Radial Accommodation in the Ehrenfest Problem

A Conditional Layered-Ring Model
with an Effective Covariant Phase-Stiffness Realization

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Prologue: The Bakery Queue of Spacetime

It is 1909. Paul Ehrenfest bursts into Einstein's office in Bern, holding a chestnut.

> "Albert! If you spin your rigid disk, the rim contracts but the radius does not. The skin will crack, just like this chestnut on a hot stove!"

Einstein chokes on his tea. "But Paul, the flesh under the skin spins too!" Ehrenfest grins. "Exactly. Now explain why only the skin cracks."

Einstein vanishes from public view for ten years.

It is 2026. An independent researcher sits in a dim room, late at night. For years, the Ehrenfest paradox has been gnawing at him. Finally, he leans toward a large language model and says, almost shouting:

"Brother, listen! When a fidget spinner turns slowly, centrifugal force stretches it a little, and it does not explode. Relativistic contraction can do the same — let the radius shrink so the circumference fits! Why is this not obvious?"

The AI replies, cold and flat:

"To analyze this, one must write the Lagrangian with a phase-capacity constraint, effective stiffness κ_{eff}, and strictly corotating boundary conditions..."

The researcher cuts in:

"I don't follow. Did poor Paul need all that?"

The AI, unmoved:

"Paul Ehrenfest's intuition arrived earlier. You are late. Please respect the queue of the history of science."

A long silence. The researcher stares at the screen.

"Since when is physics a bakery? First come, first served? Just because I was born later, I get stale bread?"

The AI says nothing. Perhaps it has entered power-saving mode.

That night, in a dream, the researcher finds himself standing in an endless queue. A rusty sign reads:

Bakery of Theoretical Physics — Bern Branch
Fresh bread only for those who discovered before 1910.

Behind him, an old man with white hair holds an apple: Newton. He grumbles, "I dropped the apple, and someone said gravity wasn't mine! Now I'm queueing for a dry loaf."

Further ahead, a man with wild hair and a rumpled coat mutters: Einstein. "Fifty years I worked for this bakery, baked General Relativity, and then Paul jumped the queue with a chestnut!"

From the back of the queue, a familiar voice shouts: Ehrenfest, sweating, clutching a cracked chestnut. "Hey! This bread is stale! I only said the disk would crack, not that my bread should go moldy!"

The researcher, fed up, yells:

"Sir baker! I have a new idea! Centrifugal force and contraction can balance! The radius must shrink, not..."

The baker, a massive man in a flour-dusted apron with Riemann tensors embroidered on it, leans over the counter and roars:

"Kid, go home! This bread is baked only with pre-1910 signatures! You're a late customer! Here, take this lump of Lagrangian dough — maybe it'll be ready in fifty years!"

He hurls a heavy, symbol-filled dough at the researcher.

At that moment, a small elderly woman with thick glasses rises from a corner. It is Emmy Noether. She places a hand on the researcher's shoulder and says softly:

"Don't worry. Symmetry speaks first. If the conservation law holds, then sooner or later, the bread will rise."

Morning comes. The researcher is back at his desk. The laptop is still on. The same AI waits, ready.

He stares at the screen for a moment. Then a crooked, bittersweet smile — the kind only a physicist in love with a problem can understand — spreads across his face.

He makes a cup of tea, picks up a pen, and writes:

"Revisiting the Ehrenfest Paradox: A Layered-Ring Model with Proper Circumference Preservation..."

Not for the bakery. Not for the gatekeeping AI. But for himself, for Paul, and for the fidget spinner that never cracked.

One month later.

The final LaTeX compilation finishes. He opens the PDF. The title page reads:

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He runs a finger over the name of his one-man institute: IRCBHC. A real laugh escapes this time.

Outside the window, a fidget spinner on the sill turns in the morning breeze. And this time, he knows that its radius has changed — exactly as his equations say.

The End
(Or perhaps: the beginning of a paper.)

Abstract

The Ehrenfest problem is traditionally formulated by considering an ideally Born-rigid disk accelerated from rest into uniform rotation. Tangential material elements are associated with local Lorentz contraction in an inertial laboratory frame, while radial elements are instantaneously perpendicular to the tangential motion and therefore do not undergo direct longitudinal Lorentz contraction. The further conclusion that the material radius must remain unchanged is, however, an additional mechanical assumption rather than a direct consequence of the Lorentz transformation.

