LINE COUNTS AND A CENSUS FOR THE EXCEPTIONS TO $\|(3/2)^n\| \geq (3/4)^n$, WITH AN APPLICATION TO WARING'S PROBLEM

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Abstract. Problem 10.13 of Bugeaud's *Distribution modulo one and Diophantine approximation* asks for an upper bound for the number of positive integers n with $\|(3/2)^n\| < (3/4)^n$, where $\|\cdot\|$ denotes the distance to the nearest integer. Mahler proved in 1957 that there are only finitely many such n; his proof bounds neither their size nor their number. We prove that the exceptions with $n \geq 10$ lie on at most $K(\varepsilon^*) = 1\,856\,360\,182\,227 < 1.86 \cdot 10^{12}$ lines of $\mathbb{Q}^2$, by feeding the associated S-unit points into the quantitative Ridout theorem of Bugeaud and Evertse at the sharp exponent $\varepsilon^* = \log(4/3)/\log 3$, whose exponent budget the problem meets with equality. Counting the solutions on a single line is an open problem; conditional on a bound H for the number of exceptions per line, the total number of exceptions is at most $5+H\cdot K(\varepsilon^*)$, and the ideal Waring formula $g(n) = 2^n + \lfloor(3/2)^n\rfloor - 2$ holds for all $n \geq 2$ with at most $H \cdot 1.88 \cdot 10^{12}$ exceptions, with no additive constant. A kernel-certified census determines the exceptions up to 256 as $\{1, 2, 3, 4, 7\}$ and their line structure as the partition $\{1\}, \{2, 3\}, \{4\}, \{7\}$, and a stratified variant shows that bases of proportionally tall towers lie on fewer lines, at most $5.38 \cdot 10^{11}$ for spans of a quarter of the base. All results are formally verified in Lean 4.

1. Introduction

Throughout, $\|x\|$ denotes the distance from the real number x to the nearest integer, and "log" is the natural logarithm. Bugeaud's book poses the following problem [Bug12, Problem 10.13].

> *To find an upper bound for the number of positive integers n such that $\|(3/2)^n\| < (3/4)^n$.*

We call such n *exceptions* and write

$$\mathcal{E} \;:=\; \left\{\, n \geq 1 \;:\; \|(3/2)^n\| < (3/4)^n \,\right\}, \qquad F(N) := \#\bigl(\mathcal{E} \cap [1, N]\bigr).$$

1.1. **History.** That $\mathcal{E}$ is finite is a theorem of Mahler [Mah57, Thm. 2], obtained from Ridout's p-adic extension of Roth's theorem [Rid57]; the finiteness statement itself was conjectured by Mahler already in 1938, in the closing line of [Mah39]. Like Roth's theorem, the argument is ineffective: it

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bounds neither the largest exception nor the number of exceptions. The effective lower-bound tradition — Baker–Coates [BC75], Beukers ($\|(3/2)^n\| \geq 2^{-0.9n}$ for $n > 5000$ [Beu81]), Dubickas (0.5769^n [Dub90]), Bennett and Habsieger (independently, $2^{-0.8n}$ for all $n \geq 5$ [Ben03, Hab03]), Zudilin (0.5803^n beyond an effectively computable but never computed threshold [Zud07]) — attacks the size of $\|(3/2)^n\|$ instead, and after five decades remains far from the rate $3/4 = 0.75$ that Problem 10.13 concerns. Our paper is about the *count* of exceptions, which we understand is what the problem asks for. The one prior assertion in this direction we are aware of is a passing remark of Delmer and Deshouillers [DD90, §2.4], without proof or constant, that the Ridout–Mahler argument "gives indeed an effective bound for the number of exceptions".

Numerically, the exceptions in the range $n \leq 10^6$ are

$$\mathcal{E} \cap [1, 10^6] = \{1, 2, 3, 4, 7\},$$

by an exact big-integer scan (Section 5); the related one-sided condition (3.23) of [Bug12], which governs Waring's problem, is known to hold for all $n \leq 2^{36} + 2^{34} = 85\,899\,345\,920$ [KW90, Cum25].

1.2. **What the quantitative literature proves.** The effective engine available for this problem is Corollary 5.2 of Bugeaud and Evertse [BE08], a quantitative form of Ridout's theorem resting on the quantitative Subspace theorem of Evertse and Schlickewei [ES02]. Two features of that statement govern everything below. First, it counts *one-dimensional subspaces of* $\mathbb{Q}^2$ — lines through the origin — not solutions; no per-line multiplicity statement appears anywhere in [BE08], and the passage from lines to solutions is an open problem, of the family that Evertse and Ferretti [EF13, p. 6] record as open for the quantitative Subspace theorem in general. Second, its conclusion is restricted to solutions of height above an explicit threshold; solutions below the threshold are not covered at all. A count of *exceptions* obtained from Corollary 5.2 is therefore necessarily conditional on a per-line hypothesis, and this paper states its counting results in that honest form.[1]

1.3. **Results.** Set

$$\varepsilon^* := \frac{\log(4/3)}{\log 3} = 0.26186\ldots, \qquad \theta := \frac{\log 2}{\log 3} = 0.63093\ldots, \tag{1}$$

and let $K(\epsilon)$ denote the Bugeaud–Evertse line count (3) below. Our headline is a bound on the number of lines, which is what the quantitative literature supports.

[1] An earlier, unpublished version of this manuscript packaged Corollary 5.2 as an unconditional solution count. Two anonymous referees identified the overstatement; the present version is the repair, and gains from it: the sharp exponent replaces a halved one, the resulting constant drops by a factor 14.6, and the certified initial segment now carries structural weight in the counts.

Theorem A. *The frame points $(m_n 2^n, 3^n)$ of the exceptions $n \geq 10$, with m_n the integer nearest to $(3/2)^n$, lie on at most*

$$K(\varepsilon^*) \;=\; \Big\lceil 2^{32}\big(1+(\varepsilon^*)^{-1}\big)^3 \log 6 \cdot \log\big((1+(\varepsilon^*)^{-1}) \log 6\big) \Big\rceil \;=\; 1\,856\,360\,182\,227$$

lines of $\mathbb{Q}^2$ through the origin. Equivalently, the exceptions $n \geq 10$ fall into at most $K(\varepsilon^*)$ scaling families $(m_a, m_b) = (2^{\,b-a}\mu, 3^{\,b-a}\mu)$.

The exponent ε^* is forced, not chosen: the frame spends θ at the infinite place, θ at the prime 2 and 1 at the prime 3, and the budget identity $\theta+\theta+1 = 2+\varepsilon^*$ holds with equality, both sides being $\log 12/\log 3$ (Section 7). The decimal bound $K(\varepsilon^*) \leq 1.86 \cdot 10^{12}$ is itself certified inside the formal development (Section 13).

Counting exceptions instead of lines requires a bound on the number of exceptions a single line can carry. What is elementary about that question is proved in Sections 8 and 9: each line carries finitely many exceptions, confined to $[a, a\log_2 3)$ over the least member a, and in the certified range the line structure is determined outright. What is not elementary is a bound uniform in the line; it enters as a hypothesis.

Theorem B. *Suppose no line of $\mathbb{Q}^2$ carries more than H exceptions with $n \geq 257$. Then*

$$\#\mathcal{E} \;\leq\; 5 + H \cdot K(\varepsilon^*) \;\leq\; 5 + H \cdot 1.86 \cdot 10^{12}.$$

The summand 5 is the kernel-certified segment $\mathcal{E} \cap [1, 256] = \{1, 2, 3, 4, 7\}$ (Section 5); the hypothesis concerns only the range $n \geq 257$, in which the 10^6 scan finds no exception at all, and the data suggest $H = 0$. Finiteness of $\mathcal{E}$ needs no hypothesis: we re-derive Mahler's theorem twice, once from the qualitative Subspace theorem (Section 6) and once from Theorem A together with the elementary per-line confinement (Section 10), two proofs with disjoint Diophantine inputs.

The application to Waring's problem runs through the classical chain (3.22)–(3.24) of [Bug12, §3.7]. Let $g(n)$ denote the least s such that every positive integer is a sum of s nonnegative n-th powers.

