SUPERLINEAR COMPLEXITY OF THE $(3/2)^n$ STEERING WORD

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ABSTRACT. Write $(3/2)^n = m_n + \varepsilon_n$ with m_n the nearest integer and $\varepsilon_n \in [-\frac{1}{2}, \frac{1}{2})$, and let $T = (t_n)$, $t_n = 2m_{n+1} - 3m_n$, be the resulting *steering word*: the step-by-step record of the map $x \mapsto \frac{3}{2}x$ on the orbit of 1, coded by nearest-integer rounding. Using results by Corvaja–Zannier and Nair–Kumar–Rout we prove that the subword complexity $p_T(k)$ of T is superlinear, $p_T(k)/k \to \infty$. The argument is completely formalized in Lean 4 and rests on a single external input, the Evertse–Schlickewei S-arithmetic subspace theorem, from which both cited results are themselves derived within the formalization.

1. INTRODUCTION

For a real number x we write $\|x\| = \min_{n\in\mathbb{Z}} |x - n|$ for its distance to the nearest integer. The distribution of the sequence $\big(\|\alpha^n\|\big)$ for a fixed real $\alpha > 1$ is poorly understood; already for $\alpha = 3/2$ it is open whether $\|(3/2)^n\| \to 0$ can fail to hold densely. Mahler [Mah57] proved that for rational $\alpha > 1$ that is not an integer, and any $0 < \ell < 1$, the fractional part of α^n exceeds ℓ^n for all but finitely many n; Corvaja and Zannier [CZ04] recast and extended this through the Schmidt subspace theorem, and it is their Main Theorem that we shall use below.

This paper concerns not the sizes $\|(3/2)^n\|$ themselves but the *symbolic dynamics* they generate. Writing $(3/2)^n = m_n + \varepsilon_n$ with $m_n \in \mathbb{Z}$ and $\varepsilon_n \in [-\frac{1}{2}, \frac{1}{2})$, the increments

$$t_n \;:=\; 2m_{n+1} - 3m_n \;\in\; \{-2, -1, 0, 1, 2\}$$

record, at each step, which integer translate the multiplication-by-$3/2$ map selects. The word $T = (t_n)_{n\geq 0}$ is the full itinerary of that autonomous dynamics; we call it the *steering word*. It is the round-to-nearest analogue of the itineraries of the $\frac{3}{2}$-dynamics studied through rational base number systems [AFS08]. Its *subword complexity* $p_T(k)$ is the number of distinct length-k factors it contains. Our main result is the following.

Theorem 1.1 (Main theorem)**.** *The subword complexity of the steering word T is superlinear: for every C there is a K with $p_T(k) > C\,k$ for all $k \geq K$; equivalently $p_T(k)/k \to \infty$.*

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The proof has the shape of a transfer of the Adamczewski–Bugeaud method [AB07] for complexity lower bounds of algebraic numbers to the present dynamical word, and it simplifies in one respect: because the $\frac{3}{2}$-dynamics is autonomous, the bookkeeping is position-uniform, and every repeated factor of T compresses into a single clean Diophantine inequality, with no prefix-anchoring. Concretely the argument runs in three stages.

Stage 0 (Section 2) is an exact identity: a length-k factor of T occurring at two positions $a < c$ forces $3^k(\varepsilon_c - \varepsilon_a) = 2^k(\varepsilon_{c+k} - \varepsilon_{a+k})$ and the divisibility $2^k \mid m_c - m_a$. This already gives two unconditional theorems — T is not eventually periodic (Theorem 2.9), and $p_T(k) \geq (41/24)\,k - 3$ (Theorem 2.10), strictly above the universal Morse–Hedlund floor $k + 1$.

Stage 1 (Section 3) reduces Theorem 1.1 by pigeonhole to a single statement:

> (K) for every $\theta < 1$, the inequality $\|(3/2)^c - (3/2)^a\| \leq \theta^{\,c}$ has only finitely many solutions $2 \leq a < c$.

This is *exponential pair repulsion* for the orbit — an improvement of the trivial repulsion floor $\|(3/2)^c - (3/2)^a\| \geq 2^{-c}$ from an exponential base 2^{-1} to an arbitrary base $\theta < 1$.

Stage 2 (Sections 4 and 5) establishes (K). Two one-parameter slices — bounded gap $c - a$, and the huge-gap band $a \leq \varepsilon' c$ — fall to the Main Theorem of Corvaja–Zannier. What remains, the middle band, is closed by a gap dichotomy: either a slice has bounded gaps (handled by Corvaja–Zannier), or one may extract one violator per gap and apply a repaired form of the S-unit pair theorem of Nair–Kumar–Rout, whose integrality conclusion is absurd for $(3/2)^c$.

The elementary consequences of Stage 0 and the Stage 1 reduction are unconditional. The two kernel slices and the middle-band-conditional form of the main theorem (Theorem 5.2) rest on the theorem of Corvaja–Zannier [CZ04]. The unconditional main theorem (Theorem 1.1, proved as Theorem 5.8) additionally uses an S-unit pair theorem in the spirit of Nair–Kumar–Rout [NKR25] (Theorem 5.3). Neither input is an axiom of the formalization: both are derived, in the $\mathbb{Q}$-specialized forms used here, from the Evertse–Schlickewei S-arithmetic subspace theorem [Sch91, BG06] in two and three variables respectively, which is the single external input of the whole development. The derivation of the second input uncovered that Theorem 1.3(i) of [NKR25] is *false as printed*: an explicit infinite family satisfies every hypothesis and violates the conclusion (Remark 5.4). We prove and use a repaired statement, obtained by restoring to inequality (1) of [NKR25] the strict-positivity hypothesis it omits. The authors, on being shown the counterexample, have confirmed the defect and propose a repair of a different kind, weakening the conclusion rather than strengthening the hypotheses; it is recorded in Remark 5.5 and would serve here equally well. An appendix lists each result together with its formal name and its precise dependency.

1.1. **Relation to the complexity landscape.** The conclusion $p_T(k)/k \to \infty$ is precisely the shape of the Adamczewski–Bugeaud theorem [AB07] for the b-ary expansions of irrational algebraic numbers, and the analogy extends to the gap that theorem leaves open: there Borel's conjecture predicts normality, that is $p(k) = b^k$, so superlinearity is the unconditional shadow of an expected exponential truth. The same is the case here, and it can be made precise.

Superlinearity already places T outside every classical low-complexity family. Automatic sequences satisfy $p(k) = O(k)$ [Cob72, AS03], as do linearly recurrent words [DHS99], Sturmian words and the codings of rotations and of interval exchanges, and fixed points of primitive substitutions; Theorem 1.1 excludes T from all of them. For the general theory of the factor complexity function we refer to [CN10], and to [Fer99, Puz25] for surveys.

What superlinearity does *not* decide is whether T is morphic, and here a rigidity theorem of Pansiot [Pan84] enters: the complexity of a pure morphic word is one of $\Theta(1)$, $\Theta(k)$, $\Theta(k \log\log k)$, $\Theta(k \log k)$ and $\Theta(k^2)$ — in particular it is polynomially bounded. Hilion and Levitt [HL25] have recently proved the analogous four-class theorem for right-infinite words representing attracting fixed points of an automorphism of a finitely generated free group. The following two statements turn this rigidity into a dichotomy for the steering word; both are consequences of Lemma 2.7, read backwards.

Proposition 1.2 (Packing bound)**.** *Let $k \in \mathbb{N}$ and let $A \subset \mathbb{N}$ be a finite set of positions whose orbit points are pairwise more than $(2/3)^k$ apart, that is $|\varepsilon_c - \varepsilon_a| > (2/3)^k$ for all $a \neq c$ in A. Then $p_T(k) \geq |A|$. Equivalently $p_T(k) \geq N\big((2/3)^k\big)$, where $N(\delta)$ is the largest cardinality of a δ-separated subset of $\{\varepsilon_n\}$.*

Proof. By Lemma 2.7, two positions carrying the same length-k factor satisfy $|\varepsilon_c - \varepsilon_a| \leq (2/3)^k$. Hence the length-$k$ factors at the positions of A are pairwise distinct. □

Corollary 1.3 (Density forces exponential complexity)**.** *If $\big((3/2)^n\big)_{n\geq 0}$ is dense modulo one, then $p_T(k) \geq \lceil (3/2)^k \rceil - 1$ for every k.*

Proof. Fix k and a positive integer $M < (3/2)^k$, so that $(2/3)^k < 1/M$. The M points $-\frac{1}{2} + j/M$, $0 \leq j < M$, lie in $[-\frac{1}{2}, \frac{1}{2})$ and are $1/M$-separated. Approximating each of them within $\delta = \frac{1}{3}\big(1/M - (2/3)^k\big) > 0$ by a point of the orbit produces M positions whose ε-values are pairwise more than $1/M - 2\delta = \frac{1}{3}\cdot\frac{1}{M} + \frac{2}{3}(2/3)^k > (2/3)^k$ apart, so $p_T(k) \geq M$ by Proposition 1.2. Since $\lceil (3/2)^k \rceil - 1 < (3/2)^k$, the claim follows. □

Two readings of Corollary 1.3 place the main theorem. First, it identifies Theorem 1.1 as the weakest nontrivial instance of what the density conjecture predicts: the expected truth is exponential, and the difficulty of the subject lies in the gap between $k \cdot \omega(1)$ and $(3/2)^k$. Second, read

contrapositively, *any* subexponential upper bound $p_T(k) = o\big((3/2)^k\big)$ would refute density of $\big((3/2)^n\big)$ modulo one. This is why no upper bound of that strength is offered here: beyond the trivial alphabet ceiling $p_T(k) \le 5^k$, the sharp known environment is Kopra's trace subshift [Kop21], which gives $p_T(k) \le 4 \cdot 3^k - 3 \cdot 2^k$ — of base 3 rather than 3/2.