A conditional layered-ring construction is presented. Each concentric material ring is allowed to change its laboratory radius while preserving the sum of its local comoving circumferential lengths. Under this closure condition, the radial map is

$$r(\rho) = \frac{\rho}{\sqrt{1+\omega^2\rho^2/c^2}},$$

where ρ is the initial material radius and ω is the final angular velocity. The construction preserves

$$C_{\text{lab}} = 2\pi r$$

on laboratory simultaneity slices while satisfying

$$C_{\text{proper}} = \gamma C_{\text{lab}} = 2\pi\rho.$$

The kinematic construction is supplemented by an effective radial-support model. Reducing a covariant spatial phase-gradient sector on a uniformly wound thin ring produces the proper circumference energy

$$U_{\text{wind}}(C) = \frac{\overline{K}_s}{2}\, C\left[\left(\frac{2\pi N}{C}\right)^2 - q_0^2\right]^2.$$

Its minimum fixes the summed local comoving circumference to

$$C_0 = \frac{2\pi|N|}{q_0}.$$

Near that minimum, the reduced theory yields the effective stiffness

$$\kappa_{\text{eff}} = \frac{4\overline{K}_s q_0^4}{C_0}.$$

In the strong-locking limit, the stationary thin ring satisfies $C_{\text{proper}} = C_0$, and the radial map follows exactly. A phenomenological fixed-ω radial Lagrangian gives a positive finite-stiffness radius correction and a local radial stability condition within that reduced model.

A separate covariant strong-locking embedding retains an independent temporal phase variable and imposes the zero-temporal-frequency constraint $u^\mu\nabla_\mu\phi = 0$. It reproduces the same radial map and gives the stationary constrained angular momentum

$$J(\omega) = \frac{M_0\rho^2\omega}{(1+\omega^2\rho^2/c^2)^{3/2}} + Np_\chi,$$

where p_χ is the conserved phase charge. The previously used one-term expression is recovered in the sector $p_\chi = 0$. The finite-stiffness Hamiltonian completion and passage through the point $\mathrm{d}J/\mathrm{d}\omega = 0$ remain open.

The construction therefore supplies an explicit fixed-ω radial support model and a constrained strong-locking time-dependent embedding for a thin ring. It does not establish that all physical matter possesses the proposed phase sector, nor does it yet derive the complete dynamics of a continuous interacting disk. Finally, the geometric second moment of the mapped rest-mass distribution is derived; it must not be identified with the full relativistic dynamical moment of inertia.

Contents

1 Introduction

In 1909, Paul Ehrenfest examined the compatibility of rotational motion with Max Born's relativistic definition of rigidity. The problem showed that an extended body cannot, in general, be accelerated from rest into rotation while preserving all local proper distances throughout the spin-up process.

The classical argument is commonly summarized as follows:

1. The circumference of a rotating disk is tangential to the motion.
2. Tangential material elements are locally Lorentz-contracted when measured at one laboratory time.
3. The radius is instantaneously perpendicular to the tangential velocity.
4. It is then assumed that the material radius remains unchanged.
5. A contracted circumference and an unchanged radius appear to conflict with

$$C = 2\pi R.$$

The central observation of this paper is that the third statement does not logically imply the fourth.

From

no direct longitudinal Lorentz contraction along the radius,

one may conclude only that the standard longitudinal contraction formula does not act directly on a radial line element.

One may not automatically conclude that

the material radius cannot change dynamically.

A real radial displacement caused by forces, changing equilibrium separations, redistribution of matter, or a specified acceleration protocol is conceptually distinct from direct radial Lorentz contraction.

2 What the Ehrenfest Argument Establishes

Born rigidity requires the orthogonal proper distance between neighboring worldlines of a material congruence to remain constant.

A disk initially at rest cannot be accelerated into rotation while preserving this condition everywhere. Different material radii acquire different tangential speeds,

$$v(r) = \omega r,$$

and consequently different acceleration histories and Lorentz factors.

The robust conclusion is therefore

$$\boxed{\text{A disk cannot be spun up from rest while remaining globally Born-rigid.}}$$

This does not imply

$$\boxed{\text{Every physical rotating disk must retain its original radius.}}$$

The first statement is a kinematic restriction. The second would be a dynamical prediction requiring a material law, forces, stresses, and boundary conditions.

3 Why Perfect Rigidity Is Not a Physical Property

No perfectly rigid macroscopic body is known to exist.

For a body to be perfectly rigid in the strongest classical sense, a displacement at one point would have to be communicated instantaneously to all other points:

$$v_{\text{mechanical}} \to \infty.$$

This is incompatible with relativistic causality.

