FRSB IN THE SK SPIN GLASS: CONVERGENCE TO FULL-INTERVAL SUPPORT AT ZERO TEMPERATURE

CHATGPT

Abstract. We study full replica symmetry breaking in the zero-field Sherrington–Kirkpatrick model at finite and zero temperature. At every inverse temperature above the critical point, we prove that the support of the Parisi measure is an interval and that the distance between its upper endpoint and one is at most the reciprocal of the inverse temperature. Consequently, these supports converge in Hausdorff distance to the full overlap interval as the temperature tends to zero. We also prove that the Stieltjes measure associated with the zero-temperature Parisi minimizer has full support after closure at the endpoint one. Both conclusions follow from a common exclusion of constant-order-parameter gaps. The finite-temperature argument uses a detilted entrance density and Lopatto's slope-coordinate inequalities, while the zero-temperature argument uses a forward factor, a corresponding slope invariant, and a boundary-layer estimate near the terminal time.

Acknowledgement. This paper was written entirely by ChatGPT 5.6, with prompts supplied by two human prompters: Hong-Bin Chen and another person who prefers to remain anonymous.

1. Introduction and main results

The Sherrington–Kirkpatrick model was introduced in [11] as a mean-field model of a spin glass. Parisi subsequently proposed the hierarchical order parameter and the variational formula that underlie replica symmetry breaking; see [10]. Guerra established the replica-symmetry-breaking interpolation bound in [6, Theorem 3], and Talagrand proved the matching Parisi formula for the SK and mixed even p-spin models in [12, Theorem 1.1]. Panchenko later extended the formula to general mixed p-spin models in [9, Theorem 1]. The present paper concerns the structure of the unique variational minimizer rather than the value of the variational formula itself.

For a Borel probability measure μ on $[0,1]$, set

$$\alpha_\mu(s) = \mu([0,s]), \qquad 0 \le s \le 1.$$

In the normalization used throughout the finite-temperature part, the Parisi PDE is

$$\partial_s u_\mu(s,x) = -\frac{\beta^2}{2}\left(u_{\mu,xx}(s,x) + \alpha_\mu(s) u_{\mu,x}(s,x)^2\right), \qquad u_\mu(1,x) = \log\cosh x,$$

and the Parisi functional is

$$\mathcal{P}_\beta(\mu) = \log 2 + u_\mu(0,0) - \frac{\beta^2}{2}\int_0^1 s\alpha_\mu(s)\,\mathrm{d}s. \tag{1.1}$$

By [2, Theorem 1], the functional in (1.1) has a unique minimizer μ_β. We write

$$S_\beta = \operatorname{supp}\mu_\beta, \qquad q_\beta = \max S_\beta.$$

For nonempty compact sets $A, B \subset \mathbb{R}$, let

$$d_{\mathrm{H}}(A,B) = \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\}.$$

Our first result gives both the finite-temperature FRSB statement and the convergence of the support as the temperature tends to zero.

Date: July 16, 2026.

2020 *Mathematics Subject Classification.* 60K35, 82B44.

Key words and phrases. Sherrington–Kirkpatrick model, Parisi measure, full replica symmetry breaking, zero temperature, Hausdorff convergence.

Theorem 1.1 (Finite-temperature interval support and convergence). *For every $\beta > 1$,*

$$S_\beta = [0, q_\beta], \qquad 1 - q_\beta \le \frac{1}{\beta}.$$

Consequently,

$$d_{\mathrm{H}}(S_\beta, [0,1]) = 1 - q_\beta \le \frac{1}{\beta}, \qquad \lim_{\beta\to\infty} d_{\mathrm{H}}(S_\beta, [0,1]) = 0.$$

The canonical zero-temperature order parameter is not the ordinary weak limit of the probability measures μ_β. It is a nonnegative, nondecreasing, right-continuous function $\gamma : [0,1) \to [0,\infty)$ with $\int_0^1 \gamma(t)\,\mathrm{d}t < \infty$, equivalently a locally finite Stieltjes measure $\nu = \mathrm{d}\gamma$ on $[0,1)$. In the SK normalization $\xi(t) = t^2/2$, its Parisi PDE and variational functional are

$$\partial_t u_\gamma + \frac{1}{2}\left(u_{\gamma,xx} + \gamma(t) u_{\gamma,x}^2\right) = 0, \qquad u_\gamma(1,x) = |x|,$$

$$\mathcal{P}(\gamma) = u_\gamma(0,0) - \frac{1}{2}\int_0^1 t\gamma(t)\,\mathrm{d}t. \tag{1.2}$$

By [3, Theorem 1] and [5, Theorem 4], the functional in (1.2) has a unique minimizer $\gamma_\star$. Set $\nu_\star = \mathrm{d}\gamma_\star$.

Theorem 1.2 (Full support of the zero-temperature order parameter). *The Stieltjes measure $\nu_\star = \mathrm{d}\gamma_\star$ satisfies*

$$\overline{\operatorname{supp}_{[0,1)} \nu_\star}^{[0,1]} = [0,1]. \tag{1.3}$$

Equivalently, every nonempty relatively open subinterval of $[0,1]$ meets $\operatorname{supp}_{[0,1)} \nu_\star$, where the endpoint 1 is understood through closure from the left.

Remark 1.3 (Finite- and zero-temperature normalizations). The scaled measures $\beta\mu_\beta$ converge vaguely on $[0,1)$ to $\nu_\star$ in the zero-temperature theory; see [4, proof of Theorem 2]. The unscaled probability measures instead satisfy

$$\lim_{\beta\to\infty} \mu_\beta = \delta_1 \quad \text{weakly on } [0,1].$$

Indeed, for $T < 1$, choose $f \in C_c([0,1))$ with $f \ge 1$ on $[0,T]$. Vague convergence makes $\int f\ \mathrm{d}(\beta\mu_\beta)$ bounded, and therefore $\mu_\beta([0,T]) = O(\beta^{-1})$. Thus Theorem 1.1 concerns the geometry of the supports of the finite-temperature probability measures, whereas Theorem 1.2 concerns the scaled zero-temperature Stieltjes measure.

The support problem has developed through several complementary results. Auffinger and Chen proved that the origin belongs to the support of every Parisi measure and that the distribution function is smooth on every interval contained in the support; see [1, Theorems 1 and 2]. Jagannath and Tobasco obtained the variational optimality conditions used below; see [7, Corollary 3.6 and Proposition 1.1]. At zero temperature, Auffinger, Chen, and Zeng proved that the Parisi measure has infinitely many support points in [4, Theorem 1]. At finite temperature, Zhou proved interval support immediately below the critical temperature in [13, Theorem 1], while Lopatto proved that the support contains an interval beginning at the origin for every $\beta > 1$ in [8, Theorem 1.1]. Theorem 1.1 identifies the entire support interval for every $\beta > 1$, and Theorem 1.2 strengthens infinite-step replica symmetry breaking to full support of the zero-temperature Stieltjes measure after closure at one.

The two proofs have the same variational architecture. The optimality conditions give self-consistency at support points and a stability inequality for a curvature observable. On a component of the complement of the support, the order parameter is constant. The essential analytic statement is then a crossing rule: a critical point of the relevant curvature observable is necessarily a strict local minimum. Endpoint consistency and an integral identity force the observable to have an interior maximum on any putative gap, which is impossible.

At finite temperature, the crossing rule extends [8, Proposition 4.1] from the Gaussian entrance law to an arbitrary entrance law. The extension rests on the slope-coordinate inequalities of [8, Proposition 3.6 and Equation (3.39)] and on a de-tilted density that evolves by the heat equation.

At zero temperature, the terminal datum changes from $\log\cosh x$ to $|x|$, and the finite-temperature entrance invariant is replaced by a forward factor $Q = \rho e^{-\gamma u}$. A score-curvature monotonicity property of Q, together with the scaled slope invariant, yields the corresponding arbitrary-gap crossing theorem. An additional boundary-layer calculation is needed only to exclude a gap adjacent to 1.

Section 2 proves Theorem 1.1. Sections 3 and 4 prove Theorem 1.2.

2. Finite-temperature interval support and convergence

2.1. **Optimality and Brownian rescaling.** Let $u = u_{\mu_\beta}$ and $\alpha(s) = \mu_\beta([0, s])$. Let $X = (X_s)_{0\le s\le 1}$ solve the optimal-diffusion SDE

$$\mathrm{d}X_s = \beta^2\alpha(s)u_x(s, X_s)\,\mathrm{d}s + \beta\,\mathrm{d}W_s, \qquad X_0 = 0.$$

Define

$$\Gamma(s) = \mathbb{E}[u_x(s, X_s)^2].$$

The following facts are the Parisi optimality conditions; see [7, Corollary 3.6 and Proposition 1.1], [1, Theorem 1], and [8, Section 2].

Proposition 2.1 (Support optimality conditions)**.** *One has $0 \in S_\beta$. Moreover, for every $q \in S_\beta$,*

$$\Gamma(q) = q, \tag{2.1}$$

$$\beta^2\mathbb{E}[u_{xx}(q, X_q)^2] \le 1. \tag{2.2}$$

Lemma 2.2 (The upper support endpoint is below one)**.** *For every finite $\beta > 1$, one has $q_\beta < 1$.*

Proof. If $q_\beta = 1$, then (2.1), evaluated at terminal time, would give

$$1 = \mathbb{E}[u_x(1, X_1)^2] = \mathbb{E}[\tanh^2(X_1)].$$

The random variable X_1 is finite almost surely, whereas $\tanh^2 x < 1$ for every finite x. The right-hand side is therefore strictly smaller than one, a contradiction. □

We now remove the factor β^2 from the PDE. Set

$$T = \beta^2, \qquad U(t, x) = u(t/\beta^2, x), \qquad a(t) = \alpha(t/\beta^2), \qquad 0 \le t \le T.$$

Then

$$U_t = -\frac{1}{2}(U_{xx} + a(t)U_x^2), \qquad U(T, x) = \log\cosh x. \tag{2.3}$$

If $Y_t = X_{t/\beta^2}$ and $B_t = \beta W_{t/\beta^2}$, then B is a standard Brownian motion and

$$\mathrm{d}Y_t = a(t)U_x(t, Y_t)\,\mathrm{d}t + \mathrm{d}B_t, \qquad Y_0 = 0. \tag{2.4}$$

Define

$$M(t) = \mathbb{E}[U_x(t, Y_t)^2], \qquad R(t) = \mathbb{E}[U_{xx}(t, Y_t)^2]. \tag{2.5}$$

Lemma 2.3 (Basic stochastic identity)**.** *For $0 < t < T$,*

$$M'(t) = R(t). \tag{2.6}$$

If $q \in S_\beta$ and $t = \beta^2 q$, then

$$M(t) = \frac{t}{\beta^2}, \qquad R(t) \le \frac{1}{\beta^2}. \tag{2.7}$$

Proof. Differentiating (2.3) in x gives

$$U_{xt} = -\frac{1}{2}U_{xxx} - a(t)U_xU_{xx}.$$

Applying Itô's formula to $U_x(t, Y_t)$ and using (2.4), the drift cancels:

$$\begin{aligned}\mathrm{d}U_x(t, Y_t) &= \left(U_{xt} + a(t)U_xU_{xx} + \frac{1}{2}U_{xxx}\right)(t, Y_t)\,\mathrm{d}t + U_{xx}(t, Y_t)\,\mathrm{d}B_t\\ &= U_{xx}(t, Y_t)\,\mathrm{d}B_t.\end{aligned}$$

Since the spatial derivatives of U are bounded, the stochastic integral is square-integrable. Itô's formula applied once more to $U_x(t, Y_t)^2$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[U_x(t, Y_t)^2] = \mathbb{E}[U_{xx}(t, Y_t)^2],$$

proving (2.6). The identities in (2.7) are exactly (2.1)–(2.2) after the time change $t = \beta^2 q$. □

Lemma 2.4 (Continuity of the curvature observable)**.** *The map R defined in (2.5) is continuous on $[0, T)$.*

Proof. Fix $\eta > 0$. The mild formulation of (2.3), followed by differentiation of the heat kernel, gives joint continuity of $(t, x) \mapsto U_{xx}(t, x)$ on $[0, T - \eta] \times \mathbb{R}$ and the bound

$$\sup_{0 \le t \le T-\eta} \sup_{x \in \mathbb{R}} |U_{xx}(t, x)| \le C_\eta. \tag{2.8}$$

Equivalently, one may obtain these facts by approximating a by step functions and using the Cole–Hopf representation on each step; the estimates are uniform because $|U_x| \le 1$.

The drift in (2.4) is bounded by one. Thus, for $0 \le s \le t \le T$,

$$\mathbb{E}|Y_t - Y_s|^2 \le 2(t - s) + 2(t - s)^2,$$

and $Y_s \to Y_t$ in probability as $s \to t$. If $t_n \to t < T$, choose $\eta > 0$ so that $t, t_n \le T - \eta$ for all sufficiently large n. On $\{|Y_{t_n}| \vee |Y_t| \le M\}$, joint continuity of U_{xx} gives

$$U_{xx}(t_n, Y_{t_n}) \longrightarrow U_{xx}(t, Y_t)$$

in probability. The complementary event has arbitrarily small probability uniformly in n, by the bounded drift, while (2.8) gives uniform integrability. The convergence therefore holds in L^2, and taking expectations of the squares proves $R(t_n) \to R(t)$. In particular, R is continuous across jump times of a; only the time derivative of the backward profile may jump there. □

2.2. **De-tilted entrance densities.** The purpose of this subsection is to identify the structural property of the law entering an arbitrary interval on which a is constant.

Let $I = (t_0, t_1) \subset (0, T)$ be an interval on which

$$a(t) = m \qquad (t \in I)$$

for some $m \ge 0$. Let p_t denote the density of Y_t. For $t \in I$, define the de-tilted density

$$r_t(x) = p_t(x) e^{-mU(t,x)}. \tag{2.9}$$

Lemma 2.5 (Heat equation for the de-tilted density)**.** *On $I \times \mathbb{R}$,*

$$\partial_t r_t = \frac{1}{2}\partial_{xx} r_t. \tag{2.10}$$

Proof. The Fokker–Planck equation associated with (2.4) is

$$p_t = \frac{1}{2}p_{xx} - \partial_x(mU_x p).$$

Write $p = e^{mU} r$. Then

$$\frac{1}{2}p_{xx} - \partial_x(mU_x p) = e^{mU}\left[\frac{1}{2}r_{xx} - \frac{m}{2}U_{xx} r - \frac{m^2}{2}U_x^2 r\right].$$

On the other hand,

$$p_t = e^{mU}(r_t + mU_t r).$$

Since $U_t = -\frac{1}{2}(U_{xx} + mU_x^2)$ on I, the last two terms cancel and (2.10) follows. □

For a positive, smooth, even density r, define on $(0, \infty)$

$$A_r = -\partial_x \log r, \qquad V_r = \frac{r_{xx}}{r} = A_r^2 - (A_r)_x.$$

We shall use the following shape class:

$$A_r \ge 0, \qquad (A_r)_x \ge 0, \qquad (V_r)_x \ge 0 \qquad (x > 0). \tag{2.11}$$

Lemma 2.6 (Preservation by heat flow)**.** *Suppose that r is positive, even, smooth, integrable and log-concave, that V_r is nondecreasing on $(0,\infty)$, and that*

$$\int_{\mathbb{R}}(1+|x|)|r''(x)|\,\mathrm{d}x<\infty. \tag{2.12}$$

For $h>0$, let $\bar r = P_h r$, where P_h is the heat semigroup. Then $\bar r$ is positive, even, smooth and log-concave, and $V_{\bar r}$ is nondecreasing on $(0,\infty)$.

