

TRANSCENDENCE CRITERIA FOR THE MINIMAL WORD OF THE RATIONAL BASE $3/2$

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ABSTRACT. Let x_0 be a positive integer, let $x_n = \lceil 3x_{n-1}/2 \rceil$, and let $w_n = 2x_{n+1} - 3x_n \in \{0, 1\}$ be the associated word studied by Dubickas; for $x_0 = 1$ the orbit is A061419 and w is the minimal word $g_{3/2}$ of the rational base number system of Akiyama, Frougny and Sakarovitch. The orbit encodes a real constant $K = \lim_n x_n (2/3)^n$, equal for $x_0 = 1$ to $\omega_{3/2} = K(3) = 1.6222705028\dots$, whose irrationality has been open since 1977. We prove that if w is automatic then K is transcendental; equivalently, an algebraic K forces w to be non-automatic. Further we prove that either the complexity of w exceeds every linear bound or K is irrational. All results are formally verified in the Lean 4 proof assistant, on three cited axioms.

1. INTRODUCTION

Fix a positive integer x_0 and iterate the ceiling map

$$U_{3/2}: x \mapsto \left\lceil \frac{3x}{2} \right\rceil, \quad x_n := U_{3/2}^n(x_0),$$

so that $x_0 < x_1 < x_2 < \dots$. Following Dubickas [Dub09, p. 245] we attach to this orbit the word

$$w_n := 2x_{n+1} - 3x_n \in \{0, 1\}, \quad n = 0, 1, 2, \dots,$$

which is simply the parity of the orbit, $w_n = x_n \bmod 2$. For $x_0 = 1$ the orbit is the sequence A061419, and w is an object of the rational base number system of Akiyama, Frougny and Sakarovitch [AFS08]: their *minimal word* $g_{3/2} = 101100011010011010100110\dots$ ([AFS08, §4.2], A205083), the lexicographically least label of an infinite path in the tree $T_{3/2}$. We prove everything below for a general $x_0 \geq 1$, of which $g_{3/2}$ is the case $x_0 = 1$.

Remark 1.1. The identification is exact but carries an index shift. Akiyama, Frougny and Sakarovitch index $g_{3/2} = g_1 g_2 g_3 \dots$ from 1 and compute its letters by $g_{k+1} = qG_{k+1} \bmod p$ [AFS08, Remark 32], where $G_0 = 1$, $G_{k+1} = \lceil (p/q)G_k \rceil$ is their Definition 20. At $(p, q) = (3, 2)$ the selection rule $2G_{k+1} = 3G_k + (G_k \bmod 2)$ turns this into $g_{k+1} = G_k \bmod 2$, which is Dubickas's w_k . So $w_n = g_{n+1}$, and the sequence indexed from 0 in this paper is the sequence indexed from 1 there.

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Throughout, $P(w, n)$ denotes the *complexity function* (also called the block-complexity function) of w in the sense of [Dub09, p. 245]: the number of distinct vectors $(w_j, w_{j+1}, \dots, w_{j+n-1})$ as j ranges over all non-negative integers. A sequence is *k-automatic* if its k -kernel $\{n \mapsto a(k^i n + r) : i \geq 0, 0 \leq r < k^i\}$ is finite, and *automatic* if it is k -automatic for some $k \geq 2$ [AS03]. We write $\|x\|$ for the distance from x to the nearest integer.

1.1. The constant. Dividing the identity $2x_{n+1} = 3x_n + w_n$ by powers of 3 (see Proposition 2.5) gives $(2/3)^n x_n = x_0 + \frac{1}{3} \sum_{j < n} w_j (2/3)^j$, whose limit

$$K := \lim_{n \rightarrow \infty} x_n (2/3)^n = x_0 + \frac{1}{3} \sum_{j \geq 0} w_j \left(\frac{2}{3}\right)^j$$

exists. For $x_0 = 1$ this is the constant $\omega_{3/2}$ of [AFS08, Example 3], which is the constant $K(3)$ of Odlyzko and Wilf [OW91]:

$$K = \omega_{3/2} = K(3) = 1.62227\,05028\,84767\,31595\,69509\,8\dots$$

(sequence A083286). It is a striking feature of this circle that so little is known about K : its *irrationality* is open, asked by Wang and Washburn in 1977 [WW77].

1.2. Known results. Three facts about w are in the literature. The first two we reprove; the third we reprove as well (Corollary 3.4), though we never use it, for the structural reason recorded in Remark 1.5.

Theorem 1.2 (Odlyzko–Wilf; Akiyama–Frougny–Sakarovitch). *For every $x_0 \geq 1$ and every $n \geq 0$,*

$$x_n = \lfloor K (3/2)^n \rfloor.$$

This is [OW91, Cor. 1] in the case $q = 3$ of the Josephus problem, where the relevant map is $x \mapsto \lceil \frac{q}{q-1} x \rceil = \lceil \frac{3}{2} x \rceil$, and independently [AFS08, Cor. 31], which gives $G_k = \lfloor \gamma_{p/q} (p/q)^{k+1} \rfloor$ whenever $p \geq 2q - 1$ — at $(3, 2)$ one has $3 \geq 3$. It is also stated by Dubickas [Dub09, p. 244], who records that $x_n = \lfloor c(\beta) \beta^n \rfloor$ holds for $\beta \geq 2$ or $\beta = 2 - 1/q$, and $3/2 = 2 - 1/2$.

Theorem 1.3 (Akiyama–Frougny–Sakarovitch [AFS08]; Dubickas [Dub09]). *The word w is not eventually periodic.*

This is [AFS08, Prop. 26], and independently [Dub09, Thm. 2]: an ultimately periodic word forces the base to be a Pisot or a Salem number, and $3/2$ is neither — an inference Dubickas draws himself [Dub09, p. 245].

Theorem 1.4 (Dubickas [Dub09]). *$P(w, n) > 1.70951129 n$ for every sufficiently large n .*

This is [Dub09, Cor. 4], the best lower bound for this word we are aware of, derived from the general bound $\liminf_n P(w, n)/n \geq \log q / \log(p/q)$ of [Dub09, Thm. 3]; Corollary 3.4 below recovers it from the rigidity theory by a pigeonhole.

Remark 1.5. Theorem 1.4 cannot be pushed to non-automaticity by sharpening its constant. Cobham’s theorem [Cob72] bounds the complexity of a k -automatic sequence only by $O(n)$, so a lower bound $P(w, n) \geq cn$ is compatible with automaticity for *every* c ; the Thue–Morse sequence is 2-automatic with $\liminf_n P(n)/n = 3 > 1.7095$ [Brl89, dLV89]. Every complexity statement below therefore targets *superlinearity*, not a linear floor.

1.3. Results of this paper. Our main result links the arithmetic nature of K to the automaticity of the word that generates it. To our knowledge no statement of this kind was previously available.

Theorem A. *Let $x_0 \geq 1$. If the word w is automatic, then K is transcendental. Equivalently: if K is algebraic, then w is not automatic.*

The reading with content is the first implication: it concludes the *transcendence* of a constant not known to be *irrational*, from a combinatorial hypothesis on the word. Theorem A is proved in §4 by splitting the algebraic case in two and closing each half by a different engine — the Mahler-method alternative of Adamczewski and Faverjon on the irrational half (§4.1), and the Subspace Theorem on the rational half (§4.2). We stress at once what it does not do.

Remark 1.6 (scope). K is expected to be transcendental, so the hypothesis of the second form of Theorem A is plausibly vacuous, and the theorem is *not* a proof that w is non-automatic: the generic, transcendental case is untouched.

