CONFINEMENT CERTIFICATES FOR POWERS OF RATIONAL NUMBERS MODULO ONE

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ABSTRACT. Let $p > q > 1$ be coprime integers and let $Z_{p/q}(s, s+t)$ denote the set of positive real ξ whose fractional parts $\{\xi(p/q)^n\}$ lie in $[s, s+t)$ for every $n \geq 0$. We describe a machine-verified development, in the Lean 4 proof assistant, of the emptiness theory of these sets at and beyond the window length $1/p$. Corollary 1.4a of Flatto, Lagarias and Pollington is regenerated as a kernel-checked finite computation, and the theorem of Dubickas — for $1 < q < p < q^2$ the parts $\{\xi(p/q)^n\}$ cannot all lie in a closed interval of length $1/p$ — is formalized in full, including proofs of its combinatorial inputs (the aperiodicity lemma of Dubickas and Novikas and the bispecial criterion for Sturmian words). We then introduce *block certificates* — finite integer witnesses, rechecked from scratch by the Lean proof kernel, whose soundness rests on the aperiodicity lemma alone — and use them to pass the length-$1/p$ line: $Z_{3/2}(1/6, 13/24) = \emptyset$ (window length $3/8 > 1/3$), $Z_{3/2}(961/3600, 2427/3600) = \emptyset$ (length $733/1800$), and explicit unions of total length up to $17/24$, against $20/39$ in print. The closed-interval union statement of Dubickas is replicated exactly as printed; no $\xi \neq 0$ keeps $\|\xi(3/2)^n\| < 1/5$ for every n, with $\|\cdot\|$ the distance to the nearest integer; and thirty windows of length $1/p$ are certified at five bases in the regime $p > q^2$ that Dubickas leaves open. The development is free of unproved assumptions beyond the classical core of Lean.

1. INTRODUCTION

Throughout, $\lfloor x \rfloor$ and $\{x\}$ denote the integer and the fractional part of a real number x. For coprime integers $p > q > 1$ and an interval $[s, s+t)$, let

$$Z_{p/q}(s, s+t) \;:=\; \big\{\xi > 0 \;:\; s \leq \{\xi(p/q)^n\} < s+t \text{ for all integers } n \geq 0\big\},$$

following Flatto, Lagarias and Pollington [FLP95, (1.4)], who call its members *generalized Z-numbers*; Mahler [Mah68] had asked whether $Z_{3/2}(0, 1/2)$ is empty, and its hypothetical members are his Z-numbers. (Dubickas [Dub09AA] defines the same sets with $\xi \neq 0$ in place of $\xi > 0$; several of our statements below hold in that generality and say so explicitly.)

Flatto, Lagarias and Pollington proved that for every $\xi > 0$,

$$\limsup_{n\to\infty} \{\xi(p/q)^n\} - \liminf_{n\to\infty} \{\xi(p/q)^n\} \;\geq\; \frac{1}{p},$$

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so no interval of length less than $1/p$ can trap an orbit [FLP95, Thm. 1.4]. At length exactly $1/p$ they proved emptiness of $Z_{p/q}(s, s+1/p)$ for a dense set of positions s, conjectured it for every $s \in [0, 1-1/p]$, and verified five explicit positions at $p/q = 3/2$:

$$Z_{3/2}(s, s+1/3) = \emptyset \qquad \text{for } s = 0,\ 1/6,\ 1/3,\ 1/2,\ 2/3$$

[FLP95, Cor. 1.4a] (the case $s = 0$ being due to Pollington, as recorded there). Bugeaud [Bug04] proved the conjecture for almost every s and enlarged the explicit list to three intervals of positions, together with the two points 0 and 2/3 [Bug04, Cor. 1]. Dubickas [Dub09AA] then settled it whenever $p < q^2$: for coprime $1 < q < p < q^2$ and *every* real s, the parts $\{\xi(p/q)^n\}$ leave the closed interval $[s, s+1/p]$ (mod 1) infinitely often, for every $\xi \neq 0$ [Dub09AA, Thm. 1]. The case $p > q^2$ remains open [Dub09AA, §4].

Beyond single intervals the printed record concerns finite unions. Dubickas [Dub08, Cor. 1.2] showed that no real ξ has all of $\{\xi(3/2)^n\}$ in the closed union $[8/39, 18/39] \cup [21/39, 31/39]$, of total length $20/39 > 1/2$. In the opposite direction, Kari and Kopra [KK17, Cor. 4.8] exhibit a *nonempty* confinement union of total length 2/3 at the same base, and prove nonconstructively that unions of total length $1 - \varepsilon$ with empty confinement set exist for every $\varepsilon > 0$ [KK17, Thm. 5.9], asking for explicit constructions [KK17, Prob. 6.1]. For $p > q^2$, Akiyama [Aki08, Thms. 2.4, 2.5] builds nonempty confinement sets of arbitrarily small measure.

1.1. **Results.** This paper reports a formalization, in the Lean 4 proof assistant, of the emptiness theory sketched above, together with new, machine-checked entries beyond the printed line. Three groups of results.

First, the two classical pillars are formally verified. Corollary 1.4a of [FLP95] is regenerated as a kernel-checked replay of the finite computation indicated there (§4), and Theorem 1 of [Dub09AA] is formalized in full (§6) — including proofs of its two word-combinatorial inputs: the aperiodicity lemma of Dubickas and Novikas [DN05, Lem. 2] and the bispecial criterion for Sturmian words [Lot02, Prop. 2.1.3, Thm. 2.1.5]. No statement in this paper rests on an unproved assumption beyond the classical core of Lean/Mathlib (§1.2).

Second, a corollary on residues (§5), which we have not found stated in the literature, although it is a folklore-grade consequence of [Dub09AA, Thm. 1]:

Theorem A. *For every real $\xi > 0$ and every integer $m \geq 3$, the sequence of residues $\lfloor \xi(3/2)^n \rfloor \bmod m$ is not eventually constant; indeed past every index it changes again. (Theorem 5.3.)*

The proof formalized here does not use [Dub09AA, Thm. 1]: it combines the five positions of [FLP95, Cor. 1.4a] with an elementary covering of all cells $[r/m, (r+1)/m)$, $m \geq 3$, by the five certified windows. The excluded modulus $m = 2$ is not an oversight: its two cells are Mahler's Z-number problem and its parity twin, both open.

Third, block certificates (§7): finite integer witnesses for statements of the form "no $\xi \neq 0$ has its whole orbit $\{\xi(p/q)^n\}$ inside the finite union U". A certificate is checked by the Lean proof kernel using integer arithmetic alone. We also give a formal algorithm for this computation, and its soundness theorem rests only on the aperiodicity lemma. The scheme has no counting step, so it is not confined to windows of length $1/p$, to single intervals, to the sign $\xi > 0$, or to the base $3/2$. Its headline entries:

Theorem B. *$Z_{3/2}\big(\frac{1}{6}, \frac{13}{24}\big) = \emptyset$, a window of length $\frac{3}{8} > \frac{1}{3}$; $Z_{3/2}\big(\frac{961}{3600}, \frac{2427}{3600}\big) = \emptyset$, a window of length $\frac{733}{1800} = 0.40722\ldots$; and $Z_{4/3}\big(\frac{1}{3}, \frac{5}{8}\big) = \emptyset$, a window of length $\frac{7}{24} > \frac{1}{4}$. Moreover no $\xi \neq 0$ has $\{\xi(3/2)^n\} \in [\frac{1}{6}, \frac{13}{24})$ for all sufficiently large n. (Theorems 7.8 and 7.15.)*

The published emptiness line for single windows stops at length $1/p$ — at five positions in [FLP95], almost everywhere in [Bug04], everywhere in [Dub09AA] — so the first and third entries pass it at two bases, and the second is the longest window our engines certify at any position.

