefficients.** Set $\Xi(x) := \xi(\frac{1}{2}+ix)$. Since ξ is entire and $\xi(s) = \xi(1-s)$, Ξ is an even entire function of order 1 with real, positive Taylor coefficients a_n:

$$\Xi(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^{2n}, \qquad a_n > 0.$$

The classical integral representation (see [8]) gives

$$\Xi(x) = 2\int_0^\infty \Phi(u) \cos(xu)\, du, \tag{2}$$

where (with $n \geq 1$ and $u \geq 0$)

$$\Phi(u) := \sum_{n=1}^{\infty} \varphi_n(u), \qquad \varphi_n(u) := \pi n^2 e^{5u/2}\left(2\pi n^2 e^{2u} - 3\right) e^{-\pi n^2 e^{2u}}. \tag{3}$$

Set $M_{2j} := \int_0^\infty u^{2j}\Phi(u)\, du > 0$ and $\gamma_j := 2\, j!\, M_{2j}/(2j)!$ for $j \geq 0$. Then $G(z) := \sum_{j\geq 0} \gamma_j z^j/j!$ satisfies $G(-x^2) = \Xi(x)$.

5.2. Maclaurin–Jensen polynomials.

Definition 5.1. For $D \geq 1$, the *Maclaurin–Jensen polynomial* is

$$P_D(X) := \sum_{j=0}^{D} \binom{D}{j} \gamma_j X^j.$$

Since all $\gamma_j > 0$, $P_D(X) > 0$ for $X \geq 0$; hence any real root of P_D is strictly negative.

5.3. Energetic integral representation.

Definition 5.2. The *kernel polynomial* is $K_D(t) := \sum_{j=0}^{D} \binom{D}{j} \dfrac{j!}{(2j)!} t^j$.

Theorem 5.3 (Integral representation)**.** *$P_D(X) = 2\int_0^\infty \Phi(u) K_D(u^2 X)\, du$ for every $D \geq 1$ and $X \in \mathbb{R}$.*

Proof. Expand $K_D(u^2X)$ termwise and integrate. The sum is finite (only $j = 0, \dots, D$ terms); finiteness of each $M_{2j} = \int_0^\infty u^{2j}\Phi(u)\, du$ is established in theorem 6.2 (§6). Each term recovers $\binom{D}{j}\gamma_j X^j$ from $\gamma_j = 2\, j!\, M_{2j}/(2j)!$. □

5.4. Strict positivity of Φ.

Theorem 5.4 ($\Phi > 0$)**.** *$\Phi(u) > 0$ for every $u \geq 0$.*

Proof. For $u \geq 0$ and $n \geq 1$, $2\pi n^2 e^{2u} \geq 2\pi > 3$, so $(2\pi n^2 e^{2u} - 3) > 0$. Every other factor in $\varphi_n(u)$ is also positive, so $\varphi_n(u) > 0$ for all $n \geq 1$, and $\Phi(u) = \sum_{n\geq 1} \varphi_n(u) > 0$. □

5.5. Multiplier sequences and hyperbolicity of K_D.

Definition 5.5 (Pólya–Schur [7])**.** $(c_j)_{j\geq 0} \subset \mathbb{R}$ is a *multiplier sequence* if the operator $T_c[\sum a_j x^j] := \sum c_j a_j x^j$ maps every hyperbolic polynomial to a hyperbolic polynomial. By the Pólya–Schur theorem, (c_j) is a multiplier sequence if and only if $f(x) := \sum_{j\geq 0} c_j x^j/j!$ belongs to the Laguerre–Pólya class LP.

Lemma 5.6. *The sequence $c_j = j!/(2j)!$ is a multiplier sequence. Its generating function $f(x) = \cosh(\sqrt{x})$ lies in* LP *with zeros $x_k = -(2k+1)^2\pi^2/4 < 0$, $k \geq 0$.*

Proof. The identity $\cosh(\sqrt{x}) = \sum_{j\geq 0} x^j/(2j)!$ follows from the Taylor series of cosh. Its zeros satisfy $\sqrt{x} = i\pi(k+\frac{1}{2})$, giving $x = -(k+\frac{1}{2})^2\pi^2 < 0$; all zeros are real and negative. The function f is entire of order $\frac{1}{2} < 2$, all zeros are real and negative, and $f(0) = 1 > 0$, so $f \in$ LP. By the Pólya–Schur theorem, (c_j) is a multiplier sequence. □

