CEILING ORBITS OF RATIONAL BASES ARE NOT P-RECURSIVE

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Abstract. Let $p > q \geq 2$ be coprime integers, let x_0 be a positive integer, and let $x_n = \lceil px_{n-1}/q \rceil$ be the ceiling orbit of the rational base p/q, with associated word $w_n = qx_{n+1} - px_n \in \{0, \ldots, q-1\}$. We give an elementary proof that w is not P-recursive: it satisfies no nontrivial linear recurrence with polynomial coefficients. At $p/q = 3/2$ we prove the same for the orbit itself, which for $x_0 = 1$ is the sequence A061419 and whose parity is the minimal word $g_{3/2}$ of the rational base number system of Akiyama, Frougny and Sakarovitch. All results are formally verified in the Lean 4 proof assistant.

1. Introduction

Fix coprime integers $p > q \geq 2$ and a positive integer x_0, and iterate the ceiling map of the rational base $\beta = p/q$,

$$U_{p/q} \colon x \longmapsto \left\lceil \frac{px}{q} \right\rceil, \qquad x_n := U_{p/q}^n(x_0),$$

so that $x_0 < x_1 < x_2 < \cdots$. Following Dubickas [Dub09, p. 245] we attach to this orbit the word

$$w_n := qx_{n+1} - px_n \in \{0, 1, \ldots, q-1\}, \qquad n = 0, 1, 2, \ldots,$$

the letter w_n being exactly the correction that makes px_n divisible by q. Since $w_n \equiv -px_n \pmod q$ and $\gcd(p, q) = 1$, the word is a recoding of the orbit modulo q (Corollary 4.6); Dubickas draws from this the equality $P(w, n) = P(X, n)$ of complexity functions, where $X_n = x_n \bmod q$ [Dub09, p. 245].

At $p/q = 3/2$ the map is $U_{3/2}(x) = 3x/2$ for even x and $(3x+1)/2$ for odd x, the word is the parity $w_n = x_n \bmod 2$, and for $x_0 = 1$ the orbit is the sequence A061419 [Dub09, p. 245], [OEIS]. In that case w is also an object of the rational base number system of Akiyama, Frougny and Sakarovitch: their sequence $G_0 = 1$, $G_{k+1} = \lceil (p/q)G_k \rceil$ [AFS08, Def. 20] is our orbit, and their *minimal word* $g_{3/2} = 101100011010011010100110\cdots$ [AFS08, §4.2], [OEIS] is our word up to an index shift.

Remark 1.1. The identification is exact but carries the shift. For $p \geq 2q-1$ Akiyama, Frougny and Sakarovitch compute the digits of the minimal word

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by $g_{k+1} = qG_{k+1} \bmod p$ [AFS08, Remark 32]. At $(p,q) = (3,2)$ one has $3 \geq 3$, and the selection rule $2G_{k+1} = 3G_k + (G_k \bmod 2)$ turns that into $g_{k+1} = G_k \bmod 2$, which is Dubickas's w_k; so $w_n = g_{n+1}$, and the sequence indexed from 0 here is the sequence indexed from 1 there. The hypothesis $p \geq 2q-1$ is genuine and fails already at $p/q = 5/4$, so outside that range we speak only of Dubickas's word.

Throughout, $P(w,n)$ is the *complexity function* (or block-complexity function) of w in the sense of [Dub09, p. 245], the number of distinct vectors $(w_j, \ldots, w_{j+n-1})$ as j ranges over $\mathbb{N}$. A sequence is *eventually periodic* if some N and $P \geq 1$ have $w_{n+P} = w_n$ for all $n \geq N$; Dubickas says *ultimately periodic.*

1.1. **The two notions.**

Definition 1.2 (Stanley [Sta80])**.** A sequence $u \colon \mathbb{N} \to \mathbb{Q}$ is *P-recursive* (polynomially recursive, or *holonomic*) if there are an $s \in \mathbb{N}$ and polynomials $Q_0, \ldots, Q_s \in \mathbb{Q}[t]$, not all zero, such that

$$\sum_{j=0}^{s} Q_j(n)\, u(n+j) = 0 \qquad \text{for every } n \in \mathbb{N}.$$

A formal power series is *D-finite* (differentiably finite) if it satisfies a nontrivial linear differential equation with polynomial coefficients [Sta80, Def. 1.1], and *algebraic* if $Q_d y^d + \cdots + Q_0 = 0$ for polynomials Q_i not all zero [Sta80, §2].