Theorem C. *No $\xi \neq 0$ has all of $\{\xi(3/2)^n\}$ in the explicit union*

$$\Big[0, \tfrac{1}{24}\Big) \cup \Big[\tfrac{1}{16}, \tfrac{3}{16}\Big) \cup \Big[\tfrac{5}{24}, \tfrac{11}{48}\Big) \cup \Big[\tfrac{1}{4}, \tfrac{1}{3}\Big) \cup \Big[\tfrac{17}{48}, \tfrac{3}{8}\Big) \cup \Big[\tfrac{5}{12}, \tfrac{7}{16}\Big) \cup \Big[\tfrac{23}{48}, \tfrac{17}{24}\Big)$$
$$\cup \Big[\tfrac{3}{4}, \tfrac{37}{48}\Big) \cup \Big[\tfrac{19}{24}, \tfrac{5}{6}\Big) \cup \Big[\tfrac{41}{48}, \tfrac{15}{16}\Big) \cup \Big[\tfrac{23}{24}, \tfrac{47}{48}\Big)$$

of total length $\frac{17}{24} = 0.7083\ldots$; likewise for explicit unions of total lengths $\frac{25}{36}$, $\frac{2}{3}$ and $\frac{7}{12}$. Corollary 1.2 of [Dub08] *— the closed union $[8/39, 18/39] \cup [21/39, 31/39]$, total length $20/39$ — is certified exactly as printed. (Theorems 7.11, 7.12 and 7.9.)*

Total length alone decides nothing here: Kari and Kopra's nonempty union [KK17, Cor. 4.8] has total length $2/3$ — exactly that of one of the empty unions above, and more than another. Our unions are records on the curve of [KK17, Prob. 6.1], not a solution of it, which asks for explicit unions of length $1 - \varepsilon$ for every ε.

A third statement concerns not a set the search chose but a named family. Write $\|x\|$ for the distance from x to the nearest integer; the set $\{y : \|y\| < c\}$ is the two-cell union $[0, c) \cup (1 - c, 1)$.

Theorem D. *For every $\xi \neq 0$ there are infinitely many n with $\|\xi(3/2)^n\| \geq \frac{1}{5}$. (Theorem 7.14.)*

Against this, Dubickas [Dub10] exhibits ξ with $\|\xi(3/2)^n\| < 1/3$ for every n, and [Dub09AA, Thm. 1] reaches only arcs of length $1/p = 1/3$, i.e. $\|\cdot\| \leq 1/6$; so the threshold for this family lies in $[1/5, 1/3]$, and our engine finds no certificate at $c = 21/100$ (§7.4).

Finally, thirty certificates live in the regime $p > q^2$, where [Dub09AA, §4] leaves the length-$1/p$ statement open: at each of the bases $5/2$, $7/2$, $9/2$, $10/3$, $11/3$ and each of the six positions $s = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$, $Z_{p/q}(s, s + \frac{1}{p}) = \emptyset$

(Theorem 7.17). We delimit carefully in §7.6 why a table of rational windows is not an approach to the all-positions statement.

1.2. **The formalization.** All statements below are theorems of Lean 4, machine-checked down to the kernel. The development uses no `sorry`, no axioms beyond Lean's classical core (propositional extensionality, choice, quotient soundness, abbreviated "std3" below), and no natively compiled arithmetic: the block certificates of §7 are verified by the kernel's own evaluator, by pure integer arithmetic, with no floating point and no appeal to the external programs that discovered them. That check is not left implicit: Algorithm 1 in §7.2 is the computation the kernel performs, and Example 7.6 carries it out by hand on one entry. In particular the two literature inputs that a paper proof would cite — [DN05, Lem. 2] and the Sturmian criterion of [Lot02] — are proved inside the development, as is the complexity floor of Morse and Hedlund [MH38] that the criterion consumes. Appendix A records the Lean name and status of every statement.

Layout. Section 2 sets up the orbit, its associated integer word, and the aperiodicity lemma. Section 3 proves the elementary dictionary between residues of integer parts and cells of fractional parts. Section 4 regenerates [FLP95, Cor. 1.4a]; Section 5 derives Theorem A; Section 6 formalizes [Dub09AA, Thm. 1]; Section 7 develops the block certificates and proves Theorems B, C and D, and closes with the table of the regime $p > q^2$.

2. The orbit and its word

Fix coprime integers $p > q > 1$, a real $\xi \neq 0$ and a real shift ν. Following [Dub09AA, §2], set

$$x_n \;:=\; \lfloor \xi(p/q)^n + \nu \rfloor, \qquad y_n \;:=\; \{\xi(p/q)^n + \nu\},$$

and attach to the orbit the integer word[1] $\mathbf{w} = s_0 s_1 s_2 \cdots$,

$$s_n \;:=\; q x_{n+1} - p x_n.$$

Its basic properties, all formalized:

Proposition 2.1 ([Dub09AA, (3)])**.** *For every n, $s_n = p y_n - q y_{n+1} - (p-q)\nu$. In particular the carry word is entirely controlled by consecutive fractional parts.*

Proof sketch. Eliminate ξ between $x_n + y_n = \xi(p/q)^n + \nu$ and the same identity at $n+1$, exactly as in [Dub09AA, §2]. □

Proposition 2.2 ([Dub09AA, (4)])**.** *For all $n, m \geq 0$,*

$$x_{n+m} \;=\; (p/q)^m x_n + \frac{(p/q)^{m-1} s_n + (p/q)^{m-2} s_{n+1} + \cdots + s_{n+m-1}}{q}.$$

[1] The word is introduced in [DN05, §2] and [Dub09AA, (2)] without a name; we call it the *carry word*, since s_n records how the integer part carries under one multiplication by p/q.

Proof sketch. Iterate $x_{n+1} = (p/q)x_n + s_n/q$, which is the definition of s_n rearranged. □

Proposition 2.3. *If $|s_n| \le C$ for every n, with $C \ge 0$, then*

$$|x_n| + C \;\le\; (|x_0| + C)\,(p/q)^n \qquad \text{for every } n.$$

Proof sketch. From $qx_{n+1} = px_n + s_n$ and $q+1 \le p$ one gets $q(|x_{n+1}|+C) \le p(|x_n|+C)$, and induction closes. [Dub09AA, §3] uses this with $C = 2p$, giving his constant $c = |x_0| + 2p$; under the confinement hypothesis of §6 the formalization obtains the slightly sharper $C = p$ (Remark 6.3). □

Call an infinite word w *periodic* if there are n_0, t with $w_{n+t} = w_n$ for all $n \ge n_0$, and *aperiodic* otherwise — Dubickas's convention [Dub09AA, §2], i.e. periodicity is ultimate periodicity. The engine of everything in this paper is:

Lemma 2.4 ([DN05, Lem. 2]; [Dub09AA, Lem. 2])**.** *For coprime $p > q > 1$, every $\xi \ne 0$ and every shift ν, the carry word $\mathbf{w} = s_0 s_1 s_2 \cdots$ is aperiodic.*

Proof sketch. The error $e_n := x_n - \xi(p/q)^n = \nu - y_n$ is bounded. If $\mathbf{w}$ were periodic with period P from index N on, then along each residue class modulo P the summation identity (Proposition 2.2) makes e satisfy an *expanding* affine recursion $e_{n+P} = (p/q)^P e_n + D_n$ with D_n eventually P-periodic; a bounded orbit of an expanding affine map is its fixed point, so e — hence y — is P-periodic from some point on. Then $x_{n+P} - x_n = \xi(p/q)^n((p/q)^P - 1)$ is an integer for all $n \ge N$, so $M := x_{N+P} - x_N$ satisfies $q^k \mid M$ for every k by coprimality, forcing $M = 0$ and therefore $\xi = 0$. The formal proof follows [DN05] in its use of the expanding affine map, but closes with this direct divisibility argument instead of reducing to [DN05, Lem. 1]. It also keeps the liberalization noted in [Dub09AA]: $\xi \ne 0$ rather than the $\xi > 0$ of [DN05] (the shift ν is arbitrary there already). □

3. Floor residues are cells

The translation between residue statements about integer parts and confinement statements about fractional parts is elementary.

Proposition 3.1. *Let $m \ge 1$ and let t be real. Then*

$$\{t/m\} \;=\; \frac{(\lfloor t \rfloor \bmod m) + \{t\}}{m},$$

and consequently, for an integer r,

$$\lfloor t \rfloor \equiv r \pmod m \quad\Longleftrightarrow\quad \{t/m\} \in \Big[\frac{r}{m}, \frac{r+1}{m}\Big),$$

where on the left r denotes the canonical residue $0 \le r < m$.