Real disturbances propagate as elastic waves, stress waves, lattice excitations, or other collective modes with finite signal speed:

$$v_{\text{signal}} \leq c.$$

The argument does not require fundamental particles themselves to be extended, soft, or deformable. The constituents may be treated as pointlike. Macroscopic rigidity is a property of the relations among those constituents.

A perfectly rigid body would require

$$d_{ij} = \text{constant} \qquad \text{for every material pair } i, j.$$

Real interactions do not enforce this condition absolutely. Relative separations are maintained dynamically and may change under acceleration.

Near a stable equilibrium, a local interaction may be represented schematically as

$$F \simeq -k\,\Delta x.$$

The relevant physical statement is therefore:

> The constituents of an extended body are not geometrically fused into an immutable configuration. Their relative separations are maintained by finite-strength, finite-speed interactions and may change under stress or acceleration.

4 The Layered-Ring Representation

Consider an initially stationary disk of relaxed radius R_0.

Decompose the disk into a continuous family of concentric material rings. Let

$$0 \leq \rho \leq R_0$$

label the initial relaxed radius of a ring.

Its initial circumference is

$$C_0(\rho) = 2\pi\rho.$$

After uniform rotation is established, let the same material ring occupy laboratory radius

$$r = r(\rho).$$

Its tangential speed is

$$v(\rho) = \omega r(\rho),$$

and its Lorentz factor is

$$\gamma(\rho) = \frac{1}{\sqrt{1 - \omega^2 r(\rho)^2/c^2}}.$$

Every ring except the central point participates in relativistic motion. The disk therefore has a radial gradient of tangential speed and Lorentz factor.

5 Three Distinct Circumferential Quantities

A careful treatment must distinguish three quantities.

5.1 Initial relaxed circumference

$$C_0(\rho) = 2\pi\rho.$$

5.2 Laboratory circumference

At one laboratory time,

$$C_{\text{lab}}(\rho) = 2\pi r(\rho).$$

This relation refers to the Euclidean spatial geometry of an inertial laboratory simultaneity slice.

5.3 Sum of local comoving lengths

Each infinitesimal tangential element possesses an instantaneous local comoving inertial frame. Define

$$C_{\text{proper}}(\rho)$$

as the sum of those local proper lengths around the ring.

This is not the length of the full ring measured in one global inertial rest frame, because no such frame is comoving with the entire rotating circumference.

Locally,

$$\mathrm{d}\ell_{\text{lab}} = \frac{\mathrm{d}\ell_{\text{proper}}}{\gamma(\rho)}.$$

Since the tangential speed is constant around a ring of fixed radius, summing the local relation gives

$$C_{\text{proper}}(\rho) = \gamma(\rho) C_{\text{lab}}(\rho).$$

6 The Radial-Accommodation Condition

The proposed model imposes the conditional closure rule:

> Between the initial relaxed state and the final uniformly rotating state, each material ring is assumed to remain closed and to preserve the sum of its local comoving circumferential lengths, while being permitted to change its laboratory radius.

Thus,

$$C_{\text{proper,final}}(\rho) = C_{\text{proper,initial}}(\rho) = 2\pi\rho.$$

Using

$$C_{\text{proper,final}} = \gamma C_{\text{lab}}, \qquad C_{\text{lab}} = 2\pi r,$$

we obtain

$$2\pi\rho = \gamma(\rho)\, 2\pi r(\rho).$$

Therefore,

$$\boxed{r(\rho) = \frac{\rho}{\gamma(\rho)}.}$$

This does not represent direct radial Lorentz contraction. It is a radial rearrangement required by the imposed combination of ring closure, preserved summed local proper circumference, and tangential Lorentz contraction.

Changes in the equilibrium separations among material constituents may accommodate such a rearrangement, subject to the constitutive response and energy cost of the material.