Remark 1.3. Three conventions deserve a word. Stanley's Definition 1.3 demands that the top coefficient Q_s be not identically zero; that is the same class, since one may truncate at the largest j with $Q_j \neq 0$. He works over $\mathbb{C}$; we work over $\mathbb{Q}$, the natural field for the integer-valued sequences here, and for a $\mathbb{Q}$-valued sequence the two agree, because the relations defining a recurrence of bounded order and degree form a $\mathbb{Q}$-linear system, which has a nonzero solution over an extension field only if it has one over $\mathbb{Q}$. (We neither use nor formalize this.) Finally, P-recursiveness is a property of the germ: altering finitely many values changes nothing [Sta80, Thm. 1.4], which is why the recurrences produced in §2 are allowed to start beyond an initial segment.

By Stanley's Theorems 2.1 and 1.5 the algebraic power series are D-finite and the D-finite ones are exactly those with P-recursive coefficients, the first inclusion being strict — e^x and $\sum n! x^n$ are D-finite and not algebraic [Sta80, p. 178]. Non-holonomy of a coefficient sequence is therefore a strictly stronger assertion than transcendence of its generating function, and it is the assertion we prove.

1.2. **Results.**

Theorem A. *Let $p > q \geq 2$ be coprime and $x_0 \geq 1$. Then the word $w_n = qx_{n+1} - px_n$ of the ceiling orbit of p/q is not P-recursive.*

At $(p, q) = (3, 2)$ this is the minimal word $g_{3/2}$ of [AFS08]. The orbit that produces it obeys the same rigidity.

Theorem B. *Let $x_0 \geq 1$ and $x_n = \lceil 3x_{n-1}/2 \rceil$. Then the sequence x itself is not P-recursive. For $x_0 = 1$: the sequence A061419 is not holonomic.*

Both are proved without any Diophantine input.

The weaker statement that f is not a *rational* function needs no axiom at all and is in fact cheaper than Theorem A rather than a consequence of it (Corollary 5.4).

Theorem B does not follow from Theorem A by closure properties: the standard route, that P-recursive sequences are closed under shifts and $\mathbb{Q}(n)$-linear combinations, is a finite-dimensionality argument over $\mathbb{Q}(n)$ [Sta80, Thm. 1.6] which we do not have. §6 replaces it by the exact identity $q^k x_{a+k} = p^k x_a + W(a, k)$ and a growth estimate.

Remark 1.4 (scope). Theorem A is not a step toward non-automaticity of w, which is the question of the companion papers [RS26, RS26b]. The generating function of an automatic sequence [AS03] satisfies a Mahler-type functional equation, and a D-finite Mahler function is rational, by a theorem of Bézivin [Bez94]; with Proposition 2.6 below, an automatic sequence over a finite alphabet that is not eventually periodic is therefore never P-recursive. Automaticity of w would *imply* Theorem A, not contradict it. What Theorem A does exclude, unconditionally, is the hypothesis that these orbits admit a closed form of P-recursive type — the standard dividing line for a sequence given by an iteration.

1.3. **Unformalized theorems.**

Only one theorem is not formalized and used in the formalization as a black box (and only by the corollaries that weaken non-holonomy to transcendence).

Theorem 1.5 (Stanley [Sta80])**.** *Let $y = \sum_{n \geq 0} f(n)x^n$ be a power series over a field of characteristic zero. If y is algebraic, then y is D-finite, and y is D-finite if and only if f is P-recursive. In particular the coefficient sequence of an algebraic power series is P-recursive.*

This is [Sta80, Thm. 2.1] composed with [Sta80, Thm. 1.5], both formal-algebraic closure properties requiring no analysis; the composite is recorded in the formalization as a cited axiom. All other statements are formally verified in Lean 4; Appendix A records the Lean name, file and axiom footprint of each.

2. Finite alphabets and P-recursive sequences

This section is about an arbitrary sequence with finitely many values; nothing about ceiling orbits enters. The goal is Theorem 2.4 and its contrapositive, which is the interface every later section consumes.

