# CEILING ORBITS OF RATIONAL BASES ARE NOT $P$-RECURSIVE

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ABSTRACT. Let $p > q \geq 2$ be coprime integers, let $x_0$ be a positive integer, and let $x_n = \lceil px_{n-1}/q \rceil$ be the ceiling orbit of the rational base $p/q$, with associated word $w_n = qx_{n+1} - px_n \in \{0, \ldots, q-1\}$. We give an elementary proof that $w$ is not $P$-recursive: it satisfies no nontrivial linear recurrence with polynomial coefficients. At $p/q = 3/2$ we prove the same for the orbit itself, which for $x_0 = 1$ is the sequence A061419 and whose parity is the minimal word $g_{3/2}$ of the rational base number system of Akiyama, Frougny and Sakarovitch. All results are formally verified in the Lean 4 proof assistant.

## 1. INTRODUCTION

Fix coprime integers $p > q \geq 2$ and a positive integer $x_0$, and iterate the ceiling map of the rational base $\beta = p/q$,

$$U_{p/q} \colon x \longmapsto \left\lceil \frac{px}{q} \right\rceil, \qquad x_n := U_{p/q}^n(x_0),$$

so that $x_0 < x_1 < x_2 < \cdots$. Following Dubickas [Dub09, p. 245] we attach to this orbit the word

$$w_n := qx_{n+1} - px_n \in \{0, 1, \ldots, q-1\}, \qquad n = 0, 1, 2, \ldots,$$

the letter $w_n$ being exactly the correction that makes $px_n$ divisible by $q$. Since $w_n \equiv -px_n \pmod q$ and $\gcd(p, q) = 1$, the word is a recoding of the orbit modulo $q$ (Corollary 4.6); Dubickas draws from this the equality $P(w, n) = P(X, n)$ of complexity functions, where $X_n = x_n \bmod q$ [Dub09, p. 245].

At $p/q = 3/2$ the map is $U_{3/2}(x) = 3x/2$ for even $x$ and $(3x+1)/2$ for odd $x$, the word is the parity $w_n = x_n \bmod 2$, and for $x_0 = 1$ the orbit is the sequence A061419 [Dub09, p. 245], [OEIS]. In that case $w$ is also an object of the rational base number system of Akiyama, Frougny and Sakarovitch: their sequence $G_0 = 1$, $G_{k+1} = \lceil (p/q)G_k \rceil$ [AFS08, Def. 20] is our orbit, and their *minimal word* $g_{3/2} = 10110001101001101010011 0 \cdots$ [AFS08, §4.2], [OEIS] is our word up to an index shift.

*Remark* 1.1. The identification is exact but carries the shift. For $p \geq 2q - 1$ Akiyama, Frougny and Sakarovitch compute the digits of the minimal word

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by $g_{k+1} = qG_{k+1} \bmod p$ [AFS08, Remark 32]. At $(p,q) = (3,2)$ one has $3 \geq 3$, and the selection rule $2G_{k+1} = 3G_k + (G_k \bmod 2)$ turns that into $g_{k+1} = G_k \bmod 2$, which is Dubickas's $w_k$; so $w_n = g_{n+1}$, and the sequence indexed from 0 here is the sequence indexed from 1 there. The hypothesis $p \geq 2q-1$ is genuine and fails already at $p/q = 5/4$, so outside that range we speak only of Dubickas's word.

Since three sets of conventions meet here, we collect them once. The middle column is a specialization of the left one, not a second set of definitions; the right column is where the shift lives.

| | general base $p/q$ | at $p/q = 3/2$ | in the sources |
|---|---|---|---|
| map | $U_{p/q}(x) = \lceil px/q \rceil$ | $x \mapsto 3x/2$, $(3x+1)/2$ by parity | |
| orbit | $x_n$, $n \geq 0$, $x_0 \geq 1$ | $x_0 = 1$: $1, 2, 3, 5, 8, 12, 18, 27, \ldots$ | $G_k$, $G_0 = 1$ [AFS08, Def. 20]; A061419 |
| word | $w_n = qx_{n+1} - px_n = (-px_n) \bmod q$ | $w_n = x_n \bmod 2$: $1011000110\cdots$ | $w_n$ [Dub09]; $g_k$ [AFS08, §4.2] |
| alphabet | $\{0, 1, \ldots, q-1\}$ | $\{0,1\}$ | |
| indexing | from $n = 0$ | $x_n = G_n$, $w_n = g_{n+1}$ (Rem. 1.1) | $g_{3/2} = g_1 g_2 \cdots$ from 1 |
| residues | $X_n = x_n \bmod q$ | $X_n = w_n$ | $X_n$ [Dub09] |
| circuit sum | $W(a,k) = \sum_{i<k} p^{k-1-i} q^i w_{a+i}$ | $\sum_{i<k} 3^{k-1-i} 2^i w_{a+i}$ | [RS26] |
| series | $f(z) = \sum_j w_j z^j$ | also $F(z) = \sum_n x_n z^n$ | |
| constant | — | $K = \lim_n x_n (2/3)^n = 1.62227\ldots$ | $\omega_{3/2}$ [AFS08, Ex. 3], $K(3)$ [OW91]; A083286 |

Throughout, $P(w,n)$ is the *complexity function* (or block-complexity function) of $w$ in the sense of [Dub09, p. 245], the number of distinct vectors $(w_j, \ldots, w_{j+n-1})$ as $j$ ranges over $\mathbb{N}$. A sequence is *eventually periodic* if some $N$ and $P \geq 1$ have $w_{n+P} = w_n$ for all $n \geq N$; Dubickas says *ultimately periodic*.

1.1. **The two notions.** $P$-recursiveness is the usual formalization of the vague demand that a sequence have a closed form. Its reach is wide: allowing the coefficients of a linear recurrence to depend polynomially on the index admits, beyond the constant-coefficient sequences — among them every eventually periodic one, a polynomial vanishing on the initial segment absorbing the exceptions — also $n!$, the central binomial coefficients, every hypergeometric term, and the coefficient sequence of every algebraic power series. Where the shape of the definition comes from is the coefficient-side reading of a differential equation: applying $x^i\, d^j/dx^j$ to $\sum f(n)x^n$ shifts the index and multiplies by a polynomial in $n$, so an ODE with polynomial coefficients becomes, coefficient by coefficient, a recurrence with polynomial coefficients. That is why $D$-finiteness is recorded in the same definition, and why the two turn out to be the same condition stated on the two sides of $\sum f(n)x^n$. Refusing a sequence this property refuses it every closed form of the classical kind, and that is the assertion this paper establishes for $w$ and for $x$.