Combining Corollary 1.3 with Pansiot's theorem: if $\big((3/2)^n\big)$ is dense modulo one, then T is not morphic, and by [HL25] it does not represent an attracting fixed point of a free group automorphism either; equivalently, exhibiting a morphism that generates T would refute the density conjecture. It is Theorem 1.1 that makes the dichotomy non-vacuous, by eliminating the classes $\Theta(1)$ and $\Theta(k)$ outright.

The implication runs one way only. Proposition 1.2 bounds the complexity below by a packing number, not conversely: the letter t_n is a function of $(\varepsilon_n, m_n \bmod 2)$, but $m_{n+1} \bmod 2$ already requires $m_n \bmod 4$, so a length-k window is governed by $(\varepsilon_a, m_a \bmod 2^k)$ and distinct factors need not arise from separated orbit points. We claim no converse to Lemma 2.7, and Theorem 1.1 accordingly transfers nothing back to the density question. The complementary unconditional information is due to Flatto, Lagarias and Pollington [FLP95], whose theorem gives the set of limit points of $\{\xi(3/2)^n\}$ a diameter of at least 1/3: the orbit closure is known to be spread out, but is not known to have positive box dimension — which by Proposition 1.2 would already force $p_T(k) \ge c^k$ for some $c > 1$.

Finally, on the shape of p_T as a function of k. The region bracketed by the bounds above is genuinely populated: Cassaigne [Cas03] constructs words whose complexity grows faster than every polynomial and slower than every exponential; and in the linear regime the pairs (α, β) realizable as $\alpha = \liminf_k p_T(k)/k$ and $\beta = \limsup_k p_T(k)/k$ constitute the Heinis spectrum, recently shown by Erazo and Moreira [EM25] to have non-empty interior.

Setup and notation. Throughout, $\|\cdot\|$ is the distance to the nearest integer, computed exactly in $\mathbb{Q}$ for all the rationals that occur. For $u \in \mathbb{Q}^\times$ written in lowest terms $u = p/q$ we write $\mathrm{H}(u) = \max(|p|, |q|)$ for its absolute Weil height; in particular $\mathrm{H}\big((3/2)^a\big) = 3^a$. We abbreviate $\Gamma = \langle 2, 3\rangle \le \mathbb{Q}^\times$, the multiplicative group generated by 2 and 3.

Definition 1.4 (The orbit decomposition and the steering word)**.** For $n \in \mathbb{N}$ put

$$m_n := \mathrm{round}\,\big((3/2)^n\big), \qquad \varepsilon_n := (3/2)^n - m_n \in [-\tfrac{1}{2}, \tfrac{1}{2}), \qquad R_n := 3^n - 2^n m_n,$$

so that $\varepsilon_n = R_n/2^n$; here round rounds half-integers up, placing ε_n in the half-open window $[-\frac{1}{2}, \frac{1}{2})$. The *steering letter* is $t_n := 2m_{n+1} - 3m_n$, the *steering word* is $T = (t_n)_{n\ge 0}$, and the *parity word* is $b = (b_n)$, $b_n := m_n \bmod 2$.

Elementary manipulations show $t_n = 3\varepsilon_n - 2\varepsilon_{n+1}$, whence $|t_n| \le 2$ (a five-letter alphabet) and $t_n \equiv m_n \pmod 2$, so that b is the letter-to-letter reduction of T modulo 2. Moreover R_n is odd for $n \ge 1$; this parity is the source

of the trivial repulsion floor. The sequence begins $m = 1, 2, 2, 3, 5, 8, \ldots$ and $t = 1, -2, 0, 1, 1, \ldots$. All objects live in $\mathbb{Z}$ or $\mathbb{Q}$; no real analysis enters before Section 4.

Definition 1.5 (Circuit sum)**.** For $a, k \in \mathbb{N}$ put

$$W(a,k) \;:=\; \sum_{i<k} 3^{\,k-1-i}\, 2^{\,i}\, t_{a+i} \;\in\; \mathbb{Z}.$$

2. The repetition identity

The whole analysis rests on one exact cancellation. Iterating the step relation $2m_{n+1} = 3m_n + t_n$ over a window telescopes the circuit sum into a closed form.

Lemma 2.1 (Closed form of the circuit sum)**.** *For all $a, k \in \mathbb{N}$,*

$$W(a,k) \;=\; 3^k\varepsilon_a - 2^k\varepsilon_{a+k}, \qquad \textit{equivalently} \qquad 2^k m_{a+k} = 3^k m_a + W(a,k).$$

Proof. Induction on k, using the recurrence $W(a,k+1) = 3\,W(a,k) + 2^k t_{a+k}$ and $t_n = 3\varepsilon_n - 2\varepsilon_{n+1}$; the main terms of the orbit cancel exactly. □

Definition 2.2 (Repetition)**.** A *repetition* of length k at positions $a < c$ is an equality of factors, $t_{a+i} = t_{c+i}$ for all $i < k$. (Occurrences may overlap; nothing anchors them to a prefix.)

Equal factors accumulate equal circuit sums, so subtracting two instances of the closed form gives the central identity.

Theorem 2.3 (The repetition identity)**.** *If there is a repetition of length k at $a < c$, then*

$$3^k(\varepsilon_c - \varepsilon_a) = 2^k(\varepsilon_{c+k} - \varepsilon_{a+k}), \qquad 3^k(m_c - m_a) = 2^k(m_{c+k} - m_{a+k}).$$

Proof. By Definition 2.2 the circuit sums agree, $W(a,k) = W(c,k)$. Substituting the two forms of Lemma 2.1 and cancelling yields both identities (the fractional form from the ε-closed form, the integer form from its integer companion). □

Corollary 2.4 (Divisibilities)**.** *Under the hypothesis of Theorem 2.3,*

$$2^k \mid m_c - m_a \qquad \textit{and} \qquad 3^k \mid m_{c+k} - m_{a+k}.$$

Proof. 2^k divides $3^k(m_c - m_a)$ by the integer identity, and $\gcd(2^k, 3^k) = 1$; symmetrically for 3^k. □

The divisibility upgrades to size once m is strictly increasing, which it is from index 2 on. This is the mechanism that forbids long repetitions late in the word.

Proposition 2.5 (Growth ceiling)**.** *If there is a repetition of length k at $2 \le a < c$, then*

$$2^{\,k+c+1} \le 3^{\,c+1},$$

that is, $k \le (c+1)\log_2(3/2) \approx 0.585\,(c+1)$.

Proof. Corollary 2.4 gives $2^k \mid m_c - m_a$, and $m_a < m_c$ (strict monotonicity of m for indices ≥ 2), so $2^k \leq m_c - m_a \leq m_c$. Combined with the a priori bound $2\,(2^c m_c) \leq 2 \cdot 3^c + 2^c$ — itself a restatement of $\varepsilon_c \geq -\frac{1}{2}$ — this yields $2^{k+c+1} \leq 3^{c+1}$ as an inequality of integers. □

Remark 2.6. The same computation bounds a $(j+1)$-fold repetition of a length-p block starting at $a \geq 2$ by $2^{jp+(a+p)+1} \leq 3^{a+p+1}$: long periodic windows cannot occur late in the word.

The identity has two further quantitative shadows, the interface to the Diophantine kernel of Section 3.

Lemma 2.7 (Contraction)**.** *A repetition of length k at $a < c$ forces $|\varepsilon_c - \varepsilon_a| \leq (2/3)^k$.*

Proof. Rewrite the fractional identity of Theorem 2.3 as

$$\varepsilon_c - \varepsilon_a = (2/3)^k(\varepsilon_{c+k} - \varepsilon_{a+k})$$

and bound the difference of two centered fractional parts by 1. □

Lemma 2.8 (Trivial repulsion floor)**.** *For $1 \leq a < c$,*

$$2^c \,\|(3/2)^c - (3/2)^a\| \geq 1.$$

Proof. The orbit difference is

$$3^a(3^{\,c-a} - 2^{\,c-a})/2^c,$$

a fraction with odd numerator over 2^c; hence its distance to the nearest integer is at least 2^{-c}. (Equivalently $2^c(\varepsilon_c - \varepsilon_a)$ is an odd integer.) □

Lemma 2.8 is the estimate the kernel (K) will improve: from the exponential base 2^{-1} to an arbitrary base $\theta < 1$. Two unconditional theorems already drop out of Stage 0.

Theorem 2.9 (Aperiodicity)**.** *The steering word T is not eventually periodic.*

Proof. If $t_{n+p} = t_n$ for all $n \geq N$ with $p \geq 1$, then at $a = \max(N, 2)$ and $c = a+p$ there is a repetition of *every* length k. Taking $k = 3^{\,c+1}$ contradicts the growth ceiling $2^{\,k+c+1} \leq 3^{\,c+1}$ of Proposition 2.5, since $3^{\,c+1} < 2^{\,3^{c+1}}$. (Aperiodicity of this family is also a special case of [DN05, Lemma 1].) □

Theorem 2.10 (Complexity floor)**.** *For all k,*

$$41\,k \;\leq\; 24\,p_T(k) + 72, \qquad \textit{i.e.} \qquad p_T(k) \;\geq\; \tfrac{41}{24}\,k - 3 \;\approx\; 1.708\,k.$$

Proof. By Proposition 2.5 and the certified estimate $3^{41} \leq 2^{65}$ (which encodes $\log_2 3 \leq 65/41$), a repetition of length k at $2 \leq a < c$ forces $41k \leq 24c+24$. Hence the windows starting at $2, 3, \ldots, C$ with $C = \lfloor (41k-25)/24 \rfloor$ are pairwise distinct as factors, giving at least $C - 1$ distinct length-k factors; comparing with the total count $p_T(k)$ and simplifying yields the stated inequality. □

Remark 2.11. The constant intrinsic to the method is $1/\log_2(3/2) \approx 1.7095$; the rational slope $41/24 \approx 1.7083$ formalized here is the certified rounding of it obtained from the integer certificate $3^{41} \le 2^{65}$. In either form the bound lies strictly above the Morse–Hedlund floor $k+1$ valid for every aperiodic word [MH38]. A lower bound of exactly this shape, with the constant $1/\log_2(3/2)$, is due to Dubickas [Dub09] for the digit words of $\lceil (p/q)^n \rceil$; the present statement is its transfer to the nearest-integer steering word of the orbit of 1.