Proposition 2.1 (pigeonhole)**.** *Let* $u\colon \mathbb{N} \to \mathbb{Q}$ *have finite range and let* $s \in \mathbb{N}$*. There is an* N *such that for every* $m > N$ *the window* $(u(m), u(m+1), \ldots, u(m+s))$ *occurs at infinitely many positions.*

Proof sketch. The windows take values in a finite set, being tuples over the finite range of u. Call a window *rare* if it occurs at only finitely many positions; the rare windows form a finite set, so the positions carrying one lie in a finite union of finite fibres and are bounded, say by N. □

Proposition 2.2 (constant coefficients)**.** *Let* $u\colon \mathbb{N} \to \mathbb{Q}$ *have finite range and be* P*-recursive. Then there are* $s \in \mathbb{N}$*, constants* $q_0, \ldots, q_s \in \mathbb{Q}$ *not all zero, and an* N*, with*

$$\sum_{j=0}^{s} q_j\, u(m+j) = 0 \qquad \text{for every } m > N.$$

Proof sketch. Let $\sum_j Q_j(n)u(n+j) = 0$ be the recurrence, $e := \max_j \deg Q_j$, and N as in Proposition 2.1. Fix $m > N$ and consider the single polynomial $\Pi(t) := \sum_j Q_j(t)\, u(m+j)$. At every position m' carrying the same window as m the recurrence at m' reads $\Pi(m') = 0$; there are infinitely many such m', so $\Pi = 0$. Reading off the coefficient of t^e gives the displayed identity with $q_j = [t^e]Q_j$, and these are not all zero because the index attaining the maximal degree contributes its leading coefficient. □

Proposition 2.3 (finite state)**.** *Let* $u\colon \mathbb{N} \to \mathbb{Q}$ *have finite range and satisfy* $\sum_{j \le s} q_j u(m+j) = 0$ *for all* $m > N$*, with the* q_j *constants not all zero. Then* u *is eventually periodic.*

Proof sketch. Let b be the largest index with $q_b \neq 0$. Solving the recurrence at $m+1$ for its top term — legitimate since $q_b \neq 0$ — expresses $u(m+1+b)$ through $u(m+1), \ldots, u(m+b)$, so the window $(u(m), \ldots, u(m+b))$ determines the window at $m+1$: the windows evolve deterministically. They live in a finite set, so two positions $N+k_1$, $N+k_2$ with $k_1 \neq k_2$ carry the same window, and determinism propagates the agreement to all later positions, giving period $|k_2 - k_1|$ from $N + \min(k_1, k_2)$ on. □

Theorem 2.4. *A* P*-recursive sequence* $u\colon \mathbb{N} \to \mathbb{Q}$ *with finite range is eventually periodic.*

Proof. Propositions 2.2 and 2.3 in succession. □

Corollary 2.5. *A sequence* $u\colon \mathbb{N} \to \mathbb{Q}$ *with finite range which is not eventually periodic is not* P*-recursive.*

Corollary 2.5 is the interface used in §5: supply a finite alphabet and aperiodicity, receive non-holonomy. It is worth stressing that neither Theorem 2.4 nor anything preceding it passes through rationality of the generating function, so no analogue of Fatou's lemma, no pole structure and no Skolem–Mahler–Lech theorem is involved.

The rational case is nevertheless available separately, and separately cheap.

Proposition 2.6. *Let $u\colon \mathbb{N} \to \mathbb{Q}$ have finite range and suppose its generating function is a rational function:*

$$\Big(\sum_n u(n)z^n\Big) \cdot Q(z) = \Pi(z)$$

in $\mathbb{Q}[[z]]$ for polynomials Π, Q with $Q \neq 0$. Then u is eventually periodic.

Proof sketch. Read the coefficient of $z^{m+\deg Q}$ for $m > \deg \Pi$: the right side contributes nothing, and on the left the terms with index below m vanish because they call for a coefficient of Q above $\deg Q$. What remains is $\sum_{j \le \deg Q} Q_{\deg Q - j}\, u(m+j) = 0$, a constant-coefficient recurrence, nontrivial at $j = 0$ since $Q_{\deg Q}$ is the leading coefficient of Q. Now apply Proposition 2.3. □

Remark 2.7. No hypothesis $Q(0) \neq 0$ is needed. Proposition 2.3 solves its recurrence forward at the largest index with $q_j \neq 0$, which under the index reversal $q_j = Q_{\deg Q - j}$ is the *lowest* nonvanishing coefficient of Q; a factor $z^m \mid Q$ merely shortens the window.

Corollary 2.8. *Let $u\colon \mathbb{N} \to \mathbb{Q}$ have finite range and algebraic generating function. Then u is eventually periodic.* (*Theorem 1.5 and Theorem 2.4.*)

Note the routing: Proposition 2.6 does *not* go through Corollary 2.8, although a rational function is algebraic. Taking that route would pay Theorem 1.5 for a strictly weaker input.

3. Ceiling orbits of a rational base

Fix coprime integers $p > q \geq 2$ and $x_0 \geq 1$.

