TRANSCENDENCE CRITERIA FOR THE MINIMAL WORD OF THE RATIONAL BASE 3/2

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ABSTRACT. Let x_0 be a positive integer, let $x_n = \lceil 3x_{n-1}/2 \rceil$, and let $w_n = 2x_{n+1} - 3x_n \in \{0,1\}$ be the associated word studied by Dubickas; for $x_0 = 1$ the orbit is A061419 and w is the minimal word $g_{3/2}$ of the rational base number system of Akiyama, Frougny and Sakarovitch. The orbit encodes a real constant $K = \lim_n x_n(2/3)^n$, equal for $x_0 = 1$ to $\omega_{3/2} = K(3) = 1.6222705028\ldots$, whose irrationality has been open since 1977. We prove that if w is automatic then K is transcendental; equivalently, an algebraic K forces w to be non-automatic. Further we prove that either the complexity of w exceeds every linear bound or K is irrational. All results are formally verified in the Lean 4 proof assistant *except* four inputs quoted from the literature, which are assumed as axioms; the transcendence input, a special case of a theorem of Adamczewski and Faverjon, is among the verified results rather than among those four.

1. INTRODUCTION

Fix a positive integer x_0 and iterate the ceiling map

$$U_{3/2}\colon x \longmapsto \left\lceil \tfrac{3x}{2} \right\rceil, \qquad x_n := U_{3/2}^n(x_0),$$

so that $x_0 < x_1 < x_2 < \cdots$. Following Dubickas [Dub09, p. 245] we attach to this orbit the word

$$w_n := 2x_{n+1} - 3x_n \in \{0,1\}, \qquad n = 0,1,2,\ldots,$$

which is simply the parity of the orbit, $w_n = x_n \bmod 2$. For $x_0 = 1$ the orbit is the sequence A061419, and w is an object of the rational base number system of Akiyama, Frougny and Sakarovitch [AFS08]: their *minimal word* $g_{3/2} = 10110001101001101010011 0\cdots$ ([AFS08, §4.2], A205083), the lexicographically least label of an infinite path in the tree $T_{3/2}$. We prove everything below for a general $x_0 \geq 1$, of which $g_{3/2}$ is the case $x_0 = 1$.

Remark 1.1. The identification is exact but carries an index shift. Akiyama, Frougny and Sakarovitch index $g_{3/2} = g_1 g_2 g_3 \cdots$ from 1 and compute its letters by $g_{k+1} = qG_{k+1} \bmod p$ [AFS08, Remark 32], where $G_0 = 1$, $G_{k+1} = \lceil (p/q)G_k \rceil$ is their Definition 20. At $(p,q) = (3,2)$ the selection rule $2G_{k+1} =$

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$3G_k + (G_k \bmod 2)$ turns this into $g_{k+1} = G_k \bmod 2$, which is Dubickas's w_k. So $w_n = g_{n+1}$, and the sequence indexed from 0 in this paper is the sequence indexed from 1 there.

Throughout, $P(w, n)$ denotes the *complexity function* (also called the block-complexity function) of w in the sense of [Dub09, p. 245]: the number of distinct vectors $(w_j, w_{j+1}, \ldots, w_{j+n-1})$ as j ranges over all non-negative integers. A sequence is *k-automatic* if its k-kernel $\{n \mapsto a(k^i n + r) : i \geq 0,\ 0 \leq r < k^i\}$ is finite, and *automatic* if it is k-automatic for some $k \geq 2$ [AS03]. We write $\|x\|$ for the distance from x to the nearest integer.

1.1. **The constant.** Dividing the identity $2x_{n+1} = 3x_n + w_n$ by powers of 3 (see Proposition 3.5) gives $(2/3)^n x_n = x_0 + \frac{1}{3}\sum_{j<n} w_j (2/3)^j$, whose limit

$$K \;:=\; \lim_{n\to\infty} x_n (2/3)^n \;=\; x_0 + \tfrac{1}{3}\sum_{j\geq 0} w_j \left(\tfrac{2}{3}\right)^j$$

exists. For $x_0 = 1$ this is the constant $\omega_{3/2}$ of [AFS08, Example 3], which is the constant $K(3)$ of Odlyzko and Wilf [OW91]:

$$K = \omega_{3/2} = K(3) = 1.62227\,05028\,84767\,31595\,69509\,8\ldots$$

(sequence A083286). It is a striking feature of this circle that so little is known about K: its *irrationality* is open, asked by Wang and Washburn in 1977 [WW77].

1.2. **Known results.** Three facts about w are in the literature. The first two we reprove; the third we reprove as well (Corollary 4.4), though we never use it, for the structural reason recorded in Remark 1.5.

Theorem 1.2 (Odlyzko–Wilf; Akiyama–Frougny–Sakarovitch)**.** *For every $x_0 \geq 1$ and every $n \geq 0$,*

$$x_n = \left\lfloor K\,(3/2)^n \right\rfloor.$$

This is [OW91, Cor. 1] in the case $q = 3$ of the Josephus problem, where the relevant map is $x \mapsto \lceil \frac{q}{q-1} x \rceil = \lceil \frac{3}{2} x \rceil$, and independently [AFS08, Cor. 31], which gives $G_k = \lfloor \gamma_{p/q} (p/q)^{k+1} \rfloor$ whenever $p \geq 2q - 1$ — at $(3, 2)$ one has $3 \geq 3$. It is also stated by Dubickas [Dub09, p. 244], who records that $x_n = \lfloor c(\beta)\beta^n \rfloor$ holds for $\beta \geq 2$ or $\beta = 2 - 1/q$, and $3/2 = 2 - 1/2$.

Theorem 1.3 (Akiyama–Frougny–Sakarovitch [AFS08]; Dubickas [Dub09])**.** *The word w is not eventually periodic.*

This is [AFS08, Prop. 26], and independently [Dub09, Thm. 2]: an ultimately periodic word forces the base to be a Pisot or a Salem number, and $3/2$ is neither — an inference Dubickas draws himself [Dub09, p. 245].

Theorem 1.4 (Dubickas [Dub09])**.** *$P(w, n) > 1.70951129\,n$ for every sufficiently large n.*

This is [Dub09, Cor. 4], the best lower bound for this word we are aware of, derived from the general bound $\liminf_n P(w,n)/n \geq \log q/\log(p/q)$ of [Dub09, Thm. 3]; Corollary 4.4 below recovers it from the rigidity theory by a pigeonhole.

Remark 1.5. Theorem 1.4 cannot be pushed to non-automaticity by sharpening its constant. Cobham's theorem [Cob72] bounds the complexity of a k-automatic sequence only by $O(n)$, so a lower bound $P(w,n) \geq cn$ is compatible with automaticity for *every* c; the Thue–Morse sequence is 2-automatic with $\liminf_n P(n)/n = 3 > 1.7095$ [Brl89, dLV89]. Every complexity statement below therefore targets *superlinearity*, not a linear floor.

1.3. **Results of this paper.** The transcendence input of the paper is the Mahler alternative of Adamczewski and Faverjon, in the following special case.

Theorem A (Adamczewski–Faverjon [AF17], Cor. 1.8; Cobham's 1968 conjecture)**.** *Let $(a_j)_{j\geq 0}$ be an automatic sequence of non-negative integers, bounded by some B, and let $\alpha \in \mathbb{Q}$ with $0 < |\alpha| < 1$. Then either $\sum_{j\geq 0} a_j \alpha^j$ is transcendental, or it is rational.*

The statement is not ours. It is [AF17, Cor. 1.8] in the specialization we need — f automatic with bounded coefficients, α rational, whence the field k of the general statement is $\mathbb{Q}$ — and in that specialization it is precisely the 1968 conjecture of Cobham, which Adamczewski and Faverjon call the origin of their paper. What is new here is its status in this paper: it is no longer assumed. The Lean development derives it, together with each of the three reductions just named — automatic $\Rightarrow$ q-Mahlerian with a *nonsingular polynomial* matrix; bounded coefficients $\Rightarrow$ radius of convergence $\geq 1 \Rightarrow \alpha$ is not a pole; and $k = \mathbb{Q}$ — from two lemmas lying one level below the corollary, Theorems 1.8 and 1.9 of §1.4. The route assembles [AF17, §4] with the new proof of the underlying lifting theorem [AF22, §2], and uses neither the analytic desingularization of [AF17, §5] nor the earlier proof of that theorem. Appendix A records what the exercise turned up: several places where the printed proof is longer than it needs to be, one printed claim that is false as stated, one step that cannot be taken in the order in which it is printed, and a handful of hypotheses that are load-bearing and invisible until one writes them down.

Remark 1.6 (read the alternative literally). The disjunct "rational" is not a degenerate corner. Adamczewski and Faverjon show it cannot be removed — *"des exemples montrent que l'on ne peut se soustraire à l'alternative ... même en supposant la fonction $f(z)$ transcendante"* — and their §8.1 exhibits a $\{0,1\}$-valued 3-automatic sequence with f transcendental over $\mathbb{Q}(z)$ and yet $f_1(\varphi) = -\varphi/2 \in \mathbb{Q}(\varphi)$. An argument that reads Theorem A as forcing a contradiction from a *known rational* value is therefore invalid: the rational value is the permitted branch. This is why Theorem A is applied below only in the one direction where both branches die at once (§5.1).

Our main result links the arithmetic nature of K to the automaticity of the word that generates it. To our knowledge no statement of this kind was previously available.

Theorem B. *Let $x_0 \geq 1$. If the word w is automatic, then K is transcendental. Equivalently: if K is algebraic, then w is not automatic.*

The reading with content is the first implication: it concludes the *transcendence* of a constant not known to be *irrational*, from a combinatorial hypothesis on the word. Theorem B is proved in §5 by splitting the algebraic case in two and closing each half by a different engine — the Mahler-method alternative of Adamczewski and Faverjon on the irrational half (§5.1), and the Subspace Theorem on the rational half (§5.2). We stress at once what it does not do.

Remark 1.7 (scope). K is expected to be transcendental, so the hypothesis of the second form of Theorem B is plausibly vacuous, and the theorem is *not* a proof that w is non-automatic: the generic, transcendental case is untouched.

The rational half yields a dichotomy of independent interest, in which both horns are statements one would like to have and neither is available alone.

Theorem C. *Let $x_0 \geq 1$. Then either the complexity of w exceeds every linear bound — for every C there is an $m \geq 1$ with $P(w,m) > Cm$, i.e. $\limsup_m P(w,m)/m = \infty$ — or K is irrational.*

Behind the first horn stands a rigidity theorem with a one-line reformulation that organizes everything in this paper: length-m factors of w agree when the orbit values agree modulo 2^m (Theorem 4.2), so $P(w,m)$ *is* the number of residues modulo 2^m that the orbit occupies (Proposition 4.3), and Theorem 1.4 becomes a pigeonhole (Corollary 4.4).