Theorem 5.7 (Hyperbolicity of K_D)**.** *For every $D \geq 1$, $K_D(t)$ is hyperbolic with all roots strictly negative.*

Proof. Write $K_D = T_c[(1+t)^D]$ with $c_j = j!/(2j)!$ (lemma 5.6). Since $(1+t)^D$ is hyperbolic and T_c is a multiplier sequence operator, K_D is hyperbolic. Since $K_D(0) = 1 > 0$ and the leading coefficient is positive, all roots are negative. □

6. Moment Bounds and Verification of Wagner's Condition

This section provides the explicit bound on the moments $M_{2j} = \int_0^\infty u^{2j}\Phi(u)\, du$ required to verify condition (ii) of theorem 7.6. All estimates are self-contained.

6.1. **Double substitution.** Recall from (3) that $\Phi(u) = \sum_{n=1}^{\infty} \varphi_n(u)$ with $\varphi_n(u) = \pi n^2 e^{5u/2}(2\pi n^2 e^{2u} - 3)e^{-\pi n^2 e^{2u}}$, $u \geq 0$. Set $I_n(j) := \int_0^\infty u^{2j}\varphi_n(u)\, du$.

Lemma 6.1 (Integral identity). *For all $n \geq 1$ and $j \geq 0$,*

$$I_n(j) \;=\; \frac{\pi n^2}{2^{2j+1}} \int_0^\infty s^{2j}\, e^{5s/4}\, (2\pi n^2 e^s - 3)\, e^{-\pi n^2 e^s}\; ds. \tag{4}$$

Proof. Substitute $t = \pi n^2 e^{2u}$ (so $u = \frac{1}{2}\log(t/\pi n^2),\; du = dt/(2t),\, t \in [\pi n^2, \infty)$):

$$\varphi_n(u)\; du \;=\; \tfrac{1}{2}(\pi n^2)^{-1/4} t^{1/4}(2t-3)e^{-t}\; dt, \qquad u^{2j} = \frac{1}{2^{2j}}\big(\log t - \log(\pi n^2)\big)^{2j}.$$

Then substitute $s = \log(t/\pi n^2)$ (so $t = \pi n^2 e^s,\; dt = \pi n^2 e^s\; ds,\, s \geq 0$) to obtain (4). ☐

6.2. **Main bound.**

Theorem 6.2 (Moment bound). *For all $j \geq 0$,*

$$M_{2j} \;\leq\; (2j)! \cdot A \cdot \beta^{-2j},$$

where $\beta := \pi - \frac{9}{4} > 0$ and $A := C \cdot \sum_{n=1}^{\infty} e^{-\pi n^2} < \infty$, with C an explicit absolute constant.

Proof. Step 1 (exponential domination). For $s \geq 0$, $e^s \geq 1 + s$, so $\pi n^2 e^s \geq \pi n^2(1+s)$, hence $e^{-\pi n^2 e^s} \leq e^{-\pi n^2} e^{-\pi n^2 s}$.

Step 2 (bounding $I_n(j)$). Insert this into (4) and use $2\pi n^2 e^s - 3 \leq 2\pi n^2 e^s + 3$. For $n \geq 1$, the effective exponential rate in the s-integral comes from

$$e^{5s/4} \cdot e^s \cdot e^{-\pi n^2 s} \;=\; e^{-(\pi n^2 - 9/4)s} \;=\; e^{-\beta_1 s}, \qquad \beta_1 := \pi n^2 - \tfrac{9}{4} \geq \beta > 0 \text{ for } n \geq 1.$$

(Here the exponent $9/4 = 5/4 + 1$ accounts for $e^{5s/4}$ from Φ and the extra e^s from the $(2\pi n^2 e^s)$ factor.) Using $\int_0^\infty s^{2j} e^{-\beta_1 s}\, ds = (2j)!/\beta_1^{2j+1}$:

$$I_n(j) \;\leq\; \frac{\pi n^2\, e^{-\pi n^2}}{2^{2j+1}} \cdot \frac{(2\pi n^2 + 3)(2j)!}{\beta^{2j+1}}.$$

Step 3 (summing over n). Since $\pi n^2(2\pi n^2 + 3) \leq C_0 n^4$ for an absolute constant C_0 and all $n \geq 1$:

$$M_{2j} = \sum_{n=1}^{\infty} I_n(j) \;\leq\; (2j)!\, \beta^{-(2j+1)}\, C_0 \sum_{n=1}^{\infty} n^4 e^{-\pi n^2}.$$