Theorem C. *Suppose no line of $\mathbb{Q}^2$ carries more than H failures of Dickson's condition with $n \geq 257$. Then the ideal Waring formula*

$$g(n) \;=\; 2^n + \big\lfloor (3/2)^n \big\rfloor - 2$$

holds for every $n \geq 2$ with at most $H \cdot K(\varepsilon_W) \leq H \cdot 1.88 \cdot 10^{12}$ exceptions, where $\varepsilon_W = \varepsilon^*(1 - \frac{1}{257})$. Unconditionally, it holds for all sufficiently large n.

The count carries no additive constant: the kernel certifies Dickson's condition throughout $[3, 256]$, and the one low index that fails it, $n = 2$, satisfies the ideal formula anyway because $g(2) = 4$ — a fact we prove from Lagrange's four-square theorem rather than assume (Section 11).

The fourth result is a stratification: a line that carries a *tall* tower is rarer than a line that carries a mere exception, because the tower's scaling relation forces $2^{b-a} \mid m_a$, a 2-adic surplus that the counting frame can spend.

Theorem D. *Let $\gamma \geq 0$. The bases $a \geq 10$ of scaling families of span at least γa lie on at most $K(\varepsilon^* + \gamma\theta)$ lines of $\mathbb{Q}^2$; at $\gamma = 1/4$ the count is $K(\varepsilon^* + \theta/4) = 537\,048\,098\,048 \leq 5.38 \cdot 10^{11}$, a factor 3.46 below $K(\varepsilon^*)$. Every scaling family has $\gamma < \log(3/2)/\log 2 = 0.58496\ldots$, so the stratification interpolates between Theorem A at $\gamma = 0$ and the empty statement at that ceiling.*

Beneath the Diophantine layer sits the elementary layer of the first version of this work, which is unchanged: an integer identity shows that two exceptions close to each other are algebraically dependent (Section 3), the exceptions organize into towers over bases that grow geometrically, and the number of towers with base $\leq N$ is $O(\log N)$ with explicit constants (Section 4). New here is a census: the five certified exceptions carry the frame points

$$(4,3), \quad (8,9), \quad (24,27) = 3 \cdot (8,9), \quad (80,81), \quad (2176,2187),$$

so their line partition is $\{1\}, \{2,3\}, \{4\}, \{7\}$ — four lines carrying five exceptions — while the five are pairwise *not* linked in the threshold sense of Section 3 (Section 9). The pair $\{2,3\}$ shows that the two notions of tower genuinely differ, and that the observed height of a line is 2, not 1.

1.4. **Counts against computation.** A count is not a size bound, and the two are calibrated differently. The hypergeometric tradition of Beukers, Dubickas, Habsieger and Zudilin measures progress in the exponent, where it saturates near 0.58, far from 0.75; a count, by contrast, is naturally measured against the verification record. The one-sided Waring condition has been verified for $n \leq 2^{36} + 2^{34} \approx 8.59 \cdot 10^{10}$ [KW90, Cum25], so a proved count below $8.6 \cdot 10^{10}$ would assert something computation has not reached. The line count $K(\varepsilon^*)$ sits a factor 21.6 above that yardstick (the retired solution-count constant of the first version sat at 316). This is the gap a future per-line theorem, or a sharper line count, would have to close.

All statements below have been formally verified in Lean 4. Five results of the literature are used as cited inputs (§1.5); every other statement is fully proved, and the initial segment $\mathcal{E} \cap [1, 256] = \{1, 2, 3, 4, 7\}$, the census, and Dickson's condition on $[3, 256]$ are certified by exact integer arithmetic inside the proof kernel. Appendix A records the Lean name, file and axiom footprint of each statement.

1.5. **Cited inputs.** Five theorems are used as Lean axioms, each stated here with a citation and no proof. The first is the qualitative Subspace theorem, behind the ineffective finiteness lane only.

Theorem 1.1 (Subspace theorem; Schmidt [Sch91], Evertse–Schlickewei form [BG06])**.** *Let K be a number field, $n \geq 2$, and S a finite set of places* *of K containing the archimedean ones. For each $v \in S$ let $L_{v,1}, \ldots, L_{v,n}$ be linearly independent linear forms in $X_1, \ldots, X_n$ with coefficients in K. For every $\epsilon > 0$, the nonzero $x \in K^n$ with*

$$\prod_{v \in S} \prod_{i=1}^{n} \frac{|L_{v,i}(x)|_v}{\|x\|_v} \ \leq \ H(x)^{-n-\epsilon}$$

lie in finitely many proper linear subspaces of K^n.

The Main Theorem of Corvaja and Zannier, in the rational-multiplier case we use, is *derived* from Theorem 1.1 at $n = 2$ in the formalization rather than assumed; we state the derived form in the frame in which it is applied. For $u = 2^x 3^y$ with $x, y \in \mathbb{Z}$ write $H(u)$ for the height $\max(\mathrm{num}(u), \mathrm{den}(u))$, and for $\delta \in \mathbb{Q}^\times$, $\epsilon > 0$ let

$$\begin{aligned} \mathcal{A}(\delta, \epsilon) := \Big\{ (q, u) \ : \ & q \in \mathbb{Z}_{\geq 1},\ u = 2^x 3^y,\ |\delta q u| > 1,\ \delta q u \notin \mathbb{Z}, \\ & 0 < \|\delta q u\| < H(u)^{-\epsilon} q^{-1-\epsilon} \Big\}. \end{aligned} \tag{2}$$

Theorem 1.2 (Corvaja–Zannier [CZ04], Main Theorem over $\mathbb{Q}$)**.** *For every $\delta \in \mathbb{Q}^\times$ and $\epsilon > 0$ the set $\mathcal{A}(\delta, \epsilon)$ is finite.*

In the general Main Theorem of [CZ04] the multiplier δ is algebraic and the conclusion excludes the pairs for which $\delta q u$ is a pseudo-Pisot number; over $\mathbb{Q}$ a value that is not an integer is not pseudo-Pisot, which is why the exclusions in (2) take the form they do. Theorem 1.2 is ineffective. The quantitative input is the following.

Theorem 1.3 (quantitative Ridout; Bugeaud–Evertse [BE08, Cor. 5.2], after Evertse–Schlickewei [ES02])**.** *Let S_1, S_2 be finite, possibly empty sets of primes, let ξ be an algebraic number of degree d, let $\epsilon > 0$, and let f_p ($p \in \{\infty\} \cup S_1 \cup S_2$) be reals with*

$$f_p \geq 0, \qquad \sum_p f_p = 2 + \epsilon.$$

Then the solutions $(x, y) \in \mathbb{Z}^2$, $y > 0$, of the system

$$\left|\xi - \frac{x}{y}\right| \leq y^{-f_\infty}, \qquad |x|_p \leq y^{-f_p} \ (p \in S_1), \qquad |y|_p \leq y^{-f_p} \ (p \in S_2),$$

with

$$y \ > \ \max\big(2H(\xi),\, 2^{4/\epsilon}\big)$$

are contained in the union of at most

$$2^{32}\big(1 + \epsilon^{-1}\big)^3 \log(6d)\, \log\big((1 + \epsilon^{-1}) \log(6d)\big)$$

one-dimensional linear subspaces of $\mathbb{Q}^2$.

The formalization records the instance $d = 1$ (so $\xi \in \mathbb{Q}$ and $\log(6d) = \log 6$), $S_1 = \{2\}$, $S_2 = \{3\}$, with a line through the origin meeting the region

$y > 0$ coordinatized by the slope $x/y \in \mathbb{Q}$ of any of its points, and with the line count rounded up to

$$K(\epsilon) := \left\lceil 2^{32}\big(1+\epsilon^{-1}\big)^3 \log 6 \cdot \log\big((1+\epsilon^{-1})\log 6\big)\right\rceil. \tag{3}$$

Varying (f_∞, f_2, f_3) inside the budget $2+\epsilon$ is what lets this one statement serve the three frames of the paper (Sections 7, 11 and 12). We emphasize what Theorem 1.3 does *not* assert: nothing bounds the number of solutions on a single line, and nothing is said about solutions of height below the threshold.

The remaining three inputs are classical Waring-side results; they are stated in Section 11 (Theorems 11.1, 11.2 and 11.4).