Definition 3.1. Let $x_0 \in \mathbb{N}$ and set

$$w_n := (-px_n) \bmod q, \qquad x_{n+1} := \frac{px_n + w_n}{q},$$

so that qx_{n+1} is the least multiple of q at or above px_n.

The word is introduced as a residue rather than as the difference $qx_{n+1} - px_n$, and the orbit as an exact quotient rather than as a ceiling, so that neither a subtraction nor a rounding of natural numbers occurs in the recursion; that these descriptions all agree is the content of the next statement, after which only the selection rule is used.

Proposition 3.2. *For every n one has the selection rule $qx_{n+1} = px_n + w_n$ with $w_n \in \{0, \dots, q-1\}$, and $x_{n+1} = \lceil px_n/q \rceil$. If $x_0 \geq 1$ then $x_0 < x_1 < x_2 < \cdots$ and every $x_n \geq 1$.*

Proof sketch. By construction $q \mid px_n + w_n$ and $0 \leq w_n < q$, so x_{n+1} is the least integer with $qx_{n+1} \geq px_n$, which is the ceiling. Strict growth follows from $qx_n < px_n \leq qx_{n+1}$ for $x_n \geq 1$. □

The whole engine of §4 is the selection rule; nothing else about Definition 3.1 is used below.

Definition 3.3. For $a, k \in \mathbb{N}$ put $W(a,k) := \sum_{i<k} p^{k-1-i} q^{i} w_{a+i}$.

We are not aware of standard terminology for $W(a,k)$; following [RS26] we call it the *circuit sum* of the length-k window at a. This is the error term in the following identity, which is the only consequence of Definition 3.1 used from here on.

Proposition 3.4 (circuit identity)**.** *For all $a, k \in \mathbb{N}$, $q^k x_{a+k} = p^k x_a + W(a,k)$.*

Proof sketch. Induction on k, feeding $q x_{a+k+1} = p x_{a+k} + w_{a+k}$ into the inductive hypothesis; the recurrence $W(a,k+1) = pW(a,k) + q^k w_{a+k}$ matches. At $a = 0$ this is $q^n x_n = p^n x_0 + W(0,n)$, whence also the lower growth bound $p^n x_0 \le q^n x_n$, the word only ever pushing the orbit up. $\square$

Proposition 3.5 (growth ceiling)**.** *For every n,*

$$q^n\big((p-q)x_n + q\big) \;\le\; p^n\big((p-q)x_0 + q\big),$$

that is, $x_n + \frac{q}{p-q} \le (p/q)^n\big(x_0 + \frac{q}{p-q}\big)$.

Proof sketch. One step: $q\big((p-q)x_{n+1} + q\big) = (p-q)(p x_n + w_n) + q^2 \le (p-q)p x_n + (p-q)q + q^2 = p\big((p-q)x_n + q\big)$, using $w_n \le q$ and $(p-q)q + q^2 = pq$. Induction on n. $\square$

Remark 3.6. The affine shift is forced and is not always 1. Since $\lceil \beta t \rceil \le \beta t + 1$, the map $t \mapsto t + c$ is submultiplicative under the ceiling map exactly when $c \ge 1/(\beta - 1) = q/(p-q)$. At $(p,q) = (3,2)$ that constant is 2, yet the sharper bound $2^n(x_n + 1) \le 3^n(x_0 + 1)$ still holds, because there $w_n + q \le p$; at $p/q = 5/4$ that inequality fails and the shift-1 bound is false. Proposition 3.5 is the form valid at every admissible base.

4. q-adic rigidity and aperiodicity

Definition 4.1. A *repeated factor* of length k at (a,c) is an equality of the length-k factors of w at a and at c: $w_{a+i} = w_{c+i}$ for all $i < k$. Occurrences may overlap.

Lemma 4.2. *If w has a repeated factor of length k at (a,c), then*

$$p^k(x_c - x_a) = q^k(x_{c+k} - x_{a+k}).$$

Proof sketch. Equal factors give equal circuit sums, $W(a,k) = W(c,k)$; subtract the two instances of Proposition 3.4. $\square$

Proposition 4.3. *If w has a repeated factor of length k at (a,c), then $q^k \mid x_c - x_a$.*

Proof sketch. Lemma 4.2 gives $q^k \mid p^k(x_c - x_a)$, and q^k is coprime to p^k. This is the only place where $\gcd(p,q) = 1$ is used, and it is used as coprimality, not as primality of q. $\square$

Lemma 4.4 (q-adic descent)**.** *If $q^m \mid x_a - x_b$, then $q^{m-i} \mid x_{a+i} - x_{b+i}$ for every $i \le m$.*