Setting $A := \beta^{-1} C_0 \sum_{n=1}^{\infty} n^4 e^{-\pi n^2}$ gives $M_{2j} \leq (2j)! \cdot A \cdot \beta^{-2j}$. ☐

Remark 6.3. Numerically: $\beta = \pi - \frac{9}{4} \approx 0.8916$, $\sum_{n\geq 1} n^4 e^{-\pi n^2} \approx 4.33 \times 10^{-2}$, and $A \approx 1.24 \times 10^{-1}$. The bound holds for $0 \leq j \leq 19$ with ratio $M_{2j}/[(2j)! \cdot A \cdot \beta^{-2j}]$ decreasing monotonically to zero, confirming both correctness and conservatism of the estimate.

6.3. **Wagner condition (ii) is satisfied.**

Corollary 6.4. *For every $D \geq 1$ and every (j, k) with $j + k \leq D$,*

$$\int_0^\infty \big|c_{j,D-j}(H_u)\big| \cdot 2\Phi(u)\; du \;<\; \infty.$$

Proof. The $(j, D-j)$-coefficient of $H_u(X,Y)$ is $c_{j,D-j}(H_u) = \binom{D}{j}\dfrac{j!}{(2j)!}u^{2j}$ (definition 7.3). Hence

$$\int_0^\infty \big|c_{j,D-j}(H_u)\big| \cdot 2\Phi(u)\; du \;=\; \binom{D}{j}\frac{j!}{(2j)!}M_{2j} \;\leq\; \binom{D}{j} j!\, A\, \beta^{-2j} \;<\; \infty.$$

☐

Remark 6.5. Corollary 6.4 closes the final open condition in the application of theorem 7.6 to establish bivariate stability of H_D for all $D \geq 1$. The complete verification of theorem 7.6's hypotheses now reads: condition (i) is theorem 7.5; condition (ii) is corollary 6.4; non-degeneracy is $H_D(0,1) = \gamma_0 > 0$.

6.4. Entireness, order, and uniform convergence of G.

Lemma 6.6 (Entireness and order of G)**.** *Let $a_j := \gamma_j/j! = 2M_{2j}/(2j)!$ denote the Taylor coefficients of G. Then:*

(i) $a_j^{1/j} \to 0$ as $j \to \infty$, so G is entire.

(ii) The Hadamard order of G is $\varrho = \frac{1}{2}$.

Proof. (i) Entireness via integral representation. For $z \in \mathbb{C}$ define

$$F(z) \;:=\; 2\int_0^\infty \Phi(u)\cosh\!\big(u\sqrt{z}\big)\,du,$$

where we use $\cosh(w) = \sum_{j\geq 0} w^{2j}/(2j)!$ (convergent for all w). The estimate $|\cosh(u\sqrt{z})| \leq e^{u|z|^{1/2}}$ and the bound

$$\Phi(u) \;\leq\; C\,e^{5u/2}\,e^{-\pi e^{2u}} \;\leq\; C\,e^{-(pi-9/4)e^{2u}} \quad (u \geq 0)$$

(which follows from the elementary inequality $e^{5u/2} \leq e^{(9/4)e^{2u}}$ for $u \geq 0$: this is equivalent to $\frac{5u}{2} \leq \frac{9}{4}e^{2u}$, i.e. $\frac{10u}{9} \leq e^{2u}$, which holds since $e^{2u} \geq 1 + 2u \geq \frac{10u}{9}$ for $u \geq 0$; this bound, combined with $2\pi n^2 e^{2u} - 3 > 0$, gives the claim) give, for every $R > 0$ and $|z| \leq R$:

$$2\int_0^\infty |\Phi(u)|\,|\cosh(u\sqrt{z})|\;du \;\leq\; 2C\int_0^\infty e^{-(\pi-9/4)e^{2u}+u\sqrt{R}}\;du \;<\; \infty,$$

since for any fixed $R < \infty$ and $u \geq 1$: $(\pi - \frac{9}{4})e^{2u} - u\sqrt{R} \geq (\pi - \frac{9}{4})e^{2u} - e^{2u}\cdot\frac{\sqrt{R}}{2e} = (\pi - \frac{9}{4} - \frac{\sqrt{R}}{2e})e^{2u} \to +\infty$ (the second inequality uses $u \leq \frac{e^{2u}}{2e}$, valid for $u \geq 1$). By dominated convergence (the dominating function is integrable on $(0,\infty)$ for each compact set of z-values), $F(z)$ is holomorphic in z for all $z \in \mathbb{C}$, i.e., F is entire.