The parity word. The primary target is the five-letter word T, not its binary reduction b. Since b is a letter-to-letter image of T we have $p_T(k) \ge p_b(k)$, so Theorem 2.10 says nothing about p_b. What survives is a dichotomy.

Proposition 2.12 (Parity dichotomy)**.** *A length-k repetition of the parity word b at positions a, c either lifts to a repetition of T (so Theorem 2.10 applies), or the two windows first disagree at some step $j < k$ at which the steering letters differ by a nonzero even amount.*

Proof. If the T-windows agree, we are in the first case. Otherwise let j be the least index of disagreement; there $b_{a+j} = b_{c+j}$ forces $t_{c+j} - t_{a+j}$ even, and it is nonzero by choice of j. □

Remark 2.13. While the two orbit windows remain T-synchronized their ε-difference expands by exactly $3/2$ per step,

$$\varepsilon_{c+j} - \varepsilon_{a+j} = (3/2)^j(\varepsilon_c - \varepsilon_a).$$

Consequently a *maximal* desynchronization $|t_{c+j} - t_{a+j}| = 4$ after j synchronized steps certifies $|\varepsilon_c - \varepsilon_a| \ge (2/3)^{j+1}$, whereas a minimal one $|t_{c+j} - t_{a+j}| = 2$ carries no such constraint. This is the precise sense in which superlinearity for b is harder; we claim no unconditional linear bound for p_b.

3. Reduction to a Diophantine kernel

Definition 3.1 (The kernel)**.** For a rational scale $\theta \in (0,1)$ let

$$V(\theta) \;:=\; \big\{(a,c) : 2 \le a < c,\ \|(3/2)^c - (3/2)^a\| \le \theta^{\,c}\big\}.$$

We say *pair repulsion* holds at θ if $V(\theta)$ is finite, and we call the assertion

$$\text{(K)} \qquad V(\theta) \text{ is finite for every rational } \theta \in (0,1)$$

the *Diophantine kernel*. The target property is that p_T be *superlinear*: for every C there is a K with $p_T(k) > C\,k$ for all $k \ge K$.

Restricting θ to the rationals loses nothing: $V(\theta)$ is monotone in θ and $\mathbb{Q}$ is dense, so the rational and real formulations of (K) are equivalent. This keeps the reduction inside $\mathbb{Q}$.

Theorem 3.2 (Reduction)**.** *The kernel* (K) *implies that p_T is superlinear.*

Proof. Fix C and suppose $p_T(k) \le C\,k$ for some large k. Among the $Ck+1$ windows starting at positions $2, \dots, Ck+2$, pigeonhole produces two equal factors, i.e. a repetition of length k at some $2 \le a < c \le (C+2)k$. Choose once and for all a rational scale $\theta < 1$ with $\theta^{C+2} \ge 2/3$ (a Bernoulli estimate provides one). By Lemma 2.7,

$$\|(3/2)^c - (3/2)^a\| \le |\varepsilon_c - \varepsilon_a| \le (2/3)^k \le \theta^{(C+2)k} \le \theta^c,$$

so $(a, c) \in V(\theta)$. As k grows, the growth ceiling (Proposition 2.5, in the form $41k \le 24c + 24$) forces $c \to \infty$, contradicting the finiteness of $V(\theta)$ granted by (K). Hence no such C bounds $p_T(k)/k$. □

The kernel is thus the sole remaining obstacle, and it is unconditional as a *reduction*: (K) enters Theorem 3.2 only as a hypothesis. The rest of the paper establishes it.

4. Two unconditional slices via Corvaja–Zannier

The kernel (K) is a two-parameter statement (in a and the gap $s = c - a$); its one-parameter slices are governed by the following theorem.

Both that theorem and its three-variable companion in Section 5 descend from the subspace theorem, and it is worth saying informally what that theorem does before the formulas begin. Its ancestor is Roth's theorem: an algebraic irrational admits no rational approximations appreciably better than the trivial ones. Schmidt's subspace theorem is the higher-dimensional form of the same phenomenon. If several linear forms take simultaneously much smaller values at an integer point than the size of that point warrants, then the point is not free to lie anywhere: all such points fall into finitely many proper subspaces, that is, they satisfy one of finitely many fixed linear relations with fixed integer coefficients. Unusually good approximation remains possible, but only along degenerate families. This bites here because the quantities we must control, $(3/2)^a$ and $(3/2)^c$, are S-units for $S = \{\infty, 2, 3\}$: huge numbers, of height growing exponentially in the exponent, and yet built from the two primes 2 and 3 alone and hence described by very few integer parameters. To say that a small integer combination of such numbers lies unexpectedly close to an integer is precisely to say that a linear form is unexpectedly small at a large integer point. The S-arithmetic version of the theorem, in which smallness is measured at the primes 2 and 3 as well as at the real place, is what makes this sparse multiplicative structure visible to the machinery, and its verdict is rigidity: an infinite family of such near-misses must satisfy one fixed linear relation. Such a relation in turn pins down the ratio of the S-units involved, or the integer they approximate, and elementary arithmetic — usually nothing more than the oddness of 3^c — then contradicts it.

Recall [CZ04, §2] that a real algebraic number α is *pseudo-Pisot* if $|\alpha| > 1$, all its conjugates have absolute value strictly below 1, and its trace $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)$ is a rational integer.

Theorem 4.1 (Corvaja–Zannier Main Theorem [CZ04])**.** *Let $\Gamma \subset \overline{\mathbb{Q}}^{\times}$ be a finitely generated multiplicative group of algebraic numbers, let $\delta \in \overline{\mathbb{Q}}^{\times}$ be fixed, and let $\epsilon > 0$. Then there are only finitely many pairs $(q, u) \in \mathbb{Z} \times \Gamma$, with $d = [\mathbb{Q}(u) : \mathbb{Q}]$, such that $|\delta q u| > 1$, $\delta q u$ is not pseudo-Pisot, and*

$$0 \;<\; \|\delta q u\| \;<\; \mathrm{H}(u)^{-\epsilon}\,|q|^{-d-\epsilon}.$$

This is the Main Theorem of [CZ04, p. 2], a consequence of the Schmidt subspace theorem (their Lemma 3, over $\mathbb{Q}$ a single two-variable application). It does not enter the formal development as an axiom: the specialization of Remark 4.2 is proved there from the Evertse–Schlickewei subspace theorem (see Remark 5.10).

Remark 4.2 (The specialization used)*.* We invoke Theorem 4.1 only for $\Gamma = \langle 2, 3\rangle$ (so $u = 2^x 3^y$, encoded by the exponent pair, and $d = 1$), with positive integer multipliers $q \geq 1$. Over $\mathbb{Q}$ the pseudo-Pisot exclusion, given $|\delta q u| > 1$, is exactly "$\delta q u \notin \mathbb{Z}$", which is automatic once $\|\delta q u\| > 0$. Each of these is a restriction of the source statement, hence harmless.

Theorem 4.3 (Bounded-gap slice)**.** *For every fixed gap $s_0 \geq 1$ and every rational $\theta \in (0, 1)$, only finitely many $a \geq 2$ satisfy*

$$\|(3/2)^{a+s_0} - (3/2)^a\| \leq \theta^{\,a+s_0}.$$

Proof. Apply Theorem 4.1 with the data

$$\delta = (3/2)^{s_0} - 1, \qquad q = 1, \qquad u = (3/2)^a \in \Gamma,$$

so that

$$\delta q u = (3/2)^{a+s_0} - (3/2)^a, \qquad d = [\mathbb{Q}(u) : \mathbb{Q}] = 1, \qquad \mathrm{H}(u) = 3^a.$$

Here δ is a fixed nonzero rational; since $s_0 \geq 1$ it satisfies $\delta \geq \frac{1}{2}$.

The side hypotheses. For $a \geq 2$ we have $(3/2)^a \geq \frac{9}{4}$, hence

$$|\delta q u| = \big((3/2)^{s_0} - 1\big)(3/2)^a \;\geq\; \tfrac{1}{2} \cdot \tfrac{9}{4} \;=\; \tfrac{9}{8} \;>\; 1.$$

Lemma 2.8 gives $\|\delta q u\| \geq 2^{-(a+s_0)} > 0$, and over $\mathbb{Q}$ that positivity is exactly the statement that $\delta q u$ is not an integer, which by Remark 4.2 discharges the pseudo-Pisot exclusion.

The threshold. Put $L := \log \theta^{-1} > 0$ and choose

$$\epsilon \;:=\; \frac{L}{2 \log 3} \;>\; 0.$$

Because $0 < \theta < 1$ and $s_0 \geq 1$ we have $\theta^{\,a+s_0} \leq \theta^a$, so it is enough to compare θ^a with $(3^a)^{-\epsilon}$. Both are positive, and taking logarithms,

$$\log\big((3^a)^{-\epsilon}\big) \;=\; -\epsilon\, a \log 3 \;=\; -\frac{aL}{2} \;>\; -aL \;=\; a \log \theta \;=\; \log\big(\theta^a\big),$$

the strict inequality holding because $aL > 0$ for $a \geq 1$. Therefore

$$\|\delta q u\| \;\leq\; \theta^{\,a+s_0} \;\leq\; \theta^a \;<\; (3^a)^{-\epsilon} \;=\; \mathrm{H}(u)^{-\epsilon}|q|^{-d-\epsilon},$$

the final equality because $q = 1$ and $d = 1$.

Transfer of finiteness. Theorem 4.1 thus admits only finitely many pairs $(q,u) = \big(1,(3/2)^a\big)$, and $a \mapsto (3/2)^a$ is injective, so only finitely many a occur. □

Corollary 4.4 (Gap-bounded slice)**.** *For every S and every rational $\theta \in (0,1)$, the set of $(a,c) \in V(\theta)$ with $c \le a + S$ is finite.*

Proof. It is the union over $1 \le s_0 \le S$ of the finite slices of Theorem 4.3, under $(a,c) \mapsto (a, a+s_0)$. □

The second slice uses the theorem's uniformity in the *integer* multiplier slot q, and is the observation that makes the huge-gap band fall.