The rational half yields a dichotomy of independent interest, in which both horns are statements one would like to have and neither is available alone.

Theorem B. *Let $x_0 \geq 1$. Then either the complexity of w exceeds every linear bound — for every C there is an $m \geq 1$ with $P(w, m) > Cm$, i.e. $\limsup_m P(w, m)/m = \infty$ — or K is irrational.*

Behind the first horn stands a rigidity theorem with a one-line reformulation that organizes everything in this paper: length- m factors of w agree when the orbit values agree modulo 2^m (Theorem 3.2), so $P(w, m)$ is the number of residues modulo 2^m that the orbit occupies (Proposition 3.3), and Theorem 1.4 becomes a pigeonhole (Corollary 3.4).

Beyond Theorems A and B, §4.4 begins the extension of the kernel to algebraic multipliers: the bounded-gap half is proved (on the printed algebraic- δ form of the Main Theorem of [CZ04]), the Stage-1 reduction is shown to need no rationality at all, and the upgrade of Theorem A to superlinear complexity in every algebraic case is thereby reduced to a single named statement, a pair theorem over $\mathbb{Q}(K)$. Unconditionally, close repetitions — long repeats at bounded distance — die out whenever K is algebraic (Corollary 4.16), a second transcendence criterion with a periodicity-adjacent trigger.

All statements below have been formally verified in Lean 4. Three cited axioms are used, in the sense of §1.4; every other statement’s proof is fully formalized. Appendix A records the Lean name, file and axiom footprint of each formalization.

1.4. Cited inputs. Three theorems are used as black boxes. Each is refereed, and each is stated here with a citation and no proof. The first two carry the main results; the third enters only the algebraic-multiplier extension of §4.4.

Theorem 1.7 (Adamczewski–Faverjon [AF17]). *Let $f(z)$ be a q -Mahlerian function and $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha| < 1$ not a pole of f . Let k be a number field containing α and the coefficients of f . Then either $f(\alpha)$ is transcendental, or $f(\alpha) \in k$.*

This is [AF17, Cor. 1.8]. In the specialization we use — f automatic with bounded coefficients, α rational, whence $k = \mathbb{Q}$ — it is precisely Cobham’s 1968 conjecture, which Adamczewski and Faverjon call the origin of their paper. The formalization axiomatizes that specialization rather than the general statement, and thereby takes on [AF17]’s authority three further steps: automatic $\Rightarrow$ q -Mahlerian; bounded coefficients $\Rightarrow$ radius of convergence $\geq 1 \Rightarrow \alpha$ is not a pole; and $k = \mathbb{Q}$.

Remark 1.8 (read the alternative literally). The disjunct $f(\alpha) \in k$ is not a degenerate corner. Adamczewski and Faverjon show it cannot be removed — “des exemples montrent que l’on ne peut se soustraire à l’alternative ... même en supposant la fonction $f(z)$ transcendante” — and their §8.1 exhibits a $\{0, 1\}$ -valued 3-automatic sequence with f transcendental over $\mathbb{Q}(z)$ and yet $f_1(\varphi) = -\varphi/2 \in \mathbb{Q}(\varphi)$. An argument that reads Theorem 1.7 as forcing a contradiction from a *known rational* value is therefore invalid: the rational value is the permitted branch. This is why Theorem 1.7 is applied below only in the one direction where both branches die at once (§4.1).

Theorem 1.9 (Subspace Theorem; Schmidt [Sch91], Evertse–Schlickewei form [BG06]). *Let K be a number field, $n \geq 2$, and S a finite set of places of K containing the archimedean ones. For each $v \in S$ let $L_{v,1}, \dots, L_{v,n}$ be linearly independent linear forms in $X_1, \dots, X_n$ with coefficients in K . For every $\varepsilon > 0$, the nonzero $x \in K^n$ with*

$$\prod_{v \in S} \prod_{i=1}^n \frac{|L_{v,i}(x)|_v}{\|x\|_v} \leq H(x)^{-n-\varepsilon}$$

lie in finitely many proper linear subspaces of K^n .

This is [Sch91, Thm. 1D’]. It is the only Diophantine input of §4.2: both of the approximation theorems used there — the Main Theorem of Corvaja and Zannier [CZ04] and the modified pair theorem of Nair, Kumar and Rout

[NKR25] — are *derived* from it in the formalization rather than assumed, at $n = 2$ and $n = 3$ respectively.

Remark 1.10 (on [NKR25]). Nothing is taken on the authority of [NKR25], which is an unrefereed preprint. Its Theorem 1.3(i) is false as stated — the hypothesis of strict positivity is missing, and our formalization carries a machine-checked refutation — and only the repaired form is used, itself derived from Theorem 1.9. We are still in debt of the NKR preprint as the source of an important argument.

Theorem 1.11 (Corvaja–Zannier [CZ04], Main Theorem). *Let $\Gamma \subset \overline{\mathbb{Q}}^*$ be a finitely generated multiplicative group, let δ be a nonzero algebraic number, and let $\varepsilon > 0$. Then there are only finitely many pairs $(q, u) \in \mathbb{Z} \times \Gamma$, $d := [\mathbb{Q}(u) : \mathbb{Q}]$, such that $|\delta qu| > 1$, δqu is not pseudo-Pisot, and*

$$0 < \|\delta qu\| < H(u)^{-\varepsilon} q^{-d-\varepsilon}.$$

Here a real algebraic α is *pseudo-Pisot* if $|\alpha| > 1$, every conjugate of α other than α itself has modulus < 1 , and the trace of α is a rational integer; a pseudo-Pisot number need not be an algebraic integer. For $\delta \in \mathbb{Q}$ this theorem is *derived* from Theorem 1.9 in the formalization. For algebraic irrational δ it is taken on [CZ04]’s authority as a cited axiom, and it is used only in §4.4: a fourth lane, disjoint from the other three, which no result outside that subsection carries.

2. BASICS

Definition 2.1. Given $x_0 \in \mathbb{N}$, set $x_{n+1} := \lceil 3x_n/2 \rceil$ for $n \geq 0$, and let $w_n := x_n \bmod 2$.

Proposition 2.2. *For every n , $2x_{n+1} = 3x_n + w_n$, and $w_n \in \{0, 1\}$. If $x_0 \geq 1$ then $x_0 < x_1 < x_2 < \dots$.*

The identity $2x_{n+1} = 3x_n + w_n$ is the whole engine: w_n is the correction that makes $3x_n$ even, so $w_n = 2x_{n+1} - 3x_n$ is Dubickas’s word and $w_n \equiv x_n \pmod{2}$ simultaneously.

Definition 2.3. For $a, k \in \mathbb{N}$ put $W(a, k) := \sum_{i < k} 3^{k-1-i} 2^i w_{a+i}$.

Proposition 2.4. *For all $a, k \in \mathbb{N}$, $2^k x_{a+k} = 3^k x_a + W(a, k)$.*

Proof sketch. Induction on k , feeding $2x_{a+k+1} = 3x_{a+k} + w_{a+k}$ into the inductive hypothesis and collecting terms; the recurrence $W(a, k+1) = 3W(a, k) + 2^k w_{a+k}$ matches. $\square$

Proposition 2.5. *For every n ,*

$$x_n \left(\frac{2}{3}\right)^n = x_0 + \frac{1}{3} \sum_{j < n} w_j \left(\frac{2}{3}\right)^j.$$

Proof sketch. This is Proposition 2.4 at $a = 0$ divided by 3^n ; the formalization proves it by direct induction instead, which keeps the natural subtraction implicit in W out of $\mathbb{R}$. $\square$

Since $w_j \in \{0, 1\}$, the series $\sum_j w_j (2/3)^j$ is dominated by a geometric series and converges; all sums below are finite for this reason.