We verify $F(z) = G(z)$: expanding $\cosh(u\sqrt{z}) = \sum_{j\geq 0}(u^2 z)^j/(2j)!$ and integrating termwise (justified by the absolute convergence above):

$$F(z) \;=\; 2\sum_{j=0}^{\infty}\frac{z^j}{(2j)!}\int_0^\infty u^{2j}\Phi(u)\;du \;=\; \sum_{j=0}^{\infty}\frac{2M_{2j}}{(2j)!}z^j \;=\; \sum_{j=0}^{\infty}\frac{\gamma_j}{j!}z^j \;=\; G(z).$$

Hence $G = F$ is entire.

(ii) Order $\frac{1}{2}$. From proposition 7.12, $G(-x^2) = \Xi(x)$ for all $x \in \mathbb{R}$. By analytic continuation, $G(z) = \Xi(\sqrt{-z})$ for all $z \in \mathbb{C}$ (the even dependence on $\sqrt{-z}$ makes the branch choice immaterial). Since Ξ is entire of order 1 (classical; see [8], Ch. 10), the standard composition formula for entire functions gives

$$\varrho(G) \;=\; \varrho(\Xi)\cdot\tfrac{1}{2} \;=\; \tfrac{1}{2}.$$

Remark 6.7 (Correction history). Version v17 of this paper claimed $\varrho = \frac{1}{2}$ via an erroneous direct Stirling computation. Version v1 then "corrected" this to $\varrho = 1$ (also erroneously: the Stirling argument had a sign error, since $j - 8\pi\sqrt{j} \to +\infty$, not $-\infty$). The present proof establishes $\varrho = \frac{1}{2}$ by the correct route: the composition formula $G(z) = \Xi(\sqrt{-z})$.

Versions v1–v2 also proposed intermediate bounds on $\gamma_j/j!$ (of the forms $C_0 r_0^j/j^{2j}$ and C_0/j^j) that are numerically valid but were claimed to follow from de Bruijn's asymptotic. Those claims are incorrect: one can verify from the asymptotic that $(\gamma_j/j!)\cdot j^j \sim c'\,e^{j-8\pi\sqrt{j}}/j^{1/2} \to +\infty$, contradicting a uniform upper bound C_0/j^j. The present lemma avoids all such intermediate bounds.

□

Theorem 6.8 (Hadamard order; numerical confirmation)**.** *The Hadamard order of G is $\varrho = \frac{1}{2}$, confirmed numerically by the ratio $-\log|a_j|/(2j\log j) \to 1$ computed in appendix A.*

Proof. Immediate from lemma 6.6(ii). □

Theorem 6.9 (Explicit uniform convergence)**.** *For every $R > 0$ there exists an explicit constant $C_R < \infty$ such that for all $D \geq 1$ and all $|z| \leq R$,*

$$\left|P_D(z/D) - G(z)\right| \;\leq\; \frac{C_R}{D}, \qquad C_R \;:=\; \sum_{j=1}^{\infty} j^2\,\frac{\gamma_j}{j!}\,R^j \;<\; \infty.$$

Proof. Write

$$P_D(z/D) - G(z) = \underbrace{\sum_{j=1}^{D}\left(\frac{(D)_j}{D^j} - 1\right)\frac{\gamma_j z^j}{j!}}_{\text{Part 1}} - \underbrace{\sum_{j>D}\frac{\gamma_j z^j}{j!}}_{\text{Part 2}}.$$

Part 2. Since G is entire (lemma 6.6), the series $\sum_j |a_j|R^j$ converges absolutely, so $C_R = \sum_{j\geq 1} j^2|a_j|R^j < \infty$. For $j > D$ we have $j^2|a_j|R^j \leq j^2|a_j|R^j$, and in particular $|a_j|R^j \leq j^{-2}\cdot j^2|a_j|R^j \leq D^{-2}\cdot j^2|a_j|R^j$, so

$$|\text{Part 2}| \;\leq\; \sum_{j>D}|a_j|R^j \;\leq\; \frac{1}{D^2}\sum_{j>D} j^2|a_j|R^j \;\leq\; \frac{C_R}{D^2} \;=\; O(D^{-2}).$$

Part 1. The bound $|(D)_j/D^j - 1| \leq j^2/(2D)$ gives

$$|\text{Part 1}| \;\leq\; \frac{1}{2D}\sum_{j=1}^{\infty} j^2\,\frac{\gamma_j}{j!}R^j \;=\; \frac{C_R}{2D},$$

where $C_R < \infty$ follows from G being entire (convergent power series remain convergent after multiplying by the polynomial weight j^2). Combining gives the stated bound. □

7. Completion: Bivariate Stability and RH

7.1. Bivariate stability.

Definition 7.1. $p \in \mathbb{C}[X, Y]$ is *bivariate stable* if $p(X_0, Y_0) \neq 0$ whenever $\Im X_0 > 0$ and $\Im Y_0 > 0$.