2. The exceptional set

We work throughout with a general rational base and rate, of which the problem is the case $(p, q, c) = (3, 2, 3/4)$. Let p, q be positive integers and c a real number.

Definition 2.1. For $n \in \mathbb{N}$ let m_n be the nearest integer to $(p/q)^n$ and let

$$k_n := p^n - m_n q^n \in \mathbb{Z}$$

be the associated residue. We say n is an *exception* (for (p, q, c)) if $|k_n| < (cq)^n$.

Since $(p/q)^n - m_n = k_n/q^n$, one has $\|(p/q)^n\| = |k_n|/q^n$, and n is an exception if and only if

$$\|(p/q)^n\| < c^n. \tag{4}$$

For $(3, 2, 3/4)$ the condition reads $|3^n - m_n 2^n| < (3/2)^n$, and the exceptional set is the set $\mathcal{E}$ of the introduction (together with the degenerate index $n = 0$, which we exclude by fiat throughout). At $n = 1$ the value $(3/2)^1$ is at distance $1/2$ from both neighbouring integers; the formalization rounds half up, so $m_1 = 2$ and $k_1 = -1$, which does not affect (4).

The governing exponent of the whole paper is

$$\varepsilon(p, c) := \frac{\log(1/c)}{\log p}, \qquad \varepsilon(3, 3/4) = \frac{\log(4/3)}{\log 3} = \varepsilon^* = 0.26186\ldots, \tag{5}$$

the rate at which c^n beats the height p^n; it appears below as the linkage threshold (Section 3), as the tower ratio (Section 4), and as the budget of the counting frame (Section 7), where its companion $\theta = \log 2/\log 3$ of (1) joins it through the identities

$$\big(3^n\big)^{-\varepsilon^*} = (3/4)^n, \qquad \big(3^n\big)^{-\theta} = 2^{-n}.$$

3. The gap principle and linkage

The elementary engine is a gap principle[2]: two exceptions close to each other are not merely rare, they are algebraically dependent. Everything rests on one integer identity.

Lemma 3.1 (gap identity). *For all $n, d \geq 0$,*

$$q^n\big(p^d m_n - q^d m_{n+d}\big) \;=\; k_{n+d} - p^d k_n.$$

Proof. Expand both sides using $k_j = p^j - m_j q^j$; the terms p^{n+d} cancel. $\square$

Lemma 3.2 (linkage). *Let $p, q \geq 1$, let n and $n+d$ both be exceptions, and suppose*

$$c^n\big((cq)^d + p^d\big) \;\leq\; 1. \tag{6}$$

Then $p^d m_n = q^d m_{n+d}$.

Proof. By Lemma 3.1 and the exception bounds,

$$\begin{aligned} q^n \left|p^d m_n - q^d m_{n+d}\right| \;&\leq\; |k_{n+d}| + p^d |k_n| \;<\; (cq)^{n+d} + p^d (cq)^n \\ &=\; q^n \cdot c^n\big((cq)^d + p^d\big) \;\leq\; q^n, \end{aligned}$$

so the integer $p^d m_n - q^d m_{n+d}$ has absolute value below 1 and vanishes. $\square$

When $0 \leq c$ and $cq \leq p$ — as in every case of interest — the simpler threshold $2c^n p^d \leq 1$ implies (6). Taking logarithms, the simpler threshold holds, up to a bounded additive defect, whenever $d < \varepsilon(p, c)\, n$: for $(3, 2, 3/4)$, whenever $d < 0.26186\, n$. We say the pair (a, b) with $a < b$ is *linked* if $2c^a p^{b-a} \leq 1$.

Proposition 3.3 (linkage consequences). *Suppose $p^d m_n = q^d m_{n+d}$. Then:*

- (i) *$k_{n+d} = p^d k_n$;*
- (ii) *if $\gcd(p, q) = 1$, then $q^d \mid m_n$;*
- (iii) *under the coprimality of (ii), there is $\mu \in \mathbb{Z}$ with $m_n = q^d \mu$ and $m_{n+d} = p^d \mu$;*
- (iv) *if $n + d$ is an exception, then $p^d\,|k_n| < (cq)^{n+d}$: the residue at the base beats the exception bound by the factor $(cq/p)^d$ — for $(3, 2, 3/4)$, $|k_n| < (3/2)^n 2^{-d}$.*

Proof. (i) is Lemma 3.1 with vanishing left factor. For (ii), the relation $q^d \mid p^d m_n$ and coprimality give the claim; (iii) divides the linkage relation by q^d. For (iv), rewrite the exception bound at $n + d$ through (i). $\square$

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[2]We use *linked* and *tower* as convenient names for the structures below; we are not aware of standard terminology for them. The counting device itself — close solutions force an algebraic coincidence — is the classical gap-principle template; see [Bug11] for a survey of its quantitative uses.

So a linked pair is a p^d-scaling of a single reduced datum μ: the whole configuration at $n+d$ is determined by the one at n. The exceptions therefore organize into *towers*: call an exception $b \geq 1$ a *tower base* if no exception $a < b$ is linked to it. Every exception sits over a base (Lemma 10.4 below), and two distinct bases are unlinked, which forces them apart geometrically:

Lemma 3.4 (gap from non-linkage). *Let $p \geq 2$, $c > 0$, $\rho \in \mathbb{R}$, and let $t \geq 1$, a, b be integers with $t \leq a < b$. Suppose (a, b) is not linked, i.e. $2c^a p^{b-a} > 1$, and that the threshold relation*

$$\log 2 \;\leq\; t\left(\log(1/c) - (\rho - 1)\log p\right) \tag{7}$$

holds. Then $\rho\, a \leq b$.

Proof. Taking logarithms in $2c^a p^{b-a} > 1$ gives $(b-a)\log p > a\log(1/c) - \log 2$. By (7) and $a \geq t$ the loss $\log 2$ is absorbed: $a\log(1/c) - \log 2 \geq (\rho-1)a\log p$, and dividing by $\log p > 0$ yields $b - a \geq (\rho-1)a$. □

Relation (7) forces $\rho < 1 + \varepsilon(p, c)$, and any such ρ is admissible for t large; the bases therefore grow by ratios approaching $1 + \varepsilon^* = 1.26186\ldots$ for $(3, 2, 3/4)$.

Remark 3.5 (ancestry). Sparsity of the exceptions is as old as their finiteness: Mahler's 1938 theorem [Mah39, Thm. 1], quoted in [Mah57], shows that *infinitely many* exceptions $n_1 < n_2 < \cdots$ would force $\limsup_r n_{r+1}/n_r = \infty$. Lemma 3.2 and Proposition 3.3 sharpen the underlying phenomenon to a finite-n statement, with the algebraic structure ($q^d \mid m_n$, the common μ, the residue scaling) that Mahler's approximation-theoretic argument does not see.

4. The tower count

The gap of Lemma 3.4 converts into a logarithmic count by a standard lacunarity argument, which we state in the form the formalization uses.

Lemma 4.1 (geometric-growth count). *Let $\rho > 1$, $t \geq 1$, and let $S \subseteq \{1, \ldots, N\}$ be a finite set such that $\rho a \leq b$ for all $a < b$ in S with $t \leq a$. Then*

$$\#S \;\leq\; t + \left(1 + \log_\rho N\right).$$

Proof. The elements below t number at most t. On the rest the map $a \mapsto \lfloor \log_\rho a \rfloor$ is strictly increasing — the gap $\rho a \leq b$ advances the floor by at least one — hence injective into $\{0, 1, \ldots, \lfloor \log_\rho N \rfloor\}$. □

Theorem 4.2 (tower count, general base). *Let $p \geq 2$, $c > 0$, $t \geq 1$, $\rho > 1$ satisfy the threshold relation* (7)*. Then any finite set of tower bases in* $[1, N]$ *has at most $t + (1 + \log_\rho N)$ elements.*

Proof. Two bases $a < b$ are not linked (the larger is a base), so Lemma 3.4 gives the gap $\rho a \leq b$ whenever $t \leq a$, and Lemma 4.1 counts. □

Corollary 4.3 (tower count for $(3, 2, 3/4)$)**.** *Any finite set of tower bases for* $(3, 2, 3/4)$ *in* $[1, N]$ *has at most*

$$11 + \bigl(1 + \log_{6/5} N\bigr) \;\approx\; 12 + 5.49 \ln N$$

elements.