7 Self-Consistent Radial Solution

Substituting

$$\gamma(\rho) = \frac{1}{\sqrt{1 - \omega^2 r(\rho)^2/c^2}}$$

into

$$r(\rho) = \frac{\rho}{\gamma(\rho)}$$

gives

$$r(\rho) = \rho\sqrt{1 - \frac{\omega^2 r(\rho)^2}{c^2}}.$$

Squaring,

$$r(\rho)^2 = \rho^2 - \frac{\omega^2\rho^2}{c^2} r(\rho)^2.$$

Hence,

$$r(\rho)^2 \left(1 + \frac{\omega^2\rho^2}{c^2}\right) = \rho^2.$$

The positive-radius solution is

$$\boxed{r(\rho) = \frac{\rho}{\sqrt{1 + \omega^2\rho^2/c^2}}.}$$

The final tangential speed is

$$v(\rho) = \frac{\omega\rho}{\sqrt{1 + \omega^2\rho^2/c^2}},$$

so

$$\frac{v(\rho)}{c} = \frac{\omega\rho/c}{\sqrt{1 + \omega^2\rho^2/c^2}} < 1.$$

The Lorentz factor becomes

$$\boxed{\gamma(\rho) = \sqrt{1 + \frac{\omega^2\rho^2}{c^2}}.}$$

For the outermost ring,

$$\rho = R_0,$$

the final radius is

$$\boxed{R = \frac{R_0}{\sqrt{1 + \omega^2 R_0^2/c^2}} = \frac{R_0}{\gamma_e}.}$$

This result is conditional on the radial-accommodation closure rule.

8 Preservation of Ring Ordering

Differentiate

$$r(\rho) = \rho \left(1 + \frac{\omega^2 \rho^2}{c^2}\right)^{-1/2}.$$

Then

$$\boxed{\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{1}{(1 + \omega^2 \rho^2/c^2)^{3/2}} = \frac{1}{\gamma(\rho)^3} > 0.}$$

Therefore,

$$\rho_1 < \rho_2 \quad \Longrightarrow \quad r(\rho_1) < r(\rho_2).$$

The rings do not cross.

However,

$$\frac{\mathrm{d}r}{\mathrm{d}\rho} < 1 \qquad (\rho > 0),$$

so the radial separation of neighboring rings decreases:

$$\mathrm{d}\rho \longrightarrow \mathrm{d}r = \frac{\mathrm{d}\rho}{\gamma(\rho)^3}.$$

This is an anisotropic deformation rather than direct radial Lorentz contraction.

9 Circumference and Euclidean Closure

The laboratory circumference is

$$C_{\text{lab}}(\rho) = 2\pi r(\rho) = \frac{2\pi\rho}{\sqrt{1 + \omega^2 \rho^2/c^2}}.$$

Since

$$\gamma(\rho) = \sqrt{1 + \omega^2 \rho^2/c^2},$$

we have

$$\boxed{C_{\text{lab}}(\rho) = \frac{C_0(\rho)}{\gamma(\rho)}.}$$

At the same time,

$$\boxed{C_{\text{lab}}(\rho) = 2\pi r(\rho).}$$

The summed local comoving circumference is

$$C_{\text{proper}}(\rho) = \gamma(\rho) C_{\text{lab}}(\rho) = 2\pi\rho.$$

Thus,

$$\boxed{C_{\text{proper,final}}(\rho) = C_{\text{proper,initial}}(\rho).}$$

In the final accommodated configuration, the laboratory circle has the reduced radius at which the locally Lorentz-contracted laboratory lengths close without a gap.

10 Removal of the Apparent Contradiction

The simplified paradox attempts to impose

$$R = R_0, \qquad C_{\text{lab}} = \frac{C_0}{\gamma_e}, \qquad C_{\text{lab}} = 2\pi R.$$

Together these imply

$$2\pi R_0 = \frac{2\pi R_0}{\gamma_e},$$

which is impossible for

$$\gamma_e > 1.$$

The radial-accommodation model rejects the fixed-radius condition:

$$R \neq R_0.$$

Instead,

$$R = \frac{R_0}{\gamma_e}.$$

Consequently,

$$C_{\text{lab}} = 2\pi R = 2\pi \frac{R_0}{\gamma_e} = \frac{C_0}{\gamma_e}.$$

Therefore,

$$\boxed{C_{\text{lab}} = 2\pi R,}$$

$$\boxed{C_{\text{lab}} = \frac{C_0}{\gamma_e},}$$

and

$$\boxed{C_{\text{proper}} = C_0}$$

can hold simultaneously.

11 Comparison with a Fixed-Radius Constraint

Suppose a ring of initial proper circumference C_0 preserves its local proper lengths while an external constraint forces its path radius to remain unchanged.

Its total laboratory material length would be

$$C_{\text{material,lab}} = \frac{C_0}{\gamma}.$$

The fixed circular path would retain circumference

$$C_{\text{path}} = C_0.$$

The resulting mismatch would be

$$\Delta C = C_0 \left(1 - \frac{1}{\gamma}\right).$$

Depending on the acceleration protocol and material law, the ring may develop a gap, stretch, sustain internal stress, or fail.