Proof sketch. Induction on i. As long as one power of q survives, $x_{a+i} \equiv x_{b+i} \pmod q$, hence $w_{a+i} = w_{b+i}$, the letters being determined by the residue of the orbit modulo q; then subtracting the two selection rules gives $q(x_{a+i+1} - x_{b+i+1}) = p(x_{a+i} - x_{b+i})$, which cancels exactly one power of q. No coprimality is needed here: the cancellation is on the left and is indifferent to p. □

Theorem 4.5 (window rigidity)**.** *For all a, b, m, the length-m factors of w at a and b agree if and only if $q^m \mid x_b - x_a$.*

Proof. Forwards is Proposition 4.3; backwards, Lemma 4.4 keeps $q \mid x_{a+i} - x_{b+i}$ alive for $i < m$, and each such divisibility fixes one letter. □

So length-m windows of w *are* the residues $x_n \bmod q^m$. The case $m = 1$ is the recoding quoted in the introduction.

Corollary 4.6. *$w_a = w_b$ if and only if $x_a \equiv x_b \pmod q$.*

Dubickas draws from this the identity $P(w, n) = P(X, n)$ between the complexity of the word and that of the residue sequence $X_n = x_n \bmod q$ [Dub09, p. 245]; Theorem 4.5 is the same identity read one window at a time. We record only the equivalences themselves, the complexity function playing no role in what follows.

Rigidity is also a cost: a repeated window forces a large power of q to divide a difference of orbit values, while the orbit has only $(p/q)^c$ room.

Theorem 4.7 (repetition ceiling)**.** *Let $x_0 \ge 1$ and $a < c$. If w has a repeated factor of length k at (a, c), then*

$$q^{k+c} \;<\; p^c\big((p-q)x_0 + q\big).$$

Proof sketch. By Theorem 4.5, $q^k \mid x_c - x_a$, and $x_a < x_c$, so $q^k \le x_c - x_a < x_c \le (p-q)x_c + q$. Multiplying by q^c and applying Proposition 3.5 chains onto the display. □

Theorem 4.8 (Dubickas [Dub09, p. 245]; cf. [AFS08, Prop. 26])**.** *Let $p > q \ge 2$ be coprime and $x_0 \ge 1$. Then w is not eventually periodic.*

Proof. Eventual periodicity with threshold N and period $P \ge 1$ gives a repeated factor at the *fixed* pair (N, c), $c := N + P$, of *every* length k. But Theorem 4.7 caps every such k by the single number $k_0 := p^c\big((p-q)x_0 + q\big)$, with c fixed. Take $k = k_0$: then $k_0 < 2^{k_0} \le q^{k_0} \le q^{k_0+c} < k_0$, using $q \ge 2$, a contradiction. □

Remark 4.9 (what this replaces). Theorem 4.8 is the parallel of [AFS08, Prop. 26] — no element of $W_{p/q}$ is eventually periodic but 0^ω — and it is also the inference Dubickas draws on [Dub09, p. 245] from his Theorem 2, that an eventually periodic w forces β to lie in $U = S \cup T$, the Pisot numbers together with the Salem numbers, together with the observation that p/q lies in neither: an element of U is an algebraic integer, and a non-integral rational is not one, $\mathbb{Z}$ being integrally closed in $\mathbb{Q}$. That observation is elementary and we record it as the next proposition.

Proposition 4.10. *If $q \geq 2$, $p > q$ and $\gcd(p, q) = 1$, then p/q is not an algebraic integer; in particular $p/q \notin U$.*

Proof sketch. $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$, so an algebraic integer in $\mathbb{Q}$ lies in $\mathbb{Z}$, and $q \nmid p$. □

5. The words are not holonomic

Proof of Theorem A. The word takes its values in the finite set $\{0, \dots, q-1\}$ by Proposition 3.2, and it is not eventually periodic by Theorem 4.8. Now apply Corollary 2.5. □

Corollary 5.1. *For every $x_0 \geq 1$ the minimal word of the base $3/2$ — the parity $w_n = x_n \bmod 2$ of the orbit $x_n = \lceil 3x_{n-1}/2 \rceil$, which at $x_0 = 1$ is $g_{3/2}$ — is not P-recursive.*

In the formalization Corollary 5.1 also carries a proof of its own, which draws its aperiodicity from the 2-adic rigidity of our companion paper [RS26b] rather than from Theorem 4.8; the two words are proved equal, so the statements are the same statement.