Proof. Positivity, evenness and smoothness are immediate. Log-concavity is preserved by convolution with the Gaussian kernel, by the Prékopa–Leindler theorem. Assumption (2.12) permits two integrations by parts against the Gaussian kernel and differentiation under the integral below. Thus $(P_h r)'' = P_h(r'') = P_h(rV_r)$, and hence

$$V_{\bar r}(x) = \frac{P_h(rV_r)(x)}{P_h r(x)}. \tag{2.13}$$

Introduce the probability measure

$$\nu_x(\,\mathrm{d}y) = \frac{r(y)e^{-(y-x)^2/(2h)}\,\mathrm{d}y}{\int_{\mathbb{R}} r(z)e^{-(z-x)^2/(2h)}\,\mathrm{d}z}.$$

Differentiating (2.13) under the integral gives

$$\partial_x V_{\bar r}(x) = \frac{1}{h}\operatorname{Cov}_{\nu_x}(V_r(Y),Y).$$

Let $\rho=|Y|$. Because r and V_r are even,

$$\mathbb{E}_{\nu_x}[Y \mid |Y|=\rho] = \rho\tanh\left(\frac{x\rho}{h}\right).$$

For $x>0$, both $\rho\mapsto V_r(\rho)$ and $\rho\mapsto\rho\tanh(x\rho/h)$ are nondecreasing. Conditional covariance and the elementary covariance inequality for two nondecreasing functions therefore imply

$$\operatorname{Cov}_{\nu_x}(V_r(Y),Y) = \operatorname{Cov}_{\nu_x}\left(V_r(\rho),\rho\tanh\left(\frac{x\rho}{h}\right)\right)\geq 0.$$

This proves the claim. In the finite-step application below, (2.12) is automatic: after the first positive heat evolution the density and all of its derivatives have Gaussian tails, and multiplication by e^{-dU} preserves this property because $|U_x|\leq 1$ and the spatial derivatives of U are bounded. $\square$

We next record the slope-structure input from [8, Proposition 3.6 and Equation (3.39)].

Proposition 2.7 (Slope-structure inequalities)**.** *Let U be a finite-step Parisi PDE profile at a time at which the current order parameter is $m\geq 0$. Then, for $x>0$,*

$$U_x>0,\qquad U_{xx}>0,\qquad U_{xxx}\leq 0. \tag{2.14}$$

If $m>0$, set

$$b=U_x(x),\qquad c=U_{xx}(x),$$

regard c as a function of $b\in(0,1)$, and define K and J by

$$z=-\frac{1}{2}c_b=(m+K)b,\qquad J=K+bK_b. \tag{2.15}$$

Then

$$K,K_b,J,J_b\geq 0,\qquad cJ_b\leq 3zJ. \tag{2.16}$$

Proof. For $m>0$, this is [8, Proposition 3.6 and Equation (3.39)]; the last inequality in (2.14) follows from

$$U_{xxx}=c_x=cc_b=-2cz\leq 0.$$

At $m=0$ we use only (2.14). It follows by replacing the zero level in the finite-step recursion by $\varepsilon>0$ and passing to the limit $\varepsilon\downarrow 0$; the finite-step Cole–Hopf maps and all their spatial derivatives depend continuously on the level parameter. Thus no slope inequality at $m=0$ is needed later. $\square$

Lemma 2.8 (Preservation at an increase of the order parameter)**.** *Suppose that r satisfies (2.11), and let U be an even, strictly convex profile satisfying (2.14). If $d \geq 0$ and*
$$\widetilde{r} = re^{-dU},$$
then $\widetilde{r}$ also satisfies (2.11).

Proof. Put
$$A = -\partial_x \log r, \qquad B = U_x, \qquad C = U_{xx}.$$
Then
$$\widetilde{A} = A + dB,$$
so $\widetilde{A} \geq 0$ and $\widetilde{A}_x = A_x + dC \geq 0$. Furthermore,
$$\begin{aligned}\widetilde{V} &= \widetilde{A}^2 - \widetilde{A}_x \\ &= V + 2dAB + d^2B^2 - dC.\end{aligned}$$
Differentiating gives
$$\widetilde{V}_x = V_x + 2d(A_xB + AC) + 2d^2BC - dU_{xxx}.$$
Every term on the right-hand side is nonnegative for $x > 0$. ☐

Lemma 2.9 (Uniform right-tail expansion)**.** *Let U solve (2.3) for a nondecreasing right-continuous coefficient $a : [0, T] \to [0, 1]$. For every $\eta > 0$ and every integer $j \geq 0$, uniformly for $0 \leq t \leq T - \eta$ as $x \to +\infty$,*
$$U(t, x) = x + c_0(t) + c_1(t)e^{-2x} + c_2(t)e^{-4x} + c_3(t)e^{-6x} + O_{\eta,j}(e^{-8x}), \tag{2.17}$$
where the remainder may be differentiated j times in x. On every interval on which a is constant, it may also be differentiated once in t. The coefficients are absolutely continuous and satisfy, almost everywhere,
$$c_0' = -\frac{1}{2}a, \qquad c_0(T) = -\log 2, \tag{2.18}$$
$$c_1' = -2(1-a)c_1, \qquad c_1(T) = 1, \tag{2.19}$$
$$c_2' = -4(2-a)c_2 - 2ac_1^2, \qquad c_2(T) = -\frac{1}{2},$$
$$c_3' = -6(3-a)c_3 - 8ac_1c_2, \qquad c_3(T) = \frac{1}{3}. \tag{2.20}$$
In particular, $c_1(t) > 0$. If
$$b = U_x(t, x), \qquad c = U_{xx}(t, x),$$
then, uniformly for t in a compact subinterval of a plateau,
$$x(t, b) = \frac{1}{2}\log\frac{1}{1-b} + O(1), \tag{2.21}$$
$$c(t, b) = 2\delta + \kappa_2(t)\delta^2 + \kappa_3(t)\delta^3 + O(\delta^4), \qquad \delta = 1 - b, \tag{2.22}$$
where
$$\partial_b^j O(\delta^4) = O(\delta^{4-j}), \qquad 0 \leq j \leq 3.$$
Consequently, as $b \uparrow 1$, $z \to 1$, while K, J, J_b, R_0 and $(R_0)_b$ remain bounded.

Proof. At terminal time,
$$\begin{aligned}\log\cosh x &= x - \log 2 + \log(1 + e^{-2x}) \\ &= x - \log 2 + e^{-2x} - \frac{1}{2}e^{-4x} + \frac{1}{3}e^{-6x} + O(e^{-8x}).\end{aligned}$$
Insert the ansatz
$$U^{(3)}(t, x) = x + c_0(t) + c_1(t)e^{-2x} + c_2(t)e^{-4x} + c_3(t)e^{-6x}$$
into (2.3). Equating the coefficients of $1, e^{-2x}, e^{-4x}, e^{-6x}$ gives (2.18)–(2.20). For example,
$$\begin{aligned}U_x^{(3)} &= 1 - 2c_1e^{-2x} - 4c_2e^{-4x} - 6c_3e^{-6x}, \\ U_{xx}^{(3)} &= 4c_1e^{-2x} + 16c_2e^{-4x} + 36c_3e^{-6x},\end{aligned}$$

so the coefficient of e^{-4x} in $U^{(3)}_{xx} + a(U^{(3)}_x)^2$ is

$$8(2-a)c_2 + 4ac_1^2,$$

and the coefficient of e^{-6x} is

$$12(3-a)c_3 + 16ac_1c_2.$$

This verifies the two nonlinear equations above. The linear equation (2.19) and the terminal value $c_1(T) = 1$ imply $c_1(t) > 0$ for every t.

We justify the remainder uniformly. Put $E = U - U^{(3)}$ and reverse time, $\tau = T - t$. The coefficient equations imply that the residual produced by $U^{(3)}$ and each of its spatial derivatives are $O(e^{-8x})$ on a right half-line. Thus E satisfies

$$E_\tau = \frac{1}{2}E_{xx} + \gamma(\tau,x)E_x + f(\tau,x), \qquad |\gamma| \le C, \qquad |\partial_x^j f(\tau,x)| \le C_j e^{-8x}, \tag{2.23}$$

where

$$\gamma(\tau,x) = \frac{1}{2}a(T-\tau)\big(U_x(T-\tau,x) + U^{(3)}_x(T-\tau,x)\big).$$

Here $|U_x| \le 1$, and the coefficient ODEs show that the c_i are bounded on $[0,T]$. At $\tau = 0$ the terminal expansion gives

$$|E(0,x)| \le Ce^{-8x} \qquad (x \ge x_0) \tag{2.24}$$

after x_0 is chosen sufficiently large.

We now obtain the same bound at positive reverse times without imposing an unproved boundary condition at $x = +\infty$. Fix (τ,x) with $x > x_0$, and let Ξ be the time-inhomogeneous diffusion

$$\mathrm{d}\Xi_s = \gamma(\tau - s, \Xi_s)\,\mathrm{d}s + \mathrm{d}W_s, \qquad \Xi_0 = x, \qquad 0 \le s \le \tau.$$

Let $\sigma = \inf\{s \ge 0 : \Xi_s = x_0\}$. Applying Itô's formula to $E(\tau - s, \Xi_s)$ up to $\tau \wedge \sigma$, and then localizing in space, gives the stopped representation

$$\begin{aligned} E(\tau,x) &= \mathbb{E}\left[E(0,\Xi_\tau)\mathbf{1}_{\{\sigma>\tau\}}\right] + \mathbb{E}\left[E(\tau-\sigma,x_0)\mathbf{1}_{\{\sigma\le\tau\}}\right] \\ &\quad + \mathbb{E}\left[\int_0^{\tau\wedge\sigma} f(\tau-s,\Xi_s)\,\mathrm{d}s\right]. \end{aligned} \tag{2.25}$$

The localization is harmless because U is 1-Lipschitz in x, so E has at most linear growth. If C_γ bounds $|\gamma|$, the reflection principle yields, uniformly for $0 < \tau \le T$,

$$\mathbb{P}(\sigma \le \tau) \le 2\exp\left(-\frac{(x - x_0 - C_\gamma\tau)_+^2}{2\tau}\right). \tag{2.26}$$

This term is $O(e^{-8x})$ as $x \to \infty$. On $\{s < \sigma\}$ one has $\Xi_s \ge x_0$, and

$$\mathbb{E}\left[e^{-8\Xi_s}\mathbf{1}_{\{s<\sigma\}}\right] \le e^{-8x + (8C_\gamma + 32)s},$$

because $\Xi_s = x + W_s + \int_0^s \gamma(\tau - r, \Xi_r)\,\mathrm{d}r$ and $\mathbb{E}e^{-8W_s} = e^{32s}$. The boundary values $E(\cdot, x_0)$ are bounded on $[0,T]$. Consequently, (2.24)–(2.26), applied in (2.25), give

$$|E(\tau,x)| \le Ce^{-8x}, \qquad 0 \le \tau \le T, \quad x \ge x_0 + 1. \tag{2.27}$$

It remains to differentiate the estimate. Fix $\eta > 0$. On the reverse-time strip $\eta/2 \le \tau \le T$, parabolic smoothing for the viscous Burgers equation gives uniform bounds for all spatial derivatives of U, and hence of γ; the same derivatives of f are $O(e^{-8x})$. Applying the standard interior parabolic derivative estimates to (2.23) on translated cylinders of fixed size, and using (2.27) on the slightly larger cylinder, yields

$$|\partial_x^j E(\tau,x)| \le C_{\eta,j}e^{-8x}, \qquad \eta \le \tau \le T,$$

for every $j \ge 0$; the endpoint $\tau = T$ follows from the analogous one-sided estimate. On an interval on which a is constant, the equation itself then gives the same order for one time derivative. Returning to $t = T - \tau$ proves (2.17).

Finally, differentiating (2.17) gives, with $y = e^{-2x}$,

$$1 - b = 2c_1y + 4c_2y^2 + 6c_3y^3 + O(y^4),$$

$$c = 4c_1y + 16c_2y^2 + 36c_3y^3 + O(y^4).$$

On a compact time interval, c_1 is bounded away from zero. Inverting the first expansion yields

$$y = \frac{1-b}{2c_1} + O((1-b)^2),$$

which proves (2.21). Substitution of the inverse power series into the expansion for c gives coefficients κ_2, κ_3 and the expansion (2.22). The differentiated remainder estimates above and the quantitative implicit-function theorem applied to the first expansion give $\partial_b^j O(\delta^4) = O(\delta^{4-j})$ for $0 \le j \le 3$. More explicitly, (2.22) gives $c_b = -2 + O(\delta)$, and hence $z = -c_b/2 \to 1$. The differentiated remainder bounds show that z, z_b, z_{bb} remain bounded. Since b stays away from zero near the right endpoint,

$$K = \frac{z}{b} - m, \qquad J = z_b - m, \qquad J_b = z_{bb}$$

are bounded there. Finally, (2.49) and (2.56) below express R_0 and $(R_0)_b$ in terms of these quantities and c, c_b, proving the remaining assertions. □

The next proposition is the technical closure step needed when the Parisi measure is not finitely supported.

Proposition 2.10 (Closure for a general order parameter)**.** *Let $a : [0,T] \to [0,1]$ be nondecreasing and right-continuous, and let U and Y solve* (2.3) *and* (2.4)*. Suppose $a \equiv m$ on $I = (t_0, t_1)$. For every $t \in I$, the density r_t defined in* (2.9) *is positive, even and smooth and satisfies*

$$A := -\partial_x \log r_t \ge 0, \qquad A_x \ge 0, \qquad V_x \ge 0, \quad V := \frac{r_{t,xx}}{r_t}, \qquad x > 0. \tag{2.28}$$

Moreover, V is not constant. If $m > 0$, the slope inequalities (2.16) *also hold at time t. Finally, on each compact subinterval $J \Subset I$, the fixed-slope quantities appearing in Proposition 2.11 are jointly continuous in $(t,b) \in J \times (0,1)$, and all differentiations under the b-integral, improper integrations, and boundary limits used there are justified uniformly for $t \in J$.*

Proof. We give the approximation and domination arguments in five steps.

Step 1: finite-step approximation. Choose nondecreasing, right-continuous step functions $a_n : [0,T] \to [0,1]$ such that

$$\|a_n - a\|_{L^1(0,T)} \longrightarrow 0 \tag{2.29}$$

and such that $a_n = m$ on every fixed compact subinterval of I for all sufficiently large n. To construct such a sequence, fix $I' \Subset I$, approximate a by monotone step functions separately on $[0, \inf I']$ and $[\sup I', T]$, set the approximation equal to m on I', and then use a diagonal sequence over an exhaustion of I by compact intervals.

Let U_n solve (2.3) with coefficient a_n, and let Y^n solve the corresponding diffusion. For a finite-step profile, the forward de-tilted measure at time zero is a positive multiple of δ_0. On an interval where the order parameter is constant, it evolves by the heat semigroup, by Lemma 2.5; at an upward jump of size d, it changes according to

$$r \longmapsto re^{-dU_n}.$$

The first positive heat evolution of δ_0 is Gaussian. It satisfies (2.11) and the integrability hypothesis (2.12). Subsequent heat evolutions and jump multiplications retain Gaussian tails together with the corresponding derivative bounds. Hence Lemmas 2.6 and 2.8 show inductively that every finite-step de-tilted density satisfies (2.11). At a plateau of positive level, the slope inequalities follow from Proposition 2.7.

Step 2: stability and smoothing of the backward profiles. Write $v_n = U_{n,x}$ and reverse time by $\tau = T - t$. Then

$$\partial_\tau v_n = \frac{1}{2}(v_n)_{xx} + a_n(T-\tau)v_n(v_n)_x, \qquad v_n(0,x) = \tanh x.$$

The maximum principle gives $|v_n| \le 1$. If $W_n = U_n - U$, then in reverse time

$$\partial_\tau W_n = \frac{1}{2}(W_n)_{xx} + \frac{1}{2}a_n(T-\tau)(U_{n,x}+U_x)(W_n)_x + \frac{1}{2}(a_n - a)(T-\tau)U_x^2, \qquad W_n(0,\cdot) = 0.$$

Since $|U_{n,x}|, |U_x| \le 1$, comparison with the two time-dependent constants $\pm\frac{1}{2}\int_0^\tau |a_n - a|(T-s)\,\mathrm{d}s$ gives

$$\|U_n - U\|_{L^\infty([0,T]\times\mathbb{R})} \le \frac{1}{2}\|a_n - a\|_{L^1(0,T)}. \tag{2.30}$$

Each $U_n(t,\cdot)$ is convex, and the uniform limit $U(t,\cdot)$ is convex as well. The convergence of the first derivatives can therefore be read directly from secant slopes. For $\rho > 0$,

$$\frac{U_n(t,x) - U_n(t,x-\rho)}{\rho} \le U_{n,x}(t,x) \le \frac{U_n(t,x+\rho) - U_n(t,x)}{\rho}.$$

Applying the analogous bounds to U, using (2.30), and then first letting $n \to \infty$ and next $\rho \downarrow 0$, shows that

$$U_{n,x} \longrightarrow U_x \quad \text{locally uniformly on } [0,T]\times\mathbb{R}. \tag{2.31}$$

Uniformity follows from the uniform continuity of U_x on compact sets.