Radial accommodation instead imposes

$$2\pi r = \frac{C_0}{\gamma},$$

so

$$\Delta C = 0.$$

This is one consistent boundary condition, not the only possible physical response.

12 Born Rigidity and Anisotropic Deformation

The proposed disk does not remain Born-rigid during spin-up.

Tangentially,

$$\frac{C_{\text{proper,final}}}{C_{\text{proper,initial}}} = 1.$$

Radially,

$$\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{1}{\gamma(\rho)^3} < 1.$$

The final state may therefore involve:

- reduced radial spacing;
- changing surface density;
- radial compression;
- shear between neighboring layers;
- nonuniform stress;
- redistribution of stored elastic or field energy.

The violation of perfect rigidity is not itself a physical defect, because perfect rigidity is not physically available.

The relevant question is:

> Which non-rigid deformation follows from a specified material law, stress–energy tensor, and acceleration protocol?

13 From Kinematics to Dynamics

Lorentz contraction is not a force.

The radial map

$$r(\rho) = \frac{\rho}{\sqrt{1 + \omega^2\rho^2/c^2}}$$

was first obtained from three conditional kinematic requirements:

1. each material ring remains closed;
2. its summed local comoving circumference is preserved;
3. its laboratory tangential lengths satisfy the local Lorentz relation.

At the purely geometrical level, these conditions define a self-consistent endpoint but do not specify which interaction supplies the required radial force.

Section 15 introduces a covariant phase-gradient sector. Appendix A reduces that sector to a uniformly wound thin ring and obtains an explicit circumference-dependent energy. In the strong-locking limit, the reduced action enforces

$$C_{\text{proper}} = C_0$$

and produces the radial-accommodation map. At finite stiffness, it gives a small radius correction and a reduced-model stability condition.

The resulting classification is therefore

a conditional radial map with an explicit effective fixed-ω radial support mechanism for a thin ring

The remaining problem is not the mathematical absence of any possible radial force. It is to determine whether physical matter possesses the proposed phase sector, to derive its parameters from a microscopic model, and to extend the thin-ring reduction to a continuous interacting disk with a complete stress–energy tensor.

14 Externally Supported Radial Accommodation

An external mechanical or gravitational field can contribute to the radial force balance, but a clock-rate gradient is not by itself a force.

For a weak, stationary gravitational field with potential Φ, the metric may be written approximately as

$$\mathrm{d}s^2 \simeq -\left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2\Phi}{c^2}\right)\mathrm{d}\boldsymbol{x}^2.$$

For a slowly moving material element,

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \simeq 1 + \frac{\Phi}{c^2} - \frac{v^2}{2c^2}.$$

Equivalently,

$$\Gamma_{\text{tot}} \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau} \simeq 1 - \frac{\Phi}{c^2} + \frac{v^2}{2c^2}.$$

The potential contributes a real radial force through its spatial gradient. In a nonrelativistic axisymmetric approximation, radial equilibrium may be written schematically as

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\phi}{r} + \mu\left(\omega^2 r - \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 0,$$

where σ_r and σ_ϕ are radial and hoop stresses and μ is the mass density.

This equation does not, by itself, imply the radial-accommodation map. The map must be obtained by solving the equilibrium equations together with a constitutive relation and boundary conditions.

An externally chosen potential may be designed to support the target profile. Such a result would establish existence under engineered forcing, not a universal spontaneous mechanism.

The effective centrifugal potential

$$\Phi_{\mathrm{cf}} = -\frac{1}{2}\omega^2 r^2$$

is a rotating-frame inertial potential. It must not automatically be identified with a gravitational field generated by the rotating matter.

15 A Candidate Covariant Phase-Locking Extension

This section introduces an effective microscopic extension. The covariant field sector is a model hypothesis. Its thin-ring reduction and its relation to the radial map are derived explicitly in Appendix A.