Theorem 4.5 (Huge-gap band)**.** *For every rational $\theta \in (0,1)$ there is a rational $\varepsilon' > 0$ such that only finitely many pairs $(a,c) \in V(\theta)$ lie in the band $a \le \varepsilon' c$.*

Proof. Abbreviate

$$L := \log \theta^{-1} > 0, \qquad D := \log 2 + 2\log 3 > 0,$$

and fix once and for all the two parameters

$$\epsilon \;:=\; \min\Big\{1,\; \frac{L}{4\log 3}\Big\} \;>\; 0, \qquad \varepsilon' \in \mathbb{Q} \;\text{ with }\; 0 < \varepsilon' < \frac{3L}{4D};$$

such a rational ε' exists because the right-hand bound is a positive real. We show that this ε' has the asserted property. Let $(a,c) \in V(\theta)$ with $a \le \varepsilon' c$, and write $s := c - a \ge 1$; recall $a \ge 2$ from Definition 3.1.

Step 1: moving the 2-part into the integer slot. Multiplying the orbit difference by 2^a clears its denominator down to 2^s,

$$2^a\big((3/2)^c - (3/2)^a\big) \;=\; 3^a(3/2)^s - 3^a,$$

so the two sides differ by the integer 3^a and have the same distance to $\mathbb{Z}$. With $\|Nx\| \le N\|x\|$ for a positive integer N this gives

$$\|3^a(3/2)^s\| \;=\; \|2^a\big((3/2)^c - (3/2)^a\big)\| \;\le\; 2^a\,\|(3/2)^c - (3/2)^a\| \;\le\; 2^a\theta^c,$$

the last step by $(a,c) \in V(\theta)$.

Step 2: the Corvaja–Zannier data. Apply Theorem 4.1 with

$$\delta = 1, \qquad q = 3^a \in \mathbb{Z}_{>0}, \qquad u = (3/2)^s \in \Gamma,$$

so that $d = 1$ and $\mathrm{H}(u) = 3^s$. Since $a \ge 2$ and $s \ge 1$,

$$|\delta q u| = 3^a(3/2)^s \;\ge\; 9 \cdot \tfrac{3}{2} \;>\; 1.$$

Moreover $3^a(3/2)^s = 3^{a+s}/2^s$ has odd numerator and $s \ge 1$, so it is not an integer and in fact $\|qu\| \ge 2^{-s} > 0$; over $\mathbb{Q}$ this simultaneously discharges the pseudo-Pisot exclusion (Remark 4.2).

Step 3: the threshold, in full. The inequality demanded by Theorem 4.1 is

$$\|qu\| \;<\; \mathrm{H}(u)^{-\epsilon}|q|^{-d-\epsilon} \;=\; (3^s)^{-\epsilon}(3^a)^{-1-\epsilon},$$

so by Step 1 it suffices to prove $2^a\theta^c < (3^s)^{-\epsilon}(3^a)^{-1-\epsilon}$. Both sides are positive; taking logarithms and using $\log\theta = -L$, this is

$$a\log 2 - cL \;<\; -\epsilon\, s\log 3 - (1+\epsilon)\, a\log 3,$$

that is, after moving all terms to the left,

$$(*) \qquad a\log 2 + (1+\epsilon)\, a\log 3 + \epsilon\, s\log 3 \;<\; c\,L.$$

We bound the left-hand side of $(*)$ in two pieces. For the first two terms, $\epsilon \le 1$ gives $(1+\epsilon)\log 3 \le 2\log 3$, and then $a \le \varepsilon' c$ together with $\varepsilon' D < \frac{3}{4}L$ gives

$$a\log 2 + (1+\epsilon)\, a\log 3 \;\le\; a\,(\log 2 + 2\log 3) \;=\; a\,D \;\le\; \varepsilon'\, c\,D \;<\; \tfrac{3}{4}\, c\,L.$$

For the third term, $\epsilon \le L/(4\log 3)$ and $s \le c$ give

$$\epsilon\, s\log 3 \;\le\; \frac{L}{4\log 3}\, s\log 3 \;=\; \frac{s\,L}{4} \;\le\; \frac{c\,L}{4}.$$

Adding the two estimates,

$$a\log 2 + (1+\epsilon)\, a\log 3 + \epsilon\, s\log 3 \;<\; \tfrac{3}{4}\, c\,L + \tfrac{1}{4}\, c\,L \;=\; c\,L,$$

which is $(*)$. Note where each parameter was used: the truncation $\epsilon \le 1$ converts the unknown exponent $1+\epsilon$ into the fixed constant D; the second branch $\epsilon \le L/(4\log 3)$ pays for the $\mathrm{H}(u)^{-\epsilon}$ factor out of one quarter of the budget cL; and ε' pays for the $|q|^{-1-\epsilon}$ factor out of the remaining three quarters.

Step 4: transfer of finiteness. Theorem 4.1 therefore admits only finitely many pairs $(q,u) = \big(3^a, (3/2)^s\big)$ arising this way. The maps $a \mapsto 3^a$ and $s \mapsto (3/2)^s$ are injective, so only finitely many (a,s) occur, hence only finitely many $(a,c) = (a, a+s)$ in the band $a \le \varepsilon' c$. □

Remark 4.6. The same input yields, as a corollary of Theorem 4.3, that fixed-gap repetitions are short in a strong sense: for each $s_0 \ge 1$ and each certified rational slope $\mu > 0$, only finitely many pairs (a,k) satisfy $k \ge \mu\, a$ together with a repetition of length k at $(a, a+s_0)$.

After these two slices, what remains open of (K) is exactly the middle band $\varepsilon' c \le a$ with gap $s = c - a \to \infty$.

5. The gap dichotomy and superlinear complexity

Definition 5.1 (Middle band)**.** For a rational scale θ, a parameter $\varepsilon' > 0$ and a threshold S, the *middle band* is

$$M(\theta, \varepsilon', S) \;:=\; \big\{(a,c) \in V(\theta) :\; \varepsilon'\, c \le a \text{ and } a + S \le c\big\}.$$

Corollary 4.4 and Theorem 4.5 reduce (K) to the finiteness of the middle band, and this reduction is already enough for a conditional form of the main theorem — the statement of record that depends only on the refereed theorem of Corvaja–Zannier.

Theorem 5.2 (Conditional main theorem)**.** *Suppose that for every rational $\theta \in (0,1)$ and every rational $\varepsilon' > 0$ there is a threshold S making the middle band $M(\theta, \varepsilon', S)$ finite. Then p_T is superlinear.*

Proof. Fix θ. By Theorem 4.5 choose $\varepsilon' > 0$ with the huge-gap band $a \le \varepsilon' c$ finite, and by hypothesis choose S with $M(\theta, \varepsilon', S)$ finite. Every violator $(a, c) \in V(\theta)$ falls into one of three finite sets — $a \le \varepsilon' c$, or $c \le a + S$ (Corollary 4.4), or the middle band — so $V(\theta)$ is finite. Thus (K) holds, and Theorem 3.2 applies. □

The middle band is closed by a dichotomy on the gap. Its infinite branch uses the following theorem, a repaired form of a recent result of Nair–Kumar–Rout (see Remarks 5.4 and 5.5).

The theorem is the three-variable counterpart of Theorem 4.1, and the informal picture drawn at the start of Section 4 applies verbatim. There a single S-unit, scaled by an integer, was required to lie close to an integer; here two independent S-units u_1 and u_2 are combined, and the integer they nearly hit is carried along as a third coordinate. The subspace theorem is applied to the resulting point and returns the same verdict of rigidity: an infinite supply of pairs whose combination hugs an integer far more closely than their size permits cannot really be a two-parameter family, and infinitely many of its members must obey one fixed linear relation among the two S-units and the nearby integer. Every way this can happen is then arithmetically impossible. A relation not involving the integer would freeze the ratio u_1/u_2, which the hypothesis of pairwise distinct ratios forbids; a relation involving it turns the distance to $\mathbb{Z}$ into an explicit rational combination of u_1 and u_2, which either has a positive floor — bounding the heights, hence the family — or drives u_1/u_2 towards a fixed rational limit, which Theorem 4.1 forbids. One hypothesis is indispensable to all of this: the combination must not already be an integer. The subspace theorem quantifies how small a nonzero quantity is, so when that quantity vanishes it has nothing to say. This hypothesis is absent from the statement as printed in [NKR25], and the counterexample of Remark 5.4 exploits its absence.

Theorem 5.3 (S-unit pairs near integers; repair of [NKR25, Thm. 1.3(i)])**.** *Let $\alpha_1, \alpha_2 \in \mathbb{Q}^\times$ and $\epsilon_1 > 0$. Let $\mathcal{N}$ be an infinite family of pairs $(u_1, u_2) \in \Gamma^2$ with $|u_i| \ge 1$ and $u_1 \ne -u_2$, whose ratios are pairwise distinct across the family (in both orders: distinct members have $u_1/u_2 \ne u_1'/u_2'$ and $u_2/u_1 \ne u_2'/u_1'$), and suppose every member satisfies*

$$0 \;<\; \big\|\alpha_1 u_1 + \alpha_2 u_2\big\| \;<\; \big(\mathrm{H}(u_1)\,\mathrm{H}(u_2)\big)^{-\epsilon_1}.$$

Then some pair in $\mathcal{N}$ has both entries in $\mathbb{Z}$.