Proof sketch. Split $\lfloor t \rfloor = m\lfloor \lfloor t \rfloor / m \rfloor + (\lfloor t \rfloor \bmod m)$ and observe that the quantity $((\lfloor t \rfloor \bmod m) + \{t\})/m$ lies in $[0, 1)$, so it is exactly the fractional part of t/m; the equivalence follows by comparing endpoints. □

Applied along an orbit, with $\eta := \xi/m$: the residue $\lfloor \xi(3/2)^n \rfloor \bmod m$ equals r for all $n \geq N$ exactly when the orbit of η is confined, from index N on, to the cell $[r/m, (r+1)/m)$ of length $1/m$. Two pieces of glue accompany the dictionary. First, the sets $Z_{p/q}$ are monotone in the window: if $s \leq s'$ and $s' + t' \leq s + t$ then $Z_{p/q}(s', s'+t') \subseteq Z_{p/q}(s, s+t)$. Second, a *shift trick* upgrades eventual confinement to genuine confinement:

Lemma 3.2. *Let $p, q \geq 1$, $\xi > 0$, and suppose $\{\xi(p/q)^n\} \in [s, s+t)$ for all $n \geq N$. Then $\xi(p/q)^N \in Z_{p/q}(s, s+t)$.*

Proof sketch. The orbit of $\xi(p/q)^N$ is the tail of the orbit of ξ from index N on. □

Hence every certified empty window kills eventual confinement, not only confinement from index 0; the same trick appears in the proof of Theorem 1 in [Dub09AA, §3].

4. Escape certificates and Corollary 1.4a

Write $f_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1$ for the *linear mod one transformation* of [FLP95, §2]. Two results of [FLP95, §3] reduce emptiness of $Z_{p/q}(s, s+1/p)$ to a finite computation. Theorem 3.2 there: with $f = f_{p/q,\{(p-q)s\}}$, if only finitely many $x \in [0,1)$ satisfy $0 \leq f^n(x) < q/p$ for all n, then $Z_{p/q}(s, s+1/p) = \emptyset$. Theorem 3.4 there: the survivor set is finite as soon as the orbit of 0 under f reaches $[q/p, 1)$ at some finite time N. Both are formalized; their combination, specialized to $p/q = 3/2$ (where $\{(p-q)s\} = s$ for $s \in [0,1)$), reads:

Proposition 4.1 ([FLP95, Thms. 3.2, 3.4])**.** *Let $0 \leq s < 1$ and suppose $f^N_{3/2,s}(0) \geq 2/3$ for some N. Then $Z_{3/2}(s, s+1/3) = \emptyset$.*

At four of the five positions of [FLP95, Cor. 1.4a] the required escape of the origin is a short exact-rational computation, reproduced in Table 1 — carrying out the finite computation indicated after Theorem 1.4 of [FLP95].

s	orbit of 0 under $f_{3/2,s}$	escape time N
0	fixed at 0 — no escape	(separate argument)
1/6	$0 \mapsto 1/6 \mapsto 5/12 \mapsto 19/24$	3
1/3	$0 \mapsto 1/3 \mapsto 5/6$	2
1/2	$0 \mapsto 1/2 \mapsto 1/4 \mapsto 7/8$	3
2/3	$0 \mapsto 2/3$	1

Table 1. The escape computations behind Theorem 4.2. Each step is one affine branch of $f_{3/2,s}$; at $s = 1/2$ the middle step uses the upper branch ($3/4 + 1/2 \geq 1$).

Theorem 4.2 ([FLP95, Cor. 1.4a])**.** *$Z_{3/2}(s, s+1/3) = \emptyset$ for $s = 0$, $1/6$, $1/3$, $1/2$ and $2/3$.*

Proof sketch. For $s \neq 0$, Proposition 4.1 with the witnesses of Table 1. At $s = 0$ the origin is a fixed point of $f_{3/2,0}$ and no escape witness exists; the formalization proves this case by the endpoint analysis of the survivor set at $\alpha = 0$, as in [FLP95], where the case is credited to Pollington. □

5. Residue non-capture

The five windows of Theorem 4.2 have length $1/3$ and sit at spacing $1/6$. They cover every cell of every modulus $m \geq 3$:

Lemma 5.1 (covering)**.** *For every $m \geq 3$ and $0 \leq r < m$ there is a $k \in \{0, 1, 2, 3, 4\}$ with*

$$km \leq 6r \qquad \text{and} \qquad 6r + 6 \leq (k + 2)m,$$

that is, $[r/m, (r + 1)/m) \subseteq [k/6,\, k/6 + 1/3)$.

Proof sketch. For $m \geq 5$, choose k by the position of $6r$ among the multiples of m; for $m = 3, 4$ the seven cells are checked one by one. The tight cases are $m = 3$, where each cell *is* one of the windows, and $m = 4$ at $r = 1, 3$, where the right endpoints coincide. □

Proposition 5.2. *For every $m \geq 3$ and $0 \leq r < m$, $Z_{3/2}\big(r/m, (r+1)/m\big) = \emptyset$.*

Proof sketch. The cell has length $1/m \leq 1/3$ and lies, by Lemma 5.1, inside one of the five windows of Theorem 4.2; window monotonicity of the sets $Z_{3/2}$ transfers the emptiness. □

Theorem 5.3. *Let $\xi > 0$ and $m \geq 3$. Then the residues $\lfloor \xi(3/2)^n \rfloor \bmod m$ are not eventually constant. Equivalently, for every N there is an $n \geq N$ with $\lfloor \xi(3/2)^n \rfloor \not\equiv \lfloor \xi(3/2)^N \rfloor \pmod m$. In particular ($\xi = 1$) the residues $\lfloor (3/2)^n \rfloor \bmod m$ do not stabilize for any $m \geq 3$.*

Proof sketch. Suppose $\lfloor \xi(3/2)^n \rfloor \equiv r \pmod m$ for all $n \geq N$. By the dictionary (Proposition 3.1) the orbit of $\eta = \xi/m$ is eventually confined to the cell $[r/m, (r + 1)/m)$, so by the shift trick (Lemma 3.2) the point $\eta(3/2)^N$ lies in $Z_{3/2}(r/m, (r + 1)/m)$, which Proposition 5.2 declares empty. □

Remark 5.4 (claim level). Theorem 5.3 is a one-line corollary of [Dub09AA, Thm. 1]: the all-positions emptiness there (Theorem 6.9 below) is strictly stronger than the five positions used here. We simply have not found the residue form stated anywhere. The proof above is nevertheless independent of [Dub09AA, Thm. 1] — it needs only Theorem 4.2 and the covering — and the content here is the verified proof object.

Remark 5.5 ($m = 2$). The modulus 2 is deliberately excluded: the two cells $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ of the rescaled orbit are exactly Mahler's Z-number problem [Mah68] and its parity twin, both open.

6. Powers of a rational number modulo one cannot lie in a small interval

This section describes the formalization of Theorem 1 of [Dub09AA], following its proof: confinement forces a two-letter carry word (§6.2), the word is Sturmian (§6.3), and $p < q^2$ is incompatible with Sturmian complexity (§6.4). Throughout, $\nu = -s$, so that confinement of $\{\xi(p/q)^n\}$ to the closed interval $[s, s + 1/p]$ (mod 1) reads $0 \le y_n \le 1/p$ for all n.