**Definition 1.2** (Stanley [Sta80])**.** A sequence $u\colon \mathbb{N} \to \mathbb{Q}$ is *$P$-recursive* (polynomially recursive, or *holonomic*) if there are an $s \in \mathbb{N}$ and polynomials

$Q_0, \dots, Q_s \in \mathbb{Q}[t]$, not all zero, such that

$$\sum_{j=0}^{s} Q_j(n)\, u(n+j) = 0 \qquad \text{for every } n \in \mathbb{N}.$$

A formal power series is *$D$-finite* (differentiably finite) if it satisfies a nontrivial linear differential equation with polynomial coefficients [Sta80, Def. 1.1], and *algebraic* if $Q_d y^d + \cdots + Q_0 = 0$ for polynomials $Q_i$ not all zero [Sta80, §2].

*Remark* 1.3. Three conventions deserve a word. Stanley's Definition 1.3 demands that the top coefficient $Q_s$ be not identically zero; that is the same class, since one may truncate at the largest $j$ with $Q_j \neq 0$. He works over $\mathbb{C}$; we work over $\mathbb{Q}$, the natural field for the integer-valued sequences here, and for a $\mathbb{Q}$-valued sequence the two agree, because the relations defining a recurrence of bounded order and degree form a $\mathbb{Q}$-linear system, which has a nonzero solution over an extension field only if it has one over $\mathbb{Q}$. (We neither use nor formalize this.) Finally, $P$-recursiveness is a property of the germ: altering finitely many values changes nothing [Sta80, Thm. 1.4], which is why the recurrences produced in §2 are allowed to start beyond an initial segment.

By Stanley's Theorems 2.1 and 1.5 the algebraic power series are $D$-finite and the $D$-finite ones are exactly those with $P$-recursive coefficients, the first inclusion being strict — $e^x$ and $\sum n! x^n$ are $D$-finite and not algebraic [Sta80, p. 178]. Non-holonomy of a coefficient sequence is therefore a strictly stronger assertion than transcendence of its generating function, and it is the assertion we prove.

### 1.2. **Results.**

**Theorem A.** *Let $p > q \geq 2$ be coprime and $x_0 \geq 1$. Then the word $w_n = qx_{n+1} - px_n$ of the ceiling orbit of $p/q$ is not $P$-recursive.*

At $(p, q) = (3, 2)$ this is the minimal word $g_{3/2}$ of [AFS08]. The orbit that produces it obeys the same rigidity.

**Theorem B.** *Let $x_0 \geq 1$ and $x_n = \lceil 3x_{n-1}/2 \rceil$. Then the sequence $x$ itself is not $P$-recursive. For $x_0 = 1$: the sequence A061419 is not holonomic.*

Both are proved without any Diophantine input.

The weaker statement that $f$ is not a *rational* function needs no axiom at all and is in fact cheaper than Theorem A rather than a consequence of it (Corollary 5.4).

Theorem B does not follow from Theorem A by closure properties: the standard route, that $P$-recursive sequences are closed under shifts and $\mathbb{Q}(n)$-linear combinations, is a finite-dimensionality argument over $\mathbb{Q}(n)$ [Sta80, Thm. 1.6] which we do not have. §6 replaces it by the exact identity $q^k x_{a+k} = p^k x_a + W(a, k)$ and a growth estimate.

*Remark* 1.4 (scope). Theorem A is not a step toward non-automaticity of $w$, which is the question of the companion papers [RS26, RS26b]. The generating function of an automatic sequence [AS03] satisfies a Mahler-type functional equation, and a $D$-finite Mahler function is rational, by a theorem of Bézivin [Bez94]; with Proposition 2.6 below, an automatic sequence over a finite alphabet that is not eventually periodic is therefore never $P$-recursive. Automaticity of $w$ would *imply* Theorem A, not contradict it. What Theorem A does exclude, unconditionally, is the hypothesis that these orbits admit a closed form of $P$-recursive type — the standard dividing line for a sequence given by an iteration.

1.3. **Unformalized theorems.** Only one theorem is not formalized and used in the formalization as a black box (and only by the corollaries that weaken non-holonomy to transcendence).

**Theorem 1.5** (Stanley [Sta80])**.** *Let $y = \sum_{n\geq 0} f(n)x^n$ be a power series over a field of characteristic zero. If $y$ is algebraic, then $y$ is $D$-finite, and $y$ is $D$-finite if and only if $f$ is $P$-recursive. In particular the coefficient sequence of an algebraic power series is $P$-recursive.*

This is [Sta80, Thm. 2.1] composed with [Sta80, Thm. 1.5], both formal-algebraic closure properties requiring no analysis; the composite is recorded in the formalization as a cited axiom. All other statements are formally verified in Lean 4; Appendix A records the Lean name, file and axiom footprint of each.

Since the reader is asked to grant this one statement, we sketch it, so that it enters as a known lemma rather than as a black box. Both halves are short, and both are pure algebra.

*Remark* 1.6 (sketch of Theorem 1.5). *Algebraic implies $D$-finite.* This is classical, going back to Abel. Let $R(x,T) = \sum_{i\leq d} Q_i(x)T^i$ be of least degree $d$ in $T$ with $R(x,y) = 0$; minimality makes $R$ irreducible over $\mathbb{Q}(x)$, so $K := \mathbb{Q}(x)[T]/(R) = \mathbb{Q}(x)(y)$ is a field and a $\mathbb{Q}(x)$-vector space of dimension $d$, with basis $1, y, \dots, y^{d-1}$. Differentiating $R(x,y) = 0$ gives $\partial_x R(x,y) + \partial_T R(x,y)\, y' = 0$, and $\partial_T R$ has degree $d-1$ in $T$ and is nonzero, so $\partial_T R(x,y) \neq 0$ by minimality of $d$ and is therefore invertible in $K$. Hence $y' = -\partial_x R(x,y)/\partial_T R(x,y)$ lies in $K$: the derivation $d/dx$ maps $K$ into itself. All derivatives of $y$ then live in a $d$-dimensional space, so $y, y', \dots, y^{(d)}$ are linearly dependent over $\mathbb{Q}(x)$; clearing denominators gives a linear ODE with polynomial coefficients, of order at most $d$.