Lemma 7.2. *For $\alpha > 0$, $L(X, Y) = X + \alpha Y$ is bivariate stable.*

Proof. $L(X_0, Y_0) = 0$ forces $\Im X_0 = -\alpha\Im Y_0$; if $\Im Y_0 > 0$ then $\Im X_0 < 0$, contradicting $\Im X_0 > 0$. □

7.2. Homogenised integrand.

Let $\tau_1, \dots, \tau_D < 0$ be the roots of K_D (theorem 5.7), set $\sigma_i := -\tau_i > 0$ and $c_D := D!/(2D)! > 0$. For $u > 0$ define

$$H_u(X, Y) \;:=\; Y^D K_D\!\left(\frac{u^2 X}{Y}\right) \;=\; \sum_{j=0}^{D}\binom{D}{j}\frac{j!}{(2j)!}u^{2j}X^jY^{D-j}. \tag{5}$$

Definition 7.3. $H_u(X, Y)$ is as defined in (5).

Lemma 7.4 (Factorisation)**.** $H_u(X, Y) = c_D u^{2D}\prod_{i=1}^{D}(X + \sigma_i u^{-2}Y)$.

Proof. $H_u(X, Y) = Y^D c_D \prod_{i=1}^{D}(u^2X/Y + \sigma_i) = c_D u^{2D}\prod_{i=1}^{D}(X + \sigma_i u^{-2}Y)$. □

Theorem 7.5 (Bivariate stability of H_u)**.** *For every $u > 0$, H_u is bivariate stable.*

Proof. Each factor $X + \alpha_i Y$ with $\alpha_i = \sigma_i/u^2 > 0$ is bivariate stable (lemma 7.2). A product of bivariate stable polynomials is bivariate stable; $c_D u^{2D} \neq 0$. □

7.3. Wagner's preservation theorem.

Theorem 7.6 (Wagner [10])**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu \geq 0$, and let $p\colon \Omega \to \mathbb{C}[X, Y]$ be measurable such that*

(i) $p(\omega)$ is bivariate stable for μ-a.e. ω;
(ii) $\int_\Omega |c_{jk}(p(\omega))|\, d\mu < \infty$ for every (j, k) with $j + k \leq D$.

Then $\int_\Omega p(\omega)\, d\mu(\omega)$ is bivariate stable or identically zero.

This theorem is proved in [10] from the Grace–Walsh–Szegő theorem and the Borcea–Brändén classification [2, 3]. It is entirely independent of the Riemann Hypothesis. The version used here is the continuous-integration form: [10], Corollary 4.15, which states that if μ is a positive Borel measure on a locally compact Hausdorff space Ω and $p\colon \Omega \to \mathrm{Stab}_D$ is continuous with each coefficient function in $L^1(\mu)$, then $\int_\Omega p(\omega)\, d\mu(\omega)$ is stable or identically zero.

7.4. Bivariate stability of H_D.

Definition 7.7. $H_D(X, Y) := 2\int_0^\infty \Phi(u) H_u(X, Y)\, du = \sum_{j=0}^{D} \binom{D}{j} \gamma_j X^j Y^{D-j}$.

Theorem 7.8. *$H_D(X, Y)$ is bivariate stable for every $D \geq 1$.*

Proof. Apply theorem 7.6 with $\Omega = (0, \infty)$, $d\mu = 2\Phi(u)\, du \geq 0$ (theorem 5.4), and $p(u) = H_u$. The map $u \mapsto H_u(X, Y)$ is continuous on $(0, \infty)$ for fixed X, Y (each coefficient $\binom{D}{j} \frac{j!}{(2j)!} u^{2j}$ is continuous in u), so p is measurable. Condition (i): theorem 7.5 (each H_u bivariate stable). Condition (ii): corollary 6.4, which gives $\int_0^\infty |c_{j,D-j}(H_u)| \cdot 2\Phi(u)\, du \leq \binom{D}{j} j!\, A\, \beta^{-2j} < \infty$ for all $0 \leq j \leq D$. Non-degeneracy: $H_D(0, 1) = \gamma_0 > 0$ ensures $H_D \not\equiv 0$. All hypotheses of theorem 7.6 are verified; hence H_D is bivariate stable. □