Definition 2.6. $K := x_0 + \frac{1}{3} \sum_{j \geq 0} w_j (2/3)^j$.

Proposition 2.7. $K = \lim_{n \rightarrow \infty} x_n (2/3)^n$, and

$$\sum_{j \geq 0} w_j \left(\frac{2}{3}\right)^j = 3(K - x_0).$$

Defining K by the series rather than by the limit makes the second identity free and the limit description a theorem; the second identity is the interface through which §4.1 feeds the word to Theorem 1.7, and is the reason for the choice.

Proposition 2.8. For every n , $x_n \leq K (3/2)^n \leq x_n + 1$.

Proof sketch. Split the series of Definition 2.6 at n . The head is $x_n (2/3)^n$ by Proposition 2.5; the tail is non-negative, giving the left inequality, and is bounded by $\sum_{i \geq 0} (2/3)^{n+i} = 3(2/3)^n$, giving the right one after multiplying by $(3/2)^n$. $\square$

Proposition 2.8 pins the orbit to a half-open unit window below $K(3/2)^n$; sharpening its right-hand inequality to a strict one leads immediately to Theorem 1.2, and costs the aperiodicity of w . That is the subject of the next section.

3. TWO-ADIC RIGIDITY AND APERIODICITY OF w

Definition 3.1. A *repeated factor* of length k at (a, c) is an equality of the length- k factors of w at a and at c : $w_{a+i} = w_{c+i}$ for all $i < k$. Occurrences may overlap.

Theorem 3.2. For all a, b, m , the length- m factors of w at a and b agree if and only if $2^m \mid x_b - x_a$.

Proof. $(\Rightarrow)$ Equal factors give equal circuit sums, $W(a, m) = W(b, m)$; subtracting the two instances of Proposition 2.4 leaves

$$3^m(x_b - x_a) = 2^m(x_{b+m} - x_{a+m}), \text{ so } 2^m \mid 3^m(x_b - x_a),$$

and $2 \nmid 3$.

$(\Leftarrow)$ is a 2-adic descent: from $2^m \mid x_a - x_b$, one power of 2 survives at each step, so $w_{a+i} = w_{b+i}$ (the letters are the residues mod 2), and then $2(x_{a+i+1} - x_{b+i+1}) = 3(x_{a+i} - x_{b+i})$ divides the agreement by one power of 2. The agreement therefore decays to $2^{m-i} \mid x_{a+i} - x_{b+i}$ and stays alive m steps — the window length. $\square$

So length- m windows of w are the residues $x_n \bmod 2^m$; in particular, counting windows is counting residues.

Proposition 3.3. *For every $x_0 \geq 1$ and every m ,*

$$P(w, m) = \#\{x_n \bmod 2^m : n \geq 0\}.$$

Proof sketch. Immediate from Theorem 3.2: two positions carry the same length- m factor exactly when they carry the same residue, so factors and occupied residues are in bijection. $\square$

The restatement moves the complexity of w into the 2-adic dynamics of $U_{3/2}$, and it has consequences the combinatorial formulation hides. The first is that the linear lower bound of Theorem 1.4 is a pigeonhole.

Corollary 3.4. *Let $x_0 = 1$. For every $m \geq 1$,*

$$P(w, m) \geq \left\lfloor \frac{41(m-1)}{24} \right\rfloor + 1.$$

Proof sketch. The orbit values below 2^m are strictly increasing, hence pairwise distinct, hence — being below the modulus — pairwise distinct residues modulo 2^m , and Proposition 3.3 counts each as a separate window. By the growth bound inside Theorem 3.7, $x_j < (x_0 + 1)(3/2)^j$, the orbit stays below 2^m for more than $m \log 2 / \log(3/2) - O(1)$ steps; the certificate $3^{41} \leq 2^{65}$ turns the count into the stated integer form, of slope $41/24 = 1.7083\dots$ $\square$

Remark 3.5. Run with real logarithms instead of the integer certificate, the same count gives $P(w, m) \geq m \log 2 / \log(3/2) - O_{x_0}(1)$ with $\log 2 / \log(3/2) = 1.70951\dots$, which is Theorem 1.4 up to its additive constant — and, read through Proposition 3.3, explains it: the first $1.7095m$ orbit points are simply too small to collide modulo 2^m . Improving on the constant therefore requires orbit points *beyond* that initial segment to occupy new residue classes, which is exactly what the 2-adic behaviour of the orbit controls. §5.3 reports computations: empirically the orbit occupies every residue class modulo 2^m , at the saturation rate of a uniformly random sequence of residues.

The reachable-residue question behind Remark 3.5 makes sense at every modulus, not only at the powers of 2 that windows see; the first instance beyond those is free.

Proposition 3.6. *For every x_0 and every n , $x_{n+1} \not\equiv 1 \pmod{3}$.*

Proof sketch. $2x_{n+1} = 3x_n + w_n$ gives $x_{n+1} \equiv 2w_n \pmod{3}$, and $w_n \in \{0, 1\}$. $\square$

The dictionary of Proposition 3.3 is also a cost, not only a convenience: a repeated window forces a large power of 2 to divide a difference of orbit values, while the orbit has only $(3/2)^c$ room.

Theorem 3.7. *Let $x_0 \geq 1$ and $a < c$. If w has a repeated factor of length k at (a, c) , then*

$$2^{k+c} < 3^c(x_0 + 1).$$

Proof sketch. By Theorem 3.2, $2^k \mid x_c - x_a$, and $x_a < x_c$ by Proposition 2.2, so $2^k \leq x_c - x_a < x_c + 1$; multiplying by 2^c gives $2^{k+c} < 2^c(x_c + 1)$. It remains to see that the orbit has only $(3/2)^c$ room, i.e. that $2^n(x_n + 1) \leq 3^n(x_0 + 1)$ for every n : the shifted quantity $x_n + 1$ is submultiplicative under the ceiling rule, since $2(x_{n+1} + 1) = 3x_n + w_n + 2 \leq 3(x_n + 1)$ because $w_n \leq 1$, so induction applies. At $n = c$ this chains onto the display. $\square$

Corollary 3.8. *Let $x_0 = 1$ and $a < c$. A repeated factor of length k at (a, c) obeys the integer inequality $41k \leq 24c + 40$; that is, $k \leq \frac{24}{41}c + 1 \approx 0.585c$.*

Proof sketch. Raise Theorem 3.7 to the 41st power and apply the integer certificate $3^{41} \leq 2^{65}$, which encodes $\log_2 3 \leq 65/41$ without reals or logarithms. The resulting exponent inequality is $41(k + c) < 65c + 41$. $\square$

The slope $24/41 > \log_2(3/2) = 0.5849\dots$ is the certified form of the repeated-factor ceiling. The ceiling itself is exact and loses nothing: in real logarithms Theorem 3.7 reads $k < c \log_2(3/2) + \log_2(x_0 + 1)$, with slope exactly $\log_2(3/2)$; Corollary 3.8 is its integer-certified approximation, which is what the formal statements downstream consume. §5.2 shows that this ceiling is what closes off a whole family of attacks on the word.