Beyond Theorems B and C, §5.4 begins the extension of the Diophantine kernel of §5.2 to algebraic multipliers: the bounded-gap half is proved (on the printed algebraic-δ form of the Main Theorem of [CZ04]); the combinatorial step that turns such a finiteness statement into a superlinear complexity bound — pigeonhole, contraction and the growth ceiling, that is, everything in §5.2 except its Diophantine input — is shown to need no rationality at all; and the upgrade of Theorem B to superlinear complexity in every algebraic case is thereby reduced to a single named statement, a pair theorem over $\mathbb{Q}(K)$. Unconditionally, close repetitions — long repeats at bounded distance — die out whenever K is algebraic (Corollary 5.16), a second transcendence criterion with a periodicity-adjacent trigger.

Every statement above and below has been formally verified in Lean 4 *except* two classes of statement: the four cited inputs of §1.4, which are *assumed* and not proved, and the computations of §6.3, which are not formalized at all. Everything else is proved in Lean on top of those four axioms.

Appendix A describes the formalization of Theorem A and Appendix B records the Lean name, file and axiom footprint of each formalization.

1.4. **Cited inputs.** Four statements are used as black boxes. Each is stated here with a citation and no proof; on their status as sources see Remark 1.10 below. The first two are what is left of Theorem A once it is proved rather than cited: they lie one level below it, and through it they carry the main results. The third carries the rational half. The fourth enters only the algebraic-multiplier extension of §5.4.

Theorem 1.8 (Adamczewski–Faverjon [AF17], Lemme 2.2)**.** *Let $q \geq 2$, let $A(z)$ be a matrix of polynomials over $\overline{\mathbb{Q}}$ with $\det A \neq 0$, and let $f_1, \ldots, f_n$ be power series over $\overline{\mathbb{Q}}$ solving the q-Mahler system $f(z) = A(z)f(z^q)$. Then the extension $\overline{\mathbb{Q}}(z)(f_1, \ldots, f_n)/\overline{\mathbb{Q}}(z)$ is regular — equivalently, in characteristic zero, $\overline{\mathbb{Q}}(z)$ is relatively algebraically closed in it.*

This is [AF17, Lemme 2.2]. It is what allows a relation whose coefficients lie in the solution field to be split into relations with coefficients in $\overline{\mathbb{Q}}(z)$, and it is the only step of the route below that reaches back into the analytic theory of Mahler functions: its own proof runs through Nishioka's theorem [Nis96, Thm. 5.1.7], in the form "an algebraic q-Mahler function is rational". Characteristic zero is not cosmetic — over $\mathbb{F}_p(z)$, Christol's theorem [Chr79] makes the algebraic power series exactly the p-automatic ones, all of them Mahlerian, and the statement is false.

Theorem 1.9 (Adamczewski–Faverjon [AF22], Lemma 2.8)**.** *Let $f_1, \ldots, f_m$ solve a q-Mahler system and converge on a disc about 0, let α be algebraic with $0 < |\alpha| < 1$, and let $\varphi(z)$ be a relation matrix for them, invertible over the ambient field of algebraic functions. Then for all sufficiently large k:* (a) *α^{q^k} lies in the disc of convergence of each f_i;* (b) *each coordinate of $\varphi(z)$ defines an analytic function on some neighbourhood of α^{q^k};* (c) *the matrix $\varphi(\alpha^{q^k})$ is invertible.*

This is [AF22, Lemma 2.8], the one point at which the algebra of the new proof meets analysis; it is cited because the ingredients of its four-line literature proof — a vanishing criterion for the resultant, Newton–Puiseux, and an implicit function theorem for algebraic function elements — are absent from the formalized library. The formalization cites a slightly wider form. The object Theorem 1.9 produces — a branch of φ near α^{q^k}, and with it a realization of the algebraic functions in play by honest analytic ones — is consumed five times, and the four further properties consumed of it (the solutions realized alongside φ, which is clause *(a)*; that realization analytic and injective; the neighbourhood connected; and the corresponding branch of the twisted matrix $\varphi(z^{q^k})$ at α, which takes the same values) all hold of that one object by construction. Folding them into the citation keeps the count of cited axioms at two rather than five; Appendix A says why none of them is a further assumption.

Remark 1.10 (on the sources). All four are refereed. Theorem 1.9 moreover asks the least trust of the four, since the statement itself is classical and is not what [AF22] is for: an element algebraic over $\overline{\mathbb{Q}}(z)$ has an analytic branch at every point outside a finite set, and a matrix with nonzero determinant is singular only at finitely many points; since the α^{q^k} are pairwise distinct and tend to 0, *(b)* and *(c)* therefore fail for at most finitely many k, and *(a)* holds for all large k. What is not classical — and what is done here rather than cited — is everything [AF22] builds on top of it.

Theorem 1.11 (Subspace Theorem; Schmidt [Sch91], Evertse–Schlickewei form [BG06])**.** *Let K be a number field, $n \geq 2$, and S a finite set of places of K containing the archimedean ones. For each $v \in S$ let $L_{v,1}, \ldots, L_{v,n}$ be linearly independent linear forms in $X_1, \ldots, X_n$ with coefficients in K. For every $\varepsilon > 0$, the nonzero $x \in K^n$ with*

$$\prod_{v \in S} \prod_{i=1}^{n} \frac{|L_{v,i}(x)|_v}{\|x\|_v} \;\leq\; H(x)^{-n-\varepsilon}$$

lie in finitely many proper linear subspaces of K^n.

This is [Sch91, Thm. 1D′]. It is the only Diophantine input of §5.2: both of the approximation theorems used there — the Main Theorem of Corvaja and Zannier [CZ04] and the modified pair theorem of Nair, Kumar and Rout [NKR25] — are *derived* from it in the formalization rather than assumed, at $n = 2$ and $n = 3$ respectively.

Remark 1.12 (on [NKR25]). Nothing is taken on the authority of [NKR25], which is an unrefereed preprint. Its Theorem 1.3(i) is false as stated — the hypothesis of strict positivity is missing, and our formalization carries a machine-checked refutation — and only the repaired form is used, itself derived from Theorem 1.11. We are still in debt of the NKR preprint as the source of an important argument.

Theorem 1.13 (Corvaja–Zannier [CZ04], Main Theorem)**.** *Let $\Gamma \subset \overline{\mathbb{Q}}^*$ be a finitely generated multiplicative group, let δ be a nonzero algebraic number, and let $\varepsilon > 0$. Then there are only finitely many pairs $(q, u) \in \mathbb{Z} \times \Gamma$, $d := [\mathbb{Q}(u) : \mathbb{Q}]$, such that $|\delta qu| > 1$, δqu is not pseudo-Pisot, and*

$$0 \;<\; \|\delta qu\| \;<\; H(u)^{-\varepsilon}\, q^{-d-\varepsilon}.$$

Here a real algebraic α is *pseudo-Pisot* if $|\alpha| > 1$, every conjugate of α other than α itself has modulus < 1, and the trace of α is a rational integer; a pseudo-Pisot number need not be an algebraic integer. For $\delta \in \mathbb{Q}$ this theorem is *derived* from Theorem 1.11 in the formalization. For algebraic irrational δ it is taken on [CZ04]'s authority as a cited axiom, and it is used only in §5.4: a *lane* of its own — an axiom footprint disjoint from those of the other three, carried by no result outside that subsection.

2. Introduction and proof overview

One object, two descriptions. Start at 1 and repeatedly multiply by 3/2, rounding up:

$$1,\ 2,\ 3,\ 5,\ 8,\ 12,\ 18,\ 27,\ 41,\ 62,\ 93,\ 140,\ 210,\ 315,\ 473,\ \ldots$$

Multiplying by 3/2 lands on an integer exactly when the number is even; when it is odd the ceiling adds a half. Record a 1 where it does and a 0 where it does not, and the record is the word

$$w \;=\; 1\,0\,1\,1\,0\,0\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\cdots$$

— which is simply the parity of the orbit, since the rounding bites exactly at the odd terms.

The word is not a lossy summary of the orbit; it is the orbit. The terms grow like $(3/2)^n$ at a definite rate

$$K \;=\; 1.6222705\ldots,$$

and the whole orbit can be read back off that single real number: $x_n = \lfloor K(3/2)^n \rfloor$ (Theorem 1.2). So w and K are two descriptions of one object — w is the digit string of K in the rational base 3/2, and K is the number that string names. Two questions about them have been open for decades: is K irrational (asked in 1977), and is w automatic, that is, produced by a finite automaton reading the digits of n? Neither is answered here. What is proved is that they cannot both go the easy way: *if w is automatic then K is transcendental* (Theorem B).

The word is an odometer reading. Everything rests on one dictionary. The m letters of w read from position n depend on nothing but $x_n \bmod 2^m$, and they determine it (Theorem 4.2). The word is the low-order end of a counter: reading a block of m letters is reading m bits of x_n, and two positions show the same block exactly when the orbit values agree modulo 2^m.

For instance $x_1 = 2$ and $x_6 = 18$ differ by $16 = 2^4$, and the four letters from position 1 and from position 6 do coincide: both read 0110. Counting blocks is therefore counting residues, and the complexity $P(w, m)$ — the number of distinct length-m blocks — *equals* the number of residues modulo 2^m that the orbit ever occupies (Proposition 4.3).

Repetitions are paid for in powers of two. That example also shows what limits the word. A repeated block of length k at positions $a < c$ forces $2^k \mid x_c - x_a$; but by step c the orbit has only reached about $(3/2)^c$, so the divisibility cannot run deep:

$$k \;\le\; \tfrac{24}{41}\,c + 1 \;\approx\; 0.585\,c \qquad \text{(Corollary 4.8)}.$$

A repetition is a coincidence that must be paid for in powers of 2, and by time c the orbit has earned only $c\log_2(3/2)$ of them. This one inequality does three jobs of increasing depth. It makes w aperiodic, since a period

would supply repetitions of *every* length at one fixed c (Theorem 4.9). It makes the known linear lower bound $P(w,n) > 1.7095\,n$ a pigeonhole: the first $1.7095\,m$ orbit points are simply too small to collide modulo 2^m (Corollary 4.4). And it closes off a whole family of attacks — periodize the digits at a repetition and feed the resulting approximants to the p-adic Subspace Theorem, as [AB07] did for Loxton–van der Poorten — because that route needs repetitions of relative length above 1.7095 where the ceiling allows 0.585 (§6.2). The two numbers are reciprocals straddling 1, so no sharpening closes the gap.

Two engines for two halves. Suppose K is algebraic and w is automatic, and derive a contradiction. K is either irrational or rational, and the halves are closed by arguments with nothing in common.

The irrational half (§5.1) is Mahler's method. An automatic sequence has a generating series satisfying $f(z) = A(z)f(z^q)$ with A a matrix of polynomials, and for such a series Adamczewski and Faverjon prove an alternative: at a nonzero rational point of the unit disc its value is transcendental or rational (Theorem A). Here that value is $3(K - x_0)$, so the alternative transfers to K verbatim, and the two exits are shut by the two hypotheses — "transcendental" contradicts the algebraicity, "rational" contradicts the irrationality. It is a trap with two doors, and each assumption closes one.