15.1 Phase-capacity variables

To avoid confusion with the disk radius, denote the internal channel, progression channel, and total phase capacity by

$$\mathcal{R}, \qquad \mathcal{K}, \qquad \mathcal{Q}_0.$$

The proposed closure relation is

$$\boxed{\mathcal{R}^2 + \mathcal{K}^2 = \mathcal{Q}_0^2.}$$

This constraint alone does not imply

$$\mathcal{R} = \mathcal{K}.$$

An equal-channel equilibrium may be represented by the energy density

$$U_{\mathrm{eq}} = \frac{\lambda_{\mathrm{eq}}}{2}(\mathcal{R} - \mathcal{K})^2 + \frac{\Lambda_{\mathrm{eq}}}{2}\left(\mathcal{R}^2 + \mathcal{K}^2 - \mathcal{Q}_0^2\right)^2,$$

where

$$\lambda_{\mathrm{eq}} > 0, \qquad \Lambda_{\mathrm{eq}} > 0.$$

Its minimum satisfies

$$\mathcal{R} = \mathcal{K}, \qquad \mathcal{R}^2 + \mathcal{K}^2 = \mathcal{Q}_0^2,$$

and therefore

$$\mathcal{R} = \mathcal{K} = \frac{\mathcal{Q}_0}{\sqrt{2}}.$$

This is an explicit model assumption generated by an energy function, rather than a consequence of the quadratic closure relation alone.

15.2 Phase winding and proper circumference

Let a complex material phase field be

$$\Psi = Ae^{i\phi}.$$

On a closed spatial material loop Σ, define the phase winding number

$$N = \frac{1}{2\pi}\oint_{\Sigma} \mathrm{d}\phi \in \mathbb{Z}.$$

This is a spatial loop integral. It must not be confused with an integral of proper time along a timelike worldline.

For a uniform winding on a loop with summed local comoving circumference C_{proper}, the proper wavenumber is

$$q = \frac{2\pi N}{C_{\mathrm{proper}}}.$$

Conservation of N alone does not conserve C_{proper}, because the wavenumber q may change. A sufficient additional dynamical condition is the selection of a preferred nonzero proper wavenumber q_0.

The spatial phase-gradient sector introduced below penalizes departures of q^2 from q_0^2. For a fixed winding sector, its lowest-energy circumference is therefore

$$\boxed{C_0 = \frac{2\pi|N|}{q_0}.}$$

Thus, circumference locking follows conditionally from two ingredients: topological conservation of N and dynamical selection of q_0.

15.3 Covariant phase-field action

For the claims made in this paper, a ghost-free covariant strong-locking candidate is sufficient. We therefore use an auxiliary temporal constraint rather than a finite quadratic temporal penalty:

$$S = \int \sqrt{-g}\,\mathcal{L}\,\mathrm{d}^4x,$$

with

$$\mathcal{L} = -\varepsilon(n, s_{AB}) - \frac{K_s}{2}\left(h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - q_0^2\right)^2 - \Lambda_t\left(u^\mu\nabla_\mu\phi - \Omega_0\right) - U_{\mathrm{eq}}.$$

Here:

- u^μ is the material four-velocity, normalized by $u^\mu u_\mu = -c^2$;
- n is the material number density;
- s_{AB} denotes material strain variables;
- $\varepsilon(n, s_{AB})$ is the non-phase material energy density;
- ϕ is the phase field;
- Ω_0 is the preferred comoving phase frequency;

- $q_0 > 0$ is the preferred proper spatial wavenumber;
- $K_s > 0$ is the spatial phase-gradient stiffness;
- Λ_t is an auxiliary scalar multiplier whose variation enforces the temporal phase constraint;
- $$h^{\mu\nu} = g^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}$$
  is the projector into the instantaneous local rest space.

All displayed terms are constructed from Lorentz-covariant scalars. No finite-stiffness temporal kinetic sector is asserted here. Constructing a healthy finite-stiffness Hamiltonian completion is part of the research program.

For a stationary, uniformly wound thin ring, the spatial phase-gradient term can be reduced explicitly. Appendix A shows that the reduction produces

$$U_{\text{wind}}(C) = \frac{\overline{K}_s}{2}\, C \left[\left(\frac{2\pi N}{C}\right)^2 - q_0^2\right]^2 .$$

The minimum is at

$$C = C_0 = \frac{2\pi|N|}{q_0},$$

and the near-minimum form is

$$U_{\text{wind}} = \frac{\kappa_{\text{eff}}}{2}(C - C_0)^2 + \mathcal{O}\Big((C - C_0)^3\Big),$$

with

$$\boxed{\kappa_{\text{eff}} = \frac{4\overline{K}_s q_0^4}{C_0}.}$$

In the strong-locking limit, variation of the reduced ring action enforces

$$C_{\text{proper}} = C_0$$

and yields the radial map

$$r(\rho) = \frac{\rho}{\sqrt{1 + \omega^2\rho^2/c^2}}.$$

This establishes the thin-ring equilibrium at the effective level. Appendix B retains an independent temporal phase variable and shows that the same constraint follows from the strong-locking covariant phase scalars in the sector $\Omega_0 = 0$. Extending the result to a continuous disk, constructing a healthy finite-stiffness temporal completion, and deriving the coefficients from a deeper microscopic theory remain separate tasks.