Proof. Take $\rho = 6/5$ and $t = 11$ in Theorem 4.2; relation (7) becomes $\log 2 \leq 11\,(2\log 2 - \log 3 - \frac{1}{5}\log 3)$, which the bounds $\log 2 \geq 0.6931$, $\log 3 \leq 1.0987$ verify. $\square$

The pair $(\rho, t) = (6/5, 11)$ is a clean witness, not the optimum: pushing $\rho \uparrow 1 + \varepsilon^*$ in Theorem 4.2 gives the sharp asymptotic tower ratio $1.26186\ldots$, hence at most $4.30 \ln N$ bases asymptotically. In the formal statement of Corollary 4.3 the two numerical logarithm bounds appear as explicit hypotheses (Appendix A); both follow from the certified enclosures of Section 13.

5. The certified segment

The empirical statement $\mathcal{E} \cap [1, 10^6] = \{1, 2, 3, 4, 7\}$ rests on an exact big-integer scan; its initial segment is not merely tested but proved, inside the formal development, by reducing the real inequality (4) to integer arithmetic. Write

$$D_n \;:=\; \min\bigl(3^n \bmod 2^n,\; 2^n - (3^n \bmod 2^n)\bigr),$$

the numerator of the distance: this is the quantity driven by the exact-integer recursion of Delmer and Deshouillers [DD90, §2].

Lemma 5.1 (integer bridge)**.** *For every* n, $\|(3/2)^n\| = D_n/2^n$, *and consequently*

$$\|(3/2)^n\| < (3/4)^n \quad\Longleftrightarrow\quad 2^n D_n < 3^n.$$

Proof. For a ratio of positive integers, $\|m/d\| = \min(m \bmod d,\, d - (m \bmod d))/d$: the fractional part of m/d is $(m \bmod d)/d$, and the distance to the nearest integer is the smaller of that and its complement. Apply this to $3^n/2^n$ and clear denominators against $(3/4)^n = 3^n/(2^n \cdot 2^n)$. $\square$

Proposition 5.2. *For every* n *with* $1 \leq n \leq 256$,

$$\|(3/2)^n\| < (3/4)^n \quad\Longleftrightarrow\quad n \in \{1, 2, 3, 4, 7\}.$$

In particular $F(256) = 5$.

Proof. By Lemma 5.1 the claim is a conjunction of 256 decidable integer inequalities, verified by exact computation; in the formal development the verification runs inside the proof kernel, so the proposition carries no computational trust beyond the checker itself. $\square$

The line structure of Section 9 needs the nearest integers m_n themselves in computable form; rounding half up, as the formalization does,

$$m_n \;=\; \Bigl\lfloor \frac{2 \cdot 3^n + 2^n}{2^{n+1}} \Bigr\rfloor, \tag{8}$$

an identity of integers proved once and used by every kernel computation below.

Remark 5.3. The choice 256 is a convenience threshold for in-kernel checking, well past the last observed exception; the range $256 < n \le 10^6$ rests on the external scan (the same integer test, run with big-integer arithmetic), and is not machine-checked. On the one-sided Waring event the record is far longer: the condition of Remark 11.3 is verified for all $n \le 2^{36} + 2^{34}$ [KW90, Cum25]. The run-length heuristic — an exception demands a run of equal binary digits of length about $0.415\,n$ in 3^n, an event of probability about $2^{-0.415n}$ — suggests $\mathcal{E} = \{1, 2, 3, 4, 7\}$ is complete; summing the tail beyond 10^6 leaves a residual expectation below 10^{-10^5}.

6. Finiteness by the Subspace theorem

From here on $(p, q, c) = (3, 2, 3/4)$ unless stated otherwise. An exception n is a near-integrality event for the S-unit $u = (3/2)^n = 2^{-n}3^n$ of the group $\Gamma = \langle 2, 3\rangle$, with multiplier $\delta = 1$ and $q = 1$ in the notation of (2): its height is $H\big((3/2)^n\big) = 3^n$, and the exception bound gives

$$\|(3/2)^n\| \;<\; (3/4)^n \;=\; \big(3^n\big)^{-\varepsilon^*}.$$

No headroom is needed: the failure inequality is already strict, so the membership below runs at the sharp exponent itself. (The first version of this development halved the exponent here; the halving was needless.)

Lemma 6.1 (membership)**.** *Let $n \ge 1$, let ϵ be real, and suppose $\|(3/2)^n\| < (3^n)^{-\epsilon}$. Then $\big(1, (3/2)^n\big) \in \mathcal{A}(1, \epsilon)$.*

Proof. The conditions of (2) hold one by one: $(3/2)^n > 1$; the value is not an integer and keeps positive distance from $\mathbb{Z}$, because $2^n(3/2)^n = 3^n$ is odd; and the smallness hypothesis is the assumed bound, with $H((3/2)^n) = 3^n$ and the factor $q^{-1-\epsilon} = 1$. □

Distinct n give distinct pairs, so the map $n \mapsto (1, (3/2)^n)$ injects exception indices into $\mathcal{A}(1, \varepsilon^*)$.

Theorem 6.2 (Mahler [Mah57, Thm. 2], re-derived)**.** *The set $\mathcal{E}$ is finite. Equivalently, there is N_0 with $\|(3/2)^n\| \ge (3/4)^n$ for all $n \ge N_0$.*

Proof. By Lemma 6.1 the exceptions inject into $\mathcal{A}(1, \varepsilon^*)$, which is finite by Theorem 1.2; in the formalization the only Diophantine input of this chain is Theorem 1.1. □

A second, independent proof of Theorem 6.2 — resting on the quantitative Theorem 1.3 instead of the qualitative Theorem 1.1 — follows from the line cover together with the elementary per-line confinement; it is given as Corollary 10.1.

7. The line cover

7.1. The frame. An exception n is turned into a solution of the system of Theorem 1.3 at $\xi = 1$, $S_1 = \{2\}$, $S_2 = \{3\}$ by the *frame point*

$$(x, y) \;:=\; \bigl(m_n 2^n,\; 3^n\bigr) \;\in\; \mathbb{Z}^2.$$

Its three local conditions are:

- *archimedean*: $\bigl|1 - x/y\bigr| = |k_n|/3^n < (3/2)^n/3^n = 2^{-n} = y^{-\theta}$ — the exception inequality itself, divided by 3^n;
- 2-*adic*: $2^n \mid x$, so $|x|_2 \leq 2^{-n} = y^{-\theta}$ — no exception hypothesis is needed, the frame point is 2-adically small by construction;
- 3-*adic*: $|y|_3 = 3^{-n} = y^{-1}$ — the whole exponent 1 is spent at 3.

The exponent budget is

$$\theta + \theta + 1 \;=\; 2 + \varepsilon^*, \tag{9}$$

an identity — both sides are $\log 12/\log 3$ — so the frame meets the budget of Theorem 1.3 at $\epsilon = \varepsilon^*$ with nothing to spare and nothing to give away: the sharp exponent is forced. The height condition reads $3^n > \max(2, 2^{4/\varepsilon^*})$, and $2^{4/\varepsilon^*} = 39659.1\ldots$ lies between $3^9 = 19683$ and $3^{10} = 59049$, so it holds from $n = 10$ on; the formalization avoids decimal logarithm bounds here, reducing the threshold to the integer certificate $16 \cdot 3^{10} = 944\,784 < 1\,048\,576 = 4^{10}$.

Encoding the line through the origin spanned by the frame point by its slope, we write

$$\ell_n \;:=\; \frac{m_n 2^n}{3^n} \;\in\; \mathbb{Q}.$$

7.2. The line theorem.

Theorem 7.1 (line theorem; Theorem A)**.** *There is a set $R \subset \mathbb{Q}$ of at most $K(\varepsilon^*)$ slopes such that every exception $n \geq 10$ has $\ell_n \in R$, where*

$$K(\varepsilon^*) \;=\; 1\,856\,360\,182\,227 \;\leq\; 1.86 \cdot 10^{12},$$

the decimal bound being certified (Section 13). Consequently the set of slopes realized by exceptions $n \geq 10$ has at most $K(\varepsilon^*)$ elements.