The rational half (§5.2) is Diophantine approximation. If $K = \delta$ is rational, a repeated block of length k at (a,c) pushes the number $\delta((3/2)^c - (3/2)^a)$ to within $(2/3)^k$ of an integer (Proposition 5.3): a repetition in the word *is* a rational number caught unusually close to an integer. Such coincidences are finite in number — that is the kernel, Theorem 5.6, and it is where the Subspace Theorem is spent. A linear bound $P(w,m) \le Cm$, on the other hand, would manufacture one at every scale by pigeonhole. So no linear bound holds (Theorem 5.9), and Cobham's theorem forbids that of an automatic word.

The simplest case of the kernel is $\delta = 1$: how close can $(3/2)^c - (3/2)^a$ come to an integer? That case is [RS26]. Allowing an arbitrary rational multiplier is not free, and the step that breaks outright is the most innocent-looking one: for $\delta \neq 1$ the value can be an integer exactly — at $\delta = 4$, $a = 1$, $c = 2$ it is 3 — and about an integer no approximation theorem has anything to say. The damage is confined to a finite box and discarded (Lemma 5.7).

Neither engine can do the other's work. The Mahler alternative sees the word only through the single real number it defines, so it yields nothing quantitative about complexity; the Diophantine kernel sees only how well that number is approximated, and needs it rational to begin with. §5.4 narrows the gap from the second side, weakening "rational" towards "algebraic" as far as the printed literature allows.

What is assumed, and where. Four statements are used without proof (§1.4), and they do not mix. Two lemmas from the Adamczewski–Faverjon circle power the irrational half and only it; the Subspace Theorem powers

the rational half and only it; the Corvaja–Zannier theorem at algebraic δ is spent entirely inside §5.4. Everything else — all of §3, §4 and §6.2 included — is proved outright and machine-checked. Figure 1 shows the shape of the dependency, Table 1 the exact footprint of each statement.

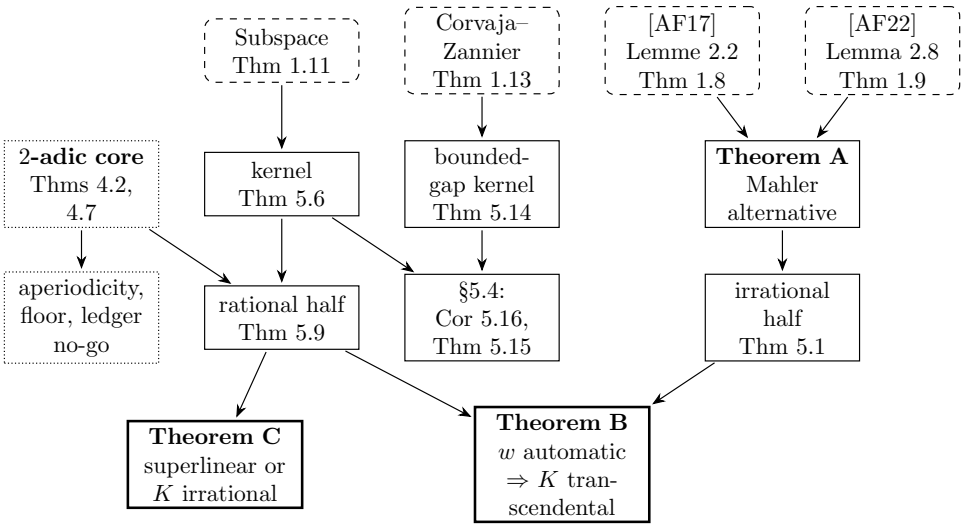

FIGURE 1. Logical dependencies. Dashed: the four cited inputs of §1.4. Dotted: the 2-adic core of §4, which uses none of them and carries aperiodicity (Theorem 4.9), the complexity floor (Corollary 4.4) and the no-go of §6.2 (Theorem 6.4) on its own. An arrow reads "is used in the proof of". The two lanes never meet: no result carries both an [AF] lemma and the Corvaja–Zannier axiom.

The route through the paper. §3 fixes the orbit, the word and the constant. §4 proves the dictionary and the ceiling, and with them aperiodicity and the closed form. §5 runs the two engines and assembles Theorem B, then pushes the Diophantine one towards algebraic multipliers. §6 collects two complements — the 2-adic value of the word, and the ledger that closes the Subspace route — and reports the computations. Appendix A records what formalizing the Mahler input turned up about the printed proofs, and Appendix B indexes every statement by its Lean name.

3. Basics

Definition 3.1. Given $x_0 \in \mathbb{N}$, set $x_{n+1} := \lceil 3x_n/2 \rceil$ for $n \geq 0$, and let $w_n := x_n \bmod 2$.

Proposition 3.2. *For every n, $2x_{n+1} = 3x_n + w_n$, and $w_n \in \{0,1\}$. If $x_0 \geq 1$ then $x_0 < x_1 < x_2 < \cdots$.*

Statement	§	[AF17]	[AF22]	[Sub]	[CZ04a]
Thm 4.2 (rigidity)	4				
Prop 4.3 (complexity = residues)	4				
Cor 4.4 (complexity floor)	4				
Thm 4.7, Cor 4.8 (ceiling)	4				
Thm 4.9 (aperiodicity)	4				
Thm 1.2 (closed form)	4				
Thm A (**Theorem A**)	1.3	✓	✓		
Thm 5.1, Cor 5.2 (irrational half)	5.1	✓	✓		
Prop 5.3, Cor 5.4 (contraction)	5.2				
Thm 5.6 (kernel)	5.2			✓	
Cor 5.12 (effective slice)	5.2				
Thm 5.9 (rational half)	5.2			✓	
Thm C (**Theorem C**)	5.2			✓	
Thm B (**Theorem B**)	5.3	✓	✓	✓	
Thm 5.14 (bounded-gap kernel)	5.4				✓
Thm 5.15, Cor 5.16	5.4			✓	✓
Thm 6.4 (ledger no-go)	6.2				
Prop 6.1 (2-adic value)	6.1				

TABLE 1. Which cited input each statement carries; a blank row is proved from nothing but classical logic. [AF17] and [AF22] are Theorems 1.8 and 1.9, [Sub] is Theorem 1.11, [CZ04a] is Theorem 1.13 at algebraic δ. Appendix B gives the same information per Lean declaration, as confirmed by `#print axioms`.

The identity $2x_{n+1} = 3x_n + w_n$ is the whole engine: w_n is the correction that makes $3x_n$ even, so $w_n = 2x_{n+1} - 3x_n$ is Dubickas's word and $w_n \equiv x_n \pmod 2$ simultaneously.

Definition 3.3. For $a, k \in \mathbb{N}$ put $W(a,k) := \sum_{i<k} 3^{k-1-i} 2^i w_{a+i}$, the *circuit sum* of the length-k factor of w at a.

Proposition 3.4. *For all $a, k \in \mathbb{N}$, $2^k x_{a+k} = 3^k x_a + W(a,k)$.*

Proof sketch. Induction on k, feeding $2x_{a+k+1} = 3x_{a+k} + w_{a+k}$ into the inductive hypothesis and collecting terms; the recurrence $W(a,k+1) = 3W(a,k) + 2^k w_{a+k}$ matches. □

Proposition 3.5. *For every n,*

$$x_n \left(\tfrac{2}{3}\right)^n = x_0 + \tfrac{1}{3} \sum_{j<n} w_j \left(\tfrac{2}{3}\right)^j.$$

Proof sketch. This is Proposition 3.4 at $a = 0$ divided by 3^n; the formalization proves it by direct induction instead, which keeps the natural subtraction implicit in W out of $\mathbb{R}$. □

Since $w_j \in \{0, 1\}$, the series $\sum_j w_j(2/3)^j$ is dominated by a geometric series and converges; all sums below are finite for this reason.

Definition 3.6. $K := x_0 + \frac{1}{3}\sum_{j\geq 0} w_j(2/3)^j$.

Proposition 3.7. $K = \lim_{n\to\infty} x_n(2/3)^n$, *and*

$$\sum_{j\geq 0} w_j\left(\tfrac{2}{3}\right)^j = 3(K - x_0).$$

Defining K by the series rather than by the limit makes the second identity free and the limit description a theorem; the second identity is the interface through which §5.1 feeds the word to Theorem A, and is the reason for the choice.

Proposition 3.8. *For every* n, $x_n \leq K\,(3/2)^n \leq x_n + 1$.

Proof sketch. Split the series of Definition 3.6 at n. The head is $x_n(2/3)^n$ by Proposition 3.5; the tail is non-negative, giving the left inequality, and is bounded by $\sum_{i\geq 0}(2/3)^{n+i} = 3(2/3)^n$, giving the right one after multiplying by $(3/2)^n$. □

Proposition 3.8 pins the orbit to a half-open unit window below $K(3/2)^n$; sharpening its right-hand inequality to a strict one leads immediaely to Theorem 1.2, and costs the aperiodicity of w. That is the subject of the next section.

4. Two-adic rigidity and aperiodicity of w

Definition 4.1. A *repeated factor* of length k at (a, c) is an equality of the length-k factors of w at a and at c: $w_{a+i} = w_{c+i}$ for all $i < k$. Occurrences may overlap.

Theorem 4.2. *For all* a, b, m, *the length-*m *factors of* w *at* a *and* b *agree if and only if* $2^m \mid x_b - x_a$.

Proof. ($\Rightarrow$) Equal factors give equal circuit sums, $W(a, m) = W(b, m)$; subtracting the two instances of Proposition 3.4 leaves

$$3^m(x_b - x_a) = 2^m(x_{b+m} - x_{a+m}), \text{ so } \; 2^m \mid 3^m(x_b - x_a),$$

and $2 \nmid 3$.

($\Leftarrow$) is a 2-adic descent: from $2^m \mid x_a - x_b$, one power of 2 survives at each step, so $w_{a+i} = w_{b+i}$ (the letters are the residues mod 2), and then $2(x_{a+i+1} - x_{b+i+1}) = 3(x_{a+i} - x_{b+i})$ divides the agreement by one power of 2. The agreement therefore decays to $2^{m-i} \mid x_{a+i} - x_{b+i}$ and stays alive m steps — the length of the factor. □

So the length-m factors of w *are* the residues $x_n \bmod 2^m$; in particular, counting factors is counting residues.

Proposition 4.3. *For every $x_0 \geq 1$ and every m,*

$$P(w,m) \;=\; \#\big\{\, x_n \bmod 2^m \;:\; n \geq 0 \,\big\}.$$

Proof sketch. Immediate from Theorem 4.2: two positions carry the same length-m factor exactly when they carry the same residue, so factors and occupied residues are in bijection. □

The restatement moves the complexity of w into the 2-adic dynamics of $U_{3/2}$, and it has consequences the combinatorial formulation hides. The first is that the linear lower bound of Theorem 1.4 is a pigeonhole.