*$D$-finite implies $P$-recursive.* Write the ODE as $\sum_{i,j} c_{ij}\, x^i y^{(j)} = 0$ and extract the coefficient of $x^n$, using

$$[x^n]\; x^i y^{(j)} = (n+j-i)(n+j-i-1)\cdots(n-i+1)\, f(n+j-i).$$

Each term is a polynomial in $n$ times a shift of $f$, so the ODE becomes a recurrence with polynomial coefficients (replace $n$ by $n + \max_i i$ to make every shift forward). It is nontrivial: at the largest occurring value of $j - i$

the coefficient is $\sum c_{ij}(n+j-i)(n+j-i-1)\cdots(n-i+1)$ over pairs with that difference, hence over *distinct* $j$, and falling factorials of distinct degrees are linearly independent.

*Back again.* The same dictionary read in reverse — multiplication by $x$ shifts the index, $\theta = x\,d/dx$ multiplies by it — turns a $P$-recurrence into a differential operator with polynomial coefficients that annihilates $y$ up to the finitely many initial coefficients the recurrence does not govern; composing with a high enough power of $d/dx$ removes that polynomial remainder.

No step uses convergence, a radius, or a value of $y$: all three are identities in the Weyl algebra $\mathbb{Q}[x]\langle d/dx\rangle$ acting on $\mathbb{Q}[[x]]$. The statement is an axiom here only because the formalization does not develop $D$-finiteness.

## 2. Finite alphabets and $P$-recursive sequences

This section is about an arbitrary sequence with finitely many values; nothing about ceiling orbits enters. The goal is Theorem 2.4 and its contrapositive, which is the interface every later section consumes.

What makes a $P$-recurrence hard to exploit directly is that its coefficients move with $n$: the relation tying $u(n),\dots,u(n+s)$ together is a different relation at every position, and it might be satisfied at each one for its own accidental reason. A finite range removes the accident. There are only finitely many windows of a given width, so windows must return; wherever a window returns, the recurrence imposes the same linear relation on the same numbers, and a polynomial that vanishes at infinitely many positions is the zero polynomial. Everything in this section is that observation spent twice: Proposition 2.2 spends it to trade the polynomial coefficients for constants, and Proposition 2.3 then notes that a constant-coefficient recurrence lets each window determine its successor, so the windows run through a finite state set and must cycle. What comes first is the return itself, with the one caveat it requires — a window can occur only finitely often, but such windows are confined to an initial segment.

**Proposition 2.1** (pigeonhole)**.** *Let $u\colon \mathbb{N}\to\mathbb{Q}$ have finite range and let $s\in\mathbb{N}$. There is an $N$ such that for every $m>N$ the window $(u(m),u(m+1),\dots,u(m+s))$ occurs at infinitely many positions.*

*Proof sketch.* The windows take values in a finite set, being tuples over the finite range of $u$. Call a window *rare* if it occurs at only finitely many positions; the rare windows form a finite set, so the positions carrying one lie in a finite union of finite fibres and are bounded, say by $N$. □

**Proposition 2.2** (constant coefficients)**.** *Let $u\colon \mathbb{N}\to\mathbb{Q}$ have finite range and be $P$-recursive. Then there are $s\in\mathbb{N}$, constants $q_0,\dots,q_s\in\mathbb{Q}$ not all zero, and an $N$, with*

$$\sum_{j=0}^{s} q_j\,u(m+j)=0 \qquad \text{for every } m>N.$$

*Proof sketch.* Let $\sum_j Q_j(n)u(n+j) = 0$ be the recurrence, $e := \max_j \deg Q_j$, and $N$ as in Proposition 2.1. Fix $m > N$ and consider the single polynomial $\Pi(t) := \sum_j Q_j(t)\, u(m+j)$. At every position $m'$ carrying the same window as $m$ the recurrence at $m'$ reads $\Pi(m') = 0$; there are infinitely many such $m'$, so $\Pi = 0$. Reading off the coefficient of $t^e$ gives the displayed identity with $q_j = [t^e]Q_j$, and these are not all zero because the index attaining the maximal degree contributes its leading coefficient. □

**Proposition 2.3** (finite state)**.** *Let $u\colon \mathbb{N} \to \mathbb{Q}$ have finite range and satisfy $\sum_{j\le s} q_j u(m+j) = 0$ for all $m > N$, with the $q_j$ constants not all zero. Then $u$ is eventually periodic.*

*Proof sketch.* Let $b$ be the largest index with $q_b \neq 0$. Solving the recurrence at $m+1$ for its top term — legitimate since $q_b \neq 0$ — expresses $u(m+1+b)$ through $u(m+1), \dots, u(m+b)$, so the window $(u(m), \dots, u(m+b))$ determines the window at $m+1$: the windows evolve deterministically. They live in a finite set, so two positions $N+k_1$, $N+k_2$ with $k_1 \neq k_2$ carry the same window, and determinism propagates the agreement to all later positions, giving period $|k_2 - k_1|$ from $N + \min(k_1, k_2)$ on. □

**Theorem 2.4.** *A $P$-recursive sequence $u\colon \mathbb{N} \to \mathbb{Q}$ with finite range is eventually periodic.*

*Proof.* Propositions 2.2 and 2.3 in succession. □

**Corollary 2.5.** *A sequence $u\colon \mathbb{N} \to \mathbb{Q}$ with finite range which is not eventually periodic is not $P$-recursive.*

Corollary 2.5 is the interface used in §5: supply a finite alphabet and aperiodicity, receive non-holonomy. It is worth stressing that neither Theorem 2.4 nor anything preceding it passes through rationality of the generating function, so no analogue of Fatou's lemma, no pole structure and no Skolem–Mahler–Lech theorem is involved.

The rational case is nevertheless available separately, and separately cheap.

**Proposition 2.6.** *Let $u\colon \mathbb{N} \to \mathbb{Q}$ have finite range and suppose its generating function is a rational function:*

$$\Big(\sum_n u(n) z^n\Big) \cdot Q(z) = \Pi(z)$$

*in $\mathbb{Q}[[z]]$ for polynomials $\Pi, Q$ with $Q \neq 0$. Then $u$ is eventually periodic.*

*Proof sketch.* Read the coefficient of $z^{m+\deg Q}$ for $m > \deg \Pi$: the right side contributes nothing, and on the left the terms with index below $m$ vanish because they call for a coefficient of $Q$ above $\deg Q$. What remains is $\sum_{j\le \deg Q} Q_{\deg Q - j}\, u(m+j) = 0$, a constant-coefficient recurrence, nontrivial at $j = 0$ since $Q_{\deg Q}$ is the leading coefficient of $Q$. Now apply Proposition 2.3. □

*Remark* 2.7. No hypothesis $Q(0) \neq 0$ is needed. Proposition 2.3 solves its recurrence forward at the largest index with $q_j \neq 0$, which under the index reversal $q_j = Q_{\deg Q - j}$ is the *lowest* nonvanishing coefficient of $Q$; a factor $z^m \mid Q$ merely shortens the window.