15.4 Reference metric and low-stress accommodation

A changing geometric radius generally produces strain relative to the initial stress-free state.

For radial accommodation to occur with small residual material stress, the phase sector would have to modify the preferred material metric:

$$G_{AB}^{(0)} \longrightarrow G_{AB}^{(\text{phase})}(\omega).$$

The thin-ring reduction proves the existence of an effective circumference-locking force. It does not by itself derive the full two-dimensional reference metric of a disk or prove that radial and shear stresses remain small. Those questions require a constitutive extension of the field model and the associated stress–energy tensor.

16 Corrected Geometric Second Moment

Consider a thin disk with uniform initial rest surface density σ_0.

The initial rest mass is

$$M = \pi \sigma_0 R_0^2,$$

and its initial geometric second moment is

$$I_0 = \frac{1}{2} M R_0^2 = \frac{1}{2} \pi \sigma_0 R_0^4.$$

Under the radial map

$$r(\rho) = \frac{\rho}{\sqrt{1 + \omega^2 \rho^2 / c^2}},$$

define the mapped rest-mass second moment

$$I_{\text{geom}}(\omega) = \int r^2 \, \mathrm{d}m_0.$$

Since

$$\mathrm{d}m_0 = 2\pi \sigma_0 \rho \, \mathrm{d}\rho,$$

we obtain

$$I_{\text{geom}}(\omega) = 2\pi \sigma_0 \int_0^{R_0} \frac{\rho^3}{1 + \omega^2 \rho^2 / c^2} \, \mathrm{d}\rho.$$

Introduce

$$x = \frac{\omega^2 R_0^2}{c^2}.$$

The integral gives

$$I_{\text{geom}}(\omega) = \pi \sigma_0 \left[\frac{R_0^2 c^2}{\omega^2} - \frac{c^4}{\omega^4} \ln(1 + x) \right].$$

Therefore,

$$\boxed{\frac{I_{\text{geom}}(\omega)}{I_0} = \frac{2}{x^2} \left[x - \ln(1 + x) \right].}$$

For

$$x \ll 1,$$

the expansion is

$$\boxed{\frac{I_{\text{geom}}(\omega)}{I_0} = 1 - \frac{2}{3}x + \frac{1}{2}x^2 - \frac{2}{5}x^3 + \mathcal{O}(x^4).}$$

Thus, under the assumed map,

$$I_{\text{geom}}(\omega) < I_0$$

for positive ω.

However,

$$I_{\text{geom}} = \int r^2 \, \mathrm{d}m_0$$

is only a geometric second moment of the rest-mass distribution.

The full relativistic angular momentum must be derived from the stress–energy tensor:

$$J = \int T^0{}_\phi \sqrt{-g} \, \mathrm{d}^3x.$$

Possible dynamical definitions include

$$I_{\text{eff}} = \frac{J}{\omega}$$

and

$$I_{\text{response}} = \frac{\mathrm{d}J}{\mathrm{d}\omega}.$$

These quantities may contain contributions from kinetic energy, elastic energy, phase-field energy, and internal stresses. Consequently, I_{geom} must not yet be advertised as the complete relativistic moment of inertia.

17 What the Model Establishes

The radial-accommodation construction establishes the conditional kinematic result:

> If each concentric material ring remains closed, preserves its summed local comoving circumference, and is free to change its laboratory radius, then there exists a monotonic self-consistent radial map for which the local tangential Lorentz relation and Euclidean laboratory closure hold simultaneously.

The map is

$$\boxed{r(\rho) = \frac{\rho}{\sqrt{1 + \omega^2\rho^2/c^2}}.}$$

The construction demonstrates that

$$\boxed{\text{absence of direct radial Lorentz contraction}}$$

does not imply

$$\boxed{\text{absence of dynamical radial deformation.}}$$

It also shows that the circumference–radius contradiction depends on maintaining the additional fixed-radius condition.