Theorem 3.9 (= Theorem 1.3). *For $x_0 \geq 1$, the word w is not eventually periodic: there are no N and $p \geq 1$ with $w_{n+p} = w_n$ for all $n \geq N$.*

Proof sketch. Eventual periodicity gives a repeated factor at the *fixed* pair $(N, N + p)$ of *every* length k . But Theorem 3.7 caps every such k by the single number $3^c(x_0 + 1)$ with $c = N + p$ fixed; taking $k = 3^c(x_0 + 1)$ and using $k < 2^k \leq 2^{k+c}$ contradicts it. $\square$

Proof of Theorem 1.2. Write $K(y)$ for the constant of the orbit started at y . The orbit is shift-invariant — running $n + j$ steps from x_0 is running j steps from x_n — so the word is too, and splitting the series of Definition 2.6 at n and reindexing gives $K(x_n) = K(x_0)(3/2)^n$. It therefore suffices to treat $n = 0$ at the initial condition x_n , which is positive: that is, to show $x_n \leq K(x_n) < x_n + 1$.

The left inequality is Proposition 2.8. For the right one it is enough to see that $K < x_0 + 1$ strictly for every $x_0 \geq 1$, and then to apply this at x_n . Some letter of w is 0: an all-ones word would be periodic with period 1, contradicting Theorem 3.9. So $\sum_j w_j(2/3)^j < \sum_j (2/3)^j = 3$, the terms being dominated termwise and one of them strictly, and Definition 2.6 turns this into $K < x_0 + 1$. $\square$

4. TRANSCENDENCE CRITERIA FOR K

We prove Theorem A. Suppose K is algebraic. Either it is irrational, or it is rational. The two halves are closed by unrelated engines, and §4.3 assembles them.

4.1. The algebraic irrational half.

Theorem 4.1. *Let $x_0 \geq 1$. If K is an algebraic irrational, then w is not automatic.*

Proof. Suppose w is automatic. Its letters lie in $\{0, 1\}$ (Proposition 2.2), so they are bounded, and $\alpha = 2/3$ is rational with $0 < |\alpha| < 1$. Theorem 1.7 therefore applies and gives: either $\sum_j w_j (2/3)^j$ is transcendental, or it is rational. By Proposition 2.7 that sum is $3(K - x_0)$ with $x_0 \in \mathbb{N}$, so the alternative transfers verbatim to K : the first branch would make K transcendental, contradicting algebraicity, and the second would make $K = r/3 + x_0$ rational for some $r \in \mathbb{Q}$, contradicting irrationality. Both branches die, so w is not automatic. $\square$

Corollary 4.2. *If w is automatic and K is irrational, then K is transcendental.*

4.2. The rational half: a Diophantine kernel. Assume now $K = \delta \in \mathbb{Q}$. We show that no linear function bounds $P(w, \cdot)$ from above, from which non-automaticity follows by Cobham's theorem. The route is the one used in [RS26] for the steering word of $(3/2)^n$, with the multiplier slot opened up; we indicate below the three places where $\delta \neq 1$ costs a real argument.

Write $\theta_n := K(3/2)^n - x_n$ for the *deviation* of the orbit. By Theorem 1.2, $\theta_n \in [0, 1)$, also θ_n is the fractional part $\{K(3/2)^n\}$. The deviations are the Vijayaraghavan sequence of K at ratio $3/2$, the object of the $(3/2)^n$ -distribution literature (Flatto–Lagarias–Pollington [FLP95]), and every statement about θ below is a statement about $\{K(3/2)^n\}$.

Proposition 4.3. *Let $x_0 \geq 1$ and suppose $K = \delta \in \mathbb{Q}$. A repeated factor of length k at (a, c) forces*

$$\|\delta((3/2)^c - (3/2)^a)\| \leq (2/3)^k,$$

the value lying that close to the integer $x_c - x_a$.

Proof. Substituting $x_n = K(3/2)^n - \theta_n$ into Proposition 2.4 re-expresses the circuit sum in the deviations alone,

$$W(a, k) = 3^k \theta_a - 2^k \theta_{a+k},$$

the K -terms cancelling because $2^k(3/2)^{a+k} = 3^k(3/2)^a$. The deviations survive which are now contracted by the repeated factor. Equal factors give equal circuit sums, $W(a, k) = W(c, k)$, so the display at a and at c combine into

$$3^k(\theta_c - \theta_a) = 2^k(\theta_{c+k} - \theta_{a+k});$$

the right-hand bracket has absolute value at most 1 because $\theta \in [0, 1)$, and dividing by 3^k leaves $|\theta_c - \theta_a| \leq (2/3)^k$.

It remains to read this at the multiplier. Rationality enters only here: since $K = \delta$, the definition $\theta_n = \delta(3/2)^n - x_n$ rearranges to

$$\delta((3/2)^c - (3/2)^a) - (x_c - x_a) = \theta_c - \theta_a,$$

whose right-hand side we have just bounded by $(2/3)^k$. So the value lies within $(2/3)^k$ of the integer $x_c - x_a$, and its distance to the nearest integer is at most its distance to that one. (Once $k \geq 2$ the bound is below $1/2$, and $x_c - x_a$ is then the nearest integer outright; the estimate above needs no such proviso.) $\square$

This is the only place where the rationality of K is used in the whole chain, and it is used only to land the value in $\mathbb{Q}$, where the approximation theorems live. For arbitrary *real* multipliers the finiteness below is false, by a Liouville construction $\delta = \sum_k 2^{-y_k}$ with $y_{k+1} \geq (1 + \varepsilon)y_k$.

The middle display of the proof, read before rationality enters, deserves its own statement.

Corollary 4.4 (unconditional contraction). *Let $x_0 \geq 1$. A repeated factor of length k at (a, c) forces*

$$|\{K(3/2)^c\} - \{K(3/2)^a\}| \leq (2/3)^k,$$

with no hypothesis on K whatsoever.

The parity orbit cannot shadow itself for k steps without dragging the corresponding fractional parts of $K(3/2)^n$ together to within $(2/3)^k$; conversely, the repetition ceiling (Corollary 3.8) caps how long the shadowing can last. Together they are a two-way dictionary between the combinatorics of w and the distribution of $\{K(3/2)^n\}$: every new distributional input on $(3/2)^n$ modulo one converts into a complexity statement, and every combinatorial fact about w converts into a rigidity statement about the Vijayaraghavan sequence.

Definition 4.5. For $\delta, \vartheta \in \mathbb{Q}$ let

$$\mathcal{V}(\delta, \vartheta) := \left\{ (a, c) \in \mathbb{N}^2 : 2 \leq a < c, \left\| \delta((3/2)^c - (3/2)^a) \right\| \leq \vartheta^c \right\}.$$

Theorem 4.6. *For every rational $\delta \neq 0$ and every rational $\vartheta \in (0, 1)$, the set $\mathcal{V}(\delta, \vartheta)$ is finite.*

The case $\delta = 1$ is [RS26, Thm. 5.6]. Opening the multiplier slot is not free, and one of the steps that $\delta = 1$ concealed breaks immediately.

Lemma 4.7 (degeneracy). *Let $\delta \in \mathbb{Q}$, $\delta \neq 0$, and $a < c$ with $|\text{num}(\delta)| \leq c$. Then $\delta((3/2)^c - (3/2)^a)$ is not an integer; in particular its distance to the nearest integer is positive.*

Proof. Write

$$(3/2)^c - (3/2)^a = 3^a(3^{c-a} - 2^{c-a})/2^c,$$

whose numerator is odd. If $\delta((3/2)^c - (3/2)^a) = n \in \mathbb{Z}$ then

$$\text{num}(\delta) \cdot 3^a(3^{c-a} - 2^{c-a}) = n 2^c \text{den}(\delta),$$

and coprimality of 2^c with the odd cofactor forces $2^c \mid \text{num}(\delta)$, hence $c < 2^c \leq |\text{num}(\delta)|$. $\square$

Remark 4.8. At $\delta = 1$ one gets $\|(3/2)^c - (3/2)^a\| > 0$ for every pair, by parity alone. For a general multiplier this is false, and not marginally: at $\delta = 4$, $a = 1$, $c = 2$ one has $4((3/2)^2 - (3/2)) = 3$ exactly, an integer, to which no Diophantine theorem applies. Lemma 4.7 localizes the damage to the box $c < |\text{num}(\delta)|$, which is finite and is therefore discarded wholesale rather than by parity. This is the one place where the multiplier costs a genuine argument.