Proof. Apply Theorem 1.3 at $d = 1$, $\xi = 1$, $S_1 = \{2\}$, $S_2 = \{3\}$ with the exponents $(f_\infty, f_2, f_3) = (\theta, \theta, 1)$, whose sum is $2 + \varepsilon^*$ by (9). The three conditions of the system hold for the frame point of every exception, as computed above, and the height condition holds for $n \geq 10$. The frame point lies on the line of slope ℓ_n, so ℓ_n belongs to the produced set of at most $K(\varepsilon^*)$ slopes. □

Nothing in Theorem 7.1 bounds the number of *exceptions*: passing from lines to a count needs a bound on the number of exceptions per line, which is the open per-line problem ([BE08, Rem. 7.4]; cf. [EF13, p. 6]). Sections 8 and 9 collect what is elementary about it, and Section 10 states the conditional counts.

8. Lines and towers

Two notions of tower now coexist, and the distinction matters. A *relation-tower* is a maximal set of exceptions whose frame points are collinear — a fibre of the slope map $n \mapsto \ell_n$; a *threshold-tower* is a chain of exceptions linked in the sense of Section 3.[3] The line cover of Theorem 7.1 counts relation-towers; the $O(\log N)$ count of Section 4 counts threshold-towers, a refinement of the partition into relation-towers.

8.1. Collinearity forces the scaling relation.

Proposition 8.1. *Let $p, q \geq 1$, $a \leq b$, and suppose the frame points of a and b are collinear, i.e. $\ell_a = \ell_b$ in the case $(3, 2)$. Then $p^{\,b-a}m_a = q^{\,b-a}m_b$ — the linkage relation of Lemma 3.2, with no threshold hypothesis. Consequently $k_b = p^{\,b-a}k_a$, and, when $\gcd(p, q) = 1$, there is $\mu \in \mathbb{Z}$ with $m_a = q^{\,b-a}\mu$ and $m_b = p^{\,b-a}\mu$; in particular $q^{\,b-a} \mid m_a$.*

Proof. The vanishing cross product reads $p^a\, m_b q^b = m_a q^a\, p^b$; cancelling $p^a q^a$ leaves the linkage relation, and Proposition 3.3 does the rest. □

So collinearity alone forces the full scaling structure: all exceptions on one line through the origin are p^d-scalings of a single reduced base datum μ. One line carries one *scaling family* — but possibly several threshold-towers, as the pair $\{2, 3\}$ shows.

8.2. **Confinement.** Collinearity is expensive for the upper member: on a line the residues scale, $k_b = 3^{\,b-a}k_a$, while $|k_a| \geq 1$ (the residue $k_a = 3^a - m_a 2^a$ is odd for $a \geq 1$, hence nonzero) and the exception at b demands $|k_b| < (3/2)^b$. We state the resulting gap for a general residue bound, since the Waring events of Section 11 carry the constant $K = 4/3$ in place of 1.

Lemma 8.2 (gap on a line)**.** *Let $1 \leq a < b$ lie on one line, and suppose $|k_b| < K\,(3/2)^b$ for a real K. Then*

$$2^{\,b-a} \;<\; K\,(3/2)^a.$$

In particular, for exceptions ($K = 1$): $(b-a)\log 2 < a\log(3/2)$, so a companion of a satisfies $b < a\log_2 3 = 1.58496\ldots \cdot a$, and crudely $b < 2a$; for $K \leq 2$, still $b \leq 2a$.

Proof. By Proposition 8.1, $|k_b| = 3^{\,b-a}|k_a| \geq 3^{\,b-a}$; comparing with the bound $K(3/2)^b$, written as $K(3/2)^a(3/2)^{\,b-a}$, and dividing by $(3/2)^{\,b-a}$ gives the display. □

Proposition 8.3. *Every line carries finitely many exceptions — with no Diophantine input: any member a of the fibre confines the whole fibre to $[0, 2a]$. Over its least member a, a fibre has at most a elements, being contained in $[a, 2a)$.*

[3] Again coined terminology. Linkage is *sufficient* for collinearity (Proposition 8.1 with Lemma 3.2), never necessary; the pair $\{2, 3\}$ of Section 9 realizes the gap between the two notions.

The span $b - a$ of a scaling family is bounded through Proposition 8.1 by the 2-adic valuation of m_a as well, the observation that powers the stratified counts of Section 12.

Remark 8.4 (the per-line problem, and a dominated alternative). What is *not* proved here is a bound on the fibres uniform in the line; that is the open problem of [BE08, Rem. 7.4] and [EF13, p. 6], and it is the hypothesis H of Theorems B and C. One alternative route was checked and is recorded to prevent its blind pursuit: bounding the multiplicity of a line through $v_2(m_a)$ by a linear form in two logarithms (Baker–type, or Yu's p-adic bounds) yields a height of order $b \log b$ — worse than the elementary $0.585\, b + 1$ of Lemma 8.2, so the transcendence route is dominated by the elementary one.

9. The census

Through the integer form (8) of m_n, the slope test $\ell_a = \ell_b$ becomes the identity $3^a(m_b 2^b) = m_a 2^a\, 3^b$ of natural numbers, decidable in the kernel. The five certified exceptions carry the frame points

n	m_n	$(x, y) = (m_n 2^n, 3^n)$	ℓ_n
1	2	$(4, 3)$	$4/3$
2	2	$(8, 9)$	$8/9$
3	3	$(24, 27)$	$8/9$
4	5	$(80, 81)$	$80/81$
7	17	$(2176, 2187)$	$2176/2187$

Theorem 9.1 (census)**.** *On the certified exceptions* $\{1, 2, 3, 4, 7\}$*:*

(i) *two members share a line if and only if they are equal or form the pair* $\{2, 3\}$*; the partition into relation-towers is* $\{1\}, \{2, 3\}, \{4\}, \{7\}$ *— four lines carrying five exceptions;*

(ii) *the five are pairwise* not *linked — five threshold-towers, each member its own base.*

Both parts are verified by kernel computation.

The pair $\{2, 3\}$ is the witness separating the two tower notions: $(24, 27) = 3 \cdot (8, 9)$ is a genuine scaling relation with $\mu = 1$ ($m_2 = 2 = 2^1\mu$, $m_3 = 3 = 3^1\mu$), while $2 \cdot (3/4)^2 \cdot 3 = 27/8 > 1$ puts the pair outside the reach of the gap principle of Section 3. The observed height of a line is therefore 2, not 1; the span 1 saturates the 2-adic cap of Proposition 8.1, since $v_2(m_2) = v_2(2) = 1$.

The confinement of Lemma 8.2 makes the census complete: a companion of an exception $b \leq 7$ would lie below $2b \leq 14$, hence inside the certified range, hence be one of the five. The known relation-towers are therefore determined outright:

Corollary 9.2. *The line of slope* $8/9$ *carries the exceptions* 2 *and* 3 *and no others; the lines of* $n = 1, 4, 7$ *carry that exception alone. No known relation-tower can grow.*

In particular, the hypothesis of Theorem B — a bound on the fibres above $n = 256$ — concerns a range in which the 10^6 scan finds no exception at all.

10. The conditional counts and the practical bound

The three layers now meet: the certified segment (5 exceptions in $[1, 256]$), the line cover (at most $K(\varepsilon^*)$ lines for $n \geq 10$), and the per-line confinement (each fibre finite). The first dividend is finiteness with a new footprint.

Corollary 10.1 (finiteness, quantitative route)**.** *The set $\mathcal{E}$ is finite. In the formalization the only Diophantine input of this proof is Theorem 1.3: the exceptions split into the certified segment $[1, 256]$ and a subset of a finite union of fibres, each finite by Proposition 8.3.*

Together with Theorem 6.2 this gives two machine-checked proofs of Mahler's theorem with disjoint Diophantine inputs — the qualitative Subspace theorem for one, the quantitative Ridout count for the other.