6.1. **Combinatorial input.** For an infinite word w let $p(w, m)$ denote the number of distinct factors of length m occurring in w; w is *Sturmian* if $p(w, m) = m + 1$ for every m [Dub09AA, §2], the least complexity an aperiodic word can have by the theorem of Morse and Hedlund [MH38]. The formalization proves the criterion quoted in [Dub09AA, §2]:

Lemma 6.1 (after [Lot02, Prop. 2.1.3, Thm. 2.1.5])**.** *An aperiodic word w on a two-letter alphabet $\{U, V\}$ such that no finite word u has both UuU and VuV as factors of w is Sturmian.*

Proof sketch. Only this direction of the classical equivalence is formalized, and by a shorter route than the balance argument of [Lot02]: a pair UuU, VuV is exactly the obstruction to uniqueness of the right special factor. If two distinct factors of length m each admitted both one-letter continuations, reading the longest common suffix of the two produces such a pair; hence at most one factor of each length is right special, truncation gives $p(w, m+1) \le p(w, m) + 1$, and induction yields $p(w, m) \le m + 1$. The matching floor $p(w, m) \ge m + 1$ is the Morse–Hedlund inequality [MH38], also proved in the development. □

6.2. **Confinement forces two letters.**

Lemma 6.2. *Let $p > q > 1$, let $0 \le y_n \le 1/p$ for all n, and put $k := \lceil (p - q)s - q/p \rceil$. Then $s_n \in \{k, k + 1\}$ for every n.*

Proof sketch. By Proposition 2.1 at $\nu = -s$, the bounds $0 \le y_n \le 1/p$ squeeze each s_n into an interval of length $1 + q/p < 2$ with left endpoint $(p-q)s-q/p$, which contains at most the two integers k, $k+1$. ([Dub09AA, Thm. 3] states the alphabet as $\{k, k + 1\}$ "for some $k \in \mathbb{Z}$"; the formal statement carries the explicit k.) □

Remark 6.3. Under the same hypotheses, and with $0 \le s < 1$ as in the source, the formalization shows $0 \le s_n \le p$, hence $|s_n| \le p$ — slightly sharper than the bound $|s_n| < 2p$ read off from the alphabet $\mathcal{A}(p, q, -s)$ in [Dub09AA, §3] — so Proposition 2.3 applies with $C = p$ and growth constant $c = |x_0| + p$.

6.3. **The carry word is Sturmian.**

Lemma 6.4 (no bispecial pair)**.** *Under the hypotheses of Lemma 6.2, no finite word u on $\{k, k+1\}$ has both kuk and $(k+1)u(k+1)$ as factors of $\mathbf{w}$.*

Proof sketch. This is the arithmetic heart of [Dub09AA, Thm. 3]. Suppose the two blocks, of common length M, occur at positions i and j. The difference of the two instances of the summation identity (Proposition 2.2), taken over the two blocks, telescopes against the letter differences — which vanish except at the two ends — and gives

$$y_{j+M} - y_{i+M} = (p/q)^M (y_j - y_i) - \frac{(p/q)^{M-1} + 1}{q}.$$

The bounds $0 \le y \le 1/p$ then force $((p/q)^{M-1}+1)/q \le ((p/q)^M+1)/p$, i.e. $p \le q$, a contradiction. □

Theorem 6.5 ([Dub09AA, Thm. 3])**.** *Let $p > q > 1$ be coprime, let $\xi \neq 0$ and let s be real. Suppose $\{\xi(p/q)^n - s\} \le 1/p$ for every $n \ge 0$ — that is, the parts $\{\xi(p/q)^n\}$ all lie in the closed interval $[s, s+1/p]$ of the torus $\mathbb{R}/\mathbb{Z}$. Then the carry word $\mathbf{w}$ is a Sturmian word on the alphabet $\{k, k+1\}$, $k = \lceil (p-q)s - q/p \rceil$.*

Proof sketch. By Lemma 6.2 the word lives on $\{k, k+1\}$; it is aperiodic by Lemma 2.4; and it has no bispecial pair by Lemma 6.4. Lemma 6.1 concludes. □

6.4. **The endgame: $p < q^2$.**

Lemma 6.6. *For $p > q \ge 1$ and $\xi \neq 0$ there is an m_0 such that $x_a \neq x_b$ whenever $m_0 \le a < b$: the integer parts are eventually strictly increasing if $\xi > 0$ and strictly decreasing if $\xi < 0$, because consecutive orbit points eventually differ by more than 1.*

Theorem 6.7. *Let $1 < q < p < q^2$ be coprime and let $\xi \neq 0$, $0 \le s < 1$. Then it is not the case that $\{\xi(p/q)^n - s\} \le 1/p$ for every $n \ge 0$; that is, no orbit is confined to the closed window $[s, s+1/p]$ modulo one.*

Proof sketch. Suppose it were. By Theorem 6.5 the carry word is Sturmian, so among the $m+2$ factors of length m read at positions $0, \dots, m+1$ two coincide, say at $i < j \le m+1$. The summation identity then gives

$$x_{j+m} - x_{i+m} = (p/q)^m (x_j - x_i),$$

and for $m \ge m_0$ the left side is nonzero (Lemma 6.6), so q^m divides the nonzero integer $x_j - x_i$ by coprimality. The growth bound (Proposition 2.3 with $c = |x_0| + p$, Remark 6.3) caps $|x_j - x_i| \le cp(p/q)^m$, whence $(q^2/p)^m \le cp$. Since $p \le q^2 - 1$, Bernoulli's inequality $(1 + 1/(q^2-1))^m \ge 1 + m/(q^2-1)$ makes this false for $m > cp\,(q^2-1)$. (Here [Dub09AA] extracts mth roots instead; the formal proof replaces that step by Bernoulli's inequality, to the same effect.) □

Theorem 6.8 ([Dub09AA, Thm. 1])**.** *Let p, q be coprime with $1 < q < p < q^2$ and let $\xi \neq 0$. Then for every real s and every N there is an $n \ge N$ with $\{\xi(p/q)^n - s\} > 1/p$; equivalently, $\{\xi(p/q)^n\}$ lies outside the closed interval $[s, s+1/p]$ (mod 1) for infinitely many n.*

Proof sketch. If not, the orbit of $\xi(p/q)^N$, at the position $\{s\}$, violates Theorem 6.7 — the shift trick again, exactly as in [Dub09AA, §3]. □

Theorem 6.9. *For coprime* $1 < q < p < q^2$ *and* every *real* s, $Z_{p/q}(s, s + 1/p) = \emptyset$. *In particular, for every real* s,

$$Z_{3/2}\big(s, s + \tfrac{1}{3}\big) = \emptyset.$$

Proof sketch. A member ξ of the set would keep $\{\xi(p/q)^n\}$ in the half-open window $[s, s + 1/p)$, which lies inside the closed one, contradicting Theorem 6.8. The position s may be any real number since the sets $Z_{p/q}$ are defined by the window $[s, s+1/p)$ itself, not by its reduction modulo one. □

This is the all-positions strengthening of Theorem 4.2 and of [Bug04, Cor. 1]; the base $3/2$ qualifies since $3 < 4 = 2^2$. The five positions of Theorem 4.2 thus receive a second, independent formal proof — a useful cross-check, since the two developments share only the definitions.

7. Block certificates

The certificates of §4 live on the length-$1/p$ line: their finiteness step (Theorem 3.4 of [FLP95]) needs the window $[s, s + 1/p)$, and the counting endgame of §6 needs $p < q^2$. This section describes a different certificate scheme with neither restriction. It applies to finite unions of rational intervals, at every admissible base, for every $\xi \neq 0$, and its soundness rests on Lemma 2.4 alone.

7.1. **The scheme.** Fix coprime $p > q > 1$ and take $\nu = 0$, so $y_n = \{\xi(p/q)^n\}$. Proposition 2.1 reads

$$qy_{n+1} \;=\; py_n - s_n, \qquad s_n \in \{-q+1, \dots, p-1\},$$

so confinement of the orbit to a finite union U of intervals is survival of y under a $(p+q-1)$-branch piecewise-affine map — four branches at $3/2$, six at $4/3$.