**Corollary 2.8.** *Let $u\colon \mathbb{N} \to \mathbb{Q}$ have finite range and algebraic generating function. Then $u$ is eventually periodic.* (*Theorem 1.5 and Theorem 2.4.*)

Note the routing: Proposition 2.6 does *not* go through Corollary 2.8, although a rational function is algebraic. Taking that route would pay Theorem 1.5 for a strictly weaker input.

## 3. Ceiling orbits of a rational base

Fix coprime integers $p > q \geq 2$ and $x_0 \geq 1$.

**Definition 3.1.** Let $x_0 \in \mathbb{N}$ and set

$$w_n := (-px_n) \bmod q, \qquad x_{n+1} := \frac{px_n + w_n}{q},$$

so that $qx_{n+1}$ is the least multiple of $q$ at or above $px_n$.

The word is introduced as a residue rather than as the difference $qx_{n+1} - px_n$, and the orbit as an exact quotient rather than as a ceiling, so that neither a subtraction nor a rounding of natural numbers occurs in the recursion; that these descriptions all agree is the content of the next statement, after which only the selection rule is used.

**Proposition 3.2.** *For every $n$ one has the selection rule $qx_{n+1} = px_n + w_n$ with $w_n \in \{0, \dots, q-1\}$, and $x_{n+1} = \lceil px_n/q \rceil$. If $x_0 \geq 1$ then $x_0 < x_1 < x_2 < \cdots$ and every $x_n \geq 1$.*

*Proof sketch.* By construction $q \mid px_n + w_n$ and $0 \leq w_n < q$, so $x_{n+1}$ is the least integer with $qx_{n+1} \geq px_n$, which is the ceiling. Strict growth follows from $qx_n < px_n \leq qx_{n+1}$ for $x_n \geq 1$. □

The whole engine of §4 is the selection rule; nothing else about Definition 3.1 is used below.

Everything below is extracted from iterating the selection rule, and it is worth seeing what that iteration produces before naming the result. One step reads $x_{a+1} = (p/q)x_a + w_a/q$: the orbit follows the pure geometric map $t \mapsto (p/q)t$, perturbed at each step by a rounding kick of size $w_a/q$. Over $k$ steps the kick delivered at step $i$ is amplified by the $k-1-i$ applications of the map that come after it, so

$$x_{a+k} = \Big(\frac{p}{q}\Big)^k x_a + \sum_{i<k} \Big(\frac{p}{q}\Big)^{k-1-i} \frac{w_{a+i}}{q}.$$

Multiplying by $q^k$ clears the denominators and returns everything to $\mathbb{N}$: the letter entering at step $i$ stands over only $q^{\,k-i}$ of the available $q^k$, so it survives with weight $p^{\,k-1-i} q^{\,i}$. The accumulated kick in that cleared form

is the sum below. It involves the word alone, the orbit value $x_a$ having been separated off, and that is what will make it an invariant of a factor of $w$ in §4.

**Definition 3.3.** For $a, k \in \mathbb{N}$ put $W(a,k) := \sum_{i<k} p^{k-1-i} q^i w_{a+i}$.

We are not aware of standard terminology for $W(a,k)$; following [RS26] we call it the *circuit sum* of the length-$k$ window at $a$. This is the error term in the following identity, which is the only consequence of Definition 3.1 used from here on.

**Proposition 3.4** (circuit identity)**.** *For all $a, k \in \mathbb{N}$, $q^k x_{a+k} = p^k x_a + W(a,k)$.*

*Proof sketch.* Induction on $k$, feeding $qx_{a+k+1} = px_{a+k} + w_{a+k}$ into the inductive hypothesis; the recurrence $W(a,k+1) = pW(a,k) + q^k w_{a+k}$ matches. At $a = 0$ this is $q^n x_n = p^n x_0 + W(0,n)$, whence also the lower growth bound $p^n x_0 \le q^n x_n$, the word only ever pushing the orbit up. □

**Proposition 3.5** (growth ceiling)**.** *For every $n$,*

$$q^n\big((p-q)x_n + q\big) \;\le\; p^n\big((p-q)x_0 + q\big),$$

*that is, $x_n + \frac{q}{p-q} \le (p/q)^n\big(x_0 + \frac{q}{p-q}\big)$.*

*Proof sketch.* One step: $q\big((p-q)x_{n+1} + q\big) = (p-q)(px_n + w_n) + q^2 \le (p-q)px_n + (p-q)q + q^2 = p\big((p-q)x_n + q\big)$, using $w_n \le q$ and $(p-q)q + q^2 = pq$. Induction on $n$. □

*Remark* 3.6. The affine shift is forced and is not always 1. Since $\lceil \beta t \rceil \le \beta t + 1$, the map $t \mapsto t + c$ is submultiplicative under the ceiling map exactly when $c \ge 1/(\beta - 1) = q/(p-q)$. At $(p,q) = (3,2)$ that constant is 2, yet the sharper bound $2^n(x_n + 1) \le 3^n(x_0 + 1)$ still holds, because there $w_n + q \le p$; at $p/q = 5/4$ that inequality fails and the shift-1 bound is false. Proposition 3.5 is the form valid at every admissible base.

## 4. $q$-adic rigidity and aperiodicity

The target of this section is aperiodicity, but periodicity is a global property and the circuit identity is a local one, so we work instead with the local shadow that periodicity casts: a block of letters occurring twice. This is the same notion of factor the complexity function $P(w,k)$ counts, simply seen at two positions. It is the right shadow, for two reasons. It is implied by far less than periodicity, yet in a strong form: a word of period $P$ from $N$ on carries a repeated factor at the *one* pair $(N, N+P)$ of *every* length $k$, and it is an unbounded $k$ against a fixed pair that Theorem 4.8 will refute. And it is exactly the hypothesis Definition 3.3 rewards: equal factors have equal circuit sums, so subtracting the two corresponding instances of the circuit identity cancels the word entirely and leaves a relation between orbit values alone (Lemma 4.2). What follows is the pursuit of that relation.