Theorem 10.2 (conditional count; Theorem B)**.** *Let $H \in \mathbb{N}$ and suppose every line of $\mathbb{Q}^2$ carries at most H exceptions with $n \geq 257$. Then*

$$\#\mathcal{E} \;\leq\; 5 + H \cdot K(\varepsilon^*) \;\leq\; 5 + H \cdot 1.86 \cdot 10^{12}.$$

Proof. Split at 256. The segment $[1, 256]$ contributes the five certified exceptions (Proposition 5.2). The exceptions $n \geq 257$ map under $n \mapsto \ell_n$ into the set R of Theorem 7.1, of size at most $K(\varepsilon^*)$, and each fibre of the map has at most H elements by hypothesis; a map with fibres of size at most H compresses cardinality by at most that factor. □

Remark 10.3 (the height hypothesis)*.* The hypothesis on H is genuine: it is the per-line problem of Remark 8.4, which this paper does not solve. The elementary confinement bounds each fibre in terms of its own least element (Proposition 8.3) but not uniformly. Empirically the fibres above 256 are all empty, and every known fibre has at most two elements (Theorem 9.1).

10.1. **The practical bound.** The elementary arm of the first version survives unchanged and combines with the new Diophantine arm. Write

$$\mathcal{F}(N) := \{\, n \in \mathcal{E} : n \leq N \,\}, \qquad \mathcal{B}(N) := \{\, 1 \leq b \leq N : \ b \text{ a tower base} \,\}.$$

Lemma 10.4 (covering)**.** *Every exception $n \geq 1$ has a tower base $b \leq n$.*

Proof. Strong induction. If n is itself a base, done. Otherwise some exception $a < n$ is linked to n; the linkage condition $2(3/4)^a 3^{n-a} \leq 1$ fails at $a = 0$, so $a \geq 1$, and the base of a serves. □

By Lemma 10.4 (and choice) there is an assignment $\sigma \colon \mathcal{E} \to \bigcup_N \mathcal{B}(N)$ with $\sigma(n) \leq n$ a tower base for every exception n; if every fibre of σ on $\mathcal{F}(N)$ has at most H_{thr} elements, then $F(N) \leq H_{\mathrm{thr}} \cdot \#\mathcal{B}(N)$, and Corollary 4.3 makes the base count explicit.

Theorem 10.5 (practical bound)**.** *Let H bound the fibres of the slope map above $n = 257$ as in Theorem 10.2, and let H_{thr} bound the fibres of σ on $\mathcal{F}(N)$. Then*

$$F(N) \;\leq\; \min\Big(5 + H \cdot K(\varepsilon^*),\; H_{\mathrm{thr}} \cdot \#\mathcal{B}(N)\Big), \qquad \#\mathcal{B}(N) \;\leq\; 12 + \log_{6/5} N.$$

Both arms are conditional, on different height parameters: the Diophantine arm on the relation-height H (uniform in N, constant astronomically large), the elementary arm on the threshold-height H_{thr} (logarithmic in N). In the certified range the threshold-height is 1 and the relation-height is 2 (Theorem 9.1); under those readings the second arm gives $F(N) \lesssim 12 + 5.49 \ln N$, against the observed $F(10^6) = 5$.

11. The application to Waring's problem

We now prove Theorem C, following the chain (3.22)–(3.24) of [Bug12, §3.7]. Write $q_n = \lfloor (3/2)^n \rfloor$ and $r_n = 3^n \bmod 2^n$, so that $3^n = q_n 2^n + r_n$, and let *Dickson's condition* be

$$(D_n) \qquad\qquad r_n + q_n + 2 \;\leq\; 2^n.$$

The two classical evaluation theorems, and the unconditional secondary bound, are the remaining cited inputs.

Theorem 11.1 (Dickson [Dic36]; see [Bug12, §3.7])**.** *Let $n \geq 2$ and suppose (D_n). Then $g(n) = 2^n + q_n - 2$.*

Theorem 11.2 (Rubugunday [Rub42], Niven [Niv44]; see [Bug12, §3.7])**.** *Let $n \geq 6$ and suppose (D_n) fails. Then, with $q'_n = \lfloor (4/3)^n \rfloor$,*

$$q_n q'_n + q_n + q'_n \;\geq\; 2^n, \qquad g(n) \;=\; 2^n + q_n + q'_n - \theta_n,$$

where $\theta_n = 2$ if $q_n q'_n + q_n + q'_n = 2^n$ and $\theta_n = 3$ otherwise.

Remark 11.3 (the frontier between the two regimes). The literature draws the frontier between Theorems 11.1 and 11.2 at slightly different places: [Bug12, (3.23)] states the ideal-formula condition as $r_n + q_n \leq 2^n$, while [KW90] and [DD90, Thm. A] use $r_n + q_n < 2^n$, and the theorem is due independently to Dickson [Dic36] and Pillai [Pil36]. The formalization adopts the two-unit buffer (D_n), which implies every variant, so Theorem 11.1 as stated is a consequence of each published form; the price is that the fallback branch, as recorded, covers the two borderline values $r_n + q_n \in \{2^n - 1, 2^n\}$, which lie beyond the letter of [Bug12, (3.23)]. No n is known, or expected, to attain the borderline, and the computations of [KW90, Cum25] cover every variant of the frontier for all $n \leq 2^{36} + 2^{34}$. The counting results use only the ideal branch and are unaffected; only the two-branch evaluation of $g(n)$ (Corollary 11.10) invokes the fallback.

Theorem 11.4 (Zou [Zou13]; Pupyrev [Pup14])**.** *For every $n \geq 6$,*

$$\|(4/3)^n\| \;\geq\; (4/9)^n.$$

This settles unconditionally the secondary condition in the regime of Theorem 11.2 (Bennett's condition on $\|(4/3)^n\|$); it was proved twice, independently, by Zou and by Pupyrev; neither text is journal-published, so we cite both. The formalization records the non-strict form, which is what is used.

11.1. **The bridge and finiteness.** The bridge from Waring to the counting frame is the observation, recorded as inequality (3.24) in [Bug12], that a failure of Dickson's condition is a near-integrality event for $(3/2)^n$.

Lemma 11.5 (the (3.24) trace)**.** *Let $n \geq 3$ and suppose (D_n) fails. Then*

$$\|(3/2)^n\| \;<\; (3/4)^{n-1}, \qquad \textit{equivalently} \qquad |k_n| \;<\; \tfrac{4}{3}\,(3/2)^n.$$

Proof. Failure of (D_n) means $2^n - r_n \leq q_n + 1$: measuring the distance to the integer $q_n + 1$,

$$\|(3/2)^n\| \;\leq\; \frac{2^n - r_n}{2^n} \;\leq\; \frac{q_n + 1}{2^n} \;\leq\; \left(\frac{3}{4}\right)^n + \left(\frac{1}{2}\right)^n \;<\; \left(\frac{3}{4}\right)^{n-1},$$

using $q_n \leq (3/2)^n$ and, in the last step, $(1/2)^n < \frac{1}{3}(3/4)^n$ for $n \geq 3$. □

Since $(3/4)^{n-1} < (3^n)^{-\varepsilon_0}$ for $n \geq 3$ with $\varepsilon_0 = \varepsilon^*/2$, Lemma 6.1 puts the failures of (D_n) into $\mathcal{A}(1, \varepsilon_0)$, and Theorem 1.2 gives:

Proposition 11.6. *The set of $n \geq 1$ failing (D_n) is finite; hence $g(n) = 2^n + q_n - 2$ for all but finitely many n, and for all sufficiently large n.*

(This qualitative lane is the one place the halved exponent ε_0 survives: the weaker event $(3/4)^{n-1}$ needs an exponent strictly below ε^*, and any such value serves for finiteness.)

11.2. **The kernel arm and $g(2) = 4$.** For the quantitative count the low range is not absorbed into a constant but certified:

Proposition 11.7. *Dickson's condition (D_n) holds for every $3 \leq n \leq 256$, by kernel computation. Below the cutoff 257 the only failures of (D_n) are $n = 1$ and $n = 2$.*

Proposition 11.8. *$g(2) = 4$. Consequently $n = 2$, which fails (D_n), satisfies the ideal formula anyway: $2^2 + \lfloor (3/2)^2 \rfloor - 2 = 4 = g(2)$.*

Proof. Every natural number is a sum of four squares (Lagrange; the four-square theorem is applied from the Mathlib library), and 7 is not a sum of three squares, by a finite check on squares up to 2^2. □

So every exception to the ideal formula with $n \geq 2$ lies above the cutoff, where the line cover applies; no additive constant remains.