Definition 7.1. A *block certificate* for a finite union U of intervals at the base p/q consists of a positive integer D such that every endpoint mentioned is a multiple of $1/D$; a convention flag declaring all intervals of the certificate half-open $[a/D, b/D)$ or all closed $[a/D, b/D]$; a *funnel* of interval lists

$$T_0 := U \;\supseteq\; T_1 \;\supseteq\; \cdots \;\supseteq\; T_K =: H,$$

whose last list H we call the *blocks*; and a partition of the blocks into *ranks* $0, 1, 2, \dots$ (by default all blocks have rank 0). The certificate is *valid* if

- (F) for every $I \in T_k$ $(0 \le k < K)$, every letter $s \in \{-q+1, \dots, p-1\}$ and every $J \in U$, the set $\{y \in J : (py - s)/q \in I\}$ is empty or contained in a *single* interval of T_{k+1};
- (D) writing $I \xrightarrow{s} J$ for blocks $I, J \in H$ when some $y \in I$ has $(py - s)/q \in J$: no edge increases the rank, and each block has at most one outgoing edge (counted with its letter s) landing in its own rank;

(B) $1 < q < p$ and $\gcd(p, q) = 1$.

All three checks are inequalities between integers over the common denominator D, decidable by the proof kernel's evaluator; §7.2 writes them out as Algorithm 1.

Theorem 7.2 (soundness)**.** *Let c be a valid block certificate for U at the base p/q. Then no $\xi \neq 0$ has $\{\xi(p/q)^n\} \in U$ for every $n \geq 0$; nor for every sufficiently large n. Consequently, if a window $[s, s+t)$ is covered by U, then $Z_{p/q}(s, s+t) = \emptyset$.*

Proof sketch. Suppose the orbit is confined to U. Condition (F) pushes it down the funnel: if $y_n \in J \in U$ and $y_{n+1} \in I \in T_k$, then y_n lies in the piece $\{y \in J : (py - s_n)/q \in I\}$, hence in T_{k+1}; by induction the whole orbit lies in the blocks H. The block itinerary follows the edges of (D), so its rank is non-increasing, hence eventually constant; from that moment each block has at most one available outgoing edge in its own rank, so the itinerary is deterministic, and by pigeonhole on the finite block list it is eventually periodic — along with the letters s_n read on the edges. An eventually periodic carry word contradicts Lemma 2.4. The eventual form follows by the shift trick, and the Z form by covering. Nothing here counts survivors, uses the sign of ξ, or compares p with q^2. □

Remark 7.3 (the checks have teeth)*.* Three formal counterexamples guard the definition. The full interval $[0, 1)$ passes the funnel condition trivially but its single block has all four outgoing edges, so (D) rejects it, and the same happens for $[0, 1]$ even with a rank of its own (a self-loop is an edge of equal rank). And (B) is not a formality: at the non-coprime "base" $4/2 = 2$ the point $\xi = 1/3$ has the genuinely periodic orbit $1/3 \to 2/3 \to 1/3$ with periodic carry word $0, 2, 0, 2, \ldots$, so a certificate at such a base would prove a falsehood; the validity check refuses it.

Remark 7.4 (what a certificate does and does not assert)*.* The certified sets are not vacuous for the interval dynamics: at $[1/6, 13/24)$ the surviving set of the four-branch map is the 3-cycle $4/19 \to 6/19 \to 9/19$ (letters $0, 0, 1$), which is nonempty. What the certificate rules out is a *real number* whose fractional parts follow the dynamics forever, and that is exactly what aperiodicity of the carry word forbids.

7.2. **What the kernel checks.** Conditions (F), (D) and (B) are stated above as assertions about real orbits. What the kernel evaluates, when it reduces the certificate's validity to `true`, is the finite integer computation of Algorithm 1. We give it in full, so that a certified statement can be audited — or re-checked in any language — without reading Lean.

One integer coordinate. Every interval of a certificate is a pair of integers (a, b) standing for $[a/D, b/D\rangle$, and the convention flag κ decides once and

Algorithm 1 ok(c): the check the proof kernel evaluates on a block certificate c. Every quantity is an integer; $\preccurlyeq_\kappa$ is the certificate's own right-endpoint convention and $\preccurlyeq_{\bar\kappa}$ its flip. The rank $\rho(I)$ is the index of the first stratum containing I, and 0 if there are no strata; $I \xrightarrow{s} J$, for $I = (a, b)$ and $J = (c, d)$, means $\max(pa,\, qc + sD) \preccurlyeq_\kappa \min(pb,\, qd + sD)$.

Input: $c = (p, q, D, \kappa, U, T_1, \dots, T_K,\, S_0, \dots, S_{R-1})$: base p/q, denominator D, convention κ, certified set U, funnel levels T_k, rank strata S_r (possibly none). All intervals are pairs of integers.

Output: **true** only if no $\xi \neq 0$ keeps $\{\xi(p/q)^n\}$ in $\bigcup U$ for all n (Theorem 7.2).

if $D \leq 0$ or $q \leq 1$ or $p \leq q$ or $\gcd(p, q) \neq 1$ **then return false** ▷ (B) admissible base
end if
$\mathcal{A} \leftarrow \{-q + 1, \dots, p - 1\}$; $T_0 \leftarrow U$; $H \leftarrow T_K$ ▷ $p + q - 1$ carry letters; H = the blocks

for all $k = 0, \dots, K-1$ and $I = (a, b) \in T_k$ and $s \in \mathcal{A}$ and $J = (c, d) \in U$ **do** ▷ (F) the funnel
    $A \leftarrow qa + sD$; $C \leftarrow qb + sD$; $B \leftarrow pc$; $E \leftarrow pd$
    if $\min(C, E) \preccurlyeq_{\bar\kappa} \max(A, B)$ **then**
        continue ▷ the piece is empty
    end if
    if no $(e, f) \in T_{k+1}$ has $pe \leq \max(A, B)$ and $\min(C, E) \leq pf$ **then**
        return false ▷ piece not inside one interval
    end if
end for

for all $I \in H$ **do** ▷ (D) the block graph
    $\mathcal{E} \leftarrow \left\{(s, J) \in \mathcal{A} \times H \;:\; I \xrightarrow{s} J\right\}$
    if $\rho(J) > \rho(I)$ for some $(s, J) \in \mathcal{E}$ **then**
        return false ▷ an edge raises the rank
    end if
    if $\#\{(s, J) \in \mathcal{E} \;:\; \rho(J) = \rho(I)\} > 1$ **then**
        return false ▷ two edges inside one rank
    end if
end for
return true

for all whether $\rangle$ means) or]. Write

$$x \preccurlyeq_\kappa y \;:\Longleftrightarrow\; \begin{cases} x \leq y, & \kappa = \text{closed}, \\ x < y, & \kappa = \text{half-open}, \end{cases} \qquad x \preccurlyeq_{\bar\kappa} y \;:\Longleftrightarrow\; \neg\,(y \preccurlyeq_\kappa x),$$

so that $[u, v\rangle$ is nonempty iff $u \preccurlyeq_\kappa v$ and empty iff $v \preccurlyeq_{\bar\kappa} u$.

Substituting

$$z \;:=\; pD\,y$$

clears every denominator at once. For intervals (a, b), (c, d) and a letter s,

$$y \in [\tfrac{c}{D}, \tfrac{d}{D}\rangle \iff pc \le z \;\text{ and }\; z \preccurlyeq_\kappa pd,$$
$$\frac{py - s}{q} \in [\tfrac{a}{D}, \tfrac{b}{D}\rangle \iff qa + sD \le z \;\text{ and }\; z \preccurlyeq_\kappa qb + sD,$$

the second because multiplying $a/D \le (py-s)/q$ by $qD > 0$ gives $qa+sD \le pDy$, and likewise on the right. Hence the piece $\{y \in J : (py - s)/q \in I\}$ of condition (F), with $I = (a, b)$ and $J = (c, d)$, is in the coordinate z the interval $[\max(A, B), \min(C, E)\rangle$ with

$$A = qa + sD, \qquad C = qb + sD, \qquad B = pc, \qquad E = pd.$$

It is empty exactly when $\min(C, E) \preccurlyeq_{\bar\kappa} \max(A, B)$, and it lies inside the interval (e, f) of the next level exactly when $pe \le \max(A, B)$ and $\min(C, E) \le pf$. The edge test of condition (D) is the same computation read forward: the branch-s image of the block $I = (a, b)$ meets the block $J = (c, d)$ iff the integer intervals $[pa, pb\rangle$ and $[qc+sD, qd+sD\rangle$ overlap. No rational number, and a fortiori no floating-point number, occurs anywhere.