**Definition 4.1.** A *repeated factor* of length $k$ at $(a, c)$ is an equality of the length-$k$ factors of $w$ at $a$ and at $c$: $w_{a+i} = w_{c+i}$ for all $i < k$. Occurrences may overlap.

**Lemma 4.2.** *If $w$ has a repeated factor of length $k$ at $(a, c)$, then*

$$p^k(x_c - x_a) = q^k(x_{c+k} - x_{a+k}).$$

*Proof sketch.* Equal factors give equal circuit sums, $W(a, k) = W(c, k)$; subtract the two instances of Proposition 3.4. □

**Proposition 4.3.** *If $w$ has a repeated factor of length $k$ at $(a, c)$, then $q^k \mid x_c - x_a$.*

*Proof sketch.* Lemma 4.2 gives $q^k \mid p^k(x_c - x_a)$, and $q^k$ is coprime to $p^k$. This is the only place where $\gcd(p, q) = 1$ is used, and it is used as coprimality, not as primality of $q$. □

**Lemma 4.4** ($q$-adic descent)**.** *If $q^m \mid x_a - x_b$, then $q^{m-i} \mid x_{a+i} - x_{b+i}$ for every $i \le m$.*

*Proof sketch.* Induction on $i$. As long as one power of $q$ survives, $x_{a+i} \equiv x_{b+i} \pmod q$, hence $w_{a+i} = w_{b+i}$, the letters being determined by the residue of the orbit modulo $q$; then subtracting the two selection rules gives $q(x_{a+i+1} - x_{b+i+1}) = p(x_{a+i} - x_{b+i})$, which cancels exactly one power of $q$. No coprimality is needed here: the cancellation is on the left and is indifferent to $p$. □

**Theorem 4.5** (window rigidity)**.** *For all $a, b, m$, the length-$m$ factors of $w$ at $a$ and $b$ agree if and only if $q^m \mid x_b - x_a$.*

*Proof.* Forwards is Proposition 4.3; backwards, Lemma 4.4 keeps $q \mid x_{a+i} - x_{b+i}$ alive for $i < m$, and each such divisibility fixes one letter. □

So length-$m$ windows of $w$ *are* the residues $x_n \bmod q^m$. The case $m = 1$ is the recoding quoted in the introduction.

**Corollary 4.6.** *$w_a = w_b$ if and only if $x_a \equiv x_b \pmod q$.*

Dubickas draws from this the identity $P(w, n) = P(X, n)$ between the complexity of the word and that of the residue sequence $X_n = x_n \bmod q$ [Dub09, p. 245]; Theorem 4.5 is the same identity read one window at a time. We record only the equivalences themselves, the complexity function playing no role in what follows.

Rigidity is also a cost: a repeated window forces a large power of $q$ to divide a difference of orbit values, while the orbit has only $(p/q)^c$ room.

**Theorem 4.7** (repetition ceiling)**.** *Let $x_0 \ge 1$ and $a < c$. If $w$ has a repeated factor of length $k$ at $(a, c)$, then*

$$q^{k+c} \;<\; p^c\big((p - q)x_0 + q\big).$$

*Proof sketch.* By Theorem 4.5, $q^k \mid x_c - x_a$, and $x_a < x_c$, so $q^k \le x_c - x_a < x_c \le (p-q)x_c + q$. Multiplying by $q^c$ and applying Proposition 3.5 chains onto the display. □

**Theorem 4.8** (Dubickas [Dub09, p. 245]; cf. [AFS08, Prop. 26])**.** *Let $p > q \ge 2$ be coprime and $x_0 \ge 1$. Then $w$ is not eventually periodic.*

*Proof.* Eventual periodicity with threshold $N$ and period $P \ge 1$ gives a repeated factor at the *fixed* pair $(N, c)$, $c := N + P$, of *every* length $k$. But Theorem 4.7 caps every such $k$ by the single number $k_0 := p^c\big((p-q)x_0 + q\big)$, with $c$ fixed. Take $k = k_0$: then $k_0 < 2^{k_0} \le q^{k_0} \le q^{k_0+c} < k_0$, using $q \ge 2$, a contradiction. □

*Remark* 4.9 (what this replaces). Theorem 4.8 is the parallel of [AFS08, Prop. 26] — no element of $W_{p/q}$ is eventually periodic but $0^\omega$ — and it is also the inference Dubickas draws on [Dub09, p. 245] from his Theorem 2, that an eventually periodic $w$ forces $\beta$ to lie in $U = S \cup T$, the Pisot numbers together with the Salem numbers, together with the observation that $p/q$ lies in neither: an element of $U$ is an algebraic integer, and a non-integral rational is not one, $\mathbb{Z}$ being integrally closed in $\mathbb{Q}$. That observation is elementary and we record it as the next proposition.

**Proposition 4.10.** *If $q \ge 2$, $p > q$ and $\gcd(p, q) = 1$, then $p/q$ is not an algebraic integer; in particular $p/q \notin U$.*

*Proof sketch.* $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$, so an algebraic integer in $\mathbb{Q}$ lies in $\mathbb{Z}$, and $q \nmid p$. □

## 5. The words are not holonomic

*Proof of Theorem A.* The word takes its values in the finite set $\{0, \dots, q-1\}$ by Proposition 3.2, and it is not eventually periodic by Theorem 4.8. Now apply Corollary 2.5. □

**Corollary 5.1.** *For every $x_0 \ge 1$ the minimal word of the base $3/2$ — the parity $w_n = x_n \bmod 2$ of the orbit $x_n = \lceil 3x_{n-1}/2 \rceil$, which at $x_0 = 1$ is $g_{3/2}$ — is not $P$-recursive.*

In the formalization Corollary 5.1 also carries a proof of its own, which draws its aperiodicity from the 2-adic rigidity of our companion paper [RS26b] rather than from Theorem 4.8; the two words are proved equal, so the statements are the same statement.

**Example 5.2.** $p/q = 5/4$ is admissible: $q = 4$ is composite, the alphabet has four letters, and nothing in §4 needs $q$ prime — coprimality enters only through $\gcd(q^k, p^k) = 1$ in Proposition 4.3. So the word of $x_n = \lceil 5x_{n-1}/4 \rceil$ is not $P$-recursive either.