11.3. **The Waring frame and the count.** A failure of (D_n) satisfies the weaker smallness event $\|(3/2)^n\| < (3/4)^{n-1}$, so its frame point loses a factor $4/3$ at the archimedean place. Buying the factor back costs an exponent $\varepsilon^*/257$: for $n \geq 257$,

$$(3^n)^{\varepsilon^*/257} = (4/3)^{n/257} \;\geq\; \tfrac{4}{3},$$

so the archimedean condition holds at the exponent $\theta_W := \theta - \varepsilon^*/257$, and the budget becomes

$$\theta_W + \theta + 1 \;=\; 2 + \varepsilon_W, \qquad \varepsilon_W \;:=\; \varepsilon^*\bigl(1 - \tfrac{1}{257}\bigr) \;=\; 0.26084\ldots.$$

The cutoff 257 is where the kernel arm stops; the 0.4% shave of the exponent costs 1.1% in the constant, $K(\varepsilon_W) = 1\,876\,339\,827\,243$ against $K(\varepsilon^*) = 1\,856\,360\,182\,227$.

Theorem 11.9 (Waring line cover and count; Theorem C)**.** *The failures of (D_n) with $n \geq 257$ lie on at most $K(\varepsilon_W) \leq 1.88 \cdot 10^{12}$ lines of $\mathbb{Q}^2$. If no line carries more than H of them, then*

$$\begin{aligned} \#\{n \geq 1 : (D_n) \textit{ fails}\} \;&\leq\; 2 + H \cdot K(\varepsilon_W), \\ \#\{n \geq 2 : g(n) \neq 2^n + q_n - 2\} \;&\leq\; H \cdot K(\varepsilon_W). \end{aligned}$$

Proof. The cover is Theorem 1.3 at the frame $(\theta_W, \theta, 1)$, the height threshold holding from $n = 257$ with enormous margin. For the first count, the two genuine low failures $\{1, 2\}$ give the summand 2 (Proposition 11.7); above the cutoff, the fibre argument of Theorem 10.2 applies, each fibre being finite by the $K = 4/3$ case of Lemma 8.2. For the second count, every exception to the ideal formula fails (D_n) (Theorem 11.1), the kernel clears $[3, 256]$, and $n = 2$ is cleared by Proposition 11.8, so every exception lies above the cutoff. □

11.4. **The value of $g(n)$, both branches.**

Corollary 11.10 (the value dichotomy, with its count)**.** *For every $n \geq 6$: either (D_n) holds and $g(n) = 2^n + q_n - 2$, or (D_n) fails and $g(n) = 2^n + q_n + q'_n - \theta_n$ with $\theta_n \in \{2, 3\}$ as in Theorem 11.2. Under the per-line hypothesis of Theorem 11.9, the second branch occurs for at most $H \cdot K(\varepsilon_W)$ values of $n \geq 2$.*

The dichotomy clause is unconditional and needs only the two cited evaluation theorems; the count clause carries the per-line hypothesis and the line-cover axiom. The formalization states the two clauses in one statement so that the footprint of each is visible (Appendix A). For the fallback branch to have its classical meaning, the secondary condition on $\|(4/3)^n\|$ must hold; by Theorem 11.4 it holds unconditionally for $n \geq 6$, with nothing to count.

12. Span-stratified line counts

The line cover of Theorem 7.1 spends the whole budget $2+\varepsilon^*$ on a bare exception. An exception that is the base of a tall relation-tower carries more: if $a<b$ lie on one line then $2^{b-a} \mid m_a$ (Proposition 8.1), so the frame point $x = m_a 2^a$ is 2-adically small to depth $a+(b-a)$, not merely a. Call $b-a$ the *span*. When the span is at least γa, the 2-adic condition sharpens from $|x|_2 \le y^{-\theta}$ to $|x|_2 \le y^{-\theta(1+\gamma)}$, and the budget becomes

$$\theta + \theta(1+\gamma) + 1 \;=\; 2 + \big(\varepsilon^* + \gamma\theta\big) :$$

the same Theorem 1.3 now counts lines at the larger exponent $\varepsilon^*+\gamma\theta$, where its constant is smaller.

Theorem 12.1 (stratified line cover; Theorem D)**.** *Let $\gamma \ge 0$. There is a set of at most $K(\varepsilon^*+\gamma\theta)$ slopes containing ℓ_a for every exception $a \ge 10$ that admits a collinear companion $b > a$ with $b-a \ge \gamma a$. In particular the set of slopes of such bases has at most $K(\varepsilon^*+\gamma\theta)$ elements, and at $\gamma = 1/4$,*

$$K(\varepsilon^* + \theta/4) \;=\; 537\,048\,098\,048 \;\le\; 5.38\cdot 10^{11},$$

a factor 3.46 *below $K(\varepsilon^*)$.*

Proof. Apply Theorem 1.3 at the frame $(\theta, \theta(1+\gamma), 1)$. The archimedean and 3-adic conditions are those of Theorem 7.1; the 2-adic condition is the sharpened one, from $2^{b-a} \mid m_a$ and $b-a \ge \gamma a$; the height threshold at the larger exponent is implied by the one at ε^*, since $2^{4/\epsilon}$ decreases in ϵ. □

Only collinearity of the companion is used — b need not itself be an exception. The useful range of γ is capped by the confinement: for a collinear pair with b an exception, Lemma 8.2 gives $(b-a)\log 2 < a\log(3/2)$, so

$$\gamma \;<\; \frac{\log(3/2)}{\log 2} \;=\; 0.58496\ldots,$$

and the family of counts interpolates between Theorem 7.1 at $\gamma = 0$ and the empty statement at the ceiling.

Remark 12.2 (the corrected form of a natural mistake). One might try to stratify by a fixed *absolute* surplus instead — counting the events $\|(3/2)^n\| < 2^{-d}(3/4)^n$ at the exponent $\varepsilon^*+\delta$. That inequality runs the wrong way: the frame demands $2^{-d} \le 3^{-n\delta}$, i.e. $n \le d\log 2/(\delta\log 3)$, an *upper* bound on n — a fixed absolute surplus is worth less and less as n grows, and the set it captures is finite for trivial reasons. What survives is the span-proportional stratification above, and it must sharpen the 2-adic exponent, not the archimedean one: the surplus is 2-adic in origin. A third variant — a count decaying in the absolute height of a tower, uniformly in the base — is equivalent to the open per-line problem and cannot come from Theorem 1.3 at all.

13. Certification of the constants

The line count (3) is a ceiling of a transcendental expression; the results above are only as quotable as the decimal bounds one can prove for it. The formal development certifies

$$K(\varepsilon^*) \leq 1.86 \cdot 10^{12}, \qquad K(\varepsilon_W) \leq 1.88 \cdot 10^{12}, \qquad K(\varepsilon^* + \theta/4) \leq 5.38 \cdot 10^{11},$$

against the true values 1 856 360 182 227, 1 876 339 827 243 and 537 048 098 048. Everything is derived from certified enclosures of $\log 2$ and $\log 3$ (nine to ten decimal digits, available in the Mathlib library) through $\log(4/3) = 2\log 2 - \log 3$: the sensitive step is $1 + 1/\varepsilon^* \leq 4.8189$, which amounts to $4.8189 \log 3 \leq 7.6378 \log 2$ and clears by $1.6 \cdot 10^{-5}$ — the chain divides by the small $\log(4/3) = 0.2876\ldots$; the nested logarithm is handled by anchoring at a power of two or at 6 and applying $\log(1+t) \leq t$, e.g. $\log 8.63431 \leq 3\log 2 + 0.63431/8$. Only rational arithmetic on the enclosures is involved; no floating-point value enters any proof, and the kernel computations of Sections 5, 9 and 11 use the proof checker's exact integer arithmetic with no compiled-code shortcuts.

14. Acknowledgements

The author thanks two anonymous reviewers, whose reports identified the overstated packaging of Corollary 5.2 of [BE08] in an earlier version of this manuscript and prompted the repairs and extensions presented here. The author utilized Claude Code as an AI coding assistant to aid in the Lean 4 formalization of the proofs presented in this paper. The author directed and reviewed all generated code and takes full responsibility for the mathematical integrity and final content of the work.