Corollary 4.4. *Let $x_0 = 1$. For every $m \geq 1$,*

$$P(w,m) \;\geq\; \left\lfloor \tfrac{41(m-1)}{24} \right\rfloor + 1.$$

Proof sketch. The orbit values below 2^m are strictly increasing, hence pairwise distinct, hence — being below the modulus — pairwise distinct residues modulo 2^m, and Proposition 4.3 counts each as a separate factor. By the growth bound inside Theorem 4.7, $x_j < (x_0+1)(3/2)^j$, the orbit stays below 2^m for more than $m \log 2/\log(3/2) - O(1)$ steps; the certificate $3^{41} \leq 2^{65}$ turns the count into the stated integer form, of slope $41/24 = 1.7083\ldots$ □

Remark 4.5. Run with real logarithms instead of the integer certificate, the same count gives $P(w,m) \geq m \log 2/\log(3/2) - O_{x_0}(1)$ with $\log 2/\log(3/2) = 1.70951\ldots$, which is Theorem 1.4 up to its additive constant — and, read through Proposition 4.3, explains it: the first $1.7095\,m$ orbit points are simply too small to collide modulo 2^m. Improving on the constant therefore requires orbit points *beyond* that initial segment to occupy new residue classes, which is exactly what the 2-adic behaviour of the orbit controls. §6.3 reports computations: empirically the orbit occupies every residue class modulo 2^m, at the saturation rate of a uniformly random sequence of residues.

The reachable-residue question behind Remark 4.5 makes sense at every modulus, not only at the powers of 2 that the factors of w see; the first instance beyond those is free.

Proposition 4.6. *For every x_0 and every n, $x_{n+1} \not\equiv 1 \pmod 3$.*

Proof sketch. $2x_{n+1} = 3x_n + w_n$ gives $x_{n+1} \equiv 2w_n \pmod 3$, and $w_n \in \{0,1\}$. □

The dictionary of Proposition 4.3 is also a cost, not only a convenience: a repeated factor forces a large power of 2 to divide a difference of orbit values, while the orbit has only $(3/2)^c$ room.

Theorem 4.7. *Let $x_0 \geq 1$ and $a < c$. If w has a repeated factor of length k at (a,c), then*

$$2^{\,k+c} < 3^c(x_0+1).$$

Proof sketch. By Theorem 4.2, $2^k \mid x_c - x_a$, and $x_a < x_c$ by Proposition 3.2, so $2^k \le x_c - x_a < x_c + 1$; multiplying by 2^c gives $2^{k+c} < 2^c(x_c+1)$. It remains to see that the orbit has only $(3/2)^c$ room, i.e. that $2^n(x_n+1) \le 3^n(x_0+1)$ for every n: the shifted quantity $x_n + 1$ is submultiplicative under the ceiling rule, since $2(x_{n+1} + 1) = 3x_n + w_n + 2 \le 3(x_n + 1)$ because $w_n \le 1$, so induction applies. At $n = c$ this chains onto the display. □

Corollary 4.8. *Let $x_0 = 1$ and $a < c$. A repeated factor of length k at (a, c) obeys the integer inequality $41k \le 24c + 40$; that is, $k \le \frac{24}{41}c + 1 \approx 0.585\,c$.*

Proof sketch. Raise Theorem 4.7 to the 41st power and apply the integer certificate $3^{41} \le 2^{65}$, which encodes $\log_2 3 \le 65/41$ without reals or logarithms. The resulting exponent inequality is $41(k + c) < 65c + 41$. □

The slope $24/41 > \log_2(3/2) = 0.5849\ldots$ is the certified form of the repeated-factor ceiling. The ceiling itself is exact and loses nothing: in real logarithms Theorem 4.7 reads $k < c\log_2(3/2) + \log_2(x_0 + 1)$, with slope exactly $\log_2(3/2)$; Corollary 4.8 is its integer-certified approximation, which is what the formal statements downstream consume. §6.2 shows that this ceiling is what closes off a whole family of attacks on the word.

Theorem 4.9 (= Theorem 1.3)**.** *For $x_0 \ge 1$, the word w is not eventually periodic: there are no N and $p \ge 1$ with $w_{n+p} = w_n$ for all $n \ge N$.*

Proof sketch. Eventual periodicity gives a repeated factor at the *fixed* pair $(N, N + p)$ of *every* length k. But Theorem 4.7 caps every such k by the single number $3^c(x_0 + 1)$ with $c = N + p$ fixed; taking $k = 3^c(x_0 + 1)$ and using $k < 2^k \le 2^{k+c}$ contradicts it. □

Proof of Theorem 1.2. Write $K(y)$ for the constant of the orbit started at y. The orbit is shift-invariant — running $n + j$ steps from x_0 is running j steps from x_n — so the word is too, and splitting the series of Definition 3.6 at n and reindexing gives $K(x_n) = K(x_0)(3/2)^n$. It therefore suffices to treat $n = 0$ at the initial condition x_n, which is positive: that is, to show $x_n \le K(x_n) < x_n + 1$.

The left inequality is Proposition 3.8. For the right one it is enough to see that $K < x_0 + 1$ strictly for every $x_0 \ge 1$, and then to apply this at x_n. Some letter of w is 0: an all-ones word would be periodic with period 1, contradicting Theorem 4.9. So $\sum_j w_j(2/3)^j < \sum_j (2/3)^j = 3$, the terms being dominated termwise and one of them strictly, and Definition 3.6 turns this into $K < x_0 + 1$. □

5. Transcendence criteria for K

We prove Theorem B. Suppose K is algebraic. Either it is irrational, or it is rational. The two halves are closed by unrelated engines, and §5.3 assembles them.

5.1. **The algebraic irrational half.**

Theorem 5.1. *Let $x_0 \geq 1$. If K is an algebraic irrational, then w is not automatic.*

Proof. Suppose w is automatic. Its letters lie in $\{0,1\}$ (Proposition 3.2), so they are bounded, and $\alpha = 2/3$ is rational with $0 < |\alpha| < 1$. Theorem A therefore applies and gives: either $\sum_j w_j(2/3)^j$ is transcendental, or it is rational. By Proposition 3.7 that sum is $3(K - x_0)$ with $x_0 \in \mathbb{N}$, so the alternative transfers verbatim to K: the first branch would make K transcendental, contradicting algebraicity, and the second would make $K = r/3 + x_0$ rational for some $r \in \mathbb{Q}$, contradicting irrationality. Both branches die, so w is not automatic. □

Corollary 5.2. *If w is automatic and K is irrational, then K is transcendental.*

5.2. **The rational half: a Diophantine kernel.** Assume now $K = \delta \in \mathbb{Q}$. We show that no linear function bounds $P(w, \cdot)$ from above, from which non-automaticity follows by Cobham's theorem. The route is the one used in [RS26] for the steering word of $(3/2)^n$, with the multiplier slot opened up; we indicate below the three places where $\delta \neq 1$ costs a real argument.

Write $\theta_n := K(3/2)^n - x_n$ for the *deviation* of the orbit. By Theorem 1.2, $\theta_n \in [0,1)$, also θ_n is the fractional part $\{K(3/2)^n\}$. The deviations are the Vijayaraghavan sequence of K at ratio $3/2$, the object of the $(3/2)^n$-distribution literature (Flatto–Lagarias–Pollington [FLP95]), and every statement about θ below is a statement about $\{K(3/2)^n\}$.

Proposition 5.3. *Let $x_0 \geq 1$ and suppose $K = \delta \in \mathbb{Q}$. A repeated factor of length k at (a, c) forces*

$$\big\| \, \delta\big((3/2)^c - (3/2)^a\big) \, \big\| \;\leq\; (2/3)^k,$$

the value lying that close to the integer $x_c - x_a$.

Proof. Substituting $x_n = K(3/2)^n - \theta_n$ into Proposition 3.4 re-expresses the circuit sum in the deviations alone,

$$W(a,k) = 3^k\theta_a - 2^k\theta_{a+k},$$

the K-terms cancelling because $2^k(3/2)^{a+k} = 3^k(3/2)^a$. The deviations survive which are now contracted by the repeated factor. Equal factors give equal circuit sums, $W(a,k) = W(c,k)$, so the display at a and at c combine into

$$3^k(\theta_c - \theta_a) = 2^k(\theta_{c+k} - \theta_{a+k});$$

the right-hand bracket has absolute value at most 1 because $\theta \in [0,1)$, and dividing by 3^k leaves $|\theta_c - \theta_a| \leq (2/3)^k$.

It remains to read this at the multiplier. Rationality enters only here: since $K = \delta$, the definition $\theta_n = \delta(3/2)^n - x_n$ rearranges to

$$\delta\big((3/2)^c - (3/2)^a\big) - (x_c - x_a) \;=\; \theta_c - \theta_a,$$

whose right-hand side we have just bounded by $(2/3)^k$. So the value lies within $(2/3)^k$ of the *integer* $x_c - x_a$, and its distance to the nearest integer is at most its distance to that one. (Once $k \geq 2$ the bound is below $1/2$, and $x_c - x_a$ is then the nearest integer outright; the estimate above needs no such proviso.) □

This is the only place where the rationality of K is used in the whole chain, and it is used only to land the value in $\mathbb{Q}$, where the approximation theorems live. For arbitrary *real* multipliers the finiteness below is false, by a Liouville construction $\delta = \sum_k 2^{-y_k}$ with $y_{k+1} \geq (1+\varepsilon)y_k$.

The middle display of the proof, read before rationality enters, deserves its own statement.

Corollary 5.4 (unconditional contraction)**.** *Let $x_0 \geq 1$. A repeated factor of length k at (a, c) forces*

$$\big|\{K(3/2)^c\} - \{K(3/2)^a\}\big| \;\leq\; (2/3)^k,$$

with no hypothesis on K whatsoever.

The parity orbit cannot shadow itself for k steps without dragging the corresponding fractional parts of $K(3/2)^n$ together to within $(2/3)^k$; conversely, the repetition ceiling (Corollary 4.8) caps how long the shadowing can last. Together they are a two-way dictionary between the combinatorics of w and the distribution of $\{K(3/2)^n\}$: every new distributional input on $(3/2)^n$ modulo one converts into a complexity statement, and every combinatorial fact about w converts into a rigidity statement about the Vijayaraghavan sequence.

Definition 5.5. For $\delta, \vartheta \in \mathbb{Q}$ let

$$\mathcal{V}(\delta, \vartheta) := \Big\{(a, c) \in \mathbb{N}^2 \;:\; 2 \leq a < c,\; \big\|\delta\big((3/2)^c - (3/2)^a\big)\big\| \leq \vartheta^{\,c}\Big\}.$$

Its elements are the *violators* at scale ϑ: the pairs at which the value lies closer to an integer than ϑ^c.

Theorem 5.6. *For every rational $\delta \neq 0$ and every rational $\vartheta \in (0, 1)$, the set $\mathcal{V}(\delta, \vartheta)$ is finite.*

Theorem 5.6 is what we call the *kernel* of the rational half, here and below; the word is used for it and its variants throughout, and has nothing to do with the k-kernel of an automatic sequence (§1). The case $\delta = 1$ is [RS26, Thm. 5.6]. Opening the multiplier slot is not free, and one of the steps that $\delta = 1$ concealed breaks immediately.