Proof sketch. One shows that the hypotheses are in fact contradictory — such a family is necessarily finite (compare Proposition 3.1 and Remark 3.2 of [NKR25]) — and the stated form follows. To each pair attach the point $x = (p_0, u_1, u_2)$ with p_0 the integer nearest to $\alpha_1 u_1 + \alpha_2 u_2$, and apply the

Evertse–Schlickewei subspace theorem in three variables with $S = \{\infty, 2, 3\}$, the forms $\alpha_1 X_1 + \alpha_2 X_2 - X_0$, X_1, X_2 at the infinite place and the coordinate forms at 2 and 3. Evaluated on the coprime integer representative of x, the double product collapses by exact p-adic cancellation to a multiple of $\|\alpha_1 u_1 + \alpha_2 u_2\|$, and the height of the representative lies between $\mathrm{H}(u_1/u_2)$ and $\big(\mathrm{H}(u_1)\mathrm{H}(u_2)\big)^2$; hence every member with $\mathrm{H}(u_1/u_2)$ above a fixed threshold satisfies the subspace inequality with exponent $-3 - \epsilon_1/4$. By pigeonhole, infinitely many members then lie in a single proper rational subspace, i.e. satisfy one fixed relation $a_0 p_0 + a_1 u_1 + a_2 u_2 = 0$. If $a_0 = 0$, the ratio u_1/u_2 is constant along that subfamily, contradicting distinctness. Otherwise the relation determines p_0, so that $\|\alpha_1 u_1 + \alpha_2 u_2\| = |\beta_1 u_1 + \beta_2 u_2|$ for fixed rationals β_i, and every sign pattern of (β_1, β_2) is impossible: $\beta_1 = \beta_2 = 0$ contradicts strict positivity; equal signs, or a single nonzero β_i, give the distance a positive floor, which bounds the heights; and opposite signs force u_1/u_2 to accumulate at $-\beta_2/\beta_1$, which the theorem of Corvaja–Zannier (Theorem 4.1) forbids along a ratio-injective family. □

Remark 5.4 (Relation to [NKR25]: a counterexample to the printed statement). Theorem 1.3(i) of [NKR25] asserts the conclusion above for general number fields, m-tuples and finitely generated Γ, but *without* the strict-positivity hypothesis $\|\sum_i \alpha_i u_i\| > 0$ (the positivity does appear in their Theorem 1.1(iv)). In that form the statement is false: for $(\alpha_1, \alpha_2) = (1, 1)$ and the family $(u_1, u_2) = \big(3^m/2,\, 3^{2m}/2\big)$, $m \geq 1$, the sum $(3^m + 3^{2m})/2$ is an integer by parity, so its distance to $\mathbb{Z}$ is 0 and every hypothesis of the printed statement holds (the ratios 3^{-m} are pairwise distinct), yet no entry $3^m/2$, $3^{2m}/2$ is ever an integer. The refutation of the printed statement is machine-checked in the formal development. The gap in the proof of [NKR25, §4.1] is that the exponent produced there depends on the tuple, while their Lemma 2.2 requires a fixed one. Theorem 5.3 as stated — with the positivity hypothesis, over $\mathbb{Q}$, for pairs in $\Gamma = \langle 2, 3\rangle$ — is the only form used here, and it is proved in the formalization as sketched above. In this setting property (P1) of [NKR25] is vacuous, property (P2) is the hypothesis $u_1 \neq -u_2$, and "algebraic integer" means "integer".

Remark 5.5 (The authors' correction). On being shown the counterexample the authors of [NKR25] confirmed the defect, and repair it in a different way (private communication; the preprint is unrevised at the time of writing). The hypotheses of their Theorem 1.3 are left untouched; conclusion (i) is weakened from integrality to a uniform bound on denominators, to the effect that some constant $C > 1$ satisfies $\max\{|u_1|_v, \ldots, |u_m|_v\} < C$ at every non-archimedean place v, along the infinite family $\mathcal{N}_1'$ of that theorem. Since "u is an algebraic integer" says exactly that $|u|_v \leq 1$ at every non-archimedean v, the printed conclusion is the limiting case $C = 1$ with non-strict inequality, and the correction is a genuine weakening — one that the counterexample above respects: there $|u_i|_2 = 2$ and $|u_i|_p \leq 1$ for every other prime, so any $C > 2$ serves.

The two repairs are logically independent: ours restricts the hypotheses and keeps the strong conclusion, theirs keeps the hypotheses and weakens the conclusion. Either closes the application made here. Indeed the pair of Lemma 5.6 has

$$\big|u_1\big|_2 = \big|(3/2)^c\big|_2 = 2^c,$$

which is unbounded along an infinite family of violators, so the corrected conclusion is contradicted just as directly as the printed one — and without appeal to strict positivity, which the corrected form no longer needs. (It is immaterial for this use whether the bound is read as holding along all of $\mathcal{N}_1'$ or only along the infinite subset the theorem produces.) We formalize our own repair rather than theirs because the integrality conclusion is what the proof from the subspace theorem sketched above delivers; on the status of either repair in the announced generality see the discussion following Problem 6.3.

Lemma 5.6 (The Nair–Kumar–Rout branch)**.** *Fix a rational $\theta \in (0,1)$. Let $\Gamma_0 \subseteq V(\theta)$ be a family of violators on which the gap map $(a,c) \mapsto c-a$ is injective. Then Γ_0 is finite.*

Proof. Suppose Γ_0 is infinite. To each $(a,c) \in \Gamma_0$ associate the pair

$$(u_1,u_2) := \big((3/2)^c, (3/2)^a\big) \in \Gamma^2, \qquad (\alpha_1,\alpha_2) := (1,-1),$$

so that $\alpha_1 u_1 + \alpha_2 u_2 = (3/2)^c - (3/2)^a$.

The tuple hypotheses. Both $|u_i| \geq 1$ and $u_1 \neq -u_2$ are immediate from $u_1, u_2 > 0$. The ratios are pairwise distinct across the family: the gap map is injective on Γ_0 and

$$\frac{u_1}{u_2} = (3/2)^{\,c-a}$$

is strictly increasing in the gap $c-a$, so distinct members of Γ_0 give distinct ratios — in both orders, since $x \mapsto x^{-1}$ is injective.

The threshold. As $\mathrm{H}\big((3/2)^n\big) = 3^n$, the height product is

$$\mathrm{H}(u_1)\,\mathrm{H}(u_2) = 3^c \cdot 3^a = 3^{\,c+a}.$$

Put $L := \log \theta^{-1} > 0$ and choose

$$\epsilon_1 \;:=\; \frac{L}{2\log 3} \;>\; 0.$$

Since $a < c$ we have $c + a < 2c$, whence

$$\log\Big(\big(3^{\,c+a}\big)^{-\epsilon_1}\Big) \;=\; -\epsilon_1(c+a)\log 3 \;=\; -\frac{(c+a)L}{2} \;>\; -cL \;=\; \log\big(\theta^c\big),$$

and therefore, using $(a,c) \in V(\theta)$,

$$\|\alpha_1 u_1 + \alpha_2 u_2\| = \|(3/2)^c - (3/2)^a\| \;\leq\; \theta^c \;<\; \big(\mathrm{H}(u_1)\mathrm{H}(u_2)\big)^{-\epsilon_1},$$

which is the required upper bound.

Strict positivity. The distance is also nonzero. Were $(3/2)^c - (3/2)^a = n$ for some $n \in \mathbb{Z}$, multiplication by 2^c would give

$$3^c - 2^{\,c-a}3^a = 2^c n,$$

whose left-hand side is odd (because $c - a \geq 1$) and whose right-hand side is even (because $c \geq 1$) — a contradiction.

Conclusion. Every hypothesis of Theorem 5.3 is met by the infinite family, so some member has both entries in $\mathbb{Z}$; in particular $(3/2)^c \in \mathbb{Z}$ for some $c \geq 2$. This is impossible, since $2^c(3/2)^c = 3^c$ is odd. Hence Γ_0 is finite. □

Theorem 5.7 (The gap dichotomy: (K) holds)**.** *For every rational $\theta \in (0,1)$ the set $V(\theta)$ is finite; that is, the kernel* (K) *holds.*

Proof. Suppose $V(\theta)$ were infinite and consider the image of $V(\theta)$ under the gap map $(a,c) \mapsto c - a$. If that image is finite, then all violators have gap $\leq S$ for some S, and $V(\theta)$ is a gap-bounded slice, finite by Corollary 4.4 — contradiction. If the image is infinite, choose one violator per attained gap; this is an infinite subfamily of $V(\theta)$ on which the gap map is injective, finite by Lemma 5.6 — contradiction. Hence $V(\theta)$ is finite. □

Theorem 5.8 (Main theorem)**.** *The subword complexity of the steering word is superlinear, $p_T(k)/k \to \infty$.*

Proof. Immediate from the gap dichotomy (Theorem 5.7), which supplies the kernel (K), and the Stage-1 reduction (Theorem 3.2). □

Corollary 5.9. *For every rational $\theta \in (0,1)$, every $\varepsilon' > 0$ and every S, the middle band $M(\theta, \varepsilon', S)$ is finite.*

Proof. It is a subset of $V(\theta)$, finite by Theorem 5.7. In particular the hypothesis of Theorem 5.2 is discharged. □

Remark 5.10 (Dependencies). The results of Sections 2 and 3 are unconditional. Every remaining result — the two slices, the gap dichotomy, and the conditional and unconditional main theorems — depends, beyond the ambient logical framework, on exactly one external input: the Evertse–Schlickewei S-arithmetic subspace theorem ([Sch91]; see [BG06, Ch. 7] for the form used), recorded once in the formal development. The theorems of Corvaja–Zannier (Theorem 4.1) and of Nair–Kumar–Rout in repaired form (Theorem 5.3) are derived from it inside the formalization, in two and three variables respectively, and contribute no independent assumptions. Every dependency is ineffective; no bound on the exceptional sets is obtained.

6. Implications and further directions

The results above sit at the junction of three programmes and leave a different residue in each.

6.1. **Diophantine approximation.** Detached from its dynamical origin, the kernel of Theorem 5.7 reads

$$\#\big\{(a,c) : 2 \le a < c,\ \|(3/2)^c - (3/2)^a\| \le \theta^{\,c}\big\} \ < \ \infty \qquad \text{for every } \theta < 1.$$

Mahler's theorem, in the form used by Zudilin [Zud07], says that $\|(3/2)^k\| \le \theta^{\,k}$ has finitely many solutions for every $\theta < 1$; the kernel is exactly its two-parameter companion. The orbit of 1 under $x \mapsto \frac{3}{2}x$ does not merely repel the integers: its points repel *each other* modulo one at every exponential rate. Only the base $\theta = \frac{1}{2}$ of that statement is elementary (Lemma 2.8); everything above it is Subspace.