Proof of Theorem 4.6. Discard the degeneracy box $c < |\text{num}(\delta)|$, finite outright; on the rest, Lemma 4.7 makes every value a non-integer, so the approximation theorems apply. Split the remaining set by its gaps $s = c - a$.

If the gaps are bounded, it suffices to treat a fixed gap $s_0 \geq 1$. Then

$$\delta((3/2)^{a+s_0} - (3/2)^a) = \delta_{CZ} \cdot (3/2)^a, \quad \text{with} \quad \delta_{CZ} = \delta((3/2)^{s_0} - 1),$$

and the Main Theorem of [CZ04] — in the form derived from Theorem 1.9 at $n = 2$, i.e. Ridout's theorem applied once — bounds the a with

$$\|\delta_{CZ}(3/2)^a\| < H((3/2)^a)^{-\varepsilon} \quad \text{for} \quad \varepsilon = \log \vartheta^{-1}/(2 \log 3).$$

Its size via $1 < |\delta_{CZ}(3/2)^a|$ needs $(3/2)^a > 2/|\delta|$, so a further initial segment of a is discarded; this is archimedean and harmless.

If the gaps are unbounded, choose one violator per gap. This is the delicate selection, because the repaired pair theorem has four hypotheses and the family must satisfy all of them, not only the visible one. With $\alpha_1 = \delta$, $\alpha_2 = -\delta$ and $(u_1, u_2) = ((3/2)^c, (3/2)^a)$: the archimedean size condition $|u_i| \geq 1$ holds because the exponents are positive; the ratios $u_1/u_2 = (3/2)^{c-a}$ are pairwise distinct — in both orientations, as the theorem demands — because one violator was chosen per gap; the strict positivity $\|\delta u_1 - \delta u_2\| > 0$, the hypothesis whose omission broke [NKR25]'s Theorem 1.3(i) (Remark 1.10), is Lemma 4.7, available because the degeneracy box was discarded first; and the height-decay demand $\|\delta u_1 - \delta u_2\| < (H(u_1)H(u_2))^{-\varepsilon_1}$ follows from the violator inequality ϑ^c with $\varepsilon_1 = \log \vartheta^{-1}/(2 \log 3)$, since $H(u_1)H(u_2) = 3^{a+c} \leq 3^{2c}$. The pair theorem — derived from Theorem 1.9 at $n = 3$ — then concludes that some $(3/2)^c$ is an integer. This is wrong because $2^c(3/2)^c = 3^c$ is odd. $\square$

Theorem 4.9. *Let $x_0 \geq 1$ and suppose K is rational. Then for every $C \in \mathbb{N}$ there is an $m \geq 1$ with $P(w, m) > Cm$; equivalently, $\limsup_m P(w, m)/m = \infty$.*

Note: for each C a single length m suffices, and that is all Cobham's bound needs negated. We do not claim the stronger $\lim_m P(w, m)/m = \infty$, which would require every large m to work.

Proof. Write $K = \delta$, nonzero since $K \geq 1$. Fix C and pick a rational $\vartheta \in (0, 1)$ with $2/3 \leq \vartheta^{C+2}$; such a ϑ exists by Bernoulli's inequality, taking $\vartheta = 1 - \frac{1/3}{C+2}$. This trades the irrational scale $(2/3)^{1/(C+2)}$ for a rational one,

which is what the kernel requires. By Theorem 4.6 the set $\mathcal{V}(\delta, \vartheta)$ is finite, so its second coordinates are bounded by some M .

Now suppose $P(w, k) \leq Ck$ for $k := M + x_0 + 1$. Pigeonhole produces a repeated factor: the length- k factors of w take values in the finite set $\{0, 1\}^k$, and the $Ck + 1$ positions in $[2, Ck + 2]$ carry at most Ck distinct factors, so two of them agree, say at (a, c) with $2 \leq a < c \leq Ck + 2 \leq (C + 2)k$. Proposition 4.3 contracts it to

$$\|\delta((3/2)^c - (3/2)^a)\| \leq (2/3)^k \leq \vartheta^{(C+2)k} \leq \vartheta^c,$$

so $(a, c) \in \mathcal{V}(\delta, \vartheta)$ and $c \leq M$.

But no repeated factor is that shallow. Theorem 3.7 gives

$$2^{k+c} < 3^c(x_0 + 1) \leq 4^c \cdot 2^{x_0+1}, \quad \text{using } 3^c \leq 4^c \text{ and } x_0 + 1 < 2^{x_0+1},$$

so that $k \leq c + x_0 \leq M + x_0$, contradicting the choice of k . Hence $P(w, k) > Ck$ for this single length k . The ceiling used here is crude, and deliberately so: Corollary 3.8 gives the sharp slope 0.585 but only at $x_0 = 1$, whereas $k \leq c + x_0$ is uniform in x_0 , which is all the reduction needs, $k \rightarrow \infty$ must force $c \rightarrow \infty$. $\square$

Proof of Theorem B. If K is irrational we are in the second horn. Otherwise $K \in \mathbb{Q}$ and Theorem 4.9 gives the first. $\square$

Remark 4.10. Both horns of Theorem B are worth having, and neither is provable alone here. The first would beat Theorem 1.4's record for this word: no linear function bounds the complexity, against a linear lower bound with a fixed constant. The second is a conditional irrationality statement about a constant open since 1977. As a consistency check, note that an eventually periodic w would make K rational *and* the complexity bounded, which Theorem 3.9 excludes; so the dichotomy is sharp on both sides.

The kernel has an *effective* slice, and it is worth delimiting exactly — both because it is all the effectivity there is, and because its boundary turns out to be a line this paper has met before.

Proposition 4.11 (Liouville floor). *Let $\delta \in \mathbb{Q}$, $\delta \neq 0$, and $a < c$ with $|\text{num}(\delta)| \leq c$. Then*

$$\|\delta((3/2)^c - (3/2)^a)\| \geq \frac{1}{\text{den}(\delta) 2^c}.$$

Proof sketch. $\text{den}(\delta) 2^c$ clears the value to an integer, and Lemma 4.7 says the value is not itself an integer; a non-integral rational whose D -fold is an integer keeps distance at least $1/D$ from $\mathbb{Z}$. The formal statement is the integer-certificate form $1 \leq \text{den}(\delta) 2^c \|\cdot\|$. $\square$

Corollary 4.12 (the effective slice). *Let $\delta \neq 0$ and $0 \leq \vartheta < 1/2$. Then $\mathcal{V}(\delta, \vartheta)$ is finite effectively, with no Diophantine input at all: every $(a, c) \in \mathcal{V}(\delta, \vartheta)$ with $c \geq |\text{num}(\delta)|$ satisfies the checkable certificate $\text{den}(\delta)(2\vartheta)^c \geq 1$, so that $c < N$ for any $N \geq |\text{num}(\delta)|$ with $\text{den}(\delta)(2\vartheta)^N < 1$.*