Lemma 5.7 (degeneracy)**.** *Let $\delta \in \mathbb{Q}$, $\delta \neq 0$, and $a < c$ with $|\mathrm{num}(\delta)| \leq c$. Then $\delta((3/2)^c - (3/2)^a)$ is not an integer; in particular its distance to the nearest integer is positive.*

Proof. Write

$$(3/2)^c - (3/2)^a = 3^a(3^{c-a} - 2^{c-a})/2^c,$$

whose numerator is odd. If $\delta((3/2)^c - (3/2)^a) = n \in \mathbb{Z}$ then

$$\mathrm{num}(\delta) \cdot 3^a(3^{c-a} - 2^{c-a}) = n\, 2^c \,\mathrm{den}(\delta),$$

and coprimality of 2^c with the odd cofactor forces $2^c \mid \mathrm{num}(\delta)$, hence $c < 2^c \le |\mathrm{num}(\delta)|$. □

Remark 5.8. At $\delta = 1$ one gets $\|(3/2)^c - (3/2)^a\| > 0$ for every pair, by parity alone. For a general multiplier this is false, and not marginally: at $\delta = 4$, $a = 1$, $c = 2$ one has $4((3/2)^2 - (3/2)) = 3$ exactly, an integer, to which no Diophantine theorem applies. Lemma 5.7 localizes the damage to the box $c < |\mathrm{num}(\delta)|$, which is finite and is therefore discarded wholesale rather than by parity. This is the one place where the multiplier costs a genuine argument.

Proof of Theorem 5.6. Discard the degeneracy box $c < |\mathrm{num}(\delta)|$, finite outright; on the rest, Lemma 5.7 makes every value a non-integer, so the approximation theorems apply. Split the remaining set by its gaps $s = c - a$.

If the gaps are bounded, it suffices to treat a fixed gap $s_0 \ge 1$. Then

$$\delta((3/2)^{a+s_0} - (3/2)^a) = \delta_{\mathrm{CZ}} \cdot (3/2)^a, \quad \text{with} \quad \delta_{\mathrm{CZ}} = \delta((3/2)^{s_0} - 1),$$

and the Main Theorem of [CZ04] — in the form derived from Theorem 1.11 at $n = 2$, i.e. Ridout's theorem applied once — bounds the a with

$$\|\delta_{\mathrm{CZ}}(3/2)^a\| < H((3/2)^a)^{-\varepsilon} \quad \text{for} \quad \varepsilon = \log \vartheta^{-1}/(2\log 3).$$

Its size via $1 < |\delta_{\mathrm{CZ}}(3/2)^a|$ needs $(3/2)^a > 2/|\delta|$, so a further initial segment of a is discarded; this is archimedean and harmless.

If the gaps are unbounded, choose one violator per gap. This is the delicate selection, because the repaired pair theorem has four hypotheses and the family must satisfy all of them, not only the visible one. With $\alpha_1 = \delta$, $\alpha_2 = -\delta$ and $(u_1, u_2) = ((3/2)^c, (3/2)^a)$: the archimedean size condition $|u_i| \ge 1$ holds because the exponents are positive; the ratios $u_1/u_2 = (3/2)^{c-a}$ are pairwise distinct — in both orientations, as the theorem demands — because one violator was chosen per gap; the strict positivity $\|\delta u_1 - \delta u_2\| > 0$, the hypothesis whose omission broke [NKR25]'s Theorem 1.3(i) (Remark 1.12), is Lemma 5.7, available because the degeneracy box was discarded first; and the height-decay demand $\|\delta u_1 - \delta u_2\| < (H(u_1)H(u_2))^{-\varepsilon_1}$ follows from the violator inequality ϑ^c with $\varepsilon_1 = \log \vartheta^{-1}/(2\log 3)$, since $H(u_1)H(u_2) = 3^{a+c} \le 3^{2c}$. The pair theorem — derived from Theorem 1.11 at $n = 3$ — then concludes that some $(3/2)^c$ is an integer. This is wrong because $2^c(3/2)^c = 3^c$ is odd. □

Theorem 5.9. *Let $x_0 \ge 1$ and suppose K is rational. Then for every $C \in \mathbb{N}$ there is an $m \ge 1$ with $P(w, m) > Cm$; equivalently,* $\limsup_m P(w, m)/m = \infty$.

Note: for each C a single length m suffices, and that is all Cobham's bound needs negated. We do not claim the stronger $\lim_m P(w, m)/m = \infty$, which would require every large m to work.

Proof. Write $K = \delta$, nonzero since $K \geq 1$. Fix C and pick a rational $\vartheta \in (0,1)$ with $2/3 \leq \vartheta^{C+2}$; such a ϑ exists by Bernoulli's inequality, taking $\vartheta = 1 - \frac{1/3}{C+2}$ — the *Bernoulli scale* for C. This trades the irrational scale $(2/3)^{1/(C+2)}$ for a rational one, which is what the kernel requires. By Theorem 5.6 the set $\mathcal{V}(\delta, \vartheta)$ is finite, so its second coordinates are bounded by some M.

Now suppose $P(w,k) \leq Ck$ for $k := M + x_0 + 1$. Pigeonhole produces a repeated factor: the length-k factors of w take values in the finite set $\{0,1\}^k$, and the $Ck+1$ positions in $[2, Ck+2]$ carry at most Ck distinct factors, so two of them agree, say at (a,c) with $2 \leq a < c \leq Ck+2 \leq (C+2)k$. Proposition 5.3 contracts it to

$$\|\delta((3/2)^c - (3/2)^a)\| \leq (2/3)^k \leq \vartheta^{(C+2)k} \leq \vartheta^c,$$

so $(a,c) \in \mathcal{V}(\delta, \vartheta)$ and $c \leq M$.

But no repeated factor is that shallow. Theorem 4.7 gives

$$2^{k+c} < 3^c(x_0+1) \leq 4^c \cdot 2^{x_0+1}, \quad \text{using } 3^c \leq 4^c \text{ and } x_0 + 1 < 2^{x_0+1},$$

so that $k \leq c + x_0 \leq M + x_0$, contradicting the choice of k. Hence $P(w,k) > Ck$ for this single length k. The ceiling used here is crude, and deliberately so: Corollary 4.8 gives the sharp slope 0.585 but only at $x_0 = 1$, whereas $k \leq c + x_0$ is uniform in x_0, which is all the reduction needs, $k \to \infty$ must force $c \to \infty$. □

Proof of Theorem C. If K is irrational we are in the second horn. Otherwise $K \in \mathbb{Q}$ and Theorem 5.9 gives the first. □

Remark 5.10. Both horns of Theorem C are worth having, and neither is provable alone here. The first would beat Theorem 1.4's record for this word: no linear function bounds the complexity, against a linear lower bound with a fixed constant. The second is a conditional irrationality statement about a constant open since 1977. As a consistency check, note that an eventually periodic w would make K rational *and* the complexity bounded, which Theorem 4.9 excludes; so the dichotomy is sharp on both sides.

The kernel has an *effective* slice, and it is worth delimiting exactly — both because it is all the effectivity there is, and because its boundary turns out to be a line this paper has met before.

Proposition 5.11 (Liouville floor)**.** *Let $\delta \in \mathbb{Q}$, $\delta \neq 0$, and $a < c$ with $|\mathrm{num}(\delta)| \leq c$. Then*

$$\big\|\delta\big((3/2)^c - (3/2)^a\big)\big\| \;\geq\; \frac{1}{\mathrm{den}(\delta)\, 2^c}.$$

Proof sketch. $\mathrm{den}(\delta)\, 2^c$ clears the value to an integer, and Lemma 5.7 says the value is not itself an integer; a non-integral rational whose D-fold is an integer keeps distance at least $1/D$ from $\mathbb{Z}$. The formal statement is the integer-certificate form $1 \leq \mathrm{den}(\delta)\, 2^c \, \|\cdot\|$. □

Corollary 5.12 (the effective slice)**.** *Let $\delta \neq 0$ and $0 \leq \vartheta < 1/2$. Then $\mathcal{V}(\delta, \vartheta)$ is finite* effectively*, with no Diophantine input at all: every $(a, c) \in \mathcal{V}(\delta, \vartheta)$ with $c \geq |\mathrm{num}(\delta)|$ satisfies the checkable certificate $\mathrm{den}(\delta)(2\vartheta)^c \geq 1$, so that $c < N$ for any $N \geq |\mathrm{num}(\delta)|$ with $\mathrm{den}(\delta)(2\vartheta)^N < 1$.*

Remark 5.13 (the effective region is the empty region)*.* Three observations delimit Corollary 5.12. First, it is uniform in the gap $c - a$: in the regime $\vartheta < 1/2$ nothing is gained by fixing the gap, and the effective statement covers the whole kernel at once — there is no need to isolate a bounded-gap family. Second, the slice cannot be widened by linear forms in logarithms: the nearest integer is a *free* third term — the relevant integer form is $\mathrm{num}(\delta)\, 3^a(3^{c-a} - 2^{c-a}) - n\,\mathrm{den}(\delta)\, 2^c$ with n unconstrained — so a two-logarithm bound must absorb n into a composite algebraic number of height $\asymp c \log(3/2)$, and to beat the floor of Proposition 5.11 its constant would have to be below $\log 2/(\log 3 \log(3/2)) \approx 1.56$, an order of magnitude beyond anything proved. (This is the archimedean twin of the familiar fact that Baker-type bounds lose to trivial estimates on three-term forms.) What lies beyond $1/2$ is the Padé frontier, and only for the multiplier 1: Beukers's $\|(3/2)^n\| \geq 2^{-0.9n}$ for $n \geq 5000$ [Beu81], improved by Zudilin to $\|(3/2)^n\| \geq 0.5803^n$ [Zud07]; no comparably strong effective bound is known with a general rational multiplier, and beyond 0.5803 nothing effective is known at all. Third — the coherence — the boundary $\vartheta = 1/2$ is the ledger of §6.2 in disguise: the floor $\asymp 2^{-c}$ beats the contraction $(2/3)^k$ of Proposition 5.3 exactly when $k/c > \log 2/\log(3/2) = 1.7095\ldots$, the demand line, while actual repeated factors obey $k/c \leq 0.585$ (Corollary 4.8). So the kernel is effective precisely in the region the word never enters, and the region the complexity application needs — $\vartheta \to 1$ in the proof of Theorem 5.9 — is ineffective essentially, not accidentally.

5.3. **Assembling.**

Proof of Theorem B. Suppose K is algebraic. If K is irrational, Theorem 5.1 gives that w is not automatic. If K is rational, Theorem 5.9 makes the complexity of w exceed every linear bound, which Cobham's bound $P(a, m) = O(m)$ for automatic a [Cob72] forbids; so w is not automatic in that case either. The two cases partition the algebraic case. The first form of the statement is the contrapositive: if w is automatic then K is not algebraic. □

Theorem B strictly strengthens Corollary 5.2, whose irrationality hypothesis was exactly the $K \in \mathbb{Q}$ horn that §5.2 removes.

5.4. **Algebraic multipliers: the Corvaja–Zannier half.**

If the kernel held for algebraic irrational multipliers as it does for rational ones, Theorem B would upgrade at once to *K algebraic $\Rightarrow$ superlinear complexity*, single-lane, with the Mahler engine of §5.1 retired and complexity content in every algebraic case. This subsection proves the half of that extension

which the printed literature supports, shows the reduction never needed rationality, and thereby compresses the remaining distance to a single named statement.