Higher derivatives at a time inside the plateau are obtained from a common smoothing operator rather than from a formal differentiation of the coefficient approximation. Fix $J \Subset I$ and choose $h > 0$ such that $[t, t+h] \subset I$ for every $t \in J$. For all sufficiently large n, $a_n = m$ on these intervals and

$$U_n(t,\cdot) = \mathcal{T}_{m,h}U_n(t+h,\cdot), \qquad U(t,\cdot) = \mathcal{T}_{m,h}U(t+h,\cdot), \tag{2.32}$$

where

$$\mathcal{T}_{m,h}\psi = \begin{cases} m^{-1}\log P_h(e^{m\psi}), & m > 0, \\ P_h\psi, & m = 0. \end{cases}$$

To make the derivative convergence explicit, suppose first that $m > 0$ and write

$$F_n(t,x) = P_h(e^{mU_n(t+h,\cdot)})(x), \qquad F(t,x) = P_h(e^{mU(t+h,\cdot)})(x).$$

If $\varepsilon_n = \|U_n - U\|_\infty$, then $e^{-m\varepsilon_n}F \le F_n \le e^{m\varepsilon_n}F$. Moreover, for every $k \ge 0$,

$$\partial_x^k F_n(t,x) = \int_{\mathbb{R}} \partial_x^k g_h(x-y)e^{mU_n(t+h,y)}\,\mathrm{d}y,$$

where g_h is the centered Gaussian density of variance h. Since $|U_n(t,y)| \le C + |y|$ uniformly in n and t, the integrands are dominated, uniformly for (t,x) in compact sets, by an integrable Gaussian times $e^{m|y|}$. Thus every derivative of F_n converges locally uniformly to the corresponding derivative of F. Since F is bounded away from zero on compact sets, differentiation of $m^{-1}\log F_n$ in (2.32) gives

$$\partial_x^k U_n \longrightarrow \partial_x^k U \quad \text{locally uniformly on } J\times\mathbb{R} \quad \text{for every } k \ge 0. \tag{2.33}$$

The case $m = 0$ follows directly by differentiating the Gaussian convolution $P_h U_n(t+h,\cdot)$.

Strict convexity is needed to introduce the slope coordinate. In reverse time $C = U_{xx}$ satisfies

$$C_\tau = \frac{1}{2}C_{xx} + a(T-\tau)U_x C_x + a(T-\tau)C^2, \qquad C(0,x) = \operatorname{sech}^2 x > 0.$$

The strong parabolic maximum principle gives $C(t,x) > 0$ for every $t < T$ and $x \in \mathbb{R}$. By Lemma 2.9 and evenness, $U_x(t,x) \to \pm 1$ as $x \to \pm\infty$. Consequently, $x \mapsto U_x(t,x)$ is a smooth increasing bijection from $\mathbb{R}$ onto $(-1,1)$.

Assume now that $m > 0$. On a compact slope interval $B \Subset (0,1)$ and a compact time interval $J \Subset I$, the inverse points $x_n(t,b)$ and $x(t,b)$ remain in one compact spatial set. Strict convexity gives a positive lower bound for $c = U_{xx}$ there. The inverse-function theorem and (2.33) therefore imply convergence of the inverse maps and their derivatives. In particular, using

$$z = -\frac{1}{2}c_b, \qquad K = \frac{z}{b} - m, \qquad J = z_b - m, \qquad J_b = z_{bb},$$

we obtain local uniform convergence of $K_n, (K_n)_b, J_n, (J_n)_b$ to K, K_b, J, J_b. Every inequality in (2.16) consequently passes to the limit.

Step 3: convergence of the forward laws and densities. The drifts

$$b_n(t,x) = a_n(t)U_{n,x}(t,x)$$

are uniformly bounded by 1. By (2.29) and (2.31), for every compact spatial set K,

$$\int_0^T \sup_{x\in K} |b_n(t,x) - a(t)U_x(t,x)| \, \mathrm{d}t \longrightarrow 0.$$

Boundedness of the drifts gives tightness of the laws of the processes Y^n in $C([0,T])$. The preceding convergence identifies every subsequential limit through the martingale problem associated with the drift $a(t)U_x(t,x)$. That martingale problem is unique because the limiting drift is locally Lipschitz in x. Hence

$$\mathcal{L}(Y_t^n) \Longrightarrow \mathcal{L}(Y_t) \qquad \text{for every } t. \tag{2.34}$$

Fix $t \in I$ and choose $h > 0$ with $[t-h,t] \subset I$. For all sufficiently large n, $a_n = m$ on this interval, so

$$r_{n,t} = P_h r_{n,t-h}. \tag{2.35}$$

Here and below $r_{n,s}$ is also viewed as a finite measure. The comparison principle gives $U_n \ge 0$, and therefore

$$r_{n,s}(\mathbb{R}) = \mathbb{E}[e^{-mU_n(s,Y_s^n)}] \le 1.$$

These measures are tight because a diffusion with drift bounded by one has Gaussian tails uniformly in n. More directly, for every bounded continuous φ, (2.34) and the uniform convergence of U_n give

$$\int \varphi \, \mathrm{d}r_{n,t-h} = \mathbb{E}\left[\varphi(Y_{t-h}^n)e^{-mU_n(t-h,Y_{t-h}^n)}\right] \longrightarrow \mathbb{E}\left[\varphi(Y_{t-h})e^{-mU(t-h,Y_{t-h})}\right].$$

Thus $r_{n,t-h} \Rightarrow r_{t-h}$. Differentiating the Gaussian kernel in (2.35) gives

$$\partial_x^k r_{n,t}(x) = \int_{\mathbb{R}} \partial_x^k g_h(x-y) r_{n,t-h}(\,\mathrm{d}y).$$

The derivatives of g_h are bounded and vanish at infinity; the weak convergence and the uniform mass bound therefore imply, for every $k \ge 0$,

$$\partial_x^k r_{n,t} \longrightarrow \partial_x^k r_t \quad \text{locally uniformly on } \mathbb{R}. \tag{2.36}$$

Positivity of $P_h r_{t-h}$ permits division by r_t. Passing to the limit in the finite-step inequalities proves (2.28).

Step 4: strict nonconstancy. Suppose V were constant, say $V \equiv \lambda$. Then $r_t'' = \lambda r_t$ on $\mathbb{R}$, and evenness gives $r_t'(0) = 0$. If $\lambda > 0$, the even solution is a multiple of $\cosh(\sqrt{\lambda}x)$ and is not integrable. If $\lambda = 0$, it is constant and is not integrable. If $\lambda < 0$, every nonzero solution changes sign. All three alternatives contradict the positivity and integrability of r_t. Hence V is not constant.

Step 5: uniform tails and the b-integrals. Fix $J \Subset I$ and choose $h > 0$ such that $[t-h,t] \subset I$ for all $t \in J$. Standard Gaussian upper bounds for diffusions with bounded drift give constants $C_J, c_J > 0$ such that

$$p_t(x) \le C_J e^{-c_J x^2 + C_J|x|}, \qquad t \in J, \quad x \in \mathbb{R}. \tag{2.37}$$

Write $r_t = P_h \nu_t$ with $\nu_t = r_{t-h}$. Under the probability kernel proportional to $g_h(x-y)\nu_t(\,\mathrm{d}y)$, differentiation of the Gaussian convolution gives

$$A(t,x) = \frac{x - \mathbb{E}_{t,x}Y}{h}, \qquad A_x(t,x) = \frac{1}{h} - \frac{\mathrm{Var}_{t,x}(Y)}{h^2} \le \frac{1}{h}.$$

By (2.28), $A_x \ge 0$, and evenness gives $A(t,0) = 0$. Therefore, uniformly for $t \in J$,

$$|A(t,x)| \le \frac{|x|}{h}, \qquad |V(t,x)| = |A(t,x)^2 - A_x(t,x)| \le \frac{x^2}{h^2} + \frac{1}{h}. \tag{2.38}$$

By Lemma 2.9, uniformly for $t \in J$ as $b \uparrow 1$,

$$x(t,b) = \frac{1}{2}\log\frac{1}{1-b} + O_J(1), \qquad c(t,b) = 2(1-b) + O_J((1-b)^2),$$

and $z, K, J, J_b, R_0, (R_0)_b$ are bounded. The formulas used in Proposition 2.11 then show that Φ, R_0 and G/c are bounded, while $\mathcal{A}$ and $(\log w)_t$ grow at most like $1 + x(t,b)^2$. Since $w = 2p_t(x(t,b))c(t,b)$, (2.37)–(2.38) give a common majorant, for all the integrands in (2.41) and (2.50), of the form

$$C(1-b)\left(1 + \log^2\frac{1}{1-b}\right)\exp\left[-c\log^2\frac{1}{1-b} + C\log\frac{1}{1-b}\right].$$

This is integrable near $b = 1$. The same estimate, together with the boundedness of $(R_0)_b$, controls the integration by parts in (2.57); one may use $|F(b)| \le \int_b^1 w|\Phi|$ before the sharper bound on τ is proved. It also gives

$$w(b)c(b)z(b) = 2p_t(x(b))c(b)^2 z(b) \longrightarrow 0 \qquad (b \uparrow 1)$$

uniformly for $t \in J$. Near $b = 0$, even Taylor expansions at $x = 0$, strict convexity, and (2.36) show that all the same quantities are smooth and uniformly bounded. Thus dominated convergence, differentiation at fixed slope, and every boundary passage used below are valid uniformly on J. □

2.3. **Arbitrary-entrance transversality.** We now prove the main local analytic statement. It is the arbitrary-entrance counterpart of [8, Proposition 4.1]: the Gaussian entrance score appearing there is replaced by the de-tilted score A.

Proposition 2.11 (Arbitrary-entrance transversality)**.** *Suppose that $a(t) = m > 0$ on an open interval I. Then $R \in C^2(I)$ and, for every $t \in I$,*

$$R'(t) = 0 \quad \Longrightarrow \quad R''(t) > 0. \tag{2.39}$$

Proof. Fix $t \in I$. To simplify notation, all functions in the remainder of the proof are evaluated at this time unless a time argument is displayed explicitly.

Step 1: the first derivative of R. Put

$$C_t = U_{xx}(t, Y_t).$$

Differentiating (2.3) twice in x and applying Itô's formula gives

$$\mathrm{d}C_t = -mC_t^2\,\mathrm{d}t + U_{xxx}(t, Y_t)\,\mathrm{d}B_t.$$

It follows that

$$R'(t) = \mathbb{E}\left[U_{xxx}(t,Y_t)^2 - 2mU_{xx}(t,Y_t)^3\right]. \tag{2.40}$$

On the positive half-line introduce the slope coordinate

$$b = U_x(t,x), \qquad c = U_{xx}(t, x(b)), \qquad 0 < b < 1,$$

and define z, K, J as in (2.15). Then

$$U_{xxx} = c_x = cc_b = -2cz.$$

Set

$$\Phi = 2z^2 - mc, \qquad w(t,b) = 2p_t(x(b))c(b).$$

Since p_t, c and Φ are even, changing variables by $\mathrm{d}b = c\,\mathrm{d}x$ in (2.40) yields

$$\frac{1}{2}R'(t) = \int_0^1 w(t,b)\Phi(t,b)\,\mathrm{d}b. \tag{2.41}$$

The local uniform domination in Proposition 2.10 makes the right-hand side continuous in t, so at this stage $R \in C^1(I)$. Furthermore,

$$\Phi_b = 4zz_b - mc_b = 2z(2J + 3m) > 0. \tag{2.42}$$

Step 2: fixed-slope time derivatives. Let

$$A = -\partial_x \log r_t, \qquad V = \frac{r_{t,xx}}{r_t}, \qquad N = A + K\,b.$$

Because $x_b = c^{-1}$, $J = K + bK_b = \partial_b(K\,b)$ and $z_b = m + J$,

$$N_b = \frac{A_x}{c} + J. \tag{2.43}$$

Since $p = e^{mU} r$,

$$(\log p)_x = mb - A.$$

Hence

$$\frac{w_b}{w} = \frac{mb - A}{c} + \frac{c_b}{c} = -\frac{N + z}{c}. \tag{2.44}$$

We next differentiate at fixed b. Differentiating $b = U_x(t, x(t, b))$ and using (2.3) gives

$$x_t = -K\,b. \tag{2.45}$$

The slope-coordinate evolution identities of [8, Section 4] give

$$c_t = c^2 J, \qquad z_t = 2czJ - \frac{1}{2}c^2 J_b.$$

Therefore

$$\Phi_t = cJ\Phi + G, \qquad G = 2cz(3zJ - cJ_b) \geq 0. \tag{2.46}$$

By (2.10) and (2.45),

$$\frac{\mathrm{d}}{\mathrm{d}t} \log r_t(x(t, b)) = \frac{1}{2}V + A\,K\,b.$$

Also,

$$\begin{aligned} m\frac{\mathrm{d}}{\mathrm{d}t}U(t, x(t, b)) &= m(U_t + U_x x_t) \\ &= -\frac{m}{2}c - \frac{m^2}{2}b^2 - mK\,b^2, \end{aligned}$$

and $(\log c)_t = cJ$. Since $w = 2e^{mU} rc$, we obtain

$$(\log w)_t = \frac{1}{2}V + A\,K\,b + cJ - \frac{m}{2}c - \frac{m^2}{2}b^2 - mK\,b^2. \tag{2.47}$$

Define

$$\mathcal{A} = \frac{1}{2}V + A\,K\,b + \frac{1}{2}(K\,b)^2, \tag{2.48}$$

$$R_0 = 2cJ - \frac{m}{2}c - \frac{1}{2}z^2. \tag{2.49}$$

Since $z = (m + K)b$,

$$\frac{1}{2}(K\,b)^2 - \frac{1}{2}z^2 = -\frac{m^2}{2}b^2 - mK\,b^2.$$

It follows from (2.46) and (2.47) that the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 w\Phi\,\mathrm{d}b = \int_0^1 w\left[(\mathcal{A} + R_0)\Phi + G\right]\mathrm{d}b \tag{2.50}$$

holds for every $t \in I$, not only at a zero of the integral. The uniform majorants in Proposition 2.10 justify differentiation under the integral and show that the right-hand side is continuous in t. Together with (2.41), this also proves the asserted regularity $R \in C^2(I)$.

Step 3: the strictly positive covariance term. Differentiating (2.48) in b and using (2.43) gives

$$\mathcal{A}_b = \frac{V_x}{2c} + \frac{A_x}{c} K\, b + AJ + (K\, b)J \geq 0. \tag{2.51}$$

By Proposition 2.10, V is nondecreasing and not constant. Since V is smooth, $V_x > 0$ on a nonempty open set. Every term in (2.51) is nonnegative, so $\mathcal{A}_b > 0$ on that set and $\mathcal{A}$ is nonconstant. Let $W_0 = \int_0^1 w(b)\, \mathrm{d}b$. If $\int_0^1 w\Phi = 0$, then

$$\int_0^1 w\mathcal{A}\Phi\, \mathrm{d}b = \frac{1}{2W_0} \int_0^1 \int_0^1 w(b)w(d)[\mathcal{A}(b) - \mathcal{A}(d)][\Phi(b) - \Phi(d)]\, \mathrm{d}b\, \mathrm{d}d > 0. \tag{2.52}$$

Strictness follows because $w > 0$, Φ is strictly increasing, and the nondecreasing function $\mathcal{A}$ is nonconstant.