Example 5.2. $p/q = 5/4$ is admissible: $q = 4$ is composite, the alphabet has four letters, and nothing in §4 needs q prime — coprimality enters only through $\gcd(q^k, p^k) = 1$ in Proposition 4.3. So the word of $x_n = \lceil 5x_{n-1}/4 \rceil$ is not P-recursive either.

Example 5.3. At $p/q = 5/3$ and $x_0 = 1$ the orbit begins $1, 2, 4, 7, 12$ and the word $w_n = 3x_{n+1} - 5x_n$ begins $1, 2, 1, 1$ over the alphabet $\{0, 1, 2\}$.

Corollary 5.4. *For $p/q = 3/2$ and every $x_0 \geq 1$, the generating function $f(z) = \sum_j w_j z^j$ is not a rational function: $f \cdot Q = \Pi$ is impossible for polynomials Π, Q over $\mathbb{Q}$ with $Q \neq 0$.*

Proof. Proposition 2.6 and Theorem 4.8. □

The same two lines give the same conclusion at every admissible base; only the case $p/q = 3/2$ is formalized. Note that Corollary 5.4 does not go through Theorem A: it consumes aperiodicity directly, and is the cheapest statement in this paper.

Corollary 5.5. *For every admissible (p, q, x_0), $f(z) = \sum_j w_j z^j$ is not algebraic over $\mathbb{Q}(z)$.*

Proof. An algebraic series has P-recursive coefficients (Theorem 1.5), which Theorem A forbids. □

Remark 5.6. Corollary 5.5 is the weaker statement and the only one here that consumes Theorem 1.5; Theorem A is stronger and free. Reading Theorem 1.5 in the other direction, Theorem A also says that f is not D-finite.

6. The orbit series

For this section fix $p/q = 3/2$ and write $F(z) = \sum_n x_n z^n$, $f(z) = \sum_j w_j z^j$.

Proposition 6.1. *As formal power series over $\mathbb{Q}$,*

$$(2-3z)\,F(z) \;=\; z\,f(z) + 2x_0.$$

Proof sketch. In degree 0 both sides are $2x_0$; in degree $n+1$ the identity reads $2x_{n+1} - 3x_n = w_n$, which is the selection rule. □

So F is a linear-fractional transform of f over $\mathbb{Q}(z)$, and in particular each of F, f is rational, algebraic or D-finite exactly when the other is.

Remark 6.2 (analytic reading; not formalized)*.* Since $0 \le w_j \le 1$ and $x_n \asymp K(3/2)^n$ with K as in Definition 6.5 below, f has radius of convergence 1 and F radius $2/3$, and Proposition 6.1 exhibits the boundary singularity of F as a simple pole at $z = 2/3$ with residue

$$\frac{(2/3)f(2/3) + 2x_0}{-3} = -\frac{2K}{3},$$

using $f(2/3) = 3(K - x_0)$ from Proposition 6.6. This paragraph is the one place where an analytic statement is made; it is outside the scope of the formalization, whose formal power series carry no analytic structure.

Proof of Theorem B. Suppose $\sum_{j\le s} Q_j(n)x_{n+j} = 0$ nontrivially and let b be the largest index with $Q_b \neq 0$. If $b = 0$ the recurrence reads $Q_0(n)x_n = 0$ with $x_n > 0$, so Q_0 vanishes at every natural number and hence is zero, a contradiction. Otherwise multiply by 2^s and substitute $2^j x_{n+j} = 3^j x_n + W(n,j)$ (Proposition 3.4) in each term. Collecting the orbit terms and the word terms separately turns the recurrence into

$$A(n)\,x_n + \sum_{i<s} B_i(n)\,w_{n+i} = 0,$$

where

$$A = \sum_j Q_j\,2^{\,s-j}3^{\,j}, \qquad B_i = \sum_{j>i} Q_j\,2^{\,s-j+i}3^{\,j-1-i},$$

polynomial identities in n. Here $B_i = 0$ for $i \ge b$, while $B_{b-1} = 2^{\,s-1}Q_b \neq 0$.

If $A = 0$, what remains is $\sum_{i<b} B_i(n)w_{n+i} = 0$, a nontrivial P-recurrence for the word, contradicting Corollary 5.1.