For real δ let $\mathcal{V}_{\mathbb{R}}(\delta,\vartheta)$ denote the violator set of Definition 5.5 with the distance to the nearest integer taken in $\mathbb{R}$; for $\delta \in \mathbb{Q}$ it coincides with $\mathcal{V}(\delta,\vartheta)$. Two of the three multiplier costs of §5.2 now behave unexpectedly well. Degeneracy *disappears*: for irrational δ the value $\delta((3/2)^c - (3/2)^a)$ is an irrational times a nonzero rational, never an integer, so $\|\cdot\| > 0$ holds for *every* pair — Lemma 5.7's box was a rational phenomenon, and the first casualty of the extension is in fact a simplification. And the bounded-gap slices survive verbatim, because Theorem 1.13 is printed for algebraic δ; what was a parity remark becomes a conjugate argument.

Theorem 5.14 (the bounded-gap kernel, algebraic multipliers)**.** *Let δ be a real algebraic irrational, let $\vartheta \in (0,1)$ be rational, and let $S \in \mathbb{N}$. Then only finitely many $(a,c) \in \mathcal{V}_{\mathbb{R}}(\delta,\vartheta)$ have gap $c-a \le S$.*

Proof sketch. Fix the gap $s_0 \le S$ and write the value as $\delta_{\mathrm{CZ}}\,(3/2)^a$ with $\delta_{\mathrm{CZ}} = \delta((3/2)^{s_0}-1)$, algebraic irrational. Apply Theorem 1.13 with $q=1$, $u=(3/2)^a$, $H(u)=3^a$, $\varepsilon = \log\vartheta^{-1}/(2\log 3)$. Three clauses are discharged along the family. The size proviso $1 < |\delta_{\mathrm{CZ}}(3/2)^a|$ fails only on an initial segment of a (archimedean). Strict positivity holds for every pair, by irrationality. The new obligation is the pseudo-Pisot exclusion: δ_{CZ}, being algebraic irrational, has a conjugate $z \neq \delta_{\mathrm{CZ}}$, $z \neq 0$; conjugates scale along rational multiples — minimal polynomials obey $p_{r\delta_{\mathrm{CZ}}}(X) = r^{\deg p}\, p_{\delta_{\mathrm{CZ}}}(X/r)$ — so $z\,(3/2)^a$ is a conjugate of the value other than the value itself, and it leaves the unit disc once $(3/2)^a \ge |z|^{-1}$. Pseudo-Pisot values, like the size proviso, are confined to an initial segment and discarded. □

The unbounded-gap branch of Theorem 5.6 used the repaired pair theorem — *derived* over $\mathbb{Q}$ from Theorem 1.11 at $n=3$. For algebraic coefficients nothing can be cited: [NKR25] is unrefereed and its Theorem 1.3(i) is false as printed (Remark 1.12), and we know of no refereed substitute. What we can state exactly is the obligation itself.

Theorem 5.15 (conditional upgrade)**.** *Let $x_0 \ge 1$ and assume the* pair branch *at $\delta = K$: for every rational $\vartheta \in (0,1)$, every family $T \subseteq \mathcal{V}_{\mathbb{R}}(K,\vartheta)$ with pairwise-distinct gaps is finite. If K is algebraic, then for every C there is an $m \ge 1$ with $P(w,m) > Cm$.*

Proof sketch. If K is rational this is Theorem 5.9, and the hypothesis is not consulted. If K is algebraic irrational, the gap dichotomy behind Theorem 5.6 reassembles: bounded gaps fall to Theorem 5.14, unbounded gaps to the assumed pair branch after selecting one violator per gap, and there is no degeneracy box to discard — so $\mathcal{V}_{\mathbb{R}}(K,\vartheta)$ is finite. Everything in the proof of Theorem 5.9 after that finiteness then runs verbatim over $\mathbb{R}$: pigeonhole, the Bernoulli scale, the contraction — Corollary 5.4, which never needed rationality — and the growth ceiling. □

The pair branch is a theorem for rational δ — supporting the proof of Theorem 5.6 —, and for algebraic irrational δ it is a Subspace argument over the number field $\mathbb{Q}(\delta)$: places above 2, 3, ∞, relative heights, S-enlargement at the places where δ is not a unit. Theorem 1.11 is already stated over an arbitrary number field, so the assumption side is ready; missing is the distance between Theorem 5.15 and an unconditional upgrade.

What the proven half buys *unconditionally* is a repetition statement. Call a pair (a, c) with $2 \le a < c \le a + S$ a *close repetition at slope* $1/L$ if the factors of w of length $\lfloor c/L \rfloor + 1$ at a and at c coincide — a period-$(c - a)$ stretch of length proportional to its position, at bounded period. By the ceiling (Theorem 4.7; $2^{2c+1} < 3^c(x_0 + 1)$ fails for large c) the class is empty for $L = 1$ and c large, and Corollary 4.8's slope 0.585 makes $L = 2$ the widest interesting one.

Corollary 5.16. *Let $x_0 \ge 1$ and suppose K is algebraic. Then for every gap bound S and every slope $1/L$, $L \ge 1$, only finitely many close repetitions occur in w. Contrapositively: infinitely many close repetitions at a single (S, L) force K to be transcendental.*

Proof sketch. A close repetition is a violator of gap $\le S$ at the Bernoulli scale $\vartheta(L)$ with $2/3 \le \vartheta^L$: Corollary 5.4 gives $\|K((3/2)^c - (3/2)^a)\| \le (2/3)^{\lfloor c/L \rfloor + 1} \le \vartheta^c$. For algebraic irrational K, Theorem 5.14 finishes; for rational K, Theorem 5.6 does. □

Remark 5.17 (scope and lanes). Under its algebraicity hypothesis, Corollary 5.16 strictly strengthens what Theorem 4.9 excludes: eventual periodicity would be one infinite family of close repetitions, while the corollary bounds every bounded-period family of proportional-length stretches. (The two are not comparable outright — Theorem 4.9 is unconditional.) It is also the first repetition-structure statement available for algebraic *irrational* K, where Theorem B gave non-automaticity through the Mahler lane and nothing quantitative. Remark 1.7 still applies unchanged: K is expected transcendental, the trigger is expected to fail (the residue data of §6.3 leaves no room for close repetitions) and the content is the criterion direction.

6. Two complements

6.1. **The 2-adic value of the word.** Proposition 3.5 is an identity between rationals, and so may be read at any place. Over $\mathbb{R}$ its limit is $3(K - x_0)$, an unknown real. Over $\mathbb{Q}_2$ the limit is an integer.

Proposition 6.1. *In $\mathbb{Q}_2$ one has $\|2/3\|_2 = 1/2$, the series $\sum_j w_j(2/3)^j$ converges, and*

$$\sum_{j \ge 0} w_j \left(\tfrac{2}{3}\right)^j = -3x_0.$$

More generally $\sum_{j \ge 0} w_{n+j}(2/3)^j = -3x_n$ for every n.

Proof sketch. The induction proving Proposition 3.5 uses only that 2 and 3 are invertible, so it runs verbatim over $\mathbb{Q}_2$. Only the limit differs:

$$\|x_n(2/3)^n\|_2 = 2^{-n}\|x_n\|_2 \le 2^{-n} \longrightarrow 0,$$

since x_n is a 2-adic integer. So the orbit term vanishes and the partial sums $3x_n(2/3)^n - 3x_0$ tend to $-3x_0$. The shifted family follows by shift-invariance of the word. □

6.2. **The exponent ledger.** A recurring proposal for this word is to periodize the digits at a repeated factor, producing S-unit approximants, and to feed those to the p-adic Subspace Theorem, as Adamczewski and Bugeaud [AB07] did for Loxton–van der Poorten. The proposal cannot work, and the reason is structural.

The two constants $\log_2(3/2) < 1 < 1/\log_2(3/2)$ are exactly the two sides of the ledger. Corollary 4.8 caps the *quality* t/c of any repeated factor of w by $\log_2(3/2) = 0.5849\ldots$, while periodizing at a repeated factor yields approximants whose per-place product over $\{\infty, 2, 3\}$ needs

$$t/c \;>\; (1-\lambda)/\lambda \;=\; 1.7095\ldots, \qquad \lambda = \log(3/2)/\log 3.$$

Since the two are reciprocal and straddle 1, no sharpening can close the gap: the very mechanism that makes the word complex — the 2-adic rigidity of Theorem 4.2, which charges $2^t \mid x_c - x_a$ for a repeated factor of length t while the orbit grows only like $(3/2)^c$ — is the mechanism that caps repeated-factor quality below 1.

Proposition 6.2 (log-free form)**.** *Let $x_0 = 1$ and $a < c$. No repeated factor of length t at (a, c) satisfies $17095\,c < 10000\,t$.*

Proof sketch. Immediate from $41t \le 24c + 40$ (Corollary 4.8); the two are incompatible already for $c \ge 1$. □

Definition 6.3. For $\tau, \varepsilon > 0$ and sequences $c, t\colon \mathbb{N} \to \mathbb{N}$, say (c_m, t_m) satisfies the *index condition* at (τ, ε) if

$$(\tau+\varepsilon)\, c_m \log 3 \;\le\; t_m \log 2 \qquad \text{for infinitely many } m.$$

This is what an application of the p-adic Subspace Theorem in the style of [AB07] needs from a family of S-unit approximants of height $\asymp 3^{c_m}$ and 2-adic quality 2^{-t_m}: the per-place product over $\{\infty, 2, 3\}$ must beat the height to the power $-\tau - \varepsilon$, for some $\tau > 1$, infinitely often. Reading a repeated factor of length t_m at position c_m as the source of such an approximant (periodize the digits there) is the one interpretive element of this subsection, and the theorem below is a statement about Definition 6.3 alone.

Theorem 6.4. *Let (a_m, c_m, t_m) be any family of repeated factors of w (at $x_0 = 1$) with $a_m < c_m$ and $c_m \to \infty$. Then (c_m, t_m) fails the index condition at (τ, ε), for every $\tau > 1$ and every $\varepsilon > 0$.*

Proof sketch. Corollary 4.8 gives $t_m \log 2 < c_m \log 3$ outright once $c_m \geq 3$, before τ is used at all; and $\tau + \varepsilon > 1$ only makes the demand worse. So the condition fails *eventually*, which contradicts its "infinitely often" shape. □

Theorem 6.4 takes the repeated-factor family as given and does not rebuild the approximant construction: the point is that no construction can help, whatever its details, because the exponents do not fit.

6.3. **Computations.** This subsection is not formally verified; it records computations and the places where the machinery above visibly bites next. It is the only part of the paper outside the scope of Appendix B.