Remark 7.5 (reading the algorithm against the source). The Lean text follows the algorithm line by line: the base test and the two loops are `Cert.ok`, the alphabet is `carries`, the funnel is `funnelOk`, `levelOk`, `pieceOk` with A, C named `pieceA`, `pieceC`, and the block graph is `hits`, `outEdges`, `rank`, `funcOk`. Two differences are cosmetic and both serve the evaluator. First, the source expands each max and min into the disjunction of its two comparisons (`leMax`, `minLe`), because the order-theoretic max on $\mathbb{Z}$ unfolds through `LinearOrder` projections that the kernel evaluator does not reduce well; so "$\min(C, E) \preccurlyeq_{\bar\kappa} \max(A, B)$" appears as the four tests $C \preccurlyeq_{\bar\kappa} A$, $C \preccurlyeq_{\bar\kappa} B$, $E \preccurlyeq_{\bar\kappa} A$, $E \preccurlyeq_{\bar\kappa} B$, and the overlap test of (D) appears as its four cross comparisons. Second, $\preccurlyeq_{\bar\kappa}$ is implemented as its own relation rather than as a negation, which is the same Boolean. Each of the ten entries of Table 2 is discharged by a one-line theorem `c.ok = true := by decide`, i.e. by kernel reduction of the Boolean above — never by `native_decide`.

Example 7.6 (a certificate checked by hand)**.** The $(5, 2)$ entry of Table 2 is small enough to run Algorithm 1 on paper. It is

$$p = 5, \quad q = 2, \quad D = 25, \quad \kappa = \text{half-open},$$
$$U = \{(5, 10)\}, \quad K = 1, \quad T_1 = \{(7, 9)\}, \quad \text{no strata},$$

that is, the window $[\frac{1}{5}, \frac{2}{5})$ with the single block $[\frac{7}{25}, \frac{9}{25})$; the alphabet is $\mathcal{A} = \{-1, 0, 1, 2, 3, 4\}$ and $z = 125\,y$. The base test passes ($25 > 0$, $1 < 2 < 5$, $\gcd(5, 2) = 1$). The funnel has one level, so (F) is six piece tests, all with $I = (7, 9)$ and $J = (5, 10)$, hence $A = 14 + 25s$, $C = 18 + 25s$, $B = 25$, $E = 50$:

s	$\max(A,B)$	$\min(C,E)$	verdict
-1	25	-7	empty
0	25	18	empty
1	39	43	piece $[39,43) \subseteq [35,45)$, the one interval of T_1
2	64	50	empty
3	89	50	empty
4	114	50	empty

For (D) the one block $I = (7,9)$ has image window $[pa, pb\rangle = [35,45)$, and $[qc+sD, qd+sD\rangle = [14+25s,\ 18+25s)$ meets it only for $s = 1$: one outgoing edge, all ranks 0, so (D) holds and $\mathrm{ok}(c) =$ **true**. Reading the surviving letter back through Theorem 7.2: a confined orbit would have the constant carry word $1, 1, 1, \ldots$, which Lemma 2.4 forbids. This is Theorem 7.16.

Remark 7.7 (the size of the checks). Algorithm 1 performs $\sum_{k<K} |T_k|\,(p+q-1)\,|U|$ piece tests and $(p+q-1)|H|^2$ edge tests, each a handful of integer comparisons. The largest entry of Table 2 is the 17/24 union, with 73 612 piece tests and 40 000 edge tests; the 25/36 union needs 7416 and 1156, and the $(5,2)$ entry of Example 7.6 six of each. The integers stay small as well: the largest denominator anywhere in the paper is $D = 48 \cdot 3^{17} = 6\,198\,727\,824$, ten digits, for that same 17/24 union. In every entry the generator chose $D = D_0\,p^K$, with D_0 the denominator of the set's own endpoints and K the funnel depth — a piece endpoint has the form $(qa + sD)/(pD)$, so each funnel level refines the grid by one factor of p, which is why depth rather than width is the binding cost. Nothing in the algorithm requires that shape: the check reads whatever denominator the certificate carries.

7.3. **The certified entries.** Table 2 lists the ten certificates verified by the kernel at the head of the atlas; the thirty further entries of the regime $p > q^2$ are Table 3. Each row is a theorem of the development. The certificates were produced by an exact-arithmetic search program, but nothing depends on it: the kernel rechecks conditions (F), (D), (B) from scratch by Algorithm 1, and each entry theorem below is Theorem 7.2 applied to its certificate.

Theorem 7.8. *$Z_{3/2}\big(\frac{1}{6}, \frac{13}{24}\big) = \emptyset$ and $Z_{3/2}\big(\frac{961}{3600}, \frac{2427}{3600}\big) = \emptyset$. Moreover, for every $\xi \neq 0$ and every N it is not the case that $\{\xi(3/2)^n\} \in [\frac{1}{6}, \frac{13}{24})$ for all $n \geq N$.*

The first window has length $3/8 > 1/3$: the published single-window line — length $1/p$, at five positions in [FLP95], almost every position in [Bug04], every position in [Dub09AA] — is passed. The second, of length $733/1800 = 0.40722\ldots$, is the longest window our search certifies at any position; the search finds nothing at any position at length 1467/3600 or beyond.

Theorem 7.9 ([Dub08, Cor. 1.2], as printed)**.** *No $\xi \neq 0$ has $\{\xi(3/2)^n\} \in [8/39, 18/39] \cup [21/39, 31/39]$ for every $n \geq 0$.*

base	certified set	total length	convention	depth K	blocks
3/2	$[\frac{1}{6}, \frac{13}{24})$	$3/8 = 0.375$	half-open	8	3
3/2	$[\frac{961}{3600}, \frac{2427}{3600})$	$\frac{733}{1800} = 0.4072\ldots$	half-open	13	2
3/2	$[\frac{8}{39}, \frac{18}{39}] \cup [\frac{21}{39}, \frac{31}{39}]$	$20/39 = 0.5128\ldots$	closed	3	12 (4 ranks)
3/2	$[0, \frac{1}{6}) \cup [\frac{1}{4}, \frac{1}{3}) \cup [\frac{5}{12}, \frac{2}{3}) \cup [\frac{3}{4}, \frac{5}{6})$	$7/12 = 0.5833\ldots$	half-open	7	1
3/2	$[0, \frac{1}{9}) \cup [\frac{1}{6}, \frac{4}{9}) \cup [\frac{1}{2}, \frac{5}{9}) \cup [\frac{11}{18}, \frac{7}{9}) \cup [\frac{5}{6}, \frac{8}{9})$	$2/3$	half-open	9	12
3/2	$[0, \frac{1}{12}) \cup [\frac{1}{9}, \frac{11}{36}) \cup [\frac{4}{9}, \frac{2}{3}) \cup [\frac{25}{36}, \frac{3}{4}) \cup [\frac{5}{6}, \frac{8}{9}) \cup [\frac{11}{12}, 1)$	$25/36 = 0.6944\ldots$	half-open	11	17
3/2	the eleven-interval union of Theorem C	$17/24 = 0.7083\ldots$	half-open	17	100
3/2	$[0, \frac{1}{5}) \cup [\frac{4}{5}, 1)$ (the arc $\|\cdot\| < \frac{1}{5}$)	$2/5$	half-open	1	2
4/3	$[\frac{1}{3}, \frac{5}{8})$	$7/24 = 0.2916\ldots$	half-open	5	2
5/2	$[\frac{1}{5}, \frac{2}{5})$	$1/5$	half-open	1	1

TABLE 2. The certified atlas entries. Each certificate forbids every $\xi \neq 0$ from keeping $\{\xi(p/q)^n\}$ in the listed set for all n; for the single windows this yields the Z-set statements of Theorem B.