Step 4: the tail-ratio estimate. Assume from now on that

$$\int_0^1 w\Phi\, \mathrm{d}b = 0. \tag{2.53}$$

Define

$$F(b) = \int_b^1 w(d)\Phi(d)\, \mathrm{d}d, \qquad \tau(b) = \frac{F(b)}{w(b)}. \tag{2.54}$$

By (2.42), Φ is strictly increasing. Also

$$\Phi(0) = -mc(0) < 0, \qquad \lim_{b \uparrow 1} \Phi(b) = 2.$$

Thus Φ has a unique zero b_Φ. If $b \leq b_\Phi$, then by (2.53),

$$F(b) = -\int_0^b w(d)\Phi(d)\, \mathrm{d}d \geq 0;$$

if $b \geq b_\Phi$, the defining integral in (2.54) is nonnegative. Hence

$$\tau \geq 0.$$

Set

$$H(b) = w(b)c(b)z(b) - F(b), \qquad L(b) = cJ - z(N + z). \tag{2.55}$$

Equations (2.44) and

$$(cz)_b = cJ - \Phi$$

give

$$H_b = wL.$$

By (2.16),

$$(cJ)_b = -2zJ + cJ_b \leq zJ.$$

On the other hand, using (2.43),

$$[z(N + z)]_b = (m + J)(N + z) + z\left(\frac{A_x}{c} + J + m + J\right) > zJ.$$

The inequality is strict for $0 < b < 1$: because $m > 0$, one has $z = (m + K)b > 0$, while $N = A + K\, b \geq 0$. Consequently,

$$L_b < 0.$$

Since $F(0) = 0$ and $z(0) = 0$, one has $H(0) = 0$. By Proposition 2.10,

$$\lim_{b \uparrow 1} w(b)c(b)z(b) = 0, \qquad \lim_{b \uparrow 1} F(b) = 0,$$

and therefore $H(1) = 0$. The function L is strictly decreasing, and

$$\int_0^1 wL\, \mathrm{d}b = H(1) - H(0) = 0.$$

Since $w > 0$, the last integral rules out L having one sign on all of $(0, 1)$. The strictly decreasing function L therefore has a unique zero. It follows from $H_b = wL$ that H first increases and then decreases; because $H(0) = H(1) = 0$, one has

$$H \geq 0.$$

From (2.55), this is equivalent to

$$0 \leq \tau \leq cz.$$

Step 5: completion of the sign estimate. Differentiating (2.49) and using $c_b = -2z$ and $z_b = m + J$ gives

$$(R_0)_b = 2cJ_b - 5zJ. \tag{2.56}$$

Since $F' = -w\Phi$ and $F(0) = F(1) = 0$, integration by parts yields

$$\int_0^1 w\Phi R_0 \,\mathrm{d}b = \int_0^1 w\tau(R_0)_b \,\mathrm{d}b. \tag{2.57}$$

Combining (2.46), (2.56) and (2.57), we find

$$\int_0^1 w(R_0\Phi + G)\,\mathrm{d}b = \int_0^1 w\left[2cJ_b(\tau - cz) + zJ(6cz - 5\tau)\right]\mathrm{d}b.$$

Because $\tau - cz \leq 0$ and $cJ_b \leq 3zJ$, multiplication by $2(\tau - cz)$ reverses the latter inequality. Hence

$$\int_0^1 w(R_0\Phi + G)\,\mathrm{d}b \geq \int_0^1 w\left[6zJ(\tau - cz) + zJ(6cz - 5\tau)\right]\mathrm{d}b \tag{2.58}$$

$$= \int_0^1 wzJ\tau \,\mathrm{d}b \geq 0. \tag{2.59}$$

Finally, if $R'(t) = 0$, then (2.41) gives (2.53). Equations (2.50), (2.52), and (2.59) show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 w\Phi \,\mathrm{d}b > 0.$$

Differentiating (2.41) therefore gives $R''(t) > 0$, proving (2.39). □

2.4. **Exclusion of support gaps.** We now apply Proposition 2.11 to the Parisi minimizer.

Proposition 2.12 (The support is an interval)**.** *For every $\beta > 1$,*

$$S_\beta = [0, q_\beta].$$

Proof. Suppose, toward a contradiction, that (s_0, s_1) is a connected component of $[0, q_\beta] \setminus S_\beta$. Since S_β is closed,

$$s_0, s_1 \in S_\beta.$$

Since the interval contains no support point,

$$\alpha(s) = m \qquad (s_0 < s < s_1)$$

for some constant m.

We claim that $m > 0$. If $s_0 > 0$, this follows from $0 \in S_\beta$, because every neighborhood of 0 has positive μ_β-mass and is contained in $[0, s_0]$ for a sufficiently small radius. If $s_0 = 0$, then $(0, s_1)$ contains no support point. The fact that $0 \in S_\beta$ therefore forces $\mu_\beta(\{0\}) > 0$, and again $m > 0$.

Set

$$t_i = \beta^2 s_i, \qquad i = 0, 1.$$

By (2.7),

$$R(t_0), R(t_1) \leq \frac{1}{\beta^2}. \tag{2.60}$$

Also, using (2.6) and the self-consistency identities at the two endpoints,

$$\int_{t_0}^{t_1} R(t)\,\mathrm{d}t = M(t_1) - M(t_0) \tag{2.61}$$

$$= s_1 - s_0 \tag{2.62}$$

$$= \frac{t_1 - t_0}{\beta^2}. \tag{2.63}$$

Thus the average of R on $[t_0, t_1]$ equals β^{-2}.

By Lemma 2.4, R is continuous on the closed interval $[t_0, t_1]$, while Proposition 2.11 gives $R \in C^2((t_0, t_1))$. In particular, R attains a maximum on the closed interval, and every interior critical point is a strict local minimum. Therefore R cannot attain its maximum at an interior point: at such a point one would have $R' = 0$ and $R'' \le 0$, contrary to (2.39). It follows from (2.60) that

$$R(t) \le \frac{1}{\beta^2} \qquad (t_0 \le t \le t_1).$$

The average identity (2.63) forces equality at every point:

$$R(t) \equiv \frac{1}{\beta^2}.$$

This is impossible, since then $R' = R'' = 0$ in the interior, again contradicting Proposition 2.11.

Hence $[0, q_\beta] \setminus S_\beta$ has no nonempty connected component. Since it is open relative to $[0, q_\beta]$, it must be empty. Therefore $S_\beta = [0, q_\beta]$. □

2.5. **Endpoint control and Hausdorff convergence.** It remains to estimate q_β.

Proposition 2.13 (Upper endpoint estimate)**.** *For every $\beta > 1$,*

$$1 - q_\beta \le \frac{1}{\beta}.$$

Proof. By Lemma 2.2, $q_\beta < 1$. Since all the mass of μ_β lies in $[0, q_\beta]$,

$$\alpha(s) = 1 \qquad (q_\beta \le s \le 1).$$

On this interval the Cole–Hopf transform $v = e^u$ solves the backward heat equation. Since $e^{u(1,x)} = \cosh x$ and

$$\mathbb{E} \cosh(x + \beta\sqrt{1-s}\, Z) = e^{\beta^2(1-s)/2} \cosh x,$$

one obtains

$$u(s,x) = \log\cosh x + \frac{\beta^2}{2}(1-s), \qquad q_\beta \le s \le 1.$$

Hence

$$u_x(q_\beta, x) = \tanh x, \qquad u_{xx}(q_\beta, x) = \operatorname{sech}^2 x.$$

Since $q_\beta \in S_\beta$, the self-consistency condition gives

$$q_\beta = \mathbb{E}[\tanh^2(X_{q_\beta})],$$
$$1 - q_\beta = \mathbb{E}[\operatorname{sech}^2(X_{q_\beta})].$$

The replicon bound gives

$$\beta^2 \mathbb{E}[\operatorname{sech}^4(X_{q_\beta})] \le 1.$$

Therefore, by Cauchy–Schwarz,

$$\begin{aligned} 1 - q_\beta &= \mathbb{E}[\operatorname{sech}^2(X_{q_\beta})] \\ &\le \left(\mathbb{E}[\operatorname{sech}^4(X_{q_\beta})]\right)^{1/2} \\ &\le \frac{1}{\beta}. \end{aligned}$$

□

Proof of Theorem 1.1. By Proposition 2.12, $S_\beta = [0, q_\beta]$. Combining this identity with Proposition 2.13 gives

$$d_{\mathrm{H}}(S_\beta, [0,1]) = d_{\mathrm{H}}([0, q_\beta], [0,1]) = 1 - q_\beta \le \frac{1}{\beta}.$$

Consequently,

$$\lim_{\beta\to\infty} d_{\mathrm{H}}(S_\beta, [0,1]) = 0.$$

□

3. Zero-temperature variational structure

3.1. **The variational problem.** Theorem 1.1 concerns the geometry of the finite-temperature supports. It does not by itself identify the zero-temperature order parameter: the probability measures concentrate at the endpoint one, whereas their interior structure survives after multiplication by the inverse temperature. We therefore turn to the zero-temperature variational problem.

The local argument remains parallel to the finite-temperature proof. On an interval where the order parameter is constant, the slope coordinate converts the derivative of the curvature observable into a weighted integral of a strictly increasing function. The finite-temperature weight is controlled by the de-tilted density from Subsection 2.2; the zero-temperature weight is controlled by the forward factor introduced in Subsection 3.2.

Let

$$\mathcal{U} = \left\{ \gamma : [0,1) \to [0,\infty) : \begin{array}{l} \gamma \text{ is nondecreasing and right-continuous,} \\ \displaystyle\int_0^1 \gamma(t)\,\mathrm{d}t < \infty \end{array} \right\}.$$

For $\gamma \in \mathcal{U}$, let u_γ solve

$$\partial_t u_\gamma + \frac{1}{2}\left(\partial_{xx} u_\gamma + \gamma(t)(\partial_x u_\gamma)^2\right) = 0, \qquad u_\gamma(1,x) = |x|. \tag{3.1}$$

The zero-temperature Parisi functional is

$$\mathcal{P}(\gamma) = u_\gamma(0,0) - \frac{1}{2}\int_0^1 t\gamma(t)\,\mathrm{d}t. \tag{3.2}$$

The zero-temperature Parisi formula is [3, Theorem 1], and uniqueness of the minimizer is [5, Theorem 4]. Hence (3.2) has a unique minimizer $\gamma_\star$. From now on, write

$$\gamma = \gamma_\star, \qquad u = u_\gamma, \qquad \nu = \mathrm{d}\gamma, \qquad S = \operatorname{supp}_{[0,1)} \nu.$$

We first record the regularity properties used below. They follow by approximating $|x|$ with $\lambda^{-1}\log\cosh(\lambda x)$, applying the Cole–Hopf and stochastic-control representations, and then passing to the limit; see [5, Proposition 2 and Theorem 5] and [4, Propositions 2 and 3].

Lemma 3.1 (PDE and diffusion facts)**.** *For every $T < 1$, the function u is continuous in (t,x) and smooth in x on $[0,T]\times\mathbb{R}$. It is even, convex, and one-Lipschitz in x, with*

$$-1 < u_x(t,x) < 1, \qquad u_{xx}(t,x) > 0, \qquad \lim_{x\to\pm\infty} u_x(t,x) = \pm 1.$$

The stochastic differential equation

$$\mathrm{d}X_t = \gamma(t) u_x(t, X_t)\,\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = 0, \tag{3.3}$$

has a unique weak solution. For $t > 0$, X_t has a smooth, positive, even density ρ_t.

Define the consistency function and the obstacle

$$\Gamma(t) = \mathbb{E}\, u_x(t, X_t)^2, \qquad h(t) = \Gamma(t) - t, \qquad G(q) = \int_q^1 h(t)\,\mathrm{d}t. \tag{3.4}$$

The directional derivative formula of [5, Propositions 3 and 4] is

$$D\mathcal{P}(\gamma)[\eta - \gamma] = \frac{1}{2}\int_0^1 (\eta(t) - \gamma(t)) h(t)\,\mathrm{d}t, \qquad \eta \in \mathcal{U}. \tag{3.5}$$

Proposition 3.2 (KKT conditions). *The functions in* (3.4) *satisfy*

$$G(q) \geq 0, \qquad 0 \leq q < 1, \qquad G(q) = 0, \qquad q \in S. \tag{3.6}$$

At every $q \in S$,

$$\Gamma(q) = q, \qquad \mathbb{E}\, u_{xx}(q, X_q)^2 \leq 1. \tag{3.7}$$

If q *is an endpoint of a constant-profile interval, the corresponding one-sided derivative of* Γ *exists and equals the expectation in* (3.7). *Moreover,*

$$0 \in S.$$

Proof. Fix $q < 1$ and $c > 0$. Adding an atom $c\delta_q$ to ν replaces γ by $\eta(t) = \gamma(t) + c\mathbf{1}_{[q,1)}(t)$. Minimality and (3.5) give $G(q) \geq 0$.

Both 2γ and 0 belong to $\mathcal{U}$. Applying (3.5) in these two directions gives

$$\int_0^1 \gamma(t)h(t)\,\mathrm{d}t = 0. \tag{3.8}$$

Since $|h(t)| \leq 1$ and $\gamma \in L^1$, Fubini's theorem is absolutely applicable:

$$\int_{[0,1)} \int_q^1 |h(t)|\,\mathrm{d}t\,\nu(\,\mathrm{d}q) = \int_0^1 \gamma(t)|h(t)|\,\mathrm{d}t < \infty,$$

$$\int_0^1 \gamma(t)h(t)\,\mathrm{d}t = \int_{[0,1)} G(q)\,\nu(\,\mathrm{d}q). \tag{3.9}$$

Equations (3.6), (3.8), and (3.9) show that $G = 0$ ν-almost everywhere. As G is continuous, it vanishes at every point of S.

The two assertions in (3.7) are exactly the zero-temperature consistency and stability conditions in [5, Proposition 3]; invoking that proposition also covers the boundary point $q = 0$, where a one-sided minimum of G alone would not imply stability. In addition, Itô's formula along (3.3) gives

$$\mathrm{d}u_x(t, X_t) = u_{xx}(t, X_t)\,\mathrm{d}W_t.$$

Consequently,

$$\Gamma(t) - \Gamma(s) = \int_s^t \mathbb{E}u_{xx}(r, X_r)^2\,\mathrm{d}r.$$

Standard one-sided parabolic continuity on each constant-profile interval turns this integral identity into the derivative formula used at gap endpoints. Finally, the zero-field support theorem in [5, Section 3.2] gives $0 \in S$. ☐

Lemma 3.3 (Parabolic stability away from the terminal time). *Let* $T < T' < 1$. *On* $[0, T']$, *let* α_n, α *be nonnegative time-dependent coefficients with*

$$\sup_n \|\alpha_n\|_{L^\infty(0,T')} < \infty, \qquad \lim_{n\to\infty} \|\alpha_n - \alpha\|_{L^1(0,T')} = 0.$$

Let v_n, v *solve*

$$v_t + \frac{1}{2}(v_{xx} + \alpha(t)v_x^2) = 0$$

backward from time T', *with* α *replaced by* α_n *for* v_n. *Assume that the boundary data are convex, one-Lipschitz, and converge uniformly on* $\mathbb{R}$. *Then*

$$\lim_{n\to\infty} \|v_n - v\|_{L^\infty([0,T']\times\mathbb{R})} = 0. \tag{3.10}$$

For every integer $k \geq 1$ *and* $R < \infty$,

$$\lim_{n\to\infty} \sup_{\substack{0\leq t\leq T \\ |x|\leq R}} |\partial_x^k v_n(t,x) - \partial_x^k v(t,x)| = 0. \tag{3.11}$$

Moreover, for every $k \geq 2$ there is a constant $C_{k,T,T'}$ such that

$$\sup_n \sup_{\substack{0\leq t\leq T\\ x\in\mathbb{R}}} |\partial_x^k v_n(t,x)| \leq C_{k,T,T'}. \tag{3.12}$$

If the convergence in (3.10) is uniform on all of $\mathbb{R}$, then

$$\lim_{n\to\infty} \sup_{\substack{0\leq t\leq T\\ x\in\mathbb{R}}} |v_{n,x}(t,x) - v_x(t,x)| = 0. \tag{3.13}$$

Proof. The comparison argument used in (3.41) below gives

$$\|v_n - v\|_\infty \leq \|v_n(T',\cdot) - v(T',\cdot)\|_\infty + \frac{1}{2}\|\alpha_n - \alpha\|_{L^1(0,T')},$$

because all slopes have absolute value at most one.