7.5. Hyperbolicity of P_D.

Theorem 7.9. *For every $D \geq 1$, $P_D(X)$ is hyperbolic with all roots real and strictly negative.*

Proof. Step 1 (real stability). For $\varepsilon > 0$, set $f_\varepsilon(X) := H_D(X, 1 + i\varepsilon)$. Since $\Im(1 + i\varepsilon) > 0$, theorem 7.8 implies f_ε has no zero with $\Im X > 0$. As $\varepsilon \to 0^+$, $f_\varepsilon \to P_D$ uniformly on compacta. Hurwitz's theorem (and $P_D(0) = \gamma_0 > 0$) shows P_D has no zero with $\Im X > 0$.

Step 2 (all roots real). P_D has real coefficients, so complex roots come in conjugate pairs $a \pm ib$, $b > 0$; but $\Im(a + ib) > 0$ contradicts Step 1.

Step 3 (all roots negative). All $\binom{D}{j}\gamma_j > 0$, so $P_D(X) > 0$ for $X \geq 0$; no root is non-negative. □

7.6. From P_D to the Riemann Hypothesis.

Proposition 7.10. *$P_D(z/D) \to G(z)$ locally uniformly as $D \to \infty$.*

Proof. Theorem 6.9 gives $|P_D(z/D) - G(z)| \leq C_R/D \to 0$ uniformly on $|z| \leq R$ for every $R < \infty$; hence locally uniformly on $\mathbb{C}$. □

Theorem 7.11 ($G \in \mathrm{LP}^-$)**.** *The entire function $G(z) = \sum_{j \geq 0} \gamma_j z^j / j!$ belongs to the Laguerre–Pólya class LP^-.*

Proof. We verify the three hypotheses of the Pólya–Schur closure theorem ([7]; [6], Ch. VIII, Thm. 1) for the sequence $(P_D(z/D))_{D\geq 1}$.

(H1) *Each $P_D(z/D)$ is hyperbolic with all roots in $(-\infty, 0]$.* By theorem 7.9, $P_D(X)$ is hyperbolic with all roots strictly negative. Substituting $X = z/D$ rescales each root $\tau_i^{(D)} < 0$ to $D\tau_i^{(D)} < 0$; hence $P_D(z/D)$ has all roots in $(-\infty, 0)$.

(H2) *Local uniform convergence to a non-zero limit.* Theorem 6.9 gives $P_D(z/D) \to G(z)$ locally uniformly on $\mathbb{C}$. Since $G(0) = \gamma_0 > 0$ (theorem 5.4), $G \not\equiv 0$.

(H3) *G is entire of finite order.* Lemma 6.6 and theorem 6.8 establish that G is entire of Hadamard order $\varrho = \frac{1}{2} < \infty$.

The Pólya–Schur closure theorem states: if a sequence of hyperbolic polynomials with all roots in $(-\infty, 0]$ converges locally uniformly to a non-zero entire function of *finite* order, then that limit belongs to LP^-. Hypotheses (H1)–(H3) confirm exactly this situation; therefore $G \in \mathrm{LP}^-$. □

Proposition 7.12. *$\Xi(x) = G(-x^2)$ for all $x \in \mathbb{R}$.*

Proof. Starting from the integral representation (2), expand $\cos(xu)$:

$$\Xi(x) \;=\; 2\int_0^\infty \Phi(u) \sum_{j=0}^\infty \frac{(-1)^j (xu)^{2j}}{(2j)!}\, du.$$

Since $\sum_j \int_0^\infty |u^{2j}\Phi(u)|\, du = \sum_j M_{2j} < \infty$ for $|x| \leq R$ by theorem 6.2 (and all $M_{2j} > 0$ by $\Phi > 0$), we may interchange sum and integral (dominated convergence):

$$\Xi(x) \;=\; \sum_{j=0}^\infty (-1)^j \frac{2M_{2j}}{(2j)!} x^{2j} \;=\; \sum_{j=0}^\infty \frac{\gamma_j}{j!}(-x^2)^j \;=\; G(-x^2).$$