If $A \neq 0$, use $|w_{n+i}| \le 1$ to get $|A(n)|\,x_n \le \sum_{i<s}|B_i(n)|$, and $x_n \ge (3/2)^n$ from the lower growth bound of Proposition 3.4 with $x_0 \ge 1$. Hence

$$|A(n)| \;\le\; \Big(\sum_{i<s}|B_i(n)|\Big)\Big(\tfrac{2}{3}\Big)^n \;\longrightarrow\; 0,$$

polynomial growth losing to geometric decay. A polynomial whose values along $\mathbb{N}$ tend to 0 is the zero polynomial, so $A = 0$ after all — a contradiction. □

Remark 6.3. Nothing in the argument is specific to $3/2$: it uses a circuit identity, a bounded word and exponential growth, all of which §3 supplies at every admissible p/q. Note also that no integrality argument is needed — one might expect to clear denominators and bound $|A(n)|$ from below off the roots of A; the limit statement replaces that step.

Corollary 6.4. *For $x_0 \geq 1$ the orbit series $F(z) = \sum_n x_n z^n$ is not algebraic over $\mathbb{Q}(z)$.*

Proof. Theorem 1.5 and Theorem B. □

6.1. **The value at the pole.** The remaining statement records what the number $f(2/3)$ is, and that it is where this circle of questions runs out.

Definition 6.5. $K := x_0 + \frac{1}{3}\sum_{j\geq 0} w_j(2/3)^j$, a convergent series since $w_j \in \{0,1\}$; equivalently $K = \lim_n x_n(2/3)^n$. For $x_0 = 1$ this is the constant $\omega_{3/2}$ of [AFS08, Ex. 3], which is the constant $K(3)$ of Odlyzko and Wilf [OW91]:

$$K = \omega_{3/2} = K(3) = 1.62227\,05028\,84767\,31595\,6\ldots$$

(sequence A083286 [OEIS]).

Proposition 6.6. $\sum_{j\geq 0} w_j(2/3)^j = 3(K - x_0)$.

Proposition 6.7. *For every $x_0 \geq 1$: $f(2/3)$ is irrational if and only if K is irrational.*

Proof sketch. Immediate from Proposition 6.6: the two differ by a nonzero rational scaling and an integer translation, neither of which affects irrationality. □

The irrationality of K has been open since Wang and Washburn asked for it in 1977 [WW77]. Proposition 6.7 says that the value of the word series at the singularity of the orbit series is exactly that open question, and Theorem B says nothing about it: non-holonomy of a coefficient sequence constrains the function, not the value.

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Appendix A. Formalization

Every statement above was verified in Lean 4, except Remark 6.2, which is analytic and is marked as such. See `https://github.com/rwst/Ceiling-Orbits`. The files live under `RB/`, with `CITED/Stanley.lean` supplying the single axiom which is Theorem 1.5. Identifiers in the namespace `RB` are written below without that prefix, and `Gen.` abbreviates `RB.Gen.`, the namespace of the general rational base. "std3" abbreviates the ambient axioms of classical Lean (propositional extensionality, choice, quotient soundness).

The engine (§2).

Statement	Lean identifier	File	Status
Def. 1.2	`Stanley.IsPRecursive`	`CITED/Stanley`	def
Thm. 1.5	`Stanley.pRecursive_of_isAlgebraic`	`CITED/Stanley`	cited axiom [Sta80]
Prop. 2.1	`exists_N_recurrent`	`EventuallyPeriodic`	std3
Prop. 2.2	`exists_constant_recurrence`	`EventuallyPeriodic`	std3
Prop. 2.3	`eventuallyPeriodic_of_constant_recurrence`	`EventuallyPeriodic`	std3
Thm. 2.4	`eventuallyPeriodic_of_isPRecursive`	`EventuallyPeriodic`	std3
Cor. 2.5	`not_isPRecursive_of_not_eventuallyPeriodic`	`EventuallyPeriodic`	std3
Prop. 2.6	`eventuallyPeriodic_of_rational_of_finite_coeffs`	`EventuallyPeriodic`	std3
Cor. 2.8	`eventuallyPeriodic_of_isAlgebraic_of_finite_coeffs`	`EventuallyPeriodic`	std3 + [Sta80]

Rational bases, rigidity, and Theorem A (§3–§5).