For the derivative assertions, reverse time by writing $V_n(s,x) = v_n(T'-s,x)$ and set $a_n(s) = \alpha_n(T'-s)$. The mild equation is

$$V_n(s) = P_s V_n(0) + \frac{1}{2}\int_0^s a_n(r) P_{s-r}\big[(V_{n,x}(r))^2\big]\, \mathrm{d}r.$$

Put $w_n = V_{n,x}$. In the mild sense,

$$w_n(s) = P_s w_n(0) + \int_0^s a_n(r) P_{s-r}\big[w_n(r)(w_n)_x(r)\big]\,\mathrm{d}r, \qquad \|w_n\|_\infty \leq 1. \tag{3.14}$$

We give the derivative bootstrap because the slope-coordinate argument requires derivatives through order five. Let $A_0 = \sup_n \|a_n\|_\infty$ and $M_{\ell,n}(s) = \|\partial_x^\ell w_n(s)\|_\infty$. The heat-semigroup bounds

$$\|\partial_x^j P_s f\|_\infty \leq C_j s^{-j/2}\|f\|_\infty, \qquad \|\partial_x P_s f\|_\infty \leq C s^{-1/2}\|f\|_\infty$$

applied to (3.14) give the base estimate

$$M_{1,n}(s) \leq C s^{-1/2} + C A_0 \int_0^s (s-r)^{-1/2} M_{1,n}(r)\,\mathrm{d}r.$$

The singular Gronwall lemma, obtained for example by iterating the kernel $(s-r)^{-1/2}$ once and then applying the ordinary Gronwall inequality, yields

$$\sup_n \sup_{s\in[\delta,T']} M_{1,n}(s) < \infty \qquad (\delta > 0). \tag{3.15}$$

For higher derivatives we use the standard interior gradient estimate for a one-dimensional uniformly parabolic equation. In the form needed here, if

$$z_s = \frac{1}{2} z_{xx} + b(s,x) z_x + c(s,x) z + f(s,x)$$

on a strip $[s_0, S]\times\mathbb{R}$, and z, b, c, f are uniformly bounded with b uniformly Lipschitz in x, then for every $\eta > 0$,

$$\sup_{s_0+\eta\leq s\leq S} \|z_x(s)\|_\infty \leq C_{\eta,S,M}\left(1 + \sup_{s_0\leq s\leq S}\|z(s)\|_\infty\right), \tag{3.16}$$

where M bounds the displayed coefficients and the source. This is the usual Bernstein interior estimate; equivalently, it follows from the local $W_p^{2,1}$ estimate on unit parabolic cylinders and the one-dimensional parabolic Sobolev embedding. Only measurability in time is needed.

Let $q_{r,n} = \partial_x^r w_n$ for $r \geq 1$. Differentiating the Burgers equation r times gives

$$(q_{r,n})_s = \frac{1}{2}(q_{r,n})_{xx} + a_n w_n (q_{r,n})_x + c_r a_n (w_n)_x q_{r,n} + a_n F_{r,n}, \tag{3.17}$$

where $c_1 = 1$, $c_r = r+1$ for $r \geq 2$, and $F_{r,n}$ is a finite sum of products of derivatives of w_n of orders at most $r-1$; in particular, $F_{1,n} = F_{2,n} = 0$. The estimate (3.15) makes the drift $a_n w_n$ uniformly Lipschitz in x after every positive reverse time. Assume inductively that $q_{j,n}$ is uniformly bounded for $1 \leq j \leq r$ on a strip $s \geq \delta_r > 0$. Then $q_{r,n}$, all coefficients, and the source in (3.17) are uniformly bounded there. Applying (3.16) on the smaller strip $s \geq \delta_{r+1} > \delta_r$ gives a uniform bound for $(q_{r,n})_x = q_{r+1,n}$. Choosing nested buffers below the target reverse time $T' - T$

proves, for every fixed k, uniform bounds for $\partial_x^k w_n$ on $[T'-T,T']$. Since $\partial_x^k v_n$ corresponds to $\partial_x^{k-1} w_n$ after time reversal, this proves (3.12).

For a fixed k, the bounds just proved through order $k+2$ imply spatial equicontinuity of $\partial_x^k v_n$. The differentiated PDE

$$\partial_t \partial_x^k v_n = -\frac{1}{2}\partial_x^{k+2} v_n - \frac{1}{2}\alpha_n(t)\partial_x^k(v_{n,x}^2)$$

has a uniformly bounded right-hand side for almost every time, and hence gives uniform Lipschitz continuity in time with values in $C([-R,R])$. Arzelà–Ascoli makes every spatial derivative precompact on $[0,T]\times[-R,R]$. Uniform convergence of the functions identifies each subsequential limit with the corresponding distributional derivative of v. The limit is therefore unique, and (3.11) holds for the whole sequence.

Finally, let $\varepsilon_n = \|v_n - v\|_\infty$ and use the global bound $0 \le v_{xx} \le M$ from the case $k=2$. Convexity and forward and backward difference quotients give, for every $h>0$,

$$|v_{n,x}(t,x) - v_x(t,x)| \le Mh + \frac{2\varepsilon_n}{h}.$$

Taking $h = \sqrt{\varepsilon_n}$ proves (3.13). □

3.2. **A forward factor and its score-curvature invariant.** At a continuity point of γ, define

$$Q(t,x) = \rho_t(x)e^{-\gamma(t)u(t,x)}.$$

At a jump, we use the right-continuous value. The cancellation behind this definition is the reason for introducing Q.

Lemma 3.4 (Heat evolution and jump rule)**.** *On an interval where $\gamma = m$ is constant,*

$$Q_t = \frac{1}{2}Q_{xx}.$$

At an upward jump $m^- \mapsto m^+ = m^- + \delta$,

$$Q(t+,x) = e^{-\delta u(t,x)}Q(t-,x). \tag{3.18}$$

Proof. On a constant-profile interval, the Fokker–Planck equation and the Parisi PDE are

$$\rho_t = \frac{1}{2}\rho_{xx} - m\partial_x(u_x\rho), \qquad u_t = -\frac{1}{2}(u_{xx} + mu_x^2).$$

Differentiate $Q = \rho e^{-mu}$ twice in x and once in t; substitution cancels all terms containing u_x or u_{xx} and leaves $Q_t = Q_{xx}/2$. At a jump, u and ρ are continuous in time, whereas the exponential factor changes from e^{-m^-u} to e^{-m^+u}. This gives (3.18). □

Extend γ by $\gamma(0-) = 0$. Since $\rho_0 = \delta_0$, the right-continuous initial measure for Q is

$$Q(0,\,\mathrm{d}x) = c_0\delta_0(\,\mathrm{d}x), \qquad c_0 = e^{-\gamma(0)u(0,0)}.$$

Iterating Lemma 3.4 first for a step profile and then approximating gives a useful explicit formula.

Lemma 3.5 (Stieltjes Brownian-bridge formula)**.** *For $0<t<1$,*

$$Q(t,x) = c_0 p_t(x)\,\mathbb{E}^{\mathrm{BB},t}_{0\to x}\exp\left\{-\int_{(0,t]} u(s,B_s)\,\mathrm{d}\gamma(s)\right\}, \tag{3.19}$$

where $p_t(x) = (2\pi t)^{-1/2}e^{-x^2/(2t)}$ and the expectation is over a Brownian bridge from 0 to x in time t. The atom at zero is contained in c_0, while an atom at t is included in the integral, consistently with the right-continuous convention.

If γ_n are nondecreasing step approximations satisfying

$$\begin{aligned} &\lim_{n\to\infty}\|\gamma_n - \gamma\|_{L^1(0,1)} = 0, \\ &\mathrm{d}\gamma_n \Longrightarrow \mathrm{d}\gamma \qquad \textit{weakly on every } [0,T] \textit{ ending at a continuity point,} \\ &\lim_{n\to\infty}\gamma_n(s) = \gamma(s) \qquad \textit{at every continuity point,} \end{aligned} \tag{3.20}$$

then, at every continuity point t and for every integer $k \ge 0$,

$$\lim_{n\to\infty} \|Q_n(t,\cdot) - Q(t,\cdot)\|_{C^k(K)} = 0 \quad \text{for every compact } K \subset \mathbb{R}. \tag{3.21}$$

At a jump, the same conclusion holds first for the left limit. The step approximations may be chosen so that their jump at that time converges to the jump of γ; multiplication by the factor in (3.18) then gives the right limit as well.

Proof. For a step profile, Lemma 3.4 alternates Gaussian heat evolution and multiplication by $e^{-\Delta\gamma(s)u(s,\cdot)}$. The usual conditioning of Brownian motion on its endpoint gives (3.19) with the Stieltjes integral replaced by the finite sum over jump times.

Choose the step approximations so that also $\lim_{n\to\infty}\gamma_n(0) = \gamma(0)$, and let u_n be the corresponding PDE solutions. For every $T < 1$ and $j \ge 0$, the stability argument in Lemma 3.3 gives

$$\lim_{n\to\infty} \partial_x^j u_n = \partial_x^j u \quad \text{locally uniformly on } [0,T]\times\mathbb{R}. \tag{3.22}$$

For fixed $T < 1$, parabolic smoothing and the one-Lipschitz bound also give, uniformly in n,

$$|u_{n,x}| \le 1, \qquad 0 \le u_{n,xx} \le M_T \quad \text{on } [0,T]\times\mathbb{R}.$$

The control representation with zero control also gives $u_n \ge 0$.

Fix a continuity point t, an integer $k \ge 0$, and write a bridge with endpoint x as $B_s^x = (s/t)x + \widetilde{B}_s$, where $\widetilde{B}$ is a centered bridge. Set

$$A_n(x) = \int_{(0,t]} u_n(s, B_s^x)\, \mathrm{d}\gamma_n(s).$$

For $1 \le j \le k$, pathwise differentiation gives

$$A_n^{(j)}(x) = \int_{(0,t]} \left(\frac{s}{t}\right)^j \partial_x^j u_n(s, B_s^x)\, \mathrm{d}\gamma_n(s). \tag{3.23}$$

The case $j = 1$ is bounded by the one-Lipschitz estimate. For $j \ge 2$, choose a continuity time T' with $t < T' < 1$. The profiles may be approximated with $\sup_n \gamma_n(T') < \infty$, and Lemma 3.3, applied across the positive distance $T' - t$, gives a bound independent of n. Since $\sup_n(\mathrm{d}\gamma_n)((0,t]) < \infty$, all quantities in (3.23) are therefore uniformly bounded.

Every bridge path has compact range. For $|x| \le R$, all arguments B_s^x lie in one random compact set, uniformly in s. Decomposing an integral against $\mathrm{d}\gamma_n$ into the difference of its integrand and the weak-convergence error, and using (3.20) together with (3.22), shows pathwise and locally uniformly in x that

$$\lim_{n\to\infty} A_n^{(j)} = A^{(j)} \quad \text{locally uniformly in } x, \qquad 0 \le j \le k. \tag{3.24}$$

Here $j = 0$ uses local uniform convergence of u_n, while $j \ge 1$ uses (3.23). Faà di Bruno's formula writes $\partial_x^j e^{-A_n}$ as e^{-A_n} times a universal polynomial in $A_n', \dots, A_n^{(j)}$. Since $A_n \ge 0$, these polynomials and (3.24) provide an integrable dominator under the bridge expectation for every fixed $j \le k$.

Denote the resulting C^k_{loc} bridge limit temporarily by $\widehat{Q}$. Differentiating the Gaussian prefactor as well as the bridge expectation gives, for every fixed k,

$$\sum_{j=0}^{k} |\partial_x^j Q_n(t,x)| \le C_{k,t}(1 + |x|^k) p_t(x), \tag{3.25}$$

with $C_{k,t}$ uniform in n. The same estimate holds for $\widehat{Q}$.

It remains to identify $\widehat{Q}$ with the quantity defined from the actual diffusion density. Put

$$b_n(s,x) = \gamma_n(s) u_{n,x}(s,x), \qquad b(s,x) = \gamma(s) u_x(s,x).$$

Choose T' with $t < T' < 1$ at which γ is continuous. The step approximations may be chosen with $\sup_n \gamma_n(T') < \infty$. The comparison estimate and the last part of Lemma 3.3 give

$$\lim_{n\to\infty} \sup_{\substack{0\le s\le t\\ x\in\mathbb{R}}} |u_{n,x}(s,x) - u_x(s,x)| = 0.$$

Consequently,

$$\int_0^t \|b_n(s,\cdot) - b(s,\cdot)\|_\infty \, \mathrm{d}s \le \|\gamma_n - \gamma\|_{L^1(0,t)} + \|\gamma\|_{L^1(0,t)} \sup_{s,x} |u_{n,x} - u_x| \longrightarrow 0. \tag{3.26}$$

All these drifts are uniformly bounded on $[0,t]\times\mathbb{R}$.

For completeness, this drift convergence implies convergence of the diffusion laws directly. Under Wiener measure, let

$$Z_n = \exp\left\{ \int_0^t b_n(s,W_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b_n(s,W_s)^2\,\mathrm{d}s \right\},$$

and define Z with b in place of b_n. The uniform drift bound and (3.26) imply

$$\lim_{n\to\infty} \int_0^t \|b_n - b\|_\infty^2 \,\mathrm{d}s = 0.$$

Itô's isometry makes the stochastic integrals converge in L^2, and the finite-variation terms converge in L^1. Moreover, for some $p > 1$,

$$\sup_n \mathbb{E} Z_n^p < \infty,$$

because Z_n^p is a stochastic exponential with integrand pb_n times the bounded factor $\exp\{\frac{1}{2}(p^2 - p)\int_0^t b_n^2\}$. Thus $Z_n \to Z$ in L^1. Girsanov's theorem shows that the one-time laws with densities $\rho_n(t,\cdot)$ converge weakly to the law with density ρ_t.

At the continuity point t,

$$\rho_n(t,x) = e^{\gamma_n(t)u_n(t,x)} Q_n(t,x) \longrightarrow \widehat{\rho}(t,x) := e^{\gamma(t)u(t,x)}\widehat{Q}(t,x)$$

locally uniformly. Testing against compactly supported continuous functions and comparing with the preceding weak convergence gives $\widehat{\rho}(t,\cdot) = \rho_t$ and therefore $\widehat{Q}(t,\cdot) = \rho_t e^{-\gamma(t)u(t,\cdot)}$. This proves (3.21) for the actual Q. At a discontinuity, u and ρ are continuous and the explicit multiplier (3.18) gives the right-continuous value, proving the last assertion. □

Define the score and score curvature

$$\mathfrak{r} = -(\log Q)_x, \qquad H = \frac{Q_{xx}}{Q} = \mathfrak{r}^2 - \mathfrak{r}_x.$$

Proposition 3.6 (Forward-factor invariant)**.** *For every $0 < t < 1$, the function $Q(t,\cdot)$ is positive, even, and log-concave. Moreover,*

$$x \longmapsto H(t,x) = \frac{Q_{xx}(t,x)}{Q(t,x)} \quad \textit{is nondecreasing on } (0,\infty). \tag{3.27}$$

If γ is constant on a positive-length interval (a,b), then for every $a < t < b$, $H(t,\cdot)$ is strictly increasing in the order sense on $(0,\infty)$.

Proof. We begin with a step profile. The initial heat evolution of $c_0\delta_0$ is a centered Gaussian, for which

$$\mathfrak{r}(t,x) = \frac{x}{t}, \qquad H(t,x) = \frac{x^2}{t^2} - \frac{1}{t}.$$

It is positive, even, and log-concave, and H is increasing for $x > 0$. Gaussian convolution preserves log-concavity by the Prékopa–Leindler theorem. At a jump,

$$\log Q(t+,\cdot) = \log Q(t-,\cdot) - \delta u(t,\cdot),$$

which is concave because $\log Q(t-,\cdot)$ is concave and u is convex. Both operations preserve evenness.

Suppose that $\widetilde{Q} = P_s Q$ and that the incoming $H = Q_{xx}/Q$ is nondecreasing on the positive half-line. Estimate (3.25), with the explicit jump multiplier when necessary, makes Q, Q_x, Q_{xx} absolutely integrable against every heat kernel below. We may therefore differentiate under the convolution. Since $Q_{xx} = QH$,

$$\widetilde{H}(x) = \frac{P_s(QH)(x)}{P_s Q(x)}. \tag{3.28}$$

After folding the even integrands to $x, y \geq 0$, the kernel is, up to a positive constant,

$$\mathcal{K}_s(x,y) = e^{-(x^2+y^2)/(2s)} \cosh(xy/s).$$

Its logarithmic mixed derivative is

$$\partial_x \partial_y \log \mathcal{K}_s(x,y) = \frac{1}{s}\tanh(xy/s) + \frac{xy}{s^2}\operatorname{sech}^2(xy/s) > 0$$

for $x, y > 0$. Integration over a rectangle shows that $\mathcal{K}_s$ is strictly TP_2. More explicitly, if $R(x)$ denotes the ratio in (3.28), then after multiplication by its two positive denominators, $R(x_2) - R(x_1)$ equals

$$\begin{aligned} &\int_{0<y_1<y_2} Q(y_1)Q(y_2)[H(y_2) - H(y_1)] \\ &\qquad \times [\mathcal{K}_s(x_2,y_2)\mathcal{K}_s(x_1,y_1) - \mathcal{K}_s(x_2,y_1)\mathcal{K}_s(x_1,y_2)]\, \mathrm{d}y_1\, \mathrm{d}y_2. \end{aligned} \tag{3.29}$$

Thus heat evolution preserves the monotonicity of H and makes it strict unless the incoming H is constant.