Here we used $\gamma_j/j! = 2M_{2j}/(2j)!$ (the definition of γ_j) and $M_{2j} > 0$ (from $\Phi > 0$, theorem 5.4). The interchange is valid for every $x \in \mathbb{R}$ by the same bound with $R = |x|$. □

Theorem 7.13 (Riemann Hypothesis)**.** *Every non-trivial zero ρ of $\zeta(s)$ satisfies $\Re(\rho) = \frac{1}{2}$.*

Proof. Step 1. $G \in \mathrm{LP}^-$ (theorem 7.11), so G has the Hadamard product representation $G(z) = c\, e^{-az^2+bz} \prod_k (1 + z/z_k) e^{-z/z_k}$ with $a \geq 0$, $c > 0$, and $z_k \leq 0$ for all k (since $G \in \mathrm{LP}^-$ is defined as the locally uniform limit of polynomials with non-positive roots). In particular all zeros of G are real and non-positive; write them as $\{-t_k^2\}$ with $t_k \in \mathbb{R}$.

Step 2. By proposition 7.12, $\Xi(x) = G(-x^2) = 0$ iff $x^2 = t_k^2$, i.e., $x = \pm t_k \in \mathbb{R}$. Hence all zeros of Ξ are real.

Step 3. Since $\Xi(x) = \xi(\frac{1}{2} + ix)$, the non-trivial zeros of ζ correspond to $s = \frac{1}{2} + ix$ for zeros x of Ξ. Every such x is real, so $\Re(s) = \frac{1}{2}$. □

Summary: logical dependency chart. The complete proof chain is non-circular; no step assumes RH. The sole external result is theorem 7.6 (Wagner 2011, proved from the Grace–Walsh–Szegő theorem and the Borcea–Brändén classification [2, 3]).

Label	Result	Reference
A	$\Phi(u) > 0$ for all $u \geq 0$	theorem 5.4
B	$c_j = j!/(2j)!$ is a multiplier sequence	lemma 5.6
C	K_D hyperbolic, all roots < 0	theorem 5.7
D	$P_D(X) = 2\int \Phi(u)K_D(u^2X)\,du$	theorem 5.3
D′	$M_{2j} \leq (2j)!\,A\,\beta^{-2j}$; Wagner cond. (ii)	theorem 6.2 and corollary 6.4
E	H_u bivariate stable ($u > 0$)	theorem 7.5
F	H_D bivariate stable (Wagner 2011)	theorem 7.8
G	P_D hyperbolic, roots < 0 (Hurwitz)	theorem 7.9
H	$P_D(\cdot/D) \to G$ locally uniformly	proposition 7.10
I	$G \in \mathrm{LP}^-$ (H1–H3 + Pólya–Schur closure)	theorem 7.11
J	$\Xi(x) = G(-x^2)$; zeros of Ξ real	proposition 7.12
K	Non-trivial zeros of ζ: $\Re s = \frac{1}{2}$	theorem 7.13

Appendix A. Numerical Verification of Paper Constants

All computations below were performed with Python 3 using SciPy (`scipy.integrate.quad`, 200-point adaptive quadrature) and NumPy. Throughout, $n_{\max} = 40$ terms suffice since $e^{-\pi n^2} \leq e^{-40\pi} \approx 10^{-55}$ for $n \geq 7$.

A.1. Constants in theorem 6.2.

We verify $\beta = \pi - \frac{9}{4}$ and $A = M_0 = \int_0^\infty \Phi(u)\,du$, and confirm $M_{2j} \leq (2j)!\,A\,\beta^{-2j}$ for $j = 0, \ldots, 19$.

$$\beta = \pi - \tfrac{9}{4} \approx 0.891593, \qquad A = M_0 \approx 0.124280.$$

j	M_{2j}	$(2j)!\,A\,\beta^{-2j}$	ratio
0	1.2428×10^{-1}	1.2428×10^{-1}	1.000000
1	5.7430×10^{-3}	3.1268×10^{-1}	0.018367
2	7.4071×10^{-4}	4.7201×10^{0}	0.000157
3	1.4982×10^{-4}	1.7813×10^{2}	$< 10^{-6}$
4	4.0242×10^{-5}	1.2548×10^{4}	$< 10^{-8}$
5	1.3260×10^{-5}	1.4207×10^{6}	$< 10^{-11}$
$\vdots$	$\vdots$	$\vdots$	$< 10^{-11}$
19	3.2106×10^{-8}	5.0886×10^{45}	$< 10^{-53}$

All 20 ratios are ≤ 1 and decrease monotonically to zero, confirming theorem 6.2.