Remark 7.10 (closed versus half-open). The convention flag of Definition 7.1 is not cosmetic here. Emptiness for a larger set is a stronger statement, so the closed form of Theorem 7.9 implies its half-open shadow and not conversely, and the gap is real rather than notational: the closed union contains the 4-cycle

$$\tfrac{6}{13} \to \tfrac{9}{13} \to \tfrac{7}{13} \to \tfrac{4}{13} \to \tfrac{6}{13}$$

of the interval map, which runs through the right endpoint $18/39 = 6/13$ and is invisible to the half-open set. That cycle is also why the rank partition of (D) exists: with the endpoints included, the hold set acquires transients feeding two disjoint cycles, and no partition of the blocks with one outgoing edge each exists at any funnel depth we examined (up to 40). What does exist, at depth 3, is the 4-rank stratification of Table 2: the 4-cycle at rank 0, the two blocks holding the 2-cycle $\{\frac{2}{5}, \frac{3}{5}\}$ at rank 2, transients at ranks 1 and 3. The other nine entries use the plain functional condition.

Theorem 7.11. *No $\xi \neq 0$ has $\{\xi(3/2)^n\}$, for every $n \geq 0$, in the union of total length $7/12$, in the union of total length $2/3$, or in the union of total length $25/36$ listed in Table 2.*

Theorem 7.12 (the union record)**.** *No $\xi \neq 0$ has $\{\xi(3/2)^n\}$, for every $n \geq 0$, in the eleven-interval union of Theorem C, of total length $\frac{17}{24} = 0.7083\ldots$*

Remark 7.13 (the largeness curve, and where the kernel stops). Against these set Corollary 4.8 of [KK17]: the union $[0, \frac{1}{6}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{5}{6}, 1]$ has a nonempty confinement set of total length $2/3$ — exactly that of the second union of Theorem 7.11, and more than the first. Total length decides nothing; position does. Theorem 7.12 exceeds the largest explicit empty union we found in print, the $20/39$ of [Dub08, Cor. 1.2], by a factor 1.38.

The search climbs further than the kernel follows, and it is worth being exact about where the two part company. Certificates exist at $43/60 = 0.7166\ldots$ (funnel depth 20, 433 blocks) and at 0.7417 (240 cells, 3783 surviving components at depth 30, merging to 2512 blocks). The first has a funnel some three times wider than the one checked here, the second wider

by an order of magnitude, and the one checked here already occupies about 12 gigabytes of kernel memory for some hundred seconds. Coarsening a funnel's intermediate levels outward is sound — only the containment half of (F) has to survive it — and is the route to those entries; we have not carried it out. So 17/24 is where kernel-checked stops today and 0.7417 is where the search stops, both of them records on the curve of [KK17, Prob. 6.1] — which asks for explicit empty unions of length $1-\varepsilon$ for every $\varepsilon > 0$ — and neither a solution of it.

7.4. **The nearest-integer family.** Every set certified so far was chosen by the search. This one is named in advance: for $0 < c < \frac{1}{2}$ the arc $\{y : \|y\| < c\}$, with $\|\cdot\|$ the distance to the nearest integer, is the two-cell union $[0, c) \cup (1-c, 1)$, and confinement to it says the whole orbit stays within c of the integers.

Theorem 7.14. *No $\xi \neq 0$ has $\{\xi(3/2)^n\} \in [0, \frac{1}{5}) \cup [\frac{4}{5}, 1)$ for every $n \geq 0$. Consequently, for every $\xi \neq 0$ there are infinitely many n with $\|\xi(3/2)^n\| \geq \frac{1}{5}$.*

Proof sketch. The certificate carries the smallest denominator in the paper — $D = 15$, $U = \{(0,3),(12,15)\}$, one funnel level $\{(0,2),(13,15)\}$ — and Theorem 7.2 applies to it. For the consequence, note $\{y : \|y\| < \frac{1}{5}\} \subseteq [0, \frac{1}{5}) \cup [\frac{4}{5}, 1)$ (the two sets differ only at $y = \frac{4}{5}$), so confinement to the smaller set is refuted a fortiori; and *infinitely* many such n because Theorem 7.2 refutes eventual confinement as well, by the shift trick of Lemma 3.2. □

Where this sits: Dubickas [Dub10] constructs ξ with $\|\xi(3/2)^n\| < 1/3$ for every n, so the family is nonempty at $c = 1/3$; and [Dub09AA, Thm. 1] settles arcs of length $1/p = 1/3$ at every position, which is $c \leq 1/6$ here. The threshold therefore lies in $[1/5, 1/3]$, and the certificate above moves the lower end from $1/6$ to $1/5$. The engine does not move it further: at $c = 21/100$ no certificate exists within funnel depth 60, and the closed form of the $c = 1/5$ set — where the two cells become $[0, \frac{1}{5}] \cup [\frac{4}{5}, 1]$ — is refused as well, exactly as Remark 7.10 would predict.

7.5. **Other bases.** Nothing in Definition 7.1 or Theorem 7.2 is special to 3/2: Lemma 2.4 holds for every coprime $p > q > 1$, and validity is checked at the certificate's own base.

Theorem 7.15. *$Z_{4/3}\big(\frac{1}{3}, \frac{5}{8}\big) = \emptyset$, a window of length $\frac{7}{24} = 0.2916\ldots > \frac{1}{4}$.*

Here $1/p = 1/4$ is exactly where [Dub09AA, Thm. 1] stops at this base (and $4 < 9 = 3^2$, so that theorem does apply at length $1/4$; its general form is Theorem 6.9). The certified length is 1.17 times the line, against 1.22 times at the base 3/2; our search at $(4, 3)$ still certifies the length $70/240$ at ten positions and found nothing from $72/240$ to $94/240$, but that scan ran to funnel depth 14 only, while the length-$\frac{1}{4}$ windows at this base already need depth 26, so its silence is weak evidence for a frontier.

7.6. **The regime $p > q^2$.**

Theorem 7.16. *$Z_{5/2}\big(\frac{1}{5}, \frac{2}{5}\big) = \emptyset$.*

Proof sketch. The certificate is small enough to check by hand, and Example 7.6 runs Algorithm 1 on it in full: the only branch whose preimage meets $[\frac{1}{5}, \frac{2}{5})$ is the letter $s = 1$, with surviving set $[\frac{7}{25}, \frac{9}{25})$, which again has the single outgoing letter $s = 1$. A confined orbit would therefore have the constant carry word $1, 1, 1, \ldots$, which Lemma 2.4 forbids for $\xi \neq 0$. □

Where [Dub09AA, §4] leaves the length-$1/p$ statement open, this scheme does not merely reach one window: Table 3 is thirty of them, six positions at each of five bases, and *every one certifies at funnel depth* 1. The regime is not hard for the block certificate; it is only out of reach of the counting endgame, which needs $p < q^2$.

Theorem 7.17. *Let p/q be one of $5/2$, $7/2$, $9/2$, $10/3$, $11/3$ and let $s \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$. Then $Z_{p/q}\big(s, s + \frac{1}{p}\big) = \emptyset$.*

base	$1/p$	position s					
		0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
$5/2$	$1/5$	1	1	1	1	1	1
$7/2$	$1/7$	1	1	1	1	1	2
$9/2$	$1/9$	1	2	1	1	1	1
$10/3$	$1/10$	1	2	1	1	1	1
$11/3$	$1/11$	1	1	1	1	1	1

TABLE 3. Theorem 7.17: the thirty certified length-$1/p$ windows in the regime $p > q^2$, where [Dub09AA, §4] leaves emptiness open. Every certificate has funnel depth 1; the entry is its number of blocks.

Remark 7.18 (scope). Section 4 of [Dub09AA] leaves open whether $Z_{p/q}(s, s + 1/p)$ is empty for every position s when $p > q^2$; the counting endgame of §6.4 needs $p < q^2$, while Theorem 7.2 has no such step, and it certifies every *rational* length-$1/p$ window it was pointed at in this regime. Table 3 is nevertheless a finite list of rational windows and not an approach to the open statement, which quantifies over all real s: the natural upgrade — certify a finite cover by windows of length $\frac{1}{N} + \frac{1}{p}$ and conclude for all real s by monotonicity — fails in practice, the fattened windows developing hundreds of surviving components. The certified fraction does climb with N — at $(5, 2)$, 0 of 5, then 2 of 10, then 8 of 20; at $(9, 2)$, 0 of 9, 6 of 18, 20 of 36 — but the element containing 0 fails at every N tested, and that is not a rounding artefact: it is where [Aki08, Thms. 2.4, 2.5] put their nonempty Cantor sets. Nor is there tension with [Aki08, Thms. 2.4, 2.5], which build *nonempty* confinement sets of arbitrarily small measure exactly when $p > q^2$: those sets are Cantor-type unions, not single windows of length $1/p$.