Appendix A. Formalization

Every statement above was verified in Lean 4. The development lives under `BB13/`, in the files `Basic`, `GapPrinciple`, `TowerCount`, `M0Verify`, `Subspace`, `LineCover`, `LineTower`, `Census`, `FailureCount`, `PracticalBound`, `Waring`, `SpanStrata` and `Constants`, with general-purpose lemmas in `ForMathlib/` and the cited inputs in `CITED/`. In the tables, "std3" abbreviates the ambient axioms of classical Lean (propositional extensionality, choice, quotient soundness), and the bracketed tags name the cited axioms of §1.5 and §11: [ES] `Subspace.evertseSchlickewei` (Theorem 1.1); [LC] `BugeaudEvertse.ridout_line_cover` (Theorem 1.3); [ID] `BB13.waringNumber_ideal_of_dickson` (Theorem 11.1); [RN] `BB13.waringNumber_fallback` (Theorem 11.2); [ZP] `BB13.distToNearestInt_four_thirds_ge` (Theorem 11.4). Theorem 1.2 is not an axiom: it is derived from [ES] in `CITED/CorvajaZannierProof.lean`. Every entry below is fully proved in Lean from the listed axioms; footprints are the output of `#print axioms`. The Lean development names exceptions "failures"; the set $\mathcal{A}$ is `CZ.approxSet`, the slope ℓ_n is `linePoint`, relation-towers are fibres of `linePoint` (`SameTower`, `lineFibre`), and (3) is `BugeaudEvertse.lineBound`.

Elementary layer and certified segment (§2–§5).

Statement	Lean identifier	File	Status
Def. 2.1	`Mnum`, `resid`, `IsFailure`	`Basic`	def
Eq. (4)	`isFailure_iff`	`Basic`	std3
Eq. (5), (1)	`epsilon`, `epsStar`, `theta`	`Basic`	def
Lem. 3.1	`gap_identity`	`GapPrinciple`	std3
Lem. 3.2	`linkage`, `linkage_of_linkable`	`GapPrinciple`	std3
Prop. 3.3	`link_resid/_dvd/_scaling/_quality`	`GapPrinciple`	std3
linked, tower base	`Linkable`, `IsTowerBase`	`Basic`	def
Lem. 3.4	`gap_of_nonlink`	`GapPrinciple`	std3
Lem. 4.1	`card_le_logb_of_geomGap(_above)`	`ForMathlib`	std3
Thm. 4.2	`towerBases_card_le`	`TowerCount`	std3
Cor. 4.3	`towerBases_card_le_three_halves`	`TowerCount`	std3†
Lem. 5.1	`isFailure_iff_failNat`	`M0Verify`	std3
Prop. 5.2	`failures_up_to_256`, `failuresUpTo_256_eq/_ncard`	`M0Verify`	std3
Eq. (8)	`mNat`, `Mnum_eq_mNat`	`M0Verify`	std3
—	`failNat_initial_segment` ($n \leq 19$)	`Basic`	propext only

†with the numerical hypotheses $\log 2 \geq 0.6931$ and $\log 3 \leq 1.0987$ carried explicitly, as stated after Corollary 4.3; both are discharged by the enclosures of Section 13 (`Real.log_two_gt_d9`, `BB13.log_three_le`).

Finiteness and the line cover (§6–§8).

Statement	Lean identifier	File	Status
Thm. 1.3	`BugeaudEvertse.ridout_line_cover`	`CITED`	cited axiom [LC]
Eq. (3)	`BugeaudEvertse.lineBound`	`CITED`	def
Lem. 6.1	`mem_approxSet_of_distBound`	`Subspace`	std3
Thm. 6.2	`failures_finite`, `failures_bounded`	`Subspace`	std3 + [ES]
frame point, ℓ_n	`frameX`, `frameY`, `linePoint`	`LineCover`	def
frame conditions	`frame_archimedean/_two_adic/_three_adic/_height`	`LineCover`	std3
Eq. (9)	`theta_add_theta_add_one`	`Basic`	std3
height certificate	`four_log_two_lt`, `two_rpow_four_div_epsStar_lt`	`Basic`	std3
Thm. 7.1	`failures_line_cover`, `towersAbove_card_le`	`LineCover`	std3 + [LC]
Prop. 8.1	`linePoint_eq_iff_collinear`, `collinear_imp_link/_scaling/_resid`, `sameTower_resid/_dvd`	`LineTower`, `Subspace`	std3
Lem. 8.2	`sameTower_gap(_of_bound)`, `sameTower_span_lt`, `sameTower_lt_two_mul`, `sameTower_le_two_mul_of_bound`	`LineTower`	std3
Prop. 8.3	`lineFibre_finite`, `sameTower_card_le_of_min`	`LineTower`	std3

Census and counts (§9–§10).

Statement	Lean identifier	File	Status
slope test	`sameTower_iff_nat`	`Census`	std3
Thm. 9.1(i)	`census_sameTower`, `sameTower_two_three`	`Census`	std3
Thm. 9.1(ii)	`census_not_linkable`, `not_linkable_two_three`	`Census`	std3
Cor. 9.2	`lineFibre_two_eq`, `lineFibre_eq_singleton`	`Census`	std3
Cor. 10.1	`failures_high_finite`, `failures_finite_of_lineCover`	`FailureCount`	std3 + [LC]
Thm. 10.2	`failures_card_le_of_heightBound`	`FailureCount`	std3 + [LC]
— (decimal form)	`failures_card_le_decimal`	`Constants`	std3 + [LC]
Lem. 10.4	`exists_towerBase_le`	`PracticalBound`	std3
fibre count	`failuresUpTo_card_le_towers`, `Set.ncard_le_mul_ncard_image`	`PracticalBound`, `ForMathlib`	std3
Thm. 10.5	`practical_bound_min`; `practical_bound_log`, `towerBasesUpTo_card_le_log`	`FailureCount`, `PracticalBound`	std3 + [LC]; std3†

Waring and strata (§11–§13).

Statement	Lean identifier	File	Status
$g(n)$, (D_n)	`Nat.waringNumber`, `DicksonCond`	`ForMathlib`, `Waring`	def
Thm. 11.1	`waringNumber_ideal_of_dickson`	`Waring`	cited axiom [ID]
Thm. 11.2	`waringNumber_fallback`	`Waring`	cited axiom [RN]
Thm. 11.4	`distToNearestInt_four_thirds_ge`	`Waring`	cited axiom [ZP]
Lem. 11.5	`notDickson_imp_distBound`, `abs_resid_lt_of_notDickson`	`Waring`	std3
Prop. 11.6	`dickson_exceptions_finite`; `waring_exceptions_finite`, `waring_ideal_formula_eventually`	`Waring`	std3 + [ES] (+ [ID])
Prop. 11.7	`dicksonCond_of_le_256`	`Waring`	std3
Prop. 11.8	`Nat.waringNumber_two`	`ForMathlib`	std3
ε_W, θ_W, budget	`epsW`, `thetaW`, `thetaW_add_theta_add_one`	`Waring`	def, std3
Thm. 11.9	`notDickson_line_cover`,	`Waring`	std3 + [LC]
	`dickson_exceptions_card_le_of_heightBound`,		std3 + [LC]
	`waring_exceptions_card_le_of_heightBound/_decimal`	`Waring`, `Constants`	std3 + [LC] + [ID]
Cor. 11.10	`waring_value_dichotomy(_count)`	`Waring`	std3 + [ID] + [RN] (+ [LC])
secondary condition	`secondary_condition_holds`	`Waring`	std3 + [ZP]
sharp 2-adic	`frame_two_adic_sharp`, `two_pow_le_rpow_of_span`	`SpanStrata`	std3
Thm. 12.1	`tall_towers_line_cover`, `tall_towerBases_card_le`, `tall_towerBases_card_le_decimal`	`SpanStrata`	std3 + [LC]
γ-ceiling	`span_ratio_lt`	`SpanStrata`	std3
§13	`lineBound_epsStar_le`, `lineBound_epsW_le`, `lineBound_epsStar_quarter_le`	`Constants`, `SpanStrata`	std3

†the logarithmic form carries the same two numerical hypotheses as Corollary 4.3.

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