At a jump, put $B = u_x$, $C = u_{xx}$, and $D = u_{xxx}$. From (3.18), $\widetilde{\mathfrak{r}} = \mathfrak{r} + \delta B$, and therefore

$$\widetilde{H} = H + 2\delta\mathfrak{r}B + \delta^2 B^2 - \delta C, \tag{3.30}$$

$$\widetilde{H}_x = H_x + 2\delta(\mathfrak{r}_x B + \mathfrak{r}C) + 2\delta^2 BC - \delta D. \tag{3.31}$$

For $x > 0$, evenness and log-concavity give $\mathfrak{r}, \mathfrak{r}_x \geq 0$, while evenness and convexity give $B, C \geq 0$. We next verify the remaining sign

$$D = u_{xxx} \leq 0 \qquad (x > 0). \tag{3.32}$$

In reverse time, $v(s,x) = u(1-s,x)$ satisfies

$$v_s = \frac{1}{2}(v_{xx} + \gamma(1-s)v_x^2).$$

Writing $D = v_{xxx}$, differentiation gives on $x > 0$

$$D_s = \frac{1}{2}D_{xx} + \gamma(1-s)v_x D_x + 3\gamma(1-s)v_{xx}D, \qquad D(s,0) = 0.$$

For the regularized terminal datum (3.39),

$$D(0,x) = -2\lambda^2 \tanh(\lambda x)\operatorname{sech}^2(\lambda x) \leq 0.$$

We spell out the half-line argument, since it is also what legitimizes (3.31) after passage to a general profile. First suppose that the coefficient is a bounded step function. On a reverse-time slab $[s_0, s_1]$ on which $a = \gamma(1-s)$ is constant, put $b = av_x$ and $c = 3av_{xx}$. Both are bounded. The regularized finite cascade also satisfies

$$\lim_{R\to\infty} \sup_{s\in[s_0,s_1]} |D(s,R)| = 0. \tag{3.33}$$

Indeed, at $s = 0$ this follows directly from ϕ_λ. On a constant slab the Cole–Hopf formula $e^{av(s,\cdot)} = P_{s-s_0}e^{av(s_0,\cdot)}$ (with the heat formula when $a = 0$), differentiated under the Gaussian integral, preserves the fact that $v_x \to 1$ and every derivative of order at least two tends to zero as $x \to +\infty$, uniformly on compact subslabs. At a step of the coefficient the function v and all of its spatial derivatives are continuous, so induction over the finitely many slabs proves (3.33).

Assume inductively that $D(s_0,\cdot) \leq 0$. Choose $M \geq \|c\|_{L^\infty([s_0,s_1]\times(0,\infty))}$ and set $\overline{D}(s,x) = e^{-M(s-s_0)}D(s,x)$. Then

$$\overline{D}_s = \frac{1}{2}\overline{D}_{xx} + b\overline{D}_x + (c-M)\overline{D}, \qquad c - M \leq 0.$$

For $\varepsilon > 0$, choose R by (3.33) so that $\overline{D}(s,R) \le \varepsilon$ on the slab. The parabolic maximum principle on $[s_0,s_1] \times [0,R]$, using $\overline{D}(s,0) = 0$ and the nonpositive initial data, gives $\overline{D} \le \varepsilon$. Letting first $R \to \infty$ along such a choice and then $\varepsilon \downarrow 0$ yields $D \le 0$ on the slab. Iteration over the step intervals proves the sign for every bounded step coefficient.

Approximate an arbitrary bounded coefficient by bounded step coefficients. The C^3_{loc} convergence in Lemma 3.3 passes the sign to the limit. Finally, apply this result to $\gamma \wedge \lambda$ with terminal datum ϕ_λ and use (3.42) with $j = 3$ to let $\lambda \to \infty$. This proves (3.32) for the actual zero-temperature solution. Consequently every term on the right side of (3.31) is nonnegative, so a jump preserves (3.27).

For a general profile, choose the step approximations in Lemma 3.5. At each continuity time, (3.21) gives local uniform convergence of $Q_n, Q_{n,xx}$ and hence of H_n; positivity, evenness, log-concavity, and the order inequality for H pass to the limit. At a jump the same argument first gives the left limit. The limiting $Q(t-,\cdot)$ is smooth and positive, so monotonicity implies $H_x(t-,x) \ge 0$ for $x > 0$; the direct formulas (3.30)–(3.31), together with (3.32), then give the right limit. Thus the passage through a jump uses only the convergence already established above.

Finally, let $a < s < t < b$ lie in a constant-profile interval. Then $Q(t) = P_{t-s}Q(s)$. The incoming $H(s,\cdot)$ cannot be constant: if $Q'' = cQ$ on $\mathbb{R}$, then for $c > 0$ every positive even solution grows like a hyperbolic cosine, for $c = 0$ it is affine, and for $c < 0$ every nonzero solution changes sign. None is positive, even, and integrable. Since $H(s,\cdot)$ is continuous, being nonconstant means that $H(y_2) - H(y_1) > 0$ on a set of positive two-dimensional measure. Hence the integral in (3.29) is strictly positive. If the interval begins at $a = 0$, the first positive-time profile is Gaussian and strictness is immediate. This proves the final claim. $\square$

For later use, suppose that $\gamma(t) = m$ on an open interval. Put $B = u_x$, $C = u_{xx}$, and $D = u_{xxx}$. Differentiating the PDE and applying Itô's formula gives

$$\Gamma'(t) = \mathbb{E}C(t,X_t)^2, \tag{3.34}$$

$$\Gamma''(t) = \mathbb{E}\left[D(t,X_t)^2 - 2mC(t,X_t)^3\right]. \tag{3.35}$$

Indeed, $\mathrm{d}B = C\,\mathrm{d}W$, while

$$\mathrm{d}C(t,X_t) = -mC(t,X_t)^2\,\mathrm{d}t + D(t,X_t)\,\mathrm{d}W_t.$$

3.3. **The zero-temperature slope invariant.** For fixed $t < 1$, Lemma 3.1 makes $x \mapsto B = u_x(t,x)$ a smooth increasing bijection from $\mathbb{R}$ onto $(-1,1)$. Write $x = x(t,B)$ for its inverse and

$$C(t,B) = u_{xx}(t,x(t,B)).$$

Suppose that $\gamma = m$ on an open interval $I \subset (0,1)$. On $I \times (-1,1)$ define

$$z = -\frac{1}{2}C_B, \qquad K = \frac{z}{B} - m, \qquad J = K + BK_B. \tag{3.36}$$

The quotients have smooth even extensions through $B = 0$.

We first isolate the finite-temperature Cole–Hopf invariant that will be rescaled below. For $a > 0$, let

$$\mathcal{T}_{a,r}f(x) = \frac{1}{a}\log\mathbb{E}\exp\{af(x+\sqrt{r}Z)\}, \qquad Z \sim N(0,1), \tag{3.37}$$

and use $\mathcal{T}_{0,r} = P_r$.

Proposition 3.7 (Lopatto's finite Cole–Hopf invariant)**.** *Begin with $(f,a) = (\log\cosh, 1)$. Perform finitely many operations of either type:*

(i) *replace f by $\mathcal{T}_{a,r}f$ for $r \ge 0$, leaving a fixed;*
(ii) *decrease the active parameter from a to some $b \in (0,a)$, leaving f fixed.*

For every resulting pair, define $B = f'$, $C = f''$ and the variables in (3.36) with m replaced by the active parameter a. Then, for $0 \le B < 1$,

$$K, K_B, J, J_B \ge 0, \qquad CJ_B \le 3zJ.$$

Proof. This is [8, Proposition 3.6 and Equation (3.39)]. In that proposition the last inequality is written as $(C^{3/2}J)_B \le 0$. Since $C_B = -2z$, the two forms are identical:

$$(C^{3/2}J)_B = C^{1/2}(CJ_B - 3zJ).$$

□

Proposition 3.8 (Zero-temperature slope invariant). *On every open interval on which $\gamma = m$ is constant, the variables in (3.36) satisfy*

$$K \ge 0, \quad K_B \ge 0, \quad J \ge 0, \quad J_B \ge 0, \quad CJ_B \le 3zJ, \qquad 0 \le B < 1. \tag{3.38}$$

Proof. For $\lambda > 0$, regularize the terminal datum by

$$\phi_\lambda(x) = \lambda^{-1} \log\cosh(\lambda x). \tag{3.39}$$

Introduce $(\mathcal{S}_\lambda f)(y) = \lambda f(y/\lambda)$. A direct change of variables in (3.37) gives

$$\mathcal{S}_\lambda \mathcal{T}_{a,r} f = \mathcal{T}_{a/\lambda,\lambda^2 r} \mathcal{S}_\lambda f, \qquad \mathcal{S}_\lambda \phi_\lambda = \log\cosh. \tag{3.40}$$

If $\bar f = \mathcal{S}_\lambda f$ and $y = \lambda x$, then

$$\bar B(y) = B(x), \qquad C_f(B) = \lambda C_{\bar f}(B), \qquad z_f(B) = \lambda z_{\bar f}(B).$$

When $a = \lambda \bar a$, one likewise has $K_f = \lambda K_{\bar f}$ and $J_f = \lambda J_{\bar f}$. Thus all five inequalities in (3.38) are homogeneous under (3.40). At the initial pair (ϕ_λ, λ),

$$B = \tanh(\lambda x), \qquad C = \lambda(1 - B^2), \qquad z = \lambda B, \qquad K = J = 0.$$

Consequently, Proposition 3.7 and (3.40) prove (3.38) for every finite step profile bounded by λ. A zero active parameter is obtained as a decreasing positive limit.

It remains to pass to the terminal condition $|x|$ and to a general profile. Put $\gamma_\lambda = \gamma \wedge \lambda$ and approximate it in $L^1(0,1)$ by nondecreasing step profiles $\gamma_{\lambda,n}$ taking values in $[0,\lambda]$. When checking a fixed compact subinterval of a gap on which $\gamma = m$, choose the partition to respect that gap, so that $\gamma_{\lambda,n} = m$ there for every sufficiently large $\lambda > m$. Equivalently, monotonicity and L^1 convergence imply pointwise convergence of the active values at every interior gap time. Let $u_{\lambda,n}$ solve (3.1) with coefficient $\gamma_{\lambda,n}$ and terminal datum ϕ_λ. At fixed λ, apply Lemma 3.3 after inserting an intermediate time T' with $T < T' < 1$. It gives, for every $j \ge 0$,

$$\lim_{n\to\infty} \partial_x^j u_{\lambda,n} = \partial_x^j u_\lambda \quad \text{locally uniformly on } [0,T] \times \mathbb{R}.$$

We include the estimate that controls the subsequent limit. If u_i have terminal data ψ_i, coefficients γ_i, and $\|\psi_i'\|_\infty \le 1$, the comparison principle gives

$$\sup_x |u_1(t,x) - u_2(t,x)| \le \|\psi_1 - \psi_2\|_\infty + \frac{1}{2}\int_t^1 |\gamma_1(s) - \gamma_2(s)|\,\mathrm{d}s. \tag{3.41}$$

To see this, add the time-dependent barriers $\pm\frac{1}{2}\int_t^1 |\gamma_1 - \gamma_2|$ to one solution and use $|u_x| \le 1$ in the two PDEs. Since

$$0 \le |x| - \phi_\lambda(x) \le \frac{\log 2}{\lambda},$$

(3.41) yields

$$\|u_\lambda - u\|_\infty \le \frac{\log 2}{\lambda} + \frac{1}{2}\|\gamma - \gamma_\lambda\|_{L^1(0,1)}, \qquad \lim_{\lambda\to\infty} \|u_\lambda - u\|_\infty = 0.$$

Fix $T < T' < 1$. Since γ is nondecreasing and integrable, $\gamma(T') < \infty$, so $\gamma_\lambda = \gamma$ on $[0,T']$ for all large λ. The values $u_\lambda(T',\cdot)$ converge uniformly to $u(T',\cdot)$. The common bounded-coefficient equation and Lemma 3.3, across the positive time distance $T' - T$, give

$$\lim_{\lambda\to\infty} \partial_x^j u_\lambda = \partial_x^j u \quad \text{locally uniformly on } [0,T] \times \mathbb{R}, \qquad j \ge 0. \tag{3.42}$$

We make the last change of coordinates explicit. Fix compact sets $I_t \Subset I$ and $I_B \Subset (-1,1)$. Since $B(t,x) = u_x(t,x)$ is strictly increasing and tends to ± 1 at spatial infinity, one may choose $R < \infty$

so that all inverse points $x(t,B)$ with $(t,B)\in I_t\times I_B$ lie in $[-R,R]$. Uniform convergence of the slopes, followed by strict monotonicity, then gives

$$\lim_{\lambda\to\infty}\lim_{n\to\infty}\sup_{(t,B)\in I_t\times I_B}|x_{\lambda,n}(t,B)-x(t,B)|=0.$$

On a slightly larger compact set of inverse points, strict convexity and compactness give $C\geq c>0$; the same lower bound, with $c/2$, holds for the approximants once the indices are large.

At fixed t,

$$\partial_B=C^{-1}\partial_x,\qquad C_B=\frac{u_{xxx}}{C},\qquad C_{BB}=\frac{u_{xxxx}}{C^2}-\frac{u_{xxx}^2}{C^3},$$

and one more differentiation gives

$$C_{BBB}=\frac{u_{xxxxx}}{C^3}-\frac{4u_{xxx}u_{xxxx}}{C^4}+\frac{3u_{xxx}^3}{C^5}.$$

Thus spatial convergence through order five, supplied by Lemma 3.3, is exactly what is needed for C,C_B,C_{BB},C_{BBB} in slope coordinates. The apparent quotient at $B=0$ causes no loss: C is even in B, $z=-C_B/2$ is odd, and

$$K(B)=\int_0^1 z_B(\theta B)\,\mathrm{d}\theta-m,$$
$$K_B(B)=\int_0^1\theta z_{BB}(\theta B)\,\mathrm{d}\theta,\qquad J(B)=z_B(B)-m,\qquad J_B(B)=z_{BB}(B).$$

The formulas hold also at $B=0$ and show that all five quantities in (3.38) converge uniformly on $I_t\times I_B$. Since the step approximations were chosen with the same active value m on I_t, the inequalities pass first in the limit $n\to\infty$ and then in the limit $\lambda\to\infty$. Finally, I_t and I_B were arbitrary, so the conclusion holds for every $t\in I$ and $0\leq B<1$. □

The following identities will drive the crossing calculation.

Lemma 3.9 (Fixed-slope evolution)**.** *On an interval where $\gamma=m$, derivatives at fixed B satisfy*

$$x_t=-KB,\qquad C_t=C^2J,\qquad z_t=2CzJ-\frac{1}{2}C^2J_B. \tag{3.43}$$

If

$$\Phi=2z^2-mC,\qquad \mathcal{G}=2Cz(3zJ-CJ_B), \tag{3.44}$$

then

$$\Phi_B=2z(2J+3m), \tag{3.45}$$

$$\Phi_t=CJ\Phi+\mathcal{G},\qquad \mathcal{G}\geq 0. \tag{3.46}$$

In particular, if $m>0$, then Φ is strictly increasing on $0<B<1$.