Effectivity. Every result from Section 4 onwards is ineffective, so no explicit K can be extracted in Theorem 1.1. For the one-parameter problem the effective theory is in better shape, though it barely clears the trivial base: after Baker and Coates, and Beukers, Zudilin [Zud07] proved $\|(3/2)^k\| > 0.5803^{\,k}$ for all k beyond an effectively computable bound, against the trivial 2^{-k}. Nothing comparable is known for pairs. The reduction of Theorem 3.2 converts any effective slice of the kernel directly into an effective complexity bound, and the exchange rate is explicit.

Problem 6.1. Exhibit a $\theta \in (0,1)$ and an effectively computable $A(\theta)$ such that every pair $(a,c) \in V(\theta)$ satisfies $c \le A(\theta)$.

Such a θ yields, by the proof of Theorem 3.2, the effective bound $p_T(k) > C\,k$ for all explicitly large k, with $C = \log(2/3)/\log\theta - 2$. The threshold to beat is therefore $\theta = (2/3)^{24/89} \approx 0.896$: below it the certified slope $41/24$ of Theorem 2.10 is already better, and at θ near Zudilin's 0.5803 the exchange rate returns nothing at all. The whole difficulty of Problem 6.1 lies in that gap.

From qualitative to quantitative. The unconditional bound $p(n)/n \to \infty$ of [AB07] was later sharpened by Bugeaud and Evertse [BE08], who replaced the Subspace Theorem by its quantitative form and obtained $p(n,\xi,b) \ge n(\log n)^{0.09}$ for infinitely many n. The corresponding upgrade here is not automatic, and the obstruction can be named. A quantitative Subspace Theorem bounds the number of *subspaces* carrying the solutions, not the number of solutions in each; in the proof of Theorem 5.3 every subspace is disposed of by an appeal to Theorem 4.1, which is itself qualitative.

Problem 6.2. Quantify Theorem 4.1 sufficiently to count, subspace by subspace, the pairs surviving in Theorem 5.3, and deduce a bound of the shape $p_T(k) \ge k(\log k)^{\delta}$ for infinitely many k.

Generality. Only two features of 3/2 were used: that $\Gamma = \langle 2,3\rangle$ is finitely generated, and that the numerator of $(3/2)^n$ is odd — the source of every positivity step (Lemma 2.8, Step 2 of Theorem 4.5, Lemma 5.6). Both survive verbatim for coprime $p > q \ge 2$ with $\Gamma = \langle q,p\rangle$, and the δ-slot of Theorem 4.1 accommodates a fixed rational multiplier.

Problem 6.3. Prove superlinearity of the steering word of $\xi\,(p/q)^n$ for every $\xi \in \mathbb{Q}^\times$ and all coprime $p > q \geq 2$. The growth ceiling becomes $k \leq (c+1)\log_q(p/q)$, so the analogue of Theorem 2.10 carries the constant $\log q/\log(p/q)$.

Finally, Theorem 5.3 is proved here only in the shape it is used: over $\mathbb{Q}$, for pairs, with $\Gamma = \langle 2, 3\rangle$. Whether the repaired statement — with the strict-positivity hypothesis restored — holds in the generality announced in [NKR25], over number fields and for m-tuples, is open; the endgame of the proof sketch is specific to pairs. The authors' correction (Remark 5.5) is asserted in that full generality, but with the weaker, denominator-bounding conclusion; we have neither verified nor formalized it. Which of the two repairs the argument of [NKR25, §4] actually supports, and whether the stronger one survives at all beyond pairs over $\mathbb{Q}$, is the natural question left by the episode. An m-tuple version would supply higher-order repulsion, but exploiting it would also require a combinatorial step the present argument lacks: the pigeonhole of Theorem 3.2 extracts coincidences two at a time.

6.2. **Symbolic dynamics.** The sharpest consequence for this field is the dichotomy of Section 1.1: if $\big((3/2)^n\big)$ is dense modulo one then T is not morphic, so a morphism generating T would refute the density conjecture. Theorem 1.1 is what makes this non-vacuous, having removed the classes $\Theta(1)$ and $\Theta(k)$ from Pansiot's list; the remaining candidates are $\Theta(k\log\log k)$, $\Theta(k\log k)$ and $\Theta(k^2)$. A systematic search for a morphism generating the observed prefix of T is therefore a well-posed experiment with an asymmetric payoff: a hit would settle a hard number-theoretic conjecture in the negative, while failure accumulates evidence for density.

The asymptotics are best phrased through the topological entropy of the subshift generated by T,

$$h(T) \;=\; \lim_{k\to\infty} \frac{\log p_T(k)}{k},$$

which exists by subadditivity. Our results say nothing about it: superlinearity is compatible with $h(T) = 0$. What is known is the bracket $h(T) \leq \log 3$ from Kopra's trace subshift [Kop21], while Corollary 1.3 gives $h(T) \geq \log(3/2)$ under density. Contrapositively, *a proof that the steering subshift has zero entropy would refute the density of* $\big((3/2)^n\big)$ *modulo one* — a purely symbolic-dynamical route into the problem, and one on which the trivial ceiling $p_T(k) \leq 5^k$ is silent.

Problem 6.4. Decide whether $h(T) > 0$. More modestly, close the gap between the base 3 of Kopra's ceiling and the base $3/2$ that density would demand.

Two structural obstacles are worth recording for anyone who takes this up. First, Proposition 1.2 is one-way: a length-k factor is governed not by ε_a alone but by the pair $(\varepsilon_a, m_a \bmod 2^k)$, so the natural phase space is

$[-\frac{1}{2},\frac{1}{2})\times\mathbb{Z}_2$ rather than the interval, and it is there that a converse — factors corresponding to separated points — would have to be sought. Establishing one would make the transfer between complexity and equidistribution two-way, which it currently is not. Second, the parity word b resists the method for a reason isolated in the remark after Proposition 2.12: a minimal desynchronization $|t_{c+j}-t_{a+j}|=2$ carries no constraint on $|\varepsilon_c-\varepsilon_a|$, so Lemma 2.7 has no b-analogue. Any unconditional superlinear bound for p_b needs a substitute — plausibly a statistical bound on how often minimal desynchronization can occur along a window.

6.3. **Formal verification of number theory.** That the formalization refuted a printed theorem is the outcome most worth generalizing. The defect in [NKR25, Thm. 1.3(i)] is a uniformity slip: the exponent produced in the proof depends on the tuple, whereas the lemma it feeds requires a fixed one, so the printed statement claims more than its argument delivers. There is accordingly more than one way to repair it — restore the missing hypothesis, as in Theorem 5.3, or weaken the conclusion, as the authors do in the correction they supplied on being shown the counterexample (Remark 5.5) — and the episode is worth recording in full for that reason: refutation identifies the false statement, not the intended one, and only the authors can say which repair their proof was reaching for. Errors of this shape are what human refereeing handles least well and what formalization catches automatically, because the proof assistant will not let a quantifier order remain implicit. Deep Diophantine machinery is almost always applied through interfaces bristling with such side conditions — fixed versus varying exponents, height normalizations, strict versus non-strict positivity — which suggests that formalizing the *statements* of the major theorems, with their hypotheses, may repay the effort faster than formalizing their proofs.

The same episode carries a warning about the practice of recording literature results as axioms. An earlier state of this development axiomatized [NKR25, Thm. 1.3(i)] verbatim; since that statement is false, the development was inconsistent while the axiom stood, and every theorem depending on it was vacuous. Faithful transcription is no protection — it is precisely what propagates the error. The discipline we ended with is the one we would recommend: reduce to a small number of canonical inputs and *derive* the rest, so that the trust boundary is a single statement (here `Subspace.evertseSchlickewei`) that `#print axioms` exhibits mechanically; and attempt to refute anything that must nonetheless be axiomatized, since a successful refutation — as here — is far more informative than a failed one.

What this required, and could not obtain, was library support. Mathlib currently offers Liouville's theorem, Dirichlet's approximation theorem, continued fractions and, recently, a theory of heights with the Northcott property; it contains neither Roth's theorem nor any form of the Subspace Theorem. The chain Roth $\to$ Schmidt $\to$ Evertse–Schlickewei is thus the

missing infrastructure, and the present work is one measure of the demand for it: two theorems from the literature and one research paper all rest on a single unformalized input. Problem 6.2 would additionally require the quantitative versions.

Problem 6.5. Formalize Roth's theorem, and beyond it the Schmidt Subspace Theorem and its S-arithmetic form, so that developments of this kind can discharge their last axiom.

One limitation deserves emphasis, since it is easily mistaken for a defect of the formalization rather than of the mathematics. The formal statements are finiteness assertions (`Set.Finite`), from which no witness can be extracted; the development therefore neither adds effectivity nor conceals its absence, and it faithfully reproduces the ineffectivity of its input. In particular Problem 6.1 lies outside the reach of the present formalization not because of how it was written, but because the Subspace Theorem does not answer it.

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The author utilized Claude Code as an AI coding assistant to aid in the Lean 4 formalization of the proofs presented in this paper. The author directed and reviewed all generated code and takes full responsibility for the mathematical integrity and final content of the work.

Appendix A. Formal provenance

Each numbered environment corresponds to a Lean 4 declaration in the accompanying repository (see `https://github.com/rwst/Superlinear-Complexity`). The table lists the formal name and the dependency beyond the ambient framework (`std3` denotes the standard logical axioms `propext`, `Classical.choice`, `Quot.sound`). "S" denotes the single cited axiom of the development, `Subspace.evertseSchlickewei`: the Evertse–Schlickewei S-arithmetic subspace theorem [Sch91, BG06]. Theorems 4.1 and 5.3 are *derived* inside the development, in the $\mathbb{Q}$-specialized forms used here: `CZ.pseudoPisot_approx_of_subspace` from S in two variables, and `NKR.sUnit_pair_integrality_of_subspace` (via the finiteness statement `NKR.pair_finite`) from S in three variables. The machine-checked refutation of Theorem 1.3(i) as printed in [NKR25] (Remark 5.4) is `NKR.thm13i_unrepaired_false`, pure `std3`; the authors' correction (Remark 5.5) is not formalized.