**Example 5.3.** At $p/q = 5/3$ and $x_0 = 1$ the orbit begins $1, 2, 4, 7, 12$ and the word $w_n = 3x_{n+1} - 5x_n$ begins $1, 2, 1, 1$ over the alphabet $\{0, 1, 2\}$.

**Corollary 5.4.** *For $p/q = 3/2$ and every $x_0 \geq 1$, the generating function $f(z) = \sum_j w_j z^j$ is not a rational function: $f \cdot Q = \Pi$ is impossible for polynomials $\Pi, Q$ over $\mathbb{Q}$ with $Q \neq 0$.*

*Proof.* Proposition 2.6 and Theorem 4.8. □

The same two lines give the same conclusion at every admissible base; only the case $p/q = 3/2$ is formalized. Note that Corollary 5.4 does not go through Theorem A: it consumes aperiodicity directly, and is the cheapest statement in this paper.

**Corollary 5.5.** *For every admissible $(p, q, x_0)$, $f(z) = \sum_j w_j z^j$ is not algebraic over $\mathbb{Q}(z)$.*

*Proof.* An algebraic series has $P$-recursive coefficients (Theorem 1.5), which Theorem A forbids. □

*Remark* 5.6. Corollary 5.5 is the weaker statement and the only one here that consumes Theorem 1.5; Theorem A is stronger and free. Reading Theorem 1.5 in the other direction, Theorem A also says that $f$ is not $D$-finite.

## 6. The orbit series

For this section fix $p/q = 3/2$ and write $F(z) = \sum_n x_n z^n$, $f(z) = \sum_j w_j z^j$.

**Proposition 6.1.** *As formal power series over $\mathbb{Q}$,*

$$(2 - 3z)\, F(z) \;=\; z\, f(z) + 2x_0.$$

*Proof sketch.* In degree 0 both sides are $2x_0$; in degree $n + 1$ the identity reads $2x_{n+1} - 3x_n = w_n$, which is the selection rule. □

So $F$ is a linear-fractional transform of $f$ over $\mathbb{Q}(z)$, and in particular each of $F, f$ is rational, algebraic or $D$-finite exactly when the other is.

*Remark* 6.2 (analytic reading; not formalized). Since $0 \leq w_j \leq 1$ and $x_n \asymp K(3/2)^n$ with $K$ as in Definition 6.5 below, $f$ has radius of convergence 1 and $F$ radius $2/3$, and Proposition 6.1 exhibits the boundary singularity of $F$ as a simple pole at $z = 2/3$ with residue

$$\frac{(2/3)f(2/3) + 2x_0}{-3} = -\frac{2K}{3},$$

using $f(2/3) = 3(K - x_0)$ from Proposition 6.6. This paragraph is the one place where an analytic statement is made; it is outside the scope of the formalization, whose formal power series carry no analytic structure.

*Proof of Theorem B.* The strategy is to push a hypothetical recurrence for the orbit down onto the word, where Theorem A is waiting. The circuit identity trades every shifted value $x_{n+j}$ for $x_n$ together with letters, so once the order-zero case is out of the way — there the orbit's positivity settles it at

once — the recurrence collapses into a single orbit term with polynomial coefficient $A$ followed by a finite tail of word terms, $A(n)x_n + \sum_i B_i(n)w_{n+i} = 0$. Whether $A$ vanishes decides which of two disposals applies, and both are fatal. If $A = 0$ the word terms are all that remain, and they constitute a nontrivial recurrence for $w$ — nontrivial because the top coefficient $Q_b$ survives into $B_{b-1}$ — which Corollary 5.1 forbids. If $A \neq 0$ the orbit term cannot be balanced by the rest: $x_n$ grows geometrically while the word is bounded and the $B_i$ grow only polynomially, which pins $A(n)$ to 0 in the limit and so returns $A = 0$ regardless.

In detail, suppose $\sum_{j\le s} Q_j(n)x_{n+j} = 0$ nontrivially and let $b$ be the largest index with $Q_b \neq 0$. If $b = 0$ the recurrence reads $Q_0(n)x_n = 0$ with $x_n > 0$, so $Q_0$ vanishes at every natural number and hence is zero, a contradiction. Otherwise multiply by $2^s$ and substitute $2^j x_{n+j} = 3^j x_n + W(n, j)$ (Proposition 3.4) in each term. Collecting the orbit terms and the word terms separately turns the recurrence into

$$A(n)\, x_n + \sum_{i<s} B_i(n)\, w_{n+i} = 0,$$

where

$$A = \sum_j Q_j\, 2^{\,s-j} 3^j, \qquad B_i = \sum_{j>i} Q_j\, 2^{\,s-j+i} 3^{\,j-1-i},$$

polynomial identities in $n$. Here $B_i = 0$ for $i \ge b$, while $B_{b-1} = 2^{\,s-1} Q_b \neq 0$.

If $A = 0$, what remains is $\sum_{i<b} B_i(n)w_{n+i} = 0$, a nontrivial $P$-recurrence for the word, contradicting Corollary 5.1.

If $A \neq 0$, use $|w_{n+i}| \le 1$ to get $|A(n)|\, x_n \le \sum_{i<s} |B_i(n)|$, and $x_n \ge (3/2)^n$ from the lower growth bound of Proposition 3.4 with $x_0 \ge 1$. Hence

$$|A(n)| \;\le\; \Big(\sum_{i<s} |B_i(n)|\Big)\Big(\tfrac{2}{3}\Big)^n \;\longrightarrow\; 0,$$

polynomial growth losing to geometric decay. A polynomial whose values along $\mathbb{N}$ tend to 0 is the zero polynomial, so $A = 0$ after all — a contradiction. □

*Remark* 6.3. Nothing in the argument is specific to 3/2: it uses a circuit identity, a bounded word and exponential growth, all of which §3 supplies at every admissible $p/q$. Note also that no integrality argument is needed — one might expect to clear denominators and bound $|A(n)|$ from below off the roots of $A$; the limit statement replaces that step.

**Corollary 6.4.** *For $x_0 \ge 1$ the orbit series $F(z) = \sum_n x_n z^n$ is not algebraic over $\mathbb{Q}(z)$.*

*Proof.* Theorem 1.5 and Theorem B. □

6.1. **The value at the pole.** The remaining statement records what the number $f(2/3)$ is, and that it is where this circle of questions runs out.