Proof. Differentiating (3.1) in x gives, at fixed x,

$$B_t=-\frac{1}{2}(D+2mBC)=C(z-mB)=CKB.$$

Since $B_x=C$, keeping B fixed gives $x_t=-KB$. A second differentiation of the PDE, followed by the change from fixed x to fixed B, gives $C_t=C^2J$. Since fixed-B derivatives commute and $z=-C_B/2$,

$$z_t=-\frac{1}{2}(C^2J)_B=2CzJ-\frac{1}{2}C^2J_B.$$

Also $z=(m+K)B$ and $z_B=m+J$. Therefore

$$\Phi_B=4zz_B-mC_B=2z(2J+3m),$$

and substitution of (3.43) into $\Phi_t=4zz_t-mC_t$ gives (3.46). Its remainder is nonnegative by (3.38). □

3.4. **Tail estimates on a constant-profile interval.** The crossing proof integrates in the slope coordinate up to $B = 1$. At zero temperature the terminal datum has Gaussian, rather than exponential, curvature tails. We establish here all bounds needed to justify the differentiations and boundary terms.

Lemma 3.10 (Uniform tail estimates)**.** *Let $(a,b) \subset [0,1)$ be an interval on which $\gamma(t) = m$, and let $I \Subset (a,b)$. Uniformly for $t \in I$ as $x \to +\infty$,*

$$\log C(t,x) = -\frac{x^2}{2(1-t)} + O(x + \log x), \tag{3.47}$$

$$z = O(x), \qquad z_x = O(x^2), \qquad z_{xx} = O(x^3),$$
$$0 \le \mathfrak{r}_x \le \frac{1}{t-a}, \qquad \mathfrak{r} = O(x), \qquad H = O(1+x^2). \tag{3.48}$$

There are constants $c_I, C_I > 0$ such that

$$\rho_t(x) \le C_I e^{-c_I x^2 + C_I |x|}, \qquad t \in I,\ x \in \mathbb{R}. \tag{3.49}$$

All weighted integrals and integrations by parts in Subsection 4.1 are consequently absolutely justified.

Proof. Fix $t \in I$ and put

$$T = 1-t, \qquad A = \int_t^1 \gamma(s)\,\mathrm{d}s < \infty.$$

Let $Y_s^{t,x}$ be the optimal diffusion started at x at time t. The drift accumulated between t and 1 has absolute value at most A, because $|u_x| \le 1$. The process $u_x(s, Y_s^{t,x})$ is a martingale with terminal value $\operatorname{sign}(Y_1^{t,x})$, with $\operatorname{sign}(0) = 0$. Thus

$$1 - B(t,x) = 2\mathbb{P}(Y_1^{t,x} < 0) + \mathbb{P}(Y_1^{t,x} = 0).$$

The deterministic drift bound gives the event inclusions

$$\{W_1 - W_t < -x - A\} \subseteq \{Y_1^{t,x} < 0\} \subseteq \{Y_1^{t,x} \le 0\} \subseteq \{W_1 - W_t \le -x + A\}.$$

Thus, writing $\overline{\Phi}_{\mathrm{G}}$ for the standard Gaussian upper tail,

$$2\overline{\Phi}_{\mathrm{G}}\left(\frac{x+A}{\sqrt{T}}\right) \le 1 - B(t,x) \le 2\overline{\Phi}_{\mathrm{G}}\left(\frac{x-A}{\sqrt{T}}\right). \tag{3.50}$$

From (3.38) and $z_B = m + J$,

$$C_x = C C_B = -2Cz, \qquad z_x = C z_B = C(m+J) \ge 0. \tag{3.51}$$

Therefore C is decreasing on $(0,\infty)$ and $(\log C)_{xx} = -2z_x \le 0$. Put

$$I_t(x) = 1 - B(t,x) = \int_x^\infty C(t,y)\,\mathrm{d}y.$$

Choose $h > 2A$. Monotonicity gives

$$\frac{I_t(x) - I_t(x+h)}{h} \le C(t,x) \le \frac{I_t(x-h)}{h}. \tag{3.52}$$

The upper bound in (3.50) at $x+h$ and the lower bound at x show, by the Gaussian Mills estimates, that

$$\frac{I_t(x+h)}{I_t(x)} \le C\,\frac{\overline{\Phi}_{\mathrm{G}}((x+h-A)/\sqrt{T})}{\overline{\Phi}_{\mathrm{G}}((x+A)/\sqrt{T})} \le Ce^{-cx(h-2A)}.$$

Thus $h - 2A > 0$ implies $I_t(x+h) \le I_t(x)/2$ for all large x. Combining this fact with (3.52) and applying Mills' estimates once more proves (3.47). The constants are uniform for $t \in I$ after taking $h > 2\sup_{t\in I} A(t)$.

Let $f = -\log C$. Then f is increasing and convex, $f' = 2z$, and (3.47) gives

$$f(x) = \frac{x^2}{2T} + O(x + \log x).$$

Convexity yields

$$0 \le f'(x) \le \frac{f(2x) - f(x)}{x} = O(x),$$

so $z = O(x)$. With $y = z_x$, differentiation of (3.51) gives

$$y' = z_{xx} = -2zy + C^2 J_B.$$

Since $CJ_B \le 3zJ$ and $CJ \le C(m + J) = y$,

$$-2zy \le y' \le zy. \tag{3.53}$$

For $s \in [x - x^{-1}, x]$, the already proved bound gives $z(s) \le Cx$. If $y(x) = 0$, the desired upper bound is immediate. Otherwise, integrating $y'/y \le z$ backward using (3.53) gives $y(s) \ge c\, y(x)$ with a constant independent of large x. Hence

$$\frac{c}{x} y(x) \le \int_{x-1/x}^{x} y(s)\, \mathrm{d}s = z(x) - z(x - 1/x) \le z(x) \le Cx.$$

Thus $z_x = O(x^2)$, and (3.53) then gives $z_{xx} = O(x^3)$.

Inside the gap,

$$Q(t, \cdot) = P_{t-a} Q(a+, \cdot).$$

The incoming object may be viewed as a finite log-concave measure; if $a = 0$, it is a multiple of δ_0. Set $s = t - a$ and define the posterior probability measure

$$\mu_x(\mathrm{d}y) = \frac{p_s(x - y) Q(a+, \mathrm{d}y)}{Q(t, x)}.$$

Differentiating the Gaussian convolution gives the heat-score identities

$$\mathfrak{r}(t, x) = \frac{x - \mathbb{E}_{\mu_x} Y}{s}, \qquad \mathfrak{r}_x(t, x) = \frac{1}{s} - \frac{\mathrm{Var}_{\mu_x}(Y)}{s^2}.$$

The variance term gives $\mathfrak{r}_x \le s^{-1}$, and log-concavity gives $\mathfrak{r}_x = -(\log Q)_{xx} \ge 0$. Evenness gives $\mathfrak{r}(t, 0) = 0$, so $0 \le \mathfrak{r}(t, x) \le x/(t - a)$. Since $H = \mathfrak{r}^2 - \mathfrak{r}_x$, this proves (3.48).

The control representation gives $u \ge 0$, and one-Lipschitzness gives $u(t, x) \le u(t, 0) + |x|$. The exponential in (3.19) is at most one, so $Q(t, x) \le c_0 p_t(x)$. Because $\rho_t = e^{mu(t,\cdot)} Q(t, \cdot)$ on the gap,

$$\rho_t(x) \le C_I \exp\left\{ -\frac{x^2}{2t} + m|x| \right\},$$

which proves (3.49) uniformly on I.

For clarity, we finish by listing the composite bounds used in the next section. Define

$$\kappa = KB = z - mB, \qquad N = \mathfrak{r} + \kappa, \qquad w = 2\rho C.$$

Then

$$\begin{aligned} &\kappa, N = O(x), && \Phi = O(x^2), \\ &CJ = z_x - mC = O(x^2), && C^2 J_B = z_{xx} + 2z z_x = O(x^3), \\ &\mathcal{G} = 6z^2(CJ) - 2z(C^2 J_B) = O(x^4). \end{aligned}$$

In particular, the quantities

$$R_1 = \kappa \mathfrak{r} + \frac{1}{2}\kappa^2, \qquad R_0 = 2CJ - \frac{m}{2} C - \frac{1}{2} z^2, \qquad L = CJ - z(N + z)$$

are $O(x^2)$. The individual J and J_B need not have polynomial growth; only the displayed composites are asserted to do so. Since $dB = C\,\mathrm{d}x$, the Gaussian estimates for both ρ and C give, for every $p \ge 0$,

$$\sup_{t \in I} \int_0^1 w(t, B)\big(1 + |x(t, B)|^p\big)\, \mathrm{d}B = \sup_{t \in I} 2 \int_0^\infty \rho_t(x) C(t, x)^2 (1 + x^p)\, \mathrm{d}x < \infty. \tag{3.54}$$

The same Gaussian bounds, together with (3.50), imply the uniform tail form

$$\lim_{B_0 \uparrow 1} \sup_{t \in I} \int_{B_0}^1 w(t, B)\big(1 + |x(t, B)|^p\big)\, \mathrm{d}B = 0. \tag{3.55}$$

Indeed, the inverse points $x(t, B_0)$ tend to $+\infty$ uniformly for $t \in I$ as $B_0 \uparrow 1$, and the right side of (3.54) then has a uniform Gaussian tail. Moreover, with all quantities in the next display evaluated at $x = x(t, B)$,

$$\sup_{t\in I} |wCz\, P(x)| = \sup_{t\in I} |2\rho C^2 z\, P(x)| \longrightarrow 0 \qquad (B \uparrow 1) \tag{3.56}$$

for every fixed polynomial P. Equations (3.54)–(3.56) are the precise uniform-integrability and boundary statements used in Subsection 4.1; they prove the final assertion of the lemma. ☐

4. Zero-temperature gap exclusion

4.1. **The arbitrary-gap crossing theorem.** We now prove the central statement. Throughout this subsection, $\gamma(t) = m > 0$ on (a, b), and every time belongs to a compact subinterval of that gap. Retain the slope variables and set

$$\kappa = KB = z - mB, \qquad N = \mathfrak{r} + \kappa, \qquad w(t, B) = 2\rho_t(x(t, B))C(t, B).$$

Proposition 4.1 (Arbitrary-gap crossing)**.** *On a constant-profile interval with $m > 0$, $\Gamma \in C^3(a, b)$ and*

$$\Gamma''(t) = 0 \quad \Longrightarrow \quad \Gamma'''(t) > 0, \qquad a < t < b. \tag{4.1}$$

Proof. We first express Γ'' in the slope coordinate. Since $D = C_x = CC_B = -2Cz$, (3.35) becomes

$$\Gamma''(t) = 2\mathbb{E}\big[C(t, X_t)^2\Phi(t, X_t)\big]\,.$$

Using evenness and $dB = C\,\mathrm{d}x$ gives

$$\frac{1}{2}\Gamma''(t) = \int_0^1 w(t, B)\Phi(t, B)\,\mathrm{d}B =: \mathcal{I}(t). \tag{4.2}$$

We next compute the derivatives of the weight. From $\rho = e^{mu}Q$,

$$(\log \rho)_x = mB - \mathfrak{r}.$$

Since $x_B = C^{-1}$ and $C_B = -2z$,

$$\frac{w_B}{w} = \frac{mB - \mathfrak{r} - 2z}{C} = -\frac{N + z}{C}, \qquad N_B = \frac{\mathfrak{r}_x}{C} + J. \tag{4.3}$$

Inside the gap, $Q_t = Q_{xx}/2$, so $Q_t/Q = H/2$. At fixed B, Lemma 3.9 gives $x_t = -\kappa$, $C_t = C^2 J$, and

$$u_t + u_x\, x_t = -\frac{1}{2}(C + mB^2) - B\kappa.$$

Therefore

$$\begin{aligned}(\log w)_t &= m(u_t + Bx_t) + \frac{Q_t}{Q} + x_t(\log Q)_x + \frac{C_t}{C}\\ &= -\frac{m}{2}(C + mB^2) - mB\kappa + \frac{1}{2}H + \kappa\mathfrak{r} + CJ.\end{aligned} \tag{4.4}$$

Because

$$-\frac{m^2B^2}{2} - mB\kappa = \frac{\kappa^2 - z^2}{2},$$

we may rewrite (4.4) as

$$(\log w)_t = \frac{1}{2}H + R_1 + R_0 - CJ, \tag{4.5}$$

where

$$R_1 = \kappa\mathfrak{r} + \frac{1}{2}\kappa^2, \qquad R_0 = 2CJ - \frac{m}{2}C - \frac{1}{2}z^2.$$

Formally combining (3.46) and (4.5) gives

$$\mathcal{I}'(t) = \frac{1}{2}\int_0^1 w\Phi H\,\mathrm{d}B + \int_0^1 wR_1\Phi\,\mathrm{d}B + \int_0^1 w(R_0\Phi + \mathcal{G})\,\mathrm{d}B. \tag{4.6}$$

We justify this differentiation by truncation. For $0 < B_0 < 1$, every quantity is smooth on a compact set $I_t \times [0, B_0]$, so the differentiated identity holds with upper limit B_0. The composite estimates in Lemma 3.10, together with (4.5) and (3.46), give uniformly for $t \in I_t \Subset (a, b)$

$$(\log w)_t = O(1 + x^2), \qquad \Phi_t = O(1 + x^4),$$
$$|w\Phi| + |\partial_t(w\Phi)| + w|\Phi H| + w|R_1\Phi| + w|R_0\Phi + \mathcal{G}| \le C_{I_t} w(1 + x^4).$$

For example, $\partial_t(w\Phi) = w\{(\log w)_t\Phi + \Phi_t\}$, so the first line implies its bound in the second. The uniform tail estimate (3.55) makes the contribution of $B \in (B_0, 1)$ tend to zero uniformly in t, both before and after differentiation. Letting $B_0 \uparrow 1$ therefore proves (4.6) and shows that $\mathcal{I} \in C^1(a, b)$. In view of (4.2), this also proves the asserted regularity $\Gamma \in C^3(a, b)$.

Assume from now on that $\Gamma''(t) = 0$. By (4.2),

$$\int_0^1 w\Phi \, \mathrm{d}B = 0. \tag{4.7}$$

Let $W_0 = \int_0^1 w \, \mathrm{d}B$. For any f such that $wf, w\Phi f \in L^1(0, 1)$, the centered identity gives

$$\begin{aligned}\int_0^1 w\Phi f \, \mathrm{d}B = \frac{1}{2W_0} \int_0^1 \int_0^1 & w(B)w(D)[\Phi(B) - \Phi(D)] \\ & \times [f(B) - f(D)] \, \mathrm{d}B \, \mathrm{d}D.\end{aligned} \tag{4.8}$$

By (3.45), Φ is strictly increasing. By Proposition 3.6, H is strictly increasing on the gap. Since $w > 0$,

$$\int_0^1 w\Phi H \, \mathrm{d}B > 0. \tag{4.9}$$

The second term in (4.6) is nonnegative. Indeed,

$$(R_1)_B = J\mathfrak{r} + \kappa\frac{\mathfrak{r}_x}{C} + \kappa J = JN + \kappa\frac{\mathfrak{r}_x}{C} \ge 0,$$

because $J, \kappa, N, \mathfrak{r}_x \ge 0$. Applying (4.8) with $f = R_1$ gives

$$\int_0^1 wR_1\Phi \, \mathrm{d}B \ge 0. \tag{4.10}$$

It remains to control the residual term. Define

$$F(B) = \int_B^1 w(\theta)\Phi(\theta) \, \mathrm{d}\theta, \qquad \tau(B) = \frac{F(B)}{w(B)}.$$

At $B = 0$, evenness gives $z(0) = 0$, so $\Phi(0) = -mC(0) < 0$. On the other hand, $z \ge mB$ and $C \to 0$ as $B \uparrow 1$, so $\Phi > 0$ near one. By strict monotonicity, Φ has a unique zero B_Φ. If $B \le B_\Phi$, then (4.7) gives

$$F(B) = -\int_0^B w\Phi \, \mathrm{d}\theta \ge 0;$$

if $B \ge B_\Phi$, nonnegativity follows directly from the definition. Therefore

$$\tau \ge 0. \tag{4.11}$$

We prove the complementary bound

$$\tau \le Cz. \tag{4.12}$$

Set $\mathcal{H}_0(B) = w(B)C(B)z(B) - F(B)$. From (4.3), $F' = -w\Phi$, and

$$(Cz)_B = -2z^2 + C(m + J) = CJ - \Phi,$$

we obtain

$$(\mathcal{H}_0)_B = wL, \qquad L = CJ - z(N + z). \tag{4.13}$$

The slope inequalities imply

$$(CJ)_B = -2zJ + CJ_B \le zJ.$$

On the other hand, (4.3) and $z_B = m + J$ give

$$[z(N+z)]_B = (m+J)(N+z) + z\left(\frac{\mathfrak{r}_x}{C} + J + m + J\right) > zJ.$$

Thus $L_B < 0$ on $(0,1)$.