Proposition 4.3 makes the complexity of w cheaply computable: $P(w, m)$ is the number of residues modulo 2^m met by the orbit, and a prefix of the orbit witnesses a lower bound. The result of running the first 10^6 orbit points (at $x_0 = 1$) is stark: *every* residue class modulo 2^m is met, for every $m \leq 16$ — that is, the witnessed counts are not merely superlinear but maximal, $P(w, m) = 2^m$ as far as the computation can see. (For $17 \leq m \leq 20$ the prefix is still producing new residues at its end, so those counts are not yet saturated; nothing suggests they stop short.) Moreover the *rate* of saturation is the generic one: the position of the last new residue — 1471 for $m = 8$, 39346 for $m = 12$, 821306 for $m = 16$ — matches the coupon-collector time $2^m(m \log 2 + \gamma) \approx 765000$ at $m = 16$ of a uniformly random residue sequence. The orbit of $U_{3/2}$ behaves, 2-adically, like a random sequence. On that model full complexity is forced — and indeed Dubickas already conjectures it at $x_0 = 1$ [Dub09, p. 246], judging it "very likely as difficult as" the corresponding conjecture $P(\alpha, n) = q^n$ for algebraic irrationals in integer bases. The computation above appears to be the first direct evidence, and supports the conjecture in the generality of this paper.

Conjecture 6.5 (Dubickas [Dub09] at $x_0 = 1$)**.** *For every $x_0 \geq 1$ and every m, $P(w, m) = 2^m$: every binary word is a factor of w.*

Two companion computations point the same way. The repeated-factor quality t/c (in the notation of §6.2) over all pairs $a < c \leq 3000$ is maximized at the early pair $(a, c, t) = (1, 6, 4)$ with $t/c = 2/3$ — an artifact of the additive constant in Corollary 4.8 — and restricted to $c \geq 100$ its maximum is 0.1089, decaying with c: the orbit sits far below the 0.585 ceiling, i.e. the word is empirically very far from *stammering* in the sense of [AB07] — from carrying the long near-repetitions that a Subspace attack would have to feed on. And the truncated kernels are as large as they can be: comparing kernel elements on their first 64 terms, the 2-kernel has 2^i distinct elements at every depth $i \leq 12$ and the 3-kernel 3^i at every depth $i \leq 8$ — no collapse anywhere, as non-automaticity predicts.

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directed and reviewed all generated code and takes full responsibility for the mathematical integrity and final content of the work.

Appendix A. Formalizing Theorem A

Theorem A was a cited axiom of this development until August 2026. Removing it is the one piece of the formalization that produced findings about the literature rather than about the word w, and this appendix records them. Nothing below strengthens a statement in the body of the paper; what changed is the trust surface, from one sequence-level black box to the two named lemmas of §1.4. The development is twenty-nine files and about 11 900 lines devoted to [AF17, AF22], together with about 2 700 lines of Mahler-system machinery on the automatic-sequence side and about 1 500 lines of general-purpose material (heights of tuples, regular field extensions, rationality of power series) written for the library rather than for this proof.

What is proved, exactly. The Lean statement is not the specialization of Theorem A but a general one: for k any field of algebraic numbers, $f_1, \dots, f_n \in k[[z]]$ summing on a disc of radius $r \leq 1$ to functions solving a q-Mahler system with polynomial matrix of nonzero determinant, and $\alpha \in k$ with $0 < |\alpha| < r$, the value $f_i(\alpha)$ is transcendental over k or lies in k. [AF17, Théorème 1.7 (i)] — linear dependence over $\overline{\mathbb{Q}}$ descends to linear dependence over k, with the support of the relation shrinking or staying put — is proved along the way, and is what makes the corollary a two-line consequence.

Two differences from the printed corollary are worth naming. The field k is *not* required to be a number field: the descent of [AF17, §4] spends one k-linear functional and no finiteness anywhere. Conversely, α is asked to lie inside the disc of convergence, rather than merely not to be a pole of the meromorphic continuation; recovering the printed hypothesis is exactly what the analytic desingularization of [AF17, §5] is for, and it is the one thing here that is deliberately not formalized. For Theorem A this costs nothing, since bounded coefficients give $r = 1$ and $\alpha = 2/3$.

Three steps that a citation of [AF17, Cor. 1.8] would have taken on its authority are proved instead. That an automatic sequence satisfies a q-Mahler system is elementary and was already available; what is *not* available from the literature is the same system with a *polynomial* matrix of nonzero determinant — [Bec94, Thm. 1] gives an invertible matrix over $\overline{\mathbb{Q}}(z)$, not over $\overline{\mathbb{Q}}[z]$ — and that gap is closed by an elementary rewriting of the system rather than by a citation. The other two, that bounded coefficients force radius of convergence ≥ 1 and hence that $|\alpha| < 1$ is not a pole, and that $k = \mathbb{Q}$ suffices when α and the coefficients are rational, are routine.

Where the printed proof is longer than it needs to be. *(i)* [AF17, §5] is avoidable: the authors give two proofs of their Théorème 1.7 (i), and the first, in §4, reaches it from three elementary lemmas and the degree-one lifting theorem, with no analytic desingularization.

(ii) The upper-bound step of [AF22, §2.3] — three pages of Taylor-coefficient bookkeeping: Cauchy–Hadamard, a binomial re-expansion at $\alpha^{q^{k_0}}$, a separate estimate for the coefficients of the auxiliary matrix, a convolution — is not needed. The quantity being bounded is a single value, so the maximum modulus principle delivers it from a supremum on one fixed circle, and the whole of their Lemma 2.13 collapses to "a power series with a zero of order p is small at a small argument". This was the step the project had budgeted as its largest risk.

(iii) [AF22, Lemma 2.2] — the eventual linearity in δ_2 of a dimension $d(\delta_1, \delta_2)$, not to be confused with Theorem 1.8 above — is proved in their Appendix A.1 by clearing denominators in a basis of an annihilator and counting linear forms in a dual space. None of that is needed: the dimensions are those of a filtration generated by iterating multiplication by z, and a five-line monotonicity gives *exact* eventual linearity, which is stronger than what is asked for.

(iv) The Puiseux field of [AF22] is a container, not a tool. The word occurs exactly twice in that paper, both times in the definition of the ambient field; no valuation, order of vanishing, or fractional exponent appears anywhere in §2. Any algebraically closed extension of $\overline{\mathbb{Q}}(z)$ in which the relation matrix is algebraic serves the same purpose.

(v) No shrinking discs, and no matrix inverse. [AF22, Rem. 2.10] shrinks the disc from step to step because $A(z^{q^j})$ acquires poles; a polynomial Mahler matrix has none, and every system here is polynomial. And the normalizing matrix $a = A_{k_0}(\alpha)\varphi(\xi)^{-1}$ — where $\xi := \alpha^{q^{k_0}}$ and $A_k(z) := A(z)A(z^q)\cdots A(z^{q^{k-1}})$ is the iterated Mahler matrix — is used only through the identity $a\varphi(\xi) = A_{k_0}(\alpha)$, so no inverse is ever formed.

(vi) The primitive element of [AF22, §2.4] can be dispensed with altogether, and this is also the fix for *(viii)* below. The coefficients of the relation generate a finitely generated torsion-free $\overline{\mathbb{Q}}[z]$-module, hence a free one; a basis expresses each of them with *polynomial* coordinates, so there is no denominator to clear and no condition on α to arrange.

One false claim, and one impossible order. *(vii)* [AF22] justify the equality of two dimensions in the auxiliary-function step by "$P(Y,z) \mapsto P(MY,z)$ is an automorphism of $\overline{\mathbb{Q}}[Y,z]_{\delta_1,\delta_2}$". It is not. Their δ_1 bounds the degree in *each* matrix variable y_{ij} separately, and a linear substitution does not preserve that: already for $m = 2$, $y_{11}y_{21} \mapsto (M_{11}y_{11} + M_{12}y_{21})(M_{21}y_{11} + M_{22}y_{21})$ has degree 2 in y_{11}. What survives is the total degree, so the substitution lands in $\overline{\mathbb{Q}}[Y,z]_{m^2\delta_1,\delta_2}$; the count is repaired by iterating their dimension lemma, at the cost of a constant depending on m alone, which is harmless because the parameter that varies is δ_1.

(viii) [AF22] let $d(z)$ be a common denominator of the matrices of their decomposition (2.36) and then assume k_0 chosen large enough that $d(\alpha^{q^{k_0}}) \neq 0$. This is legitimate for them, because their d is read off the untwisted

matrix and is fixed *before* k_0 is chosen. It cannot be imitated in the order the argument must actually be run: §2.4 never evaluates φ at $\alpha^{q^{k_0}}$, it evaluates the twisted matrix $\varphi(z^{q^{k_0}})$ at α, so the decomposition is of the twisted matrix, its d depends on k_0, and there is no second "for $k_0 \gg 1$" left to spend. Nor can the untwisted degree be used instead: $\varphi(z^{q^{k_0}})$ may have *strictly smaller* degree than φ — for $\varphi = \sqrt{z}$ and q even it is rational. The free-module argument of *(vi)* removes the denominator, and with it the difficulty.

Hypotheses that are load-bearing and invisible until written down. Characteristic zero in Theorem 1.8, as recorded in §1.4: over $\mathbb{F}_p(z)$ the statement is false, so an axiom stated for an arbitrary field would have been a false axiom.

Algebraic closedness of the base field in Theorem 1.9*(c)*. It is invisible in the printed statement and becomes visible the moment one asks where the normalizing matrix lives: the value of an algebraic function at an algebraic point is algebraic, and lands in the image of the base field only when that field is algebraically closed — true of [AF22]'s $\overline{\mathbb{Q}}$, and needed because the auxiliary-function step substitutes $Y \mapsto MY$ with M a matrix over that field.

The algebraicity in Theorem 1.9 must be required of the *entries* of φ, not of the ambient field, and this is the difference between a true statement and a false one. The ambient field is relatively algebraically closed, so for every ξ it contains $\sqrt{z-\xi}$, which has no analytic branch at ξ; a hypothesis asserting that the ambient field is realized by analytic functions would be refutable. Only finitely many elements can be realized at once — which is why the interface realizes a subring.

$\alpha \neq 0$, $|\alpha| < 1$ and $q \geq 2$ are what make "for $k \gg 1$" meaningful: the points α^{q^k} then have strictly decreasing moduli, so they are pairwise distinct and all but finitely many avoid any fixed finite bad set. At $q = 1$ the sequence is constant and the statement is false at a bad point. The same $q \geq 2$ is load-bearing a second time, in the descent below.

The two-field obstruction. Theorem 1.9 wants an algebraically closed base field. The lower-bound step is Liouville's inequality and wants a number field, because the height machinery is available only for fields carrying a finite family of archimedean absolute values — and that is a fact about $\overline{\mathbb{Q}}$, not an accident of packaging. No field is both. [AF22] never meet this, because their heights are absolute rather than relative.

The resolution is a descent to a single number field, fixed once and never enlarged. It must be a single one: the heights available are relative, $[F : \mathbb{Q}]$ times the absolute ones, so letting each auxiliary polynomial choose its own field would let the Liouville constant grow with the degree parameter and destroy the contradiction that the whole argument is aimed at. Finitely many numbers generate it — the coefficients of A, the coefficients of the

given relation, the entries of the branch value, the constant terms $f_i(0)$, and α — and the point worth recording is that the solutions contribute only their *constant* terms: comparing coefficients in $f(z) = A(z)f(z^q)$ expresses $[z^n]f_i$ through coefficients of index $< n$, an induction that exists only because $q \geq 2$.