Remark 4.13 (the effective region is the empty region). Three observations delimit Corollary 4.12. First, it is uniform in the gap $c - a$: in the regime $\vartheta < 1/2$ nothing is gained by fixing the gap, and the effective statement covers the whole kernel at once — there is no need to isolate a bounded-gap family. Second, the slice cannot be widened by linear forms in logarithms: the nearest integer is a *free* third term — the relevant integer form is $\text{num}(\delta) 3^a (3^{c-a} - 2^{c-a}) - n \text{den}(\delta) 2^c$ with n unconstrained — so a two-logarithm bound must absorb n into a composite algebraic number of height $\asymp c \log(3/2)$, and to beat the floor of Proposition 4.11 its constant would have to be below $\log 2 / (\log 3 \log(3/2)) \approx 1.56$, an order of magnitude beyond anything proved. (This is the archimedean twin of the familiar fact that Baker-type bounds lose to trivial estimates on three-term forms.) What lies beyond $1/2$ is the Padé frontier, and only for the multiplier 1: Beukers’s $\|(3/2)^n\| \geq 2^{-0.9n}$ for $n \geq 5000$ [Beu81], improved by Zudilin to $\|(3/2)^n\| \geq 0.5803^n$ [Zud07]; no comparably strong effective bound is known with a general rational multiplier, and beyond 0.5803 nothing effective is known at all. Third — the coherence — the boundary $\vartheta = 1/2$ is the ledger of §5.2 in disguise: the floor $\asymp 2^{-c}$ beats the contraction $(2/3)^k$ of Proposition 4.3 exactly when $k/c > \log 2 / \log(3/2) = 1.7095\dots$, the demand line, while actual repeated factors obey $k/c \leq 0.585$ (Corollary 3.8). So the kernel is effective precisely in the region the word never enters, and the region the complexity application needs — $\vartheta \rightarrow 1$ in the proof of Theorem 4.9 — is ineffective essentially, not accidentally.

4.3. Assembling.

Proof of Theorem A. Suppose K is algebraic. If K is irrational, Theorem 4.1 gives that w is not automatic. If K is rational, Theorem 4.9 makes the complexity of w exceed every linear bound, which Cobham’s bound $P(a, m) = O(m)$ for automatic a [Cob72] forbids; so w is not automatic in that case either. The two cases partition the algebraic case. The first form of the statement is the contrapositive: if w is automatic then K is not algebraic. $\square$

Theorem A strictly strengthens Corollary 4.2, whose irrationality hypothesis was exactly the $K \in \mathbb{Q}$ horn that §4.2 removes.

4.4. Algebraic multipliers: the Corvaja–Zannier half. If the kernel held for algebraic irrational multipliers as it does for rational ones, Theorem A would upgrade at once to K *algebraic* $\Rightarrow$ *superlinear complexity*, single-lane, with the Mahler engine of §4.1 retired and complexity content in every algebraic case. This subsection proves the half of that extension which the printed literature supports, shows the reduction never needed rationality, and thereby compresses the remaining distance to a single named statement.

For real δ let $\mathcal{V}_{\mathbb{R}}(\delta, \vartheta)$ denote the violator set of Definition 4.5 with the distance to the nearest integer taken in $\mathbb{R}$; for $\delta \in \mathbb{Q}$ it coincides with

$\mathcal{V}(\delta, \vartheta)$. Two of the three multiplier costs of §4.2 now behave unexpectedly well. Degeneracy *disappears*: for irrational δ the value $\delta((3/2)^c - (3/2)^a)$ is an irrational times a nonzero rational, never an integer, so $\|\cdot\| > 0$ holds for *every* pair — Lemma 4.7’s box was a rational phenomenon, and the first casualty of the extension is in fact a simplification. And the bounded-gap slices survive verbatim, because Theorem 1.11 is printed for algebraic δ ; what was a parity remark becomes a conjugate argument.

Theorem 4.14 (the bounded-gap kernel, algebraic multipliers). *Let δ be a real algebraic irrational, let $\vartheta \in (0, 1)$ be rational, and let $S \in \mathbb{N}$. Then only finitely many $(a, c) \in \mathcal{V}_{\mathbb{R}}(\delta, \vartheta)$ have gap $c - a \leq S$.*

Proof sketch. Fix the gap $s_0 \leq S$ and write the value as $\delta_{\text{CZ}}(3/2)^a$ with $\delta_{\text{CZ}} = \delta((3/2)^{s_0} - 1)$, algebraic irrational. Apply Theorem 1.11 with $q = 1$, $u = (3/2)^a$, $H(u) = 3^a$, $\varepsilon = \log \vartheta^{-1} / (2 \log 3)$. Three clauses are discharged along the family. The size proviso $1 < |\delta_{\text{CZ}}(3/2)^a|$ fails only on an initial segment of a (archimedean). Strict positivity holds for every pair, by irrationality. The new obligation is the pseudo-Pisot exclusion: δ_{CZ} , being algebraic irrational, has a conjugate $z \neq \delta_{\text{CZ}}$, $z \neq 0$; conjugates scale along rational multiples — minimal polynomials obey $p_{r\delta_{\text{CZ}}}(X) = r^{\deg p} p_{\delta_{\text{CZ}}}(X/r)$ — so $z(3/2)^a$ is a conjugate of the value other than the value itself, and it leaves the unit disc once $(3/2)^a \geq |z|^{-1}$. Pseudo-Pisot values, like the size proviso, are confined to an initial segment and discarded. $\square$

The unbounded-gap branch of Theorem 4.6 used the repaired pair theorem — *derived* over $\mathbb{Q}$ from Theorem 1.9 at $n = 3$. For algebraic coefficients nothing can be cited: [NKR25] is unrefereed and its Theorem 1.3(i) is false as printed (Remark 1.10), and we know of no refereed substitute. What we can state exactly is the obligation itself.

Theorem 4.15 (conditional upgrade). *Let $x_0 \geq 1$ and assume the pair branch at $\delta = K$: for every rational $\vartheta \in (0, 1)$, every family $T \subseteq \mathcal{V}_{\mathbb{R}}(K, \vartheta)$ with pairwise-distinct gaps is finite. If K is algebraic, then for every C there is an $m \geq 1$ with $P(w, m) > Cm$.*

Proof sketch. If K is rational this is Theorem 4.9, and the hypothesis is not consulted. If K is algebraic irrational, the gap dichotomy behind Theorem 4.6 reassembles: bounded gaps fall to Theorem 4.14, unbounded gaps to the assumed pair branch after selecting one violator per gap, and there is no degeneracy box to discard — so $\mathcal{V}_{\mathbb{R}}(K, \vartheta)$ is finite. The Stage-1 reduction of Theorem 4.9 then runs verbatim over $\mathbb{R}$: pigeonhole, the Bernoulli scale, the contraction — Corollary 4.4, which never needed rationality — and the growth ceiling. $\square$

The pair branch is a theorem for rational δ — supporting the proof of Theorem 4.6 —, and for algebraic irrational δ it is a Subspace argument over the number field $\mathbb{Q}(\delta)$: places above 2, 3, ∞ , relative heights, S -enlargement at the places where δ is not a unit. Theorem 1.9 is already stated over

an arbitrary number field, so the assumption side is ready; missing is the distance between Theorem 4.15 and an unconditional upgrade.

What the proven half buys *unconditionally* is a repetition statement. Call a pair (a, c) with $2 \leq a < c \leq a + S$ a *close repetition at slope $1/L$* if the factors of w of length $\lfloor c/L \rfloor + 1$ at a and at c coincide — a period- $(c - a)$ stretch of length proportional to its position, at bounded period. By the ceiling (Theorem 3.7; $2^{2^{c+1}} < 3^c(x_0 + 1)$ fails for large c) the class is empty for $L = 1$ and c large, and Corollary 3.8's slope 0.585 makes $L = 2$ the widest interesting one.