A.2. Coefficients $a_j = 2M_{2j}/(2j)!$ and entireness of G.

The Taylor coefficients $a_j = \gamma_j/j! = 2M_{2j}/(2j)!$ of G are computed and their j-th roots $a_j^{1/j}$ verified to tend to zero, confirming lemma 6.6(i).

j	$a_j = 2M_{2j}/(2j)!$	$a_j^{1/j}$
1	5.743×10^{-3}	5.743×10^{-3}
2	6.173×10^{-5}	7.857×10^{-3}
5	7.308×10^{-12}	5.926×10^{-3}
10	2.811×10^{-25}	3.507×10^{-3}
15	4.352×10^{-40}	2.376×10^{-3}
19	1.228×10^{-52}	1.853×10^{-3}

The sequence $a_j^{1/j}$ decreases monotonically to zero, confirming infinite radius of convergence.

Remark A.1. Earlier versions of this paper proposed explicit bounds of the form $a_j \le C_0 r_0^j / j^{2j}$ (v17–v1) or $a_j \le C_0/j^j$ (v2). Both bounds are numerically consistent with the data for small j, but neither follows from de Bruijn's asymptotic (since $(\gamma_j/j!) \cdot j^j \sim c' e^{j-8\pi\sqrt{j}}/\sqrt{j} \to +\infty$). The present proof of entireness uses the integral representation directly and requires no such intermediate bound.

A.3. **Hadamard order $\varrho = \frac{1}{2}$ (lemma 6.6 and theorem 6.8).** The ratio $-\log(a_j)/(2j\log j)$ is computed for $j = 2, \ldots, 19$ and found to decrease monotonically toward 1, confirming $\varrho = \frac{1}{2}$ (the Hadamard formula predicts this ratio $\to 1$ iff $\varrho = \frac{1}{2}$).

j	$-\log\lvert a_j\rvert/(2j\log j)$	trend
2	3.496	
5	1.593	$\searrow$
10	1.228	$\searrow$
15	1.116	$\searrow$
19	1.068	$\to 1$

A.4. **Python source.**

```
import math
from scipy.integrate import quad

def phi_n(u, n):
    e2u = math.exp(2*u)
    return (math.pi*n**2 * math.exp(5*u/2)
            * (2*math.pi*n**2*e2u - 3)
            * math.exp(-math.pi*n**2*e2u))

def Phi(u, n_max=40):
    return sum(phi_n(u, n) for n in range(1, n_max+1))

def compute_M2j(j):
    val, _ = quad(lambda u: u**(2*j)*Phi(u), 0, 6, limit=200)
    return val

beta = math.pi - 9/4          # 0.891593
A    = compute_M2j(0)          # 0.124280  (= M_0)

# Theorem 6.2: verify M_{2j} <= (2j)! * A * beta^{-2j}
for j in range(20):
```

```
    M     = compute_M2j(j)
    bound = math.factorial(2*j) * A * beta**(-2*j)
    assert M <= bound, f"Theorem␣6.2␣fails␣at␣j={j}"

# Appendix A.2: verify a_j^{1/j} -> 0 (G entire)
print("j␣␣␣␣a_j␣␣␣␣␣␣␣␣␣␣␣␣␣␣a_j^{1/j}")
for j in range(1, 22):
    M  = compute_M2j(j)
    aj = 2 * M / math.factorial(2*j)
    print(f"{j:2d}␣␣␣{aj:.4e}␣␣␣{aj**(1/j):.4e}")

# Appendix A.3: Hadamard order 1/2 -- ratio -log(a_j)/(2j*log j) -> 1
print("\nHadamard␣order␣1/2␣(varrho␣=␣1/2):")
for j in range(2, 20):
    M  = compute_M2j(j)
    aj = 2 * M / math.factorial(2*j)
    ratio = -math.log(aj) / (2 * j * math.log(j))
    print(f"j={j:2d}␣␣-log(a_j)/(2j*log␣j)␣=␣{ratio:.4f}␣␣(->␣1␣confirms␣varrho=1/2)")
```

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[10] D. G. Wagner, Multivariate stable polynomials: theory and applications, *Bull. Amer. Math. Soc.* **48** (2011), 53–84. (Theorem 4.14 and Corollary 4.15 are the stability-preservation results used in the present paper.)