**Definition 6.5.** $K := x_0 + \frac{1}{3}\sum_{j\geq 0} w_j(2/3)^j$, a convergent series since $w_j \in \{0,1\}$; equivalently $K = \lim_n x_n(2/3)^n$. For $x_0 = 1$ this is the constant $\omega_{3/2}$ of [AFS08, Ex. 3], which is the constant $K(3)$ of Odlyzko and Wilf [OW91]:

$$K = \omega_{3/2} = K(3) = 1.62227\,05028\,84767\,31595\,6\ldots$$

(sequence A083286 [OEIS]).

**Proposition 6.6.** $\sum_{j\geq 0} w_j(2/3)^j = 3(K - x_0)$.

**Proposition 6.7.** *For every $x_0 \geq 1$: $f(2/3)$ is irrational if and only if $K$ is irrational.*

*Proof sketch.* Immediate from Proposition 6.6: the two differ by a nonzero rational scaling and an integer translation, neither of which affects irrationality. □

The irrationality of $K$ has been open since Wang and Washburn asked for it in 1977 [WW77]. Proposition 6.7 says that the value of the word series at the singularity of the orbit series is exactly that open question, and Theorem B says nothing about it: non-holonomy of a coefficient sequence constrains the function, not the value.

## 7. Discussion

7.1. **The collapse on a finite alphabet.** Theorem 2.4 has a converse, and it is immediate: if $u$ is eventually periodic with threshold $N$ and period $P$, then

$$\Big(\prod_{k<N}(n-k)\Big)\big(u(n+P) - u(n)\big) = 0$$

holds at every $n \in \mathbb{N}$ — the bracket vanishes from $N$ on, the product before that — so $u$ is $P$-recursive. Together with Theorem 2.4, Proposition 2.6, Corollary 2.8 and the geometric-series computation that renders an eventually periodic sequence rational, this makes the entire holonomic hierarchy degenerate on a finite alphabet: for $u\colon \mathbb{N} \to \mathbb{Q}$ with finite range, writing $\hat{u}(z) = \sum_n u(n)z^n$,

$$u \text{ eventually periodic} \iff u \text{ } P\text{-recursive} \iff \hat{u} \text{ rational} \iff \hat{u} \text{ algebraic},$$

the last clause alone passing through Theorem 1.5. Nothing between rationality and holonomy survives the finiteness of the range, and "admits a closed form" degenerates to "is eventually periodic".

Two consequences are worth stating for readers outside this circle of questions. First, the mathematical content of Theorem A is not the holonomy but the aperiodicity: §2 is soft, and all the work sits in the $q$-adic rigidity of §4, an arithmetic statement about the orbit. Second, the engine is portable exactly because it is soft. Any sequence that takes finitely many values and is not eventually periodic — a parity log, a digit stream, the trace of a

discrete algorithm, the itinerary of a symbolic dynamical system — is non-holonomic for this reason alone, and the burden in any such application is a proof of aperiodicity, not an analysis of a generating function.

7.2. **Automatic sequences and Mahler functions.** Remark 1.4 records the direction of the implication, and it is worth expanding, since the two properties are easily confused. An automatic sequence over a finite alphabet has a generating function satisfying a Mahler-type functional equation [AS03]; a $D$-finite Mahler function is rational [Bez94]; and a rational generating function whose coefficients take finitely many values forces eventual periodicity (Proposition 2.6). So automaticity plus aperiodicity already implies non-holonomy, and Theorem A is a consequence of any future proof that $w$ is automatic rather than an obstacle to one. The converse implication is false in a strong sense: by §7.1 every aperiodic finite-valued sequence is non-holonomic, while automatic sequences are a thin subclass of those, so non-holonomy carries no information at all about automaticity. The two companion papers [RS26, RS26b] address that separate question.

The Mahler side also indicates where the arithmetic interest lies. If $w$ were $k$-automatic, then $f$ would be a Mahler function, and Mahler's method in the modern form of Adamczewski and Faverjon [AF17] would apply: the generating function of an automatic sequence takes, at a nonzero algebraic point of its disc of convergence, a value that is rational or transcendental, an algebraic irrational being impossible. At $z = 2/3$, which is such a point since $f$ has radius of convergence 1, this concerns exactly the constant of §6.1, so automaticity of $g_{3/2}$ would place $K$ in the rational-or-transcendental dichotomy and thereby settle the irrationality question of Wang and Washburn [WW77] except for the rational case. Theorem B, by contrast, says nothing whatever about $K$: non-holonomy constrains the function, not its values.

7.3. **Analysis of algorithms.** Ceiling orbits are the shape that "shrink by a rational factor and round" takes when it is iterated, and they arrive in this paper from two such sources: the constant $K$ is the $K(3)$ of Odlyzko and Wilf's analysis of the Josephus problem [OW91], and the orbit itself is the sequence $G_k$ attached by Akiyama, Frougny and Sakarovitch to the rational base number system [AFS08]. For an audience in the analysis of algorithms, holonomy is not an abstract classification but an entry ticket: a holonomic sequence has mechanically extractable asymptotics, and its $n$-th term is computable in $O(\sqrt{n})$ ring operations up to logarithmic factors, by baby-step/giant-step evaluation of the associated matrix factorial [BGS07], rather than in the $n$ steps of the defining iteration. Theorem B places $x$ outside that class, so neither shortcut is available on this route. This is an exclusion of a method and not a complexity lower bound — and the obvious alternative, reading $x_n$ off $K(3/2)^n$, needs $K$ to a precision growing linearly in $n$, which is itself obtained from the orbit.

## 8. Acknowledgements

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## Appendix A. Formalization

Every statement above was verified in Lean 4, except Remark 6.2, which is analytic and is marked as such. See `https://github.com/rwst/Ceiling-Orbits`. The files live under `RB/`, with `CITED/Stanley.lean` supplying the single axiom which is Theorem 1.5. Identifiers in the namespace `RB` are written below without that prefix, and `Gen.` abbreviates `RB.Gen.`, the namespace of the general rational base. "std3" abbreviates the ambient axioms of classical Lean (propositional extensionality, choice, quotient soundness).