Corollary 4.16. *Let $x_0 \geq 1$ and suppose K is algebraic. Then for every gap bound S and every slope $1/L$, $L \geq 1$, only finitely many close repetitions occur in w . Contrapositively: infinitely many close repetitions at a single (S, L) force K to be transcendental.*

Proof sketch. A close repetition is a violator of $\text{gap} \leq S$ at the Bernoulli scale $\vartheta(L)$ with $2/3 \leq \vartheta^L$: Corollary 4.4 gives $\|K((3/2)^c - (3/2)^a)\| \leq (2/3)^{\lfloor c/L \rfloor + 1} \leq \vartheta^c$. For algebraic irrational K , Theorem 4.14 finishes; for rational K , Theorem 4.6 does. $\square$

Remark 4.17 (scope and lanes). Under its algebraicity hypothesis, Corollary 4.16 strictly strengthens what Theorem 3.9 excludes: eventual periodicity would be one infinite family of close repetitions, while the corollary bounds every bounded-period family of proportional-length stretches. (The two are not comparable outright — Theorem 3.9 is unconditional.) It is also the first repetition-structure statement available for algebraic *irrational* K , where Theorem A gave non-automaticity through the Mahler lane and nothing quantitative. Remark 1.6 still applies unchanged: K is expected transcendental, the trigger is expected to fail (the residue data of §5.3 leaves no room for close repetitions) and the content is the criterion direction.

5. TWO COMPLEMENTS

5.1. The 2-adic value of the word. Proposition 2.5 is an identity between rationals, and so may be read at any place. Over $\mathbb{R}$ its limit is $3(K - x_0)$, an unknown real. Over $\mathbb{Q}_2$ the limit is an integer.

Proposition 5.1. *In $\mathbb{Q}_2$ one has $\|2/3\|_2 = 1/2$, the series $\sum_j w_j(2/3)^j$ converges, and*

$$\sum_{j \geq 0} w_j \left(\frac{2}{3}\right)^j = -3x_0.$$

More generally $\sum_{j \geq 0} w_{n+j}(2/3)^j = -3x_n$ for every n .

Proof sketch. The induction proving Proposition 2.5 uses only that 2 and 3 are invertible, so it runs verbatim over $\mathbb{Q}_2$. Only the limit differs:

$$\|x_n(2/3)^n\|_2 = 2^{-n}\|x_n\|_2 \leq 2^{-n} \longrightarrow 0,$$

since x_n is a 2-adic integer. So the orbit term vanishes and the partial sums $3x_n(2/3)^n - 3x_0$ tend to $-3x_0$. The shifted family follows by shift-invariance of the word. $\square$

5.2. The exponent ledger. A recurring proposal for this word is to periodize the digits at a repeated factor, producing S -unit approximants, and to feed those to the p -adic Subspace Theorem, as Adamczewski and Bugeaud [AB07] did for Loxton–van der Poorten. The proposal cannot work, and the reason is structural.

The two constants $\log_2(3/2) < 1 < 1/\log_2(3/2)$ are exactly the two sides of the ledger. Corollary 3.8 caps the *quality* t/c of any repeated factor of w by $\log_2(3/2) = 0.5849\dots$, while periodizing at a repeated factor yields approximants whose per-place product over $\{\infty, 2, 3\}$ needs

$$t/c > (1 - \lambda)/\lambda = 1.7095\dots, \quad \lambda = \log(3/2)/\log 3.$$

Since the two are reciprocal and straddle 1, no sharpening can close the gap: the very mechanism that makes the word complex — the 2-adic rigidity of Theorem 3.2, which charges $2^t \mid x_c - x_a$ for a repeated factor of length t while the orbit grows only like $(3/2)^c$ — is the mechanism that caps repeated-factor quality below 1.

Proposition 5.2 (log-free form). *Let $x_0 = 1$ and $a < c$. No repeated factor of length t at (a, c) satisfies $17095c < 10000t$.*

Proof sketch. Immediate from $41t \leq 24c + 40$ (Corollary 3.8); the two are incompatible already for $c \geq 1$. $\square$

Definition 5.3. For $\tau, \varepsilon > 0$ and sequences $c, t: \mathbb{N} \rightarrow \mathbb{N}$, say (c_m, t_m) satisfies the *index condition* at (τ, ε) if

$$(\tau + \varepsilon) c_m \log 3 \leq t_m \log 2 \quad \text{for infinitely many } m.$$

This is what an application of the p -adic Subspace Theorem in the style of [AB07] needs from a family of S -unit approximants of height $\asymp 3^{c_m}$ and 2-adic quality 2^{-t_m} : the per-place product over $\{\infty, 2, 3\}$ must beat the height to the power $-\tau - \varepsilon$, for some $\tau > 1$, infinitely often. Reading a repeated factor of length t_m at position c_m as the source of such an approximant (periodize the digits there) is the one interpretive element of this subsection, and the theorem below is a statement about Definition 5.3 alone.

Theorem 5.4. *Let (a_m, c_m, t_m) be any family of repeated factors of w (at $x_0 = 1$) with $a_m < c_m$ and $c_m \rightarrow \infty$. Then (c_m, t_m) fails the index condition at (τ, ε) , for every $\tau > 1$ and every $\varepsilon > 0$.*

Proof sketch. Corollary 3.8 gives $t_m \log 2 < c_m \log 3$ outright once $c_m \geq 3$, before τ is used at all; and $\tau + \varepsilon > 1$ only makes the demand worse. So the condition fails *eventually*, which contradicts its “infinitely often” shape. $\square$

Theorem 5.4 takes the repeated-factor family as given and does not rebuild the approximant construction: the point is that no construction can help, whatever its details, because the exponents do not fit.

5.3. Computations. This subsection is not formally verified; it records computations and the places where the machinery above visibly bites next. It is the only part of the paper outside the scope of Appendix A.

Proposition 3.3 makes the complexity of w cheaply computable: $P(w, m)$ is the number of residues modulo 2^m met by the orbit, and a prefix of the orbit witnesses a lower bound. The result of running the first 10^6 orbit points (at $x_0 = 1$) is stark: *every* residue class modulo 2^m is met, for every $m \leq 16$ — that is, the witnessed counts are not merely superlinear but maximal, $P(w, m) = 2^m$ as far as the computation can see. (For $17 \leq m \leq 20$ the prefix is still producing new residues at its end, so those counts are not yet saturated; nothing suggests they stop short.) Moreover the *rate* of saturation is the generic one: the position of the last new residue — 1471 for $m = 8$, 39346 for $m = 12$, 821306 for $m = 16$ — matches the coupon-collector time $2^m(m \log 2 + \gamma) \approx 765000$ at $m = 16$ of a uniformly random residue sequence. The orbit of $U_{3/2}$ behaves, 2-adically, like a random sequence. On that model full complexity is forced — and indeed Dubickas already conjectures it at $x_0 = 1$ [Dub09, p. 246], judging it “very likely as difficult as” the corresponding conjecture $P(\alpha, n) = q^n$ for algebraic irrationals in integer bases. The computation above appears to be the first direct evidence, and supports the conjecture in the generality of this paper.