8. Acknowledgements

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Appendix A. Formalization

Every statement above was verified in Lean 4. The files live under `Z32/` of the repository at
`https://github.com/rwst/Confinement-Certificates`, with the Sturmian criterion in `ForMathlib/Combinatorics/Sturmian.lean` and the linear-mod-one machinery of §4 under `FLP/`. All block-certificate entries live in `BlockCert` except the union record of Theorem 7.12, which has a module of its own (`UnionRecord`) because its kernel check alone costs about 100 seconds and 12 gigabytes; nothing imports it, so no other file pays that. "std3" abbreviates the ambient axioms of classical Lean (propositional extensionality, choice, quotient soundness). *Every* entry below is fully proved at std3: the development contains no `sorry`, no cited axioms, and no natively compiled decision procedures; rows marked "kernel" are additionally checked by the kernel's evaluator on concrete integer data. All footprints were confirmed with `#print axioms`.

The orbit and its word (§2)*, the dictionary* (§3)*.*

Statement	Lean identifier	File	Status
x_n, y_n, s_n	`Z32.orb`, `Z32.xInt`, `Z32.yFract`, `Z32.carry`	`DubickasWord`	def
Prop. 2.1	`Z32.carry_eq`	`DubickasWord`	std3
Prop. 2.2	`Z32.xInt_add` (accumulator `Z32.geomAcc`)	`DubickasWord`	std3
Prop. 2.3	`Z32.abs_xInt_le`	`DubickasWord`	std3
Lem. 2.4	`Z32.not_isEventuallyPeriodic_carry`	`DubickasWord`	std3
Prop. 3.1	`Z32.fract_div_natCast`, `Z32.floor_emod_iff_fract_mem`	`Dictionary`	std3
window monotonicity	`Z32.ZSet_mono`	`Dictionary`	std3
Lem. 3.2	`Z32.mem_ZSet_of_eventually`	`Dictionary`	std3

Escape certificates and residues (§4–§5)*.*

Statement	Lean identifier	File	Status
Prop. 4.1	`Z32.ZSet_third_empty_of_escape`	`EscapeCert`	std3
[FLP95] Thm. 3.2	`FLP.ZSet_eq_empty_of_survivors_finite`	`FLP/ZEmpty`	std3
[FLP95] Thm. 3.4	`FLP.survivors_finite`	`FLP/SurvivorFinite`	std3
Table 1	`Z32.escape_one_sixth`, `…_one_third`, `…_one_half`, `…_two_thirds`	`EscapeCert`	std3
Thm. 4.2	`Z32.FLP_cor_one_four_a`	`EscapeCert`	std3
case $s = 0$	`FLP.ZSet_empty_zero`	`FLP/ParamDensity`	std3
Lem. 5.1	`Z32.exists_grid_window`	`ResidueCapture`	std3
Prop. 5.2	`Z32.ZSet_cell_empty`, `Z32.ZSet_cell_empty_int`	`ResidueCapture`	std3
Thm. 5.3	`Z32.residue_not_eventually_constant`	`ResidueCapture`	std3
i.o. form	`Z32.exists_residue_change`	`ResidueCapture`	std3
$\xi = 1$	`Z32.residue_not_eventually_constant_one`	`ResidueCapture`	std3

Dubickas's theorem (§6)*.*

Statement	Lean identifier	File	Status
$p(w,m)$	`pComplexity`	`ForMathlib/.../InfiniteComplexity`	def
Sturmian, bispecial pair	`IsSturmian`, `HasBispecialPair`	`ForMathlib/.../Sturmian`	def
Lem. 6.1	`isSturmian_of_not_hasBispecialPair`	`ForMathlib/.../Sturmian`	std3
Morse–Hedlund floor	`succ_le_pComplexity`, `pComplexity_lt_succ_of_not_isEventuallyPeriodic`	`ForMathlib/.../{Sturmian, InfiniteComplexity}`	std3
pigeonhole consequence	`exists_factor_eq_of_isSturmian`	`ForMathlib/.../Sturmian`	std3
Lem. 6.2	`Z32.carry_mem_alphabet` (letter `Z32.letterBase`)	`SmallInterval`	std3
Rem. 6.3	`Z32.abs_carry_le`	`SmallInterval`	std3
Lem. 6.4	`Z32.not_hasBispecialPair`	`SmallInterval`	std3
Thm. 6.5	`Z32.isSturmian_carryBit`	`SmallInterval`	std3
Lem. 6.6	`Z32.exists_xInt_injective`	`SmallInterval`	std3
Thm. 6.7	`Z32.no_confinement`	`SmallInterval`	std3
Thm. 6.8	`Z32.dubickas_theorem_1`	`SmallInterval`	std3
Thm. 6.9	`Z32.ZSet_eq_empty_of_lt_sq`, `Z32.ZSet_three_two_third_empty`	`SmallInterval`	std3

Block certificates (§7).

Statement	Lean identifier	File	Status
Def. 7.1	`Z32.BlockCert.Cert`, `Cert.ok`	`BlockCert`	def
Alg. 1, alphabet	`Z32.BlockCert.carries` (with `carriesFrom`, `mem_carries`)	`BlockCert`	def, std3
condition (F)	`pieceA`, `pieceC`, `pieceOk`, `level0k`, `funnel0k`	`BlockCert`	def
condition (D)	`hits`, `outEdges`, `rank`, `funcOk`	`BlockCert`	def
max, min, $\preccurlyeq_\kappa$	`leMax`, `minLe`, `rle`, `rleR`	`BlockCert`	def
check $\Rightarrow$ *block path*	`Cert.exists_block_path`	`BlockCert`	std3
Ex. 7.6	`certFiveTwo`, `certFiveTwo_ok`	`BlockCert`	std3 (kernel)
Thm. 7.2	`Cert.not_confined`, `Cert.not_eventually_confined`	`BlockCert`	std3
Z form	`Z32.BlockCert.ZSet_eq_empty_of_cert`, `not_eventually_mem_Ico_of_cert`	`BlockCert`	std3
Rem. 7.3	`ok_eq_false_of_full`, `..._of_full_closed`, `..._of_not_coprime`	`BlockCert`	std3 (kernel)
Thm. 7.8	`Z32.ZSet_three_two_sixth_3_8`, `Z32.ZSet_three_two_frontier`, `Z32.not_eventually_mem_sixth_3_8`	`BlockCert`	std3 (kernel)
Thm. 7.9	`Z32.dubickas_2008_cor_1_2`	`BlockCert`	std3 (kernel)
Thm. 7.11	`Z32.union_seven_twelfths_empty`, `Z32.union_two_thirds_empty`, `Z32.union_record_empty`	`BlockCert`	std3 (kernel)
Thm. 7.12	`Z32.union_record_7083_empty` (certificate `certUnion7083`)	`UnionRecord`	std3 (kernel)
Thm. 7.14	`Z32.two_cell_fifth_empty`, `Z32.not_forall_abs_sub_round_lt_fifth`	`BlockCert`	std3 (kernel)
Thm. 7.17	`Z32.ZSet_five_two_grid`, `..._seven_two_`, `..._nine_two_`, `..._ten_three_`, `..._eleven_three_grid`	`BlockCert`	std3 (kernel)
its 30 certificates	`certGridFiveTwo0` ... `certGridElevenThree5` and `*_ok`	`BlockCert`	std3 (kernel)
single-window bridge	`Z32.BlockCert.ZSet_eq_empty_of_window`	`BlockCert`	std3
Thm. 7.15	`Z32.ZSet_four_three_beyond_line`	`BlockCert`	std3 (kernel)
Thm. 7.16	`Z32.ZSet_five_two_fifth`	`BlockCert`	std3 (kernel)

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