Env.	Lean name	Status
Prop. 1.2	`TH.card_le_complexity_of_separated`	proved (std3)
—	`TH.OrbitDense`	definition
Cor. 1.3	`TH.complexity_ge_ceil_of_orbitDense`	proved (std3)
Def. 1.4	`TH.m`, `TH.eps`, `TH.R`, `TH.t`, `TH.b`	definition
Def. 1.5	`TH.W`	definition
Lem. 2.1	`TH.W_closed`, `TH.circuit_sum`	proved (std3)
Def. 2.2	`TH.IsRepetition`	definition
Thm. 2.3	`TH.lemmaR_eps`, `TH.lemmaR_int`	proved (std3)
Cor. 2.4	`TH.two_pow_dvd_of_repetition`	proved (std3)
	`TH.three_pow_dvd_of_repetition`	proved (std3)
Prop. 2.5	`TH.repetition_pow_le`	proved (std3)
Lem. 2.7	`TH.abs_eps_sub_le_of_repetition`	proved (std3)
Lem. 2.8	`TH.one_le_two_pow_mul_distToNearestInt_orbit`	proved (std3)
Thm. 2.9	`TH.not_eventually_periodic`	proved (std3)
Thm. 2.10	`TH.factor_complexity_lower`	proved (std3)
Prop. 2.12	`TH.b_repetition_dichotomy`	proved (std3)
Def. 3.1	`TH.kernelViolators`, `TH.Kernel`, `TH.Superlinear`	definition
Thm. 3.2	`TH.superlinear_of_kernel`	proved (std3)
—	`Subspace.evertseSchlickewei`	cited axiom (S)
Thm. 4.1	`CZ.pseudoPisot_approx_of_subspace`	proved, std3 + S
Rem. 4.2	(`CZ.height23`, `CZ.sval`)	definition
Thm. 4.3	`TH.boundedGap_slice_finite`	proved, std3 + S
Cor. 4.4	`TH.gapBounded_slice_finite`	proved, std3 + S
Thm. 4.5	`TH.hugeGap_slice_finite`	proved, std3 + S
Def. 5.1	`TH.middleBandViolators`	definition
Thm. 5.2	`TH.superlinear_of_middleBand`	proved, std3 + S
Thm. 5.3	`NKR.sUnit_pair_integrality_of_subspace`	proved, std3 + S
Lem. 5.6	`TH.finite_of_gap_injOn`	proved, std3 + S
Thm. 5.7	`TH.pairRepulsion_all`, `TH.kernel_holds`	proved, std3 + S
Thm. 5.8	`TH.complexity_superlinear`	proved, std3 + S
Cor. 5.9	`TH.middleBandViolators_finite`	proved, std3 + S

The remark after Theorem 4.5 corresponds to `TH.boundedGap_repetition_short` (proved, std3 + S); the remarks around Theorem 2.10 and Proposition 2.12 to `TH.power_repetition_bound`, `TH.eps_sub_of_sync` and `TH.desync_eps_lower` (all proved, std3). The material of Section 1.1 lives in `TH/OrbitPacking.lean`: besides the two entries above it contains the intermediate form `TH.complexity_ge_of_orbitDense` ($M \le p_T(k)$ for every integer $M < (3/2)^k$) and the trivial alphabet ceiling `TH.complexity_le_five_pow` ($p_T(k) \le 5^k$), both proved and pure `std3`. The derivations of Theorems 4.1 and 5.3 from S live in `CITED/CorvajaZannierProof.lean`, `CITED/NairKumarRoutLemmas.lean` and `CITED/NairKumarRoutProof.lean`; their supporting lemmas are documented there and not itemized here.

Appendix B. Numerical exploration

The steering word is exactly computable, and it repays a look. Everything in this appendix was produced from the division-free integer recurrence

$$m_{n+1} = \frac{3m_n + t_n}{2}, \qquad R_{n+1} = 3R_n - 2^n t_n, \qquad \varepsilon_n = \frac{R_n}{2^n},$$

in which t_n is the unique integer with $\varepsilon_{n+1} \in [-\frac{1}{2}, \frac{1}{2})$ and $t_n \equiv m_n \pmod 2$; no floating-point arithmetic enters the generation of T. We use the first $N = 10^6$ letters. Write $p_T^{(N)}(k)$ for the number of distinct length-k factors of the prefix $t_0 t_1 \cdots t_{N-1}$. Two properties of this statistic govern how it may be read: it is a *lower* bound for $p_T(k)$, every factor of a prefix being a factor of T; and it can never exceed $N - k + 1$, so it stops being informative as soon as $p_T(k)$ approaches N. Nothing in this appendix belongs to the formal development, and Conjecture B.2 is a conjecture, not a theorem.

Letters and pairs. Suppose ε_n equidistributes in $[-\frac{1}{2}, \frac{1}{2})$ and the parity $m_n \bmod 2$ behaves like an independent fair coin. Since t_n is one of the two integers in $(3\varepsilon_n - 1, 3\varepsilon_n + 1]$, selected between them by the parity constraint, the model predicts the letter 0 with probability $\frac{1}{3}$, the letters ± 1 with probability $\frac{1}{4}$ and the letters ± 2 with probability $\frac{1}{12}$. Measured over 10^6 letters:

letter	-2	-1	0	1	2
measured	0.08302	0.25017	0.33362	0.25028	0.08292
model	1/12	1/4	1/3	1/4	1/12

Each cell agrees to within $5 \cdot 10^{-4}$. The equidistribution the model assumes is visible directly in the orbit: a 50-bin histogram of ε_n, $n < 10^6$, has occupancies between 19 732 and 20 305 against an expected 20 000. Pairs of letters, by contrast, are constrained outright.

Proposition B.1. *For every n, $|t_n + t_{n+1}| \le 2$. Consequently the six pairs $(\pm 2, \pm 2)$, $(\pm 2, \pm 1)$, $(\pm 1, \pm 2)$ with equal signs never occur, and $p_T(2) = 19$.*

Proof. Applying $t_n = 3\varepsilon_n - 2\varepsilon_{n+1}$ twice,

$$t_n + t_{n+1} = 3\varepsilon_n + \varepsilon_{n+1} - 2\varepsilon_{n+2}.$$

Since $-\frac{1}{2} \le \varepsilon_j < \frac{1}{2}$ for every j, the right-hand side is strictly below $\frac{3}{2} + \frac{1}{2} + 1 = 3$ and strictly above $-\frac{3}{2} - \frac{1}{2} - 1 = -3$; being an integer, it lies in $[-2, 2]$. The excluded pairs are exactly the six elements of $\{-2, \dots, 2\}^2$ with $|t_n + t_{n+1}| \ge 3$, and $25 - 6 = 19$ is the measured value. □

The rule is not the whole story. The subshift of finite type it defines has 75 words of length 3, whereas $p_T(3) = 65$, so constraints of range 3 are present as well; and it yields no better ceiling than Section 1.1 already has, its entropy being $\log 3.935\ldots$ against Kopra's $\log 3$.

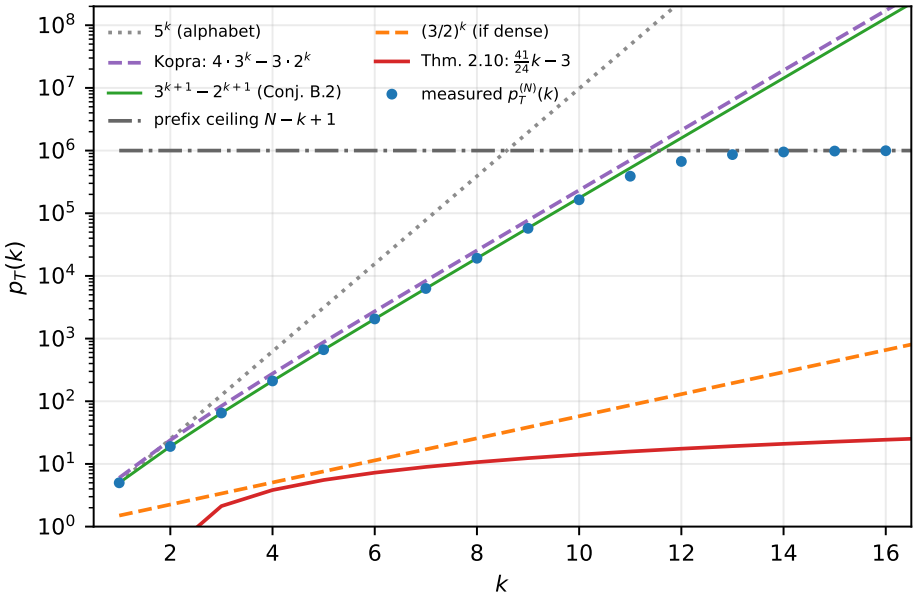

FIGURE 1. Measured $p_T^{(N)}(k)$ at $N = 10^6$ against the proven and conjectural environment. The data follow the curve $3^{k+1} - 2^{k+1}$ for as long as 10^6 letters suffice to exhibit every factor, fall away from it as sampling fails, and finally saturate at the prefix ceiling $N - k + 1$; both effects are artefacts of the finite sample. Note the distance to the certified floor of Theorem 2.10 and to the bound $(3/2)^k$ that density would force.