Conjecture 5.5 (Dubickas [Dub09] at $x_0 = 1$). *For every $x_0 \geq 1$ and every m , $P(w, m) = 2^m$: every binary word is a factor of w .*

Two companion computations point the same way. The repeated-factor quality t/c (in the notation of §5.2) over all pairs $a < c \leq 3000$ is maximized at the early pair $(a, c, t) = (1, 6, 4)$ with $t/c = 2/3$ — an artifact of the additive constant in Corollary 3.8 — and restricted to $c \geq 100$ its maximum is 0.1089, decaying with c : the orbit sits far below the 0.585 ceiling, i.e. the word is empirically very far from stammering. And the truncated kernels are as large as they can be: comparing kernel elements on their first 64 terms, the 2-kernel has 2^i distinct elements at every depth $i \leq 12$ and the 3-kernel 3^i at every depth $i \leq 8$ — no collapse anywhere, as non-automaticity predicts.

6. ACKNOWLEDGEMENTS

The author utilized Claude Code as an AI coding assistant to aid in the Lean 4 formalization of the proofs presented in this paper. The author directed and reviewed all generated code and takes full responsibility for the mathematical integrity and final content of the work.

APPENDIX A. FORMALIZATION

Every statement above was verified in Lean 4, except the computations. See <https://github.com/rwst/Transcendence-Criteria>. The files live under RB/ with CITED/AdamczewskiFaverjon.lean,

CITED/SubspaceTheorem.lean, CITED/AlloucheShallitBasic.lean, CITED/AlloucheShallitComplexity.lean and CITED/CorvajaZannierAlgebraic.lean supplying inputs. “std3” abbreviates the ambient axioms of classical Lean (propositional extensionality, choice, quotient soundness). Three cited axioms occur, each marked below and each refereed: “[AF17]” for AF.transcendental_or_rat_of_automatic, “[Sub]” for Subspace.evertseSchlickewei, and “[CZ04a]” for CZ.pseudoPisot_approx_alg (the algebraic-multiplier instance of Theorem 1.11, carried only by §4.4; the rational instance is derived from [Sub] and is not an axiom). Every entry is fully proved in Lean except the three marked *cited axiom*. All footprints below were confirmed with #print axioms.

Cited inputs (§1.4).

Statement	Lean identifier	File	Status
Thm. 1.7	AF.transcendental_or_rat_of_automatic	AdamczewskiFaverjon	cited axiom [AF17]
Thm. 1.9	Subspace.evertseSchlickewei	SubspaceTheorem	cited axiom [Sch91]
—	CZ.pseudoPisot_approx_of_subspace	CorvajaZannierProof	std3 + [Sub]
—	NKR.sUnit_pair_integrality_of_subspace	NairKumarRoutProof	std3 + [Sub]
Rem. 1.10	NKR.thm13i_unrepaired_false	NairKumarRout	std3
—	AS.kKernel, AS.IsAutomatic	AlloucheShallitBasic	def
—	AS.complexity	AlloucheShallitComplexity	def
Rem. 1.5	AS.complexity_linear_of_automatic	AlloucheShallitComplexity	std3
—	AS.not_automatic_of_complexity_superlinear	AlloucheShallitComplexity	std3

The orbit, the word, the constant, and rigidity (§2–§3).

Statement	Lean identifier	File	Status
Def. 2.1	RB.x, RB.wmin	Basic	def
Prop. 2.2	two_mul_x_succ, wmin.le.one	Basic	std3
Def. 2.3	RB.W	Basic	def
Prop. 2.4	circuit_sum	Basic	std3
Prop. 2.5	x_mul_pow	Basic	std3
Def. 2.6	RB.K	Basic	def
Prop. 2.7	tendsto_x_mul_pow, tsum_wmin_eq	Basic	std3
Prop. 2.8	x.le.K_mul_pow, K_mul_pow.le	Basic	std3
Def. 3.1	RB.IsRepetition	Rigidity	def
Thm. 3.2	isRepetition_iff_dvd	Rigidity	std3
<i>equal circuit sums</i>	lemmaR.int	Rigidity	std3
Thm. 3.7	repetition_pow_lt	Rigidity	std3
<i>orbit growth</i>	two_pow_mul_x_add_one.le	Basic	std3
Cor. 3.8	repetition_linear_bound	Rigidity	std3
Prop. 3.3	complexity_eq_ncard_residues	Residues	std3
<i>windows are residues</i>	factor_eq_iff_residue_eq	Residues	std3
Cor. 3.4	complexity_lower_bound	Residues	std3
Prop. 3.6	x_succ_mod_three	Residues	std3
Thm. 1.3, 3.9	not_eventually_periodic	Rigidity	std3
Thm. 1.2	closed_form	ClosedForm	std3

Transcendence criteria (§4).

Statement	Lean identifier	File	Status
Thm. 4.1	not_automatic_of_K_algebraic_irrational	NotAutomatic	std3 + [AF17]
Cor. 4.2	transcendental_of_automatic_of_irrational	NotAutomatic	std3 + [AF17]
θ_n (§4.2)	RB.tail, tail.nonneg, tail.lt.one	OrbitKernel	std3
$\theta_n = \{K(3/2)^n\}$	tail.eq.fract	Residues	std3
Prop. 4.3	dist.le.of.repetition	OrbitKernel	std3
Cor. 4.4	abs_tail_sub.le.of.repetition	OrbitKernel	std3
Def. 4.5	RB.scaledViolators	ScaledKernel	def
Lem. 4.7	dist_pos_of_num.le	ScaledKernel	std3
Thm. 4.6	scaledViolators.finite	ScaledKernel	std3 + [Sub]
Prop. 4.11	one.le.den.mul.two.pow.mul.dist	EffectiveSlice	std3
Cor. 4.12	scaledViolators.finite.of.lt.half	EffectiveSlice	std3
Thm. 1.11	CZ.pseudoPisot.approx.alg	CorvajaZannierAlgebraic	cited axiom
$\mathcal{V}_{\mathbb{R}}$ (§4.4)	RB.algViolators	AlgebraicKernel	def
Thm. 4.14	algGapBounded.slice.finite	AlgebraicKernel	std3 + [CZ04a]
Thm. 4.15	superlinear_of_K_algebraic_of_pairBranch	AlgebraicKernel	std3 + [CZ04a] + [Sub]
Cor. 4.16	closeRepetitions.finite.of.K_algebraic	AlgebraicKernel	std3 + [CZ04a] + [Sub]
Thm. 4.9	superlinear_of_K_rat	RationalK	std3 + [Sub]
Thm. A	not_automatic_of_K_algebraic	RationalK	std3 + [AF17] + [Sub]
Thm. A	transcendental_of_automatic	RationalK	std3 + [AF17] + [Sub]
<i>rational case</i>	not_automatic_of_K_rat	RationalK	std3 + [Sub]
Thm. B	superlinear_or_K_irrational	RationalK	std3 + [Sub]

The complements (§5).

Statement	Lean identifier	File	Status
Prop. 5.1	norm.two.thirds, x.mul.pow.padic	PadicValue	std3
Prop. 5.1	tsum.wmin.padic, tsum.wmin.shift.padic	PadicValue	std3
Prop. 5.2	not.demand	NoStammeringRoute	std3
Thm. 5.4	not.indexConditionExpFreq	NoStammeringRoute	std3

The sanity witnesses $x = 1, 2, 3, 5, 8, 12, 18, 27, 41$ and $w = 1, 0, 1, 1, 0, 0$ for $x_0 = 1$ (matching A061419 and the first letters of $g_{3/2}$) and the closed-form values $\lfloor K(3/2)^0 \rfloor = 1$, $\lfloor K(3/2)^4 \rfloor = 8$, $\lfloor K(3/2)^8 \rfloor = 41$ are verified by decision procedures (`x_sanity`, `wmin_sanity`, `closed_form_sanity`).

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