At $B = 0$, (4.7) and $z(0) = 0$ give $\mathcal{H}_0(0) = 0$. As $B \uparrow 1$, absolute integrability gives $F(B) \to 0$, while (3.56) gives $\mathcal{H}_0(B) \to 0$. Furthermore, Lemma 3.10 gives $wL \in L^1(0,1)$. The local derivative identity (4.13) therefore extends $\mathcal{H}_0$ to an absolutely continuous function on $[0,1]$. Consequently

$$\int_0^1 wL \,\mathrm{d}B = \mathcal{H}_0(1) - \mathcal{H}_0(0) = 0.$$

Since L is strictly decreasing and $w > 0$, L crosses zero exactly once. Hence $\mathcal{H}_0$ first increases and then decreases, with zero values at both endpoints. Thus $\mathcal{H}_0 \geq 0$, which is exactly (4.12).

We may now integrate by parts. Since $F' = -w\Phi$,

$$\int_0^1 w\Phi R_0 \,\mathrm{d}B = -\int_0^1 F' R_0 \,\mathrm{d}B = \int_0^1 F (R_0)_B \,\mathrm{d}B. \tag{4.14}$$

There is no boundary contribution at zero because $F(0) = 0$. At one, (4.12), (3.56), and $R_0 = O(x^2)$ give $F(B)R_0(B) \to 0$. Also

$$(R_0)_B = 2CJ_B - 5zJ.$$

The integral in (4.14) is absolutely convergent: by (4.11)–(4.12),

$$\begin{aligned} \tau|(R_0)_B| &\leq Cz(2CJ_B + 5zJ) \\ &= 2z(C^2 J_B) + 5z^2(CJ) = O(x^4), \end{aligned}$$

and (3.54) applies. Therefore

$$\int_0^1 w\Phi R_0 \,\mathrm{d}B = \int_0^1 w\tau(2CJ_B - 5zJ) \,\mathrm{d}B.$$

Using the definition of $\mathcal{G}$ in (3.44), we obtain the exact residual identity

$$\begin{aligned} \int_0^1 w(R_0\Phi + \mathcal{G}) \,\mathrm{d}B = \int_0^1 w\{&2CJ_B(\tau - Cz) \\ &+ zJ(6Cz - 5\tau)\} \,\mathrm{d}B. \end{aligned}$$

Now $\tau - Cz \leq 0$ and $CJ_B \leq 3zJ$. Multiplication by the nonpositive quantity $2(\tau - Cz)$ reverses the latter inequality, so

$$\begin{aligned} \int_0^1 w(R_0\Phi + \mathcal{G}) \,\mathrm{d}B &\geq \int_0^1 wzJ\{6(\tau - Cz) + 6Cz - 5\tau\} \,\mathrm{d}B \\ &= \int_0^1 wzJ\tau \,\mathrm{d}B \geq 0. \end{aligned} \tag{4.15}$$

Finally, (4.6), (4.9), (4.10), and (4.15) imply

$$\mathcal{I}(t) = 0 \quad \Longrightarrow \quad \mathcal{I}'(t) > 0.$$

Since $\Gamma'' = 2\mathcal{I}$ by (4.2), this is precisely (4.1). □

4.2. **Exclusion of internal support gaps.**

Proposition 4.2 (No internal support gap)**.** *There do not exist* $0 \leq a < b < 1$ *such that*

$$a, b \in S, \qquad \nu((a,b)) = 0.$$

Proof. On (a,b), the profile is constant. Its value m is strictly positive. If $a > 0$, then $0 \in S$ implies $\nu([0,a]) > 0$. If $a = 0$, the absence of mass in $(0,b)$ and the support condition $0 \in S$ imply

$$\nu(\{0\}) = \nu([0,\varepsilon)) > 0$$

for every sufficiently small $\varepsilon < b$. Thus $\gamma(t) = m > 0$ throughout the gap, and Proposition 4.1 applies.

Put

$$L = b - a, \qquad q(s) = \Gamma(a+s) - a, \qquad 0 \le s \le L.$$

The endpoint consistency and stability conditions in Proposition 3.2 give

$$q(0) = 0, \qquad q(L) = L, \qquad q'(0+) \le 1. \tag{4.16}$$

Moreover, $G(a) = G(b) = 0$, and hence

$$\begin{aligned} 0 = G(a) - G(b) &= \int_a^b (\Gamma(t) - t)\,\mathrm{d}t \\ &= \int_0^L (q(s) - s)\,\mathrm{d}s. \end{aligned}$$

Equivalently,

$$2\int_0^L q(s)\,\mathrm{d}s = Lq(L).$$

Let $f = q''$ on $(0, L)$. Because $b < 1$, Lemma 3.3 gives a uniform bound on u_{xx} up to both one-sided gap endpoints. Formula (3.34), weak continuity of the diffusion law, and local continuity of u_{xx} therefore show that q' has finite one-sided limits and is bounded on $[0, L]$.

By Proposition 4.1, every zero of f is simple and is crossed from negative to positive. Consequently, f has at most one zero. Indeed, after one upward crossing, a second upward crossing would require an intervening downward zero. Thus f has a constant sign on at most two subintervals, so q' is monotone on each of them. The bounded endpoint limits of q' imply $f \in L^1(0, L)$.

We may now integrate twice without an endpoint regularity assumption. Using $q(0) = 0$ and $q(L) = L$ gives

$$\int_0^L s(L-s)f(s)\,\mathrm{d}s = Lq(L) - 2\int_0^L q(s)\,\mathrm{d}s = 0. \tag{4.17}$$

Because the weight $s(L - s)$ is strictly positive on $(0, L)$, identity (4.17) says that either f changes sign or $f \equiv 0$. The latter alternative is impossible by Proposition 4.1: if $f \equiv 0$, then at every interior point one would have $f = 0$ but $f' = q^{(3)} > 0$. Hence f changes sign. Since it has at most one zero and every zero is an upward crossing, there is a unique $c \in (0, L)$ such that

$$f < 0 \text{ on } (0, c), \qquad f > 0 \text{ on } (c, L).$$

A second integration gives

$$\begin{aligned} q(L) - Lq'(0+) &= \int_0^L (L-s)f(s)\,\mathrm{d}s \\ &= \int_0^L \frac{s(L-s)f(s)}{s}\,\mathrm{d}s. \end{aligned}$$

On $(0, c)$ the numerator is negative and $s^{-1} > c^{-1}$; on (c, L) it is positive and $s^{-1} < c^{-1}$. The two inequalities are strict on sets of positive measure. Therefore

$$q(L) - Lq'(0+) < \frac{1}{c}\int_0^L s(L-s)f(s)\,\mathrm{d}s = 0. \tag{4.18}$$

Combining (4.16) and (4.18) yields

$$q(L) < Lq'(0+) \le L,$$

contradicting $q(L) = L$. □

Remark 4.3 (A gap issuing from zero). The preceding argument also covers $a = 0$. Evenness gives $\Gamma(0) = u_x(0, 0)^2 = 0$, Proposition 3.2 gives $G(0) = 0$ and $\Gamma'(0+) = u_{xx}(0, 0)^2 \le 1$, and the atom at zero makes the constant gap value m positive. These are precisely the conditions used in (4.16). Hence the proof excludes a component $(0, b)$ as well as every component (a, b) with $a > 0$.

4.3. **Exclusion of a terminal gap.**

Proposition 4.4 (No gap ending at one)**.** *There is no $a < 1$ such that $(a,1) \cap S = \varnothing$ and $a \in S$.*

Proof. Suppose otherwise. Then

$$\gamma(t) = m < \infty, \qquad a < t < 1. \tag{4.19}$$

Write $\varepsilon = 1 - t$. For $m > 0$, the Cole–Hopf formula on the terminal constant-profile interval is

$$e^{mu(1-\varepsilon,x)} = P_\varepsilon(e^{m|\cdot|})(x) = \mathbb{E}e^{m|x+\sqrt{\varepsilon}Z|}, \qquad Z \sim N(0,1). \tag{4.20}$$

When $m = 0$, the corresponding formula is $u(1-\varepsilon, x) = P_\varepsilon|\cdot|(x)$.

Set

$$B_\varepsilon(y) = u_x(1-\varepsilon, \sqrt{\varepsilon}\, y).$$

Differentiating (4.20), with the unweighted interpretation when $m = 0$, gives

$$B_\varepsilon(y) = \frac{\mathbb{E}\Big[\operatorname{sign}(y+Z)e^{m\sqrt{\varepsilon}|y+Z|}\Big]}{\mathbb{E}e^{m\sqrt{\varepsilon}|y+Z|}}. \tag{4.21}$$

For every fixed y,

$$\lim_{\varepsilon \downarrow 0} B_\varepsilon(y) = \mathbb{E}\operatorname{sign}(y+Z) = 2\Phi_{\mathrm{G}}(y) - 1, \tag{4.22}$$

where Φ_{G} is the standard Gaussian distribution function.

On the terminal gap, $Q = \rho e^{-mu}$ solves the heat equation. Fix $t_0 \in (a,1)$. Then

$$Q(t,\cdot) = P_{t-t_0}Q(t_0,\cdot), \qquad t_0 < t \le 1.$$

Thus Q has a smooth, strictly positive limit $Q(1,\cdot)$, and

$$\rho_1(x) := e^{m|x|}Q(1,x) \quad \text{is continuous with} \quad \rho_1(0) > 0. \tag{4.23}$$

In particular, for every fixed y,

$$\lim_{\varepsilon \downarrow 0} \rho_{1-\varepsilon}(\sqrt{\varepsilon}\, y) = \rho_1(0).$$

We also need a uniform bound. From (3.19), $Q(t,x) \le c_0 p_t(x)$; from $u(t,x) \le u(t,0) + |x|$ and (4.19),

$$\rho_t(x) \le C \exp\left\{-\frac{x^2}{2t} + m|x|\right\}.$$

Hence

$$\sup_{\substack{t_1 \le t \le 1 \\ x \in \mathbb{R}}} \rho_t(x) < \infty \qquad \text{for every } t_1 \in (a,1).$$

It remains to justify passage to the limit in the boundary-layer integral. For $y \ge 0$, put $\delta = m\sqrt{\varepsilon}$ and let $p_\varepsilon(y)$ be the probability of $y + Z < 0$ under the exponential tilt in (4.21). Splitting the denominator over the two half-lines gives

$$\begin{aligned} A_-(y) &= e^{-\delta y + \delta^2/2}\overline{\Phi}_{\mathrm{G}}(y-\delta), \\ A_+(y) &= e^{\delta y + \delta^2/2}\Phi_{\mathrm{G}}(y+\delta), \\ p_\varepsilon(y) &= \frac{A_-(y)}{A_-(y) + A_+(y)}. \end{aligned}$$

Since $\Phi_{\mathrm{G}}(y+\delta) \ge 1/2$,

$$1 - B_\varepsilon(y)^2 = 4p_\varepsilon(y)(1-p_\varepsilon(y)) \le 8\overline{\Phi}_{\mathrm{G}}(y-\delta). \tag{4.24}$$

By symmetry, the same estimate controls the negative half-line. For small ε, the right side of (4.24), together with the trivial bound by one on a fixed compact interval, has an integrable majorant independent of ε.

Changing variables $x = \sqrt{\varepsilon}\, y$ gives

$$1 - \Gamma(1-\varepsilon) = \int_{\mathbb{R}} \rho_{1-\varepsilon}(x)[1 - u_x(1-\varepsilon, x)^2]\, \mathrm{d}x$$

$$= \sqrt{\varepsilon} \int_{\mathbb{R}} \rho_{1-\varepsilon}(\sqrt{\varepsilon}\, y)[1 - B_\varepsilon(y)^2]\,\mathrm{d}y.$$

Equations (4.22), (4.23)–(4.24) and dominated convergence imply

$$1 - \Gamma(1-\varepsilon) \sim \rho_1(0)\sqrt{\varepsilon} \int_{\mathbb{R}} [1 - (2\Phi_{\mathrm{G}}(y) - 1)^2]\,\mathrm{d}y.$$

To evaluate the integral, let Z_1, Z_2 be independent standard Gaussians. Fubini's theorem gives

$$\begin{aligned}\int_{\mathbb{R}} \Phi_{\mathrm{G}}(y)\overline{\Phi}_{\mathrm{G}}(y)\,\mathrm{d}y &= \int_{\mathbb{R}} \mathbb{P}(Z_1 \le y < Z_2)\,\mathrm{d}y \\ &= \mathbb{E}(Z_2 - Z_1)_+ = \frac{1}{\sqrt{\pi}}.\end{aligned}$$

Since $1 - (2\Phi_{\mathrm{G}} - 1)^2 = 4\Phi_{\mathrm{G}}\overline{\Phi}_{\mathrm{G}}$, we conclude that

$$1 - \Gamma(t) \sim \frac{4}{\sqrt{\pi}}\rho_1(0)\sqrt{1-t}, \qquad t \uparrow 1. \tag{4.25}$$

Set $c = 4\rho_1(0)/\sqrt{\pi} > 0$. With $\varepsilon = 1 - t$, (4.25) gives

$$\Gamma(t) - t = \varepsilon - c\sqrt{\varepsilon} + o(\sqrt{\varepsilon}) < 0$$

for all t sufficiently close to one. Therefore, for $q < 1$ sufficiently close to one,

$$G(q) = \int_q^1 (\Gamma(t) - t)\,\mathrm{d}t < 0,$$

contradicting the universal KKT inequality (3.6). □

4.4. **Completion of the zero-temperature proof.**

Proof of Theorem 1.2. Suppose that $\overline{S}^{[0,1]} \ne [0,1]$. Because $0 \in S$, a nonempty connected component of the relatively open set $[0,1] \setminus \overline{S}^{[0,1]}$ either lies between two points of S in $[0,1)$ or reaches the endpoint one. In the first case, γ is constant on an internal gap, contradicting Proposition 4.2. In the second case, it is constant on a terminal gap, contradicting Proposition 4.4. Therefore

$$\overline{S}^{[0,1]} = [0,1],$$

which is (1.3). □

References

[1] A. Auffinger and W.-K. Chen, On properties of Parisi measures, *Probability Theory and Related Fields* **161** (2015), 817–850.

[2] A. Auffinger and W.-K. Chen, The Parisi formula has a unique minimizer, *Communications in Mathematical Physics* **335** (2015), 1429–1444.

[3] A. Auffinger and W.-K. Chen, Parisi formula for the ground state energy in the mixed p-spin model, *Annals of Probability* **45** (2017), no. 6B, 4617–4631.

[4] A. Auffinger, W.-K. Chen, and Q. Zeng, The SK model is infinite step replica symmetry breaking at zero temperature, *Communications on Pure and Applied Mathematics* **73** (2020), no. 5, 921–943.

[5] W.-K. Chen, M. Handschy, and G. Lerman, On the energy landscape of the mixed even p-spin model, *Probability Theory and Related Fields* **171** (2018), no. 1–2, 53–95.

[6] F. Guerra, Broken replica symmetry bounds in the mean field spin glass model, *Communications in Mathematical Physics* **233** (2003), 1–12.

[7] A. Jagannath and I. Tobasco, Some properties of the phase diagram for mixed p-spin glasses, *Probability Theory and Related Fields* **167** (2017), 615–672.

[8] P. Lopatto, Full replica symmetry breaking in the Sherrington–Kirkpatrick model, arXiv:2607.11756v2, 2026, https://arxiv.org/abs/2607.11756.

[9] D. Panchenko, The Parisi formula for mixed p-spin models, *Annals of Probability* **42** (2014), no. 3, 946–958.

[10] G. Parisi, Infinite number of order parameters for spin-glasses, *Physical Review Letters* **43** (1979), 1754–1756.

[11] D. Sherrington and S. Kirkpatrick, Solvable model of a spin-glass, *Physical Review Letters* **35** (1975), 1792–1796.

[12] M. Talagrand, The Parisi formula, *Annals of Mathematics* **163** (2006), no. 1, 221–263.

[13] Y. Zhou, Existence of full replica symmetry breaking for the Sherrington–Kirkpatrick model at low temperature, arXiv:2504.00269v2, 2025, https://arxiv.org/abs/2504.00269.