Statement	Lean identifier	File	Status
Def. 3.1	`Gen.x`, `Gen.word`	`GeneralBase`	def
Prop. 3.2	`Gen.q_mul_x_succ`, `Gen.word_lt`, `Gen.x_succ_eq_ceil`	`GeneralBase`	std3
monotonicity	`Gen.x_pos`, `Gen.x_strictMono`	`GeneralBase`	std3
Def. 3.3	`Gen.W`	`GeneralBase`	def
Prop. 3.4	`Gen.circuit_sum`, `Gen.circuit_sum_zero`, `Gen.p_pow_mul_le`	`GeneralBase`	std3
Prop. 3.5	`Gen.q_pow_mul_le`	`GeneralBase`	std3
Def. 4.1	`Gen.IsRepetition`	`GeneralBase`	def
Lem. 4.2	`Gen.lemmaR_int`	`GeneralBase`	std3
Prop. 4.3	`Gen.q_pow_dvd_of_repetition`	`GeneralBase`	std3
Lem. 4.4	`Gen.dvd_sub_descend`, `Gen.word_eq_of_dvd`	`GeneralBase`	std3
Thm. 4.5	`Gen.window_eq_of_dvd`, `Gen.isRepetition_iff_dvd`	`GeneralBase`	std3
Cor. 4.6	`Gen.word_determines_residue`	`GeneralBase`	std3
Thm. 4.7	`Gen.repetition_pow_lt`	`GeneralBase`	std3
Thm. 4.8	`Gen.not_eventually_periodic`	`GeneralBase`	std3
Prop. 4.10	`Gen.not_isIntegral_rational`, `Gen.not_mem_Bertin_U`	`GeneralBase`	std3
Thm. A	`Gen.not_isPRecursive_word`	`GeneralBase`	std3
Cor. 5.1	`not_isPRecursive_wmin`	`EventuallyPeriodic`	std3
the $(3,2)$ *bridge*	`Gen.x_eq`, `Gen.wmin_eq`	`GeneralBase`	std3
Ex. 5.2	`Gen.not_isPRecursive_word_five_fourths`	`GeneralBase`	std3
Ex. 5.3	`Gen.x_sanity`	`GeneralBase`	no axioms
Cor. 5.4	`not_rational_wminSeries`	`EventuallyPeriodic`	std3
Cor. 5.5	`Gen.not_isAlgebraic_wordSeries`	`GeneralBase`	std3 + [Sta80]
at $3/2$	`not_isAlgebraic_wminSeries`	`EventuallyPeriodic`	std3 + [Sta80]

The orbit series (§6).

Statement	Lean identifier	File	Status
F, f	`orbitSeries`, `wordSeries`	`OrbitSeries`	def
Prop. 6.1	`orbitSeries_mobius`	`OrbitSeries`	std3
Thm. B	`not_isPRecursive_orbit`	`OrbitSeries`	std3
Cor. 6.4	`not_isAlgebraic_orbitSeries`	`OrbitSeries`	std3 + [Sta80]
Def. 6.5	`K`, `tendsto_x_mul_pow`	`Basic`	def, std3
Prop. 6.6	`tsum_wmin_eq`	`Basic`	std3
Prop. 6.7	`irrational_tsum_wmin_iff`	`OrbitSeries`	std3

References

[AF17] B. Adamczewski, C. Faverjon, *Méthode de Mahler: relations linéaires, transcendance et applications aux nombres automatiques*, Proc. London Math. Soc. **115** (2017), 55–90; arXiv:1508.07158.

[AFS08] S. Akiyama, C. Frougny, J. Sakarovitch, *Powers of rationals modulo 1 and rational base number systems*, Israel J. Math. **168** (2008), 53–91.

[AS03] J.-P. Allouche, J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.

[Ber92] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.-P. Schreiber, *Pisot and Salem Numbers*, Birkhäuser, Basel, 1992.

[Bez94] J.-P. Bézivin, *Sur une classe d'équations fonctionnelles non linéaires*, Funkcial. Ekvac. **37** (1994), 263–271.

[Dub09] A. Dubickas, *On integer sequences generated by linear maps*, Glasgow Math. J. **51** (2009), 243–252.

[OEIS] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`. Sequences A061419, A083286, A205083.

[OW91] A. M. Odlyzko, H. S. Wilf, *Functional iteration and the Josephus problem*, Glasgow Math. J. **33** (1991), 235–240.

[RS26] R. Stephan, *Superlinear complexity of the* $(3/2)^n$ *steering word*, arXiv:2607.11648v2 (2026).

[RS26b] R. Stephan, *Transcendence criteria for the minimal word of the rational base* 3/2, see `https://www.researchgate.net/publication/410864178_Transcendence_Criteria_for_the_Minimal_Word_of_the_Rational_Base_32`.

[Sta80] R. P. Stanley, *Differentiably finite power series*, European J. Combin. **1** (1980), 175–188.

[WW77] E. T. H. Wang, P. C. Washburn, *Problem E2604*, Amer. Math. Monthly **84** (1977), 821–822.