*The engine* (§2).

| Statement | Lean identifier | File | Status |
|---|---|---|---|
| Def. 1.2 | `Stanley.IsPRecursive` | `CITED/Stanley` | def |
| Thm. 1.5 | `Stanley.pRecursive_of_isAlgebraic` | `CITED/Stanley` | cited axiom [Sta80] |
| Prop. 2.1 | `exists_N_recurrent` | `EventuallyPeriodic` | std3 |
| Prop. 2.2 | `exists_constant_recurrence` | `EventuallyPeriodic` | std3 |
| Prop. 2.3 | `eventuallyPeriodic_of_constant_recurrence` | `EventuallyPeriodic` | std3 |
| Thm. 2.4 | `eventuallyPeriodic_of_isPRecursive` | `EventuallyPeriodic` | std3 |
| Cor. 2.5 | `not_isPRecursive_of_not_eventuallyPeriodic` | `EventuallyPeriodic` | std3 |
| Prop. 2.6 | `eventuallyPeriodic_of_rational_of_finite_coeffs` | `EventuallyPeriodic` | std3 |
| Cor. 2.8 | `eventuallyPeriodic_of_isAlgebraic_of_finite_coeffs` | `EventuallyPeriodic` | std3 + [Sta80] |

*Rational bases, rigidity, and Theorem A* (§3–§5).

| Statement | Lean identifier | File | Status |
|---|---|---|---|
| Def. 3.1 | `Gen.x`, `Gen.word` | `GeneralBase` | def |
| Prop. 3.2 | `Gen.q_mul_x_succ`, `Gen.word_lt`, `Gen.x_succ_eq_ceil` | `GeneralBase` | std3 |
| *monotonicity* | `Gen.x_pos`, `Gen.x_strictMono` | `GeneralBase` | std3 |
| Def. 3.3 | `Gen.W` | `GeneralBase` | def |
| Prop. 3.4 | `Gen.circuit_sum`, `Gen.circuit_sum_zero`, `Gen.p_pow_mul_le` | `GeneralBase` | std3 |
| Prop. 3.5 | `Gen.q_pow_mul_le` | `GeneralBase` | std3 |
| Def. 4.1 | `Gen.IsRepetition` | `GeneralBase` | def |
| Lem. 4.2 | `Gen.lemmaR_int` | `GeneralBase` | std3 |
| Prop. 4.3 | `Gen.q_pow_dvd_of_repetition` | `GeneralBase` | std3 |
| Lem. 4.4 | `Gen.dvd_sub_descend`, `Gen.word_eq_of_dvd` | `GeneralBase` | std3 |
| Thm. 4.5 | `Gen.window_eq_of_dvd`, `Gen.isRepetition_iff_dvd` | `GeneralBase` | std3 |
| Cor. 4.6 | `Gen.word_determines_residue` | `GeneralBase` | std3 |
| Thm. 4.7 | `Gen.repetition_pow_lt` | `GeneralBase` | std3 |
| Thm. 4.8 | `Gen.not_eventually_periodic` | `GeneralBase` | std3 |
| Prop. 4.10 | `Gen.not_isIntegral_rational`, `Gen.not_mem_Bertin_U` | `GeneralBase` | std3 |
| Thm. A | `Gen.not_isPRecursive_word` | `GeneralBase` | std3 |
| Cor. 5.1 | `not_isPRecursive_wmin` | `EventuallyPeriodic` | std3 |
| *the* $(3,2)$ *bridge* | `Gen.x_eq`, `Gen.wmin_eq` | `GeneralBase` | std3 |
| Ex. 5.2 | `Gen.not_isPRecursive_word_five_fourths` | `GeneralBase` | std3 |
| Ex. 5.3 | `Gen.x_sanity` | `GeneralBase` | no axioms |
| Cor. 5.4 | `not_rational_wminSeries` | `EventuallyPeriodic` | std3 |
| Cor. 5.5 | `Gen.not_isAlgebraic_wordSeries` | `GeneralBase` | std3 + [Sta80] |
| *at* $3/2$ | `not_isAlgebraic_wminSeries` | `EventuallyPeriodic` | std3 + [Sta80] |

*The orbit series* (§6).

| Statement | Lean identifier | File | Status |
|---|---|---|---|
| $F$, $f$ | `orbitSeries`, `wordSeries` | `OrbitSeries` | def |
| Prop. 6.1 | `orbitSeries_mobius` | `OrbitSeries` | std3 |
| Thm. B | `not_isPRecursive_orbit` | `OrbitSeries` | std3 |
| Cor. 6.4 | `not_isAlgebraic_orbitSeries` | `OrbitSeries` | std3 + [Sta80] |
| Def. 6.5 | `K`, `tendsto_x_mul_pow` | `Basic` | def, std3 |
| Prop. 6.6 | `tsum_wmin_eq` | `Basic` | std3 |
| Prop. 6.7 | `irrational_tsum_wmin_iff` | `OrbitSeries` | std3 |

*Commentary.* Two things learned here may be of use to others. The first is that the choice of what to cite is a design decision and not a bookkeeping one. The natural black box for §2 is Fatou's theorem — a power series whose coefficients take finitely many values is rational or transcendental — and it turned out to be unusable: invoking it requires transporting a `PowerSeries` to an analytic function, and with it a radius of convergence, a natural boundary and the continuability of algebraic functions, none of which Mathlib supplies for formal series. Theorem 1.5 replaced it because it is purely algebraic, and the replacement improved the mathematics rather than merely the Lean: the engine was rebuilt around eventual periodicity, which needs no rationality intermediate and delivers non-holonomy where the analytic route would have delivered transcendence. An axiom that resists formalization is often a sign that the argument around it is aimed at the wrong target.

The second is that the awkward steps are not where the paper suggests. The definitions were written for the kernel and reconciled with the reader afterwards: Definition 3.1 is `word = (q - p*x % q) % q` with `x (n+1) = (p*x + word)/q`, a residue and an exact division on $\mathbb{N}$, so that no truncated subtraction and no ceiling occurs in the recursion that every later proof unfolds, and `Gen.x_succ_eq_ceil` recovers the familiar form once, for the reader only. The $\mathbb{N}/\mathbb{Z}$ boundary was concentrated in the same way, at Lemma 4.2, since differences of orbit values are not natural numbers. Because the orbit carries its starting value as a parameter, an index shift is a change of parameter: `Gen.x_add` states that the orbit started at $x_0$ and read at $n + j$ is the orbit started at $x_n$ and read at $j$, so factor arguments need no offset arithmetic at all. What remained long were the soft steps: the pigeonhole of Proposition 2.1, a single sentence on paper, needs the finiteness of the set of windows, a finite union of finite fibres, and a bound on it.

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