The complexity profile. The measured profile is, for $k \le 9$,

k	1	2	3	4	5	6	7	8	9
$p_T^{(N)}(k)$	5	19	65	211	665	2059	6304	19138	57447
$3^{k+1} - 2^{k+1}$	5	19	65	211	665	2059	6305	19171	58025

Conjecture B.2. *$p_T(k) = 3^{k+1} - 2^{k+1}$ for every $k \ge 1$.*

Agreement is exact for $1 \le k \le 6$. At $k = 7$ and $k = 8$ the prefix falls short by 1 and by 33, which is the expected effect of rarity rather than evidence against the formula: the letter frequencies are far from uniform, so a factor built from extreme letters has probability about 12^{-k} per position — at $k = 7$, some three hundredths of an occurrence in 10^6 positions. Figure 2 exhibits the effect: for each N the ratio $p_T^{(N)}(k)/(3^{k+1} - 2^{k+1})$ sits at exactly 1 up to a threshold that moves right as N grows, and collapses beyond it.

Three things would follow from Conjecture B.2. The topological entropy of the steering subshift would be exactly $\log 3$, so the base of Kopra's ceiling

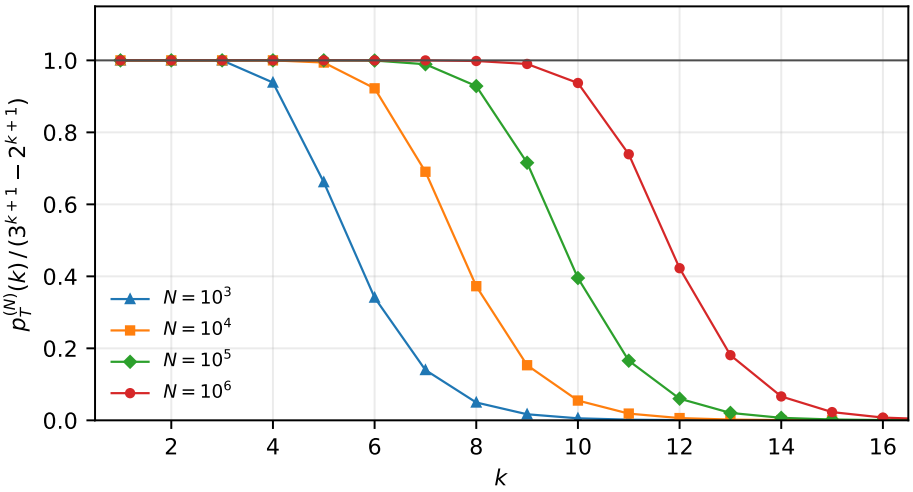

FIGURE 2. $p_T^{(N)}(k)$ normalized by the conjectured value, for four prefix lengths. The plateau at 1 lengthens with N; the collapse to its right is under-sampling, not a failure of the formula.

is sharp and only its constant (4 against 3) is not; this would settle Problem 6.4 in both its parts, with $h(T) = \log 3 > 0$. The complexity would exceed by the exponential factor 2^k the bound $\lceil (3/2)^k \rceil - 1$ that density forces through Corollary 1.3; the $(3/2)^k$ there counts the ε-coordinate alone, and the missing 2^k is exactly the parity coordinate $m_a \bmod 2^k$, their product 3^k being one more reason to regard $[-\frac{1}{2}, \frac{1}{2}) \times \mathbb{Z}_2$ as the right phase space (Section 6). And the distance to what is proved is stark: at $k = 8$ the certified floor of Theorem 2.10 reads 10.7 against a measured 19 138.

Repetitions. Proposition 2.5 bounds repetition lengths linearly in the position: inside a prefix of length N it permits repeated factors of length up to $0.585\,(N+1)$, that is 585 000 at $N = 10^6$. The longest repeated factor actually present there has length 24. What the measurements track instead is

$$2\log_3 N - \log_3 2 - 1,$$

the length at which, for a word of complexity 3^{k+1}, the expected number of coinciding pairs among N windows falls below one; the fit is within one letter over three decades. This tests Conjecture B.2 at $k \approx 24$, far outside the range in which $p_T(k)$ can be counted directly — though it tests the exponential base only, since replacing 3^{k+1} by Kopra's $4 \cdot 3^k$ would move the prediction by $\log_3(4/3) \approx 0.26$, well below the resolution of the data.

The four orders of magnitude between the two curves of Figure 3 are the visual form of the gap that Sections 3–5 do not close. The mechanism

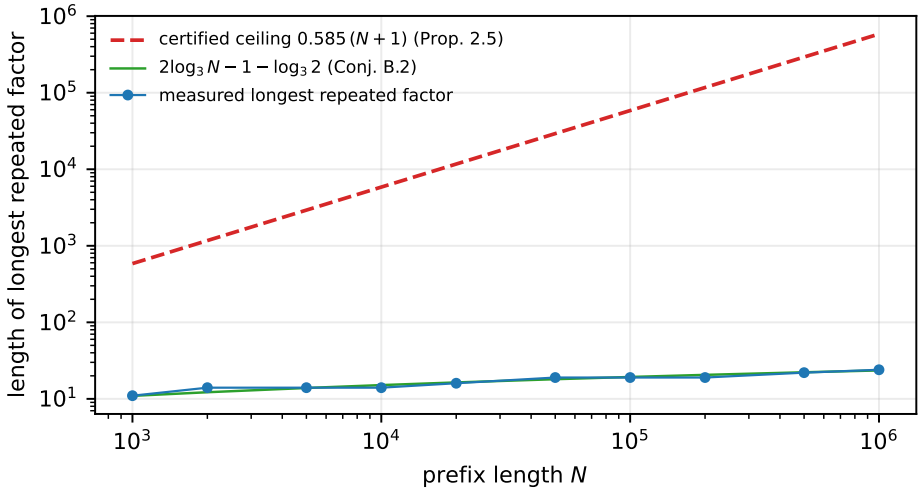

FIGURE 3. Longest repeated factor in a prefix of length N, against the ceiling certified by Proposition 2.5 and against the birthday prediction of Conjecture B.2. The certified ceiling is linear in N; the repetitions actually present die out logarithmically.

of Section 2 forbids only very long repetitions, whereas in the word itself repetitions die out logarithmically; an argument that saw the true decay rate would deliver an exponential complexity bound, and the Diophantine input used here sees only the linear ceiling.

REFERENCES

[AB07] B. Adamczewski and Y. Bugeaud, *On the complexity of algebraic numbers I. Expansions in integer bases*, Ann. of Math. (2) **165** (2007), no. 2, 547–565.

[AS03] J.-P. Allouche and J. Shallit, *Automatic sequences: theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003.

[BE08] Y. Bugeaud and J.-H. Evertse, *On two notions of complexity of algebraic numbers*, Acta Arith. **133** (2008), no. 3, 221–250.

[BG06] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs **4**, Cambridge University Press, Cambridge, 2006.

[Cas03] J. Cassaigne, *Constructing infinite words of intermediate complexity*, Developments in Language Theory (DLT 2002), Lecture Notes in Comput. Sci. **2450**, Springer, Berlin, 2003, 173–184.

[CN10] J. Cassaigne and F. Nicolas, *Factor complexity*, in: V. Berthé and M. Rigo (eds.), Combinatorics, automata and number theory, Encyclopedia Math. Appl. **135**, Cambridge University Press, Cambridge, 2010, 163–247.

[Cob72] A. Cobham, *Uniform tag sequences*, Math. Systems Theory **6** (1972), 164–192.

[CZ04] P. Corvaja and U. Zannier, *On the rational approximations to the powers of an algebraic number: solution of two problems of Mahler and Mendès France*, Acta Math. **193** (2004), no. 2, 175–191. arXiv:math/0403522.

[Dub09] A. Dubickas, *On integer sequences generated by linear maps*, Glasg. Math. J. **51** (2009), no. 2, 243–252.

[DN05] A. Dubickas and A. Novikas, *Integer parts of powers of rational numbers*, Math. Z. **251** (2005), no. 3, 635–648.

[AFS08] S. Akiyama, C. Frougny, and J. Sakarovitch, *Powers of rationals modulo* 1 *and rational base number systems*, Israel J. Math. **168** (2008), 53–91.

[DHS99] F. Durand, B. Host, and C. Skau, *Substitution dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory Dynam. Systems **19** (1999), no. 4, 953–993.

[EM25] H. Erazo and C. G. Moreira, *The Heinis spectrum has non-empty interior*, Combinatorics on Words (WORDS 2025), Lecture Notes in Comput. Sci., vol 15729. Springer, Cham, 2025.

[Fer99] S. Ferenczi, *Complexity of sequences and dynamical systems*, Discrete Math. **206** (1999), no. 1–3, 145–154.

[FLP95] L. Flatto, J. C. Lagarias, and A. D. Pollington, *On the range of fractional parts* $\{\xi(p/q)^n\}$, Acta Arith. **70** (1995), no. 2, 125–147.

[HL25] A. Hilion and G. Levitt, *A Pansiot-type subword complexity theorem for automorphisms of free groups*, Israel J. Math. 270, 59–93 (2025). arXiv:2208.00676.

[Kop21] J. Kopra, *On the trace subshifts of fractional multiplication automata*, Theoret. Comput. Sci. **851** (2021), 92–110.

[Mah57] K. Mahler, *On the fractional parts of the powers of a rational number II*, Mathematika **4** (1957), 122–124.

[MH38] M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. **60** (1938), no. 4, 815–866.

[Pan84] J.-J. Pansiot, *Complexité des facteurs des mots infinis engendrés par morphismes itérés*, Automata, Languages and Programming (ICALP 1984), Lecture Notes in Comput. Sci. **172**, Springer, Berlin, 1984, 380–389.

[Puz25] S. Puzynina, *Complexity of infinite words*, Machines, Computations, and Universality (MCU 2024), Lecture Notes in Comput. Sci. **15270**, Springer, Cham, 2025, 1–16.

[NKR25] P. S. Nair, V. Kumar, and S. S. Rout, *Algebraic approximations to linear combinations of S-units*, arXiv:2506.02898 (v3, 18 Nov 2025).

[Sch91] Schmidt, Wolfgang M., *Diophantine approximations and Diophantine equations.* Springer, 2006.

[Zud07] W. Zudilin, *A new lower bound for* $\|(3/2)^k\|$, J. Théor. Nombres Bordeaux **19** (2007), no. 1, 311–323.