What the library lacked. Liouville's inequality, which turns out not to be a product-formula computation at all — the product formula is already spent inside the height of an inverse — and costs fifteen lines. A sum bound for the projective height of tuples: the naive analogue of the one-coordinate bound is *false*, since the projective height is scaling-invariant and a sum is not (over $\mathbb{Q}$, $H(c,0) = H(0,1) = 1$ for every $c \neq 0$, while $H\big((c,0)+(0,1)\big) = H(c)$ is unbounded), and what a matrix product actually needs is the bound for sums whose terms are all read off one tuple. A sharp evaluation bound in which each degree is charged once, the naive one carrying a factor equal to the number of monomials — exponential in the degree parameter, and therefore worthless where the constant must be independent of it. "Regular extension $\Rightarrow$ linearly disjoint", against an arbitrary finite-dimensional subextension and not merely a simple one. And, last, the Mahler substitution itself on the ambient field: it has to be built from the substitution on power series, extended to the field of formal Laurent series and then to its algebraic closure, and the final step needs two algebra structures on one type at once — which in Lean is what a type synonym is for.

Interfaces. Two structural facts came out of the analytic layer and are worth stating independently of Lean. The realization of algebraic functions by analytic ones cannot be a homomorphism defined on the whole ambient field: $z-c$ is a unit there, and its realization vanishes at c. So it is defined on a subring, and then its injectivity is no longer free — that subring is never a field, for the same reason — and has to be part of what is assumed rather than derived. And taking germs along the principal filter of the domain, rather than at the point, makes a germ an honest function and evaluation at a point an honest ring homomorphism; this is what turns [AF22]'s "well-defined for $k \gg 1$", the one genuinely awkward phrase in §2.2, into a total operation, and makes their radicality lemma a five-line consequence of "the germ ring of a reduced ring is reduced".

What remains cited. Theorems 1.8 and 1.9, and nothing else: `#print axioms` on the Lean form of Theorem A, and on every result of §5.1 that consumes it, returns those two together with the ambient axioms of classical logic. The reasons they are cited are different in kind. Theorem 1.8 is cited because its own proof descends into the analytic theory of Mahler functions — natural boundaries, non-polar singularities — which is a project rather than a step; the deduction [AF22] make *from* it is not, and is proved here. Theorem 1.9 is cited because none of the three ingredients of its short

literature proof is available in the formalized library: no vanishing criterion for resultants, hence no "finitely many singularities"; no field of convergent power series and no Newton–Puiseux theorem, hence no way to produce a branch; and no analytic implicit function theorem for algebraic function elements. A model is exhibited for its clauses *(b)* and *(c)*, so that the cited form is known to be satisfiable rather than vacuous; the further clauses folded into it, as described in §1.4, are argued from [AF22]'s construction and are not separately modelled.

Appendix B. Formalization

Every statement above was verified in Lean 4, except the four cited axioms, which are assumed, and the computations of §6.3, which are not formalized. See `https://github.com/rwst/Transcendence-Criteria`. The files live under `RB/`, with the twenty-nine files `CITED/AdamczewskiFaverjon*.lean` (Appendix A) and with `CITED/SubspaceTheorem.lean`, `CITED/AlloucheShallitBasic.lean`, `CITED/AlloucheShallitComplexity.lean` and `CITED/CorvajaZannierAlgebraic.lean` supplying inputs. "std3" abbreviates the ambient axioms of classical Lean (propositional extensionality, choice, quotient soundness). Four cited axioms occur, each marked below: "[AF17]" for `AF.lemme_2_2`, "[AF22]" for `AF.lemma_2_8`, "[Sub]" for `Subspace.evertseSchlickewei`, and "[CZ04a]" for `CZ.pseudoPisot_approx_alg` (the algebraic-multiplier instance of Theorem 1.13, carried only by §5.4; the rational instance is derived from [Sub] and is not an axiom). Every entry is fully proved in Lean except the four marked *cited axiom*. All footprints below were confirmed with `#print axioms`.

Cited inputs (§1.4).

Statement	Lean identifier	File	Status
Thm. 1.8	`AF.lemme_2_2`	`AdamczewskiFaverjonRegular`	cited axiom [AF17]
Thm. 1.9	`AF.lemma_2_8`	`AdamczewskiFaverjonBranchFull`	cited axiom [AF22]
Thm. A	`AF.transcendental_or_rat_of_automatic`	`AdamczewskiFaverjon`	std3 + [AF17] + [AF22]
[AF22] Thm 2.1, the lifting thm	`AF.theoreme_2_1`	`AdamczewskiFaverjonTheoreme21`	std3 + [AF17] + [AF22]
[AF17] Thm 1.7 (i)	`AF.theoreme_1_7_i`	`AdamczewskiFaverjonTheoreme17`	std3 + [AF17] + [AF22]
[AF17] Cor 1.8, general form	`AF.corollaire_1_8`	`AdamczewskiFaverjonTheoreme17`	std3 + [AF17] + [AF22]
nonsingular Mahler system	`RB.exists_nonsingular_mahlerSystem`	`MahlerNonsingular`	std3
radius ≥ 1*, not a pole*	`RB.one_le_genFunSeries_radius`	`MahlerAnalytic`	std3
Thm. 1.11	`Subspace.evertseSchlickewei`	`SubspaceTheorem`	cited axiom [Sch91]
—	`CZ.pseudoPisot_approx_of_subspace`	`CorvajaZannierProof`	std3 + [Sub]
—	`NKR.sUnit_pair_integrality_of_subspace`	`NairKumarRoutProof`	std3 + [Sub]
Rem. 1.12	`NKR.thm13i_unrepaired_false`	`NairKumarRout`	std3
—	`AS.kKernel`, `AS.IsAutomatic`	`AlloucheShallitBasic`	def
—	`AS.complexity`	`AlloucheShallitComplexity`	def
Rem. 1.5	`AS.complexity_linear_of_automatic`	`AlloucheShallitComplexity`	std3
—	`AS.not_automatic_of_complexity_superlinear`	`AlloucheShallitComplexity`	std3

The orbit, the word, the constant, and rigidity (§3–§4).

Statement	Lean identifier	File	Status
Def. 3.1	`RB.x`, `RB.wmin`	`Basic`	def
Prop. 3.2	`two_mul_x_succ`, `wmin_le_one`	`Basic`	std3
Def. 3.3	`RB.W`	`Basic`	def
Prop. 3.4	`circuit_sum`	`Basic`	std3
Prop. 3.5	`x_mul_pow`	`Basic`	std3
Def. 3.6	`RB.K`	`Basic`	def
Prop. 3.7	`tendsto_x_mul_pow`, `tsum_wmin_eq`	`Basic`	std3
Prop. 3.8	`x_le_K_mul_pow`, `K_mul_pow_le`	`Basic`	std3
Def. 4.1	`RB.IsRepetition`	`Rigidity`	def
Thm. 4.2	`isRepetition_iff_dvd`	`Rigidity`	std3
equal circuit sums	`lemmaR_int`	`Rigidity`	std3
Thm. 4.7	`repetition_pow_lt`	`Rigidity`	std3
orbit growth	`two_pow_mul_x_add_one_le`	`Basic`	std3
Cor. 4.8	`repetition_linear_bound`	`Rigidity`	std3
Prop. 4.3	`complexity_eq_ncard_residues`	`Residues`	std3
factors are residues	`factor_eq_iff_residue_eq`	`Residues`	std3
Cor. 4.4	`complexity_lower_bound`	`Residues`	std3
Prop. 4.6	`x_succ_mod_three`	`Residues`	std3
Thm. 1.3, 4.9	`not_eventually_periodic`	`Rigidity`	std3
Thm. 1.2	`closed_form`	`ClosedForm`	std3

Transcendence criteria (§5).

Statement	Lean identifier	File	Status
Thm. 5.1	`not_automatic_of_K_algebraic_irrational`	`NotAutomatic`	std3 + [AF17] + [AF22]
Cor. 5.2	`transcendental_of_automatic_of_irrational`	`NotAutomatic`	std3 + [AF17] + [AF22]
θ_n (§5.2)	`RB.tail`, `tail_nonneg`, `tail_lt_one`	`OrbitKernel`	std3
$\theta_n = \{K(3/2)^n\}$	`tail_eq_fract`	`Residues`	std3
Prop. 5.3	`dist_le_of_repetition`	`OrbitKernel`	std3
Cor. 5.4	`abs_tail_sub_le_of_repetition`	`OrbitKernel`	std3
Def. 5.5	`RB.scaledViolators`	`ScaledKernel`	def
Lem. 5.7	`dist_pos_of_num_le`	`ScaledKernel`	std3
Thm. 5.6	`scaledViolators_finite`	`ScaledKernel`	std3 + [Sub]
Prop. 5.11	`one_le_den_mul_two_pow_mul_dist`	`EffectiveSlice`	std3
Cor. 5.12	`scaledViolators_finite_of_lt_half`	`EffectiveSlice`	std3
Thm. 1.13	`CZ.pseudoPisot_approx_alg`	`CorvajaZannierAlgebraic`	cited axiom
$\mathcal{V}_{\mathbb{R}}$ (§5.4)	`RB.algViolators`	`AlgebraicKernel`	def
Thm. 5.14	`algGapBounded_slice_finite`	`AlgebraicKernel`	std3 + [CZ04a]
Thm. 5.15	`superlinear_of_K_algebraic_of_pairBranch`	`AlgebraicKernel`	std3 + [CZ04a] + [Sub]
Cor. 5.16	`closeRepetitions_finite_of_K_algebraic`	`AlgebraicKernel`	std3 + [CZ04a] + [Sub]
Thm. 5.9	`superlinear_of_K_rat`	`RationalK`	std3 + [Sub]
Thm. B	`not_automatic_of_K_algebraic`	`RationalK`	std3 + [AF17] + [AF22] + [Sub]
Thm. B	`transcendental_of_automatic`	`RationalK`	std3 + [AF17] + [AF22] + [Sub]
rational case	`not_automatic_of_K_rat`	`RationalK`	std3 + [Sub]
Thm. C	`superlinear_or_K_irrational`	`RationalK`	std3 + [Sub]

The complements (§6).

Statement	Lean identifier	File	Status
Prop. 6.1	`norm_two_thirds`, `x_mul_pow_padic`	`PadicValue`	std3
Prop. 6.1	`tsum_wmin_padic`, `tsum_wmin_shift_padic`	`PadicValue`	std3
Prop. 6.2	`not_demand`	`NoStammeringRoute`	std3
Thm. 6.4	`not_indexConditionExpFreq`	`NoStammeringRoute`	std3

The sanity witnesses $x = 1, 2, 3, 5, 8, 12, 18, 27, 41$ and $w = 1, 0, 1, 1, 0, 0$ for $x_0 = 1$ (matching A061419 and the first letters of $g_{3/2}$) and the closed-form values $\lfloor K(3/2)^0 \rfloor = 1$, $\lfloor K(3/2)^4 \rfloor = 8$, $\lfloor K(3/2)^8 \rfloor = 41$ are verified by decision procedures (`x_sanity`, `wmin_sanity`, `closed_form_sanity`).

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