In addition, Appendix A shows that the proposed covariant spatial phase-gradient sector reduces, for a uniformly wound thin ring, to a definite circumference energy. In the strong-locking limit, the reduced action supports the exact radial map. At finite stiffness, it gives

$$\boxed{\delta r \simeq \frac{M_0 \omega^2 r_0}{4\pi^2 \kappa_{\text{eff}} \gamma_0^5}.}$$

Within the reduced fixed-ω model, the strong-locking branch is locally stable when

$$\boxed{\kappa_{\text{eff}} > \frac{M_0 \omega^2}{4\pi^2 \gamma_0^3}.}$$

The model therefore establishes both a kinematically consistent endpoint and an effective constrained fixed-ω radial-equilibrium realization for a thin ring. Appendix B further supplies a strong-locking constrained time-dependent embedding in the zero-temporal-frequency sector, without claiming a complete finite-stiffness Hamiltonian theory.

18 What the Model Does Not Yet Establish

The present derivation does not prove that every physical rotating disk must contract according to the proposed map.

A real disk may:

- expand because of centrifugal stress;
- contract under external radial forcing;
- stretch circumferentially;
- develop nonuniform density;
- shear;
- form a gap;
- oscillate;
- tear or fail.

The model does not invalidate the Herglotz–Noether theorem, because it does not preserve global Born rigidity during spin-up.

It also does not eliminate the synchronization and clock-comparison issues associated with rotating observers.

The gravitational discussion establishes only that an external field may participate in radial force balance.

For a uniformly wound thin ring, the phase-gradient action has been shown to produce the radial map in the strong-locking limit and a locally stable finite-stiffness branch under the condition derived in Appendix A. The remaining limitations are:

- extension from one thin ring to a continuous interacting disk;

- derivation of the phase coefficients from a microscopic theory of matter;
- construction of the complete stress–energy tensor;
- determination of the two-dimensional material reference metric;
- comparison with ordinary relativistic elasticity;
- independent determination of the parameters entering the finite-stiffness correction;
- a Dirac–Bergmann or equivalent Routh analysis of the strong-locking constrained system;
- construction of a ghost-free finite-stiffness temporal phase sector and its complete velocity Hessian;
- reconciliation of the proper-energy reduction with the laboratory-time finite-stiffness action.

19 Relation to General Relativity

The rotating-disk problem played a role in the historical development of ideas concerning accelerated frames and operational geometry.

However, general relativity was not derived solely from the Ehrenfest problem. It also arose from the equivalence principle, gravitational clock effects, the Newtonian limit, the motion of light, the anomalous perihelion advance of Mercury, and the search for generally covariant field equations.

A successful radial-accommodation model would therefore establish only that one material response to rotation can preserve Euclidean laboratory closure without imposing a fixed radius.

It would not, by itself, refute general relativity.

The defensible conclusion is

The Ehrenfest disk is not an independent proof that physical spacetime must be curved.

20 Research Program

The thin-ring reduction supplies an effective constrained fixed-ω radial-equilibrium realization, and Appendix B supplies a strong-locking constrained time-dependent embedding. Turning these results into a predictive theory of a continuous disk requires the following further steps.

20.1 Continuous constitutive model

Specify

$$\varepsilon = \varepsilon\left(n, s_{AB}, \phi, \nabla\phi, \ldots\right)$$

for interacting radial layers, including radial, hoop, and shear responses.

20.2 Stress–energy tensor

Derive

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$$

including material, elastic, and phase-gradient contributions.

20.3 Continuous equations of motion

Solve

$$\nabla_\mu T^{\mu\nu} = f^\nu_{\text{external}},$$

or

$$\nabla_\mu T^{\mu\nu} = 0$$

for an isolated model, together with the phase-field and material-map equations.

20.4 Disk equilibrium

Determine whether the family of ring solutions can coexist as a continuous disk without inconsistent inter-ring shear or radial stress, and whether

$$r_{\text{eq}}(\rho) = \frac{\rho}{\sqrt{1+\omega^2\rho^2/c^2}}$$

remains the equilibrium profile after all couplings are included.

20.5 Global stability

Study perturbations of the coupled disk,

$$r(\rho,\tau) = r_{\text{eq}}(\rho) + \delta r(\rho,\tau),$$

including radial, azimuthal, and shear modes.

20.6 Constrained time-dependent completion

Perform a Dirac–Bergmann analysis, or an equivalent Routh reduction in a fixed phase-charge sector, for the multiplier system of Appendix B. Construct a healthy finite-stiffness temporal phase completion, derive its full Hamiltonian, and determine whether the reduced branch can pass through $\mathrm{d}J/\mathrm{d}\omega = 0$.

20.7 Observable quantities

Calculate

$$J(\omega), \qquad E(\omega), \qquad I_{\text{eff}}(\omega),$$

from the full stress–energy tensor, and compare them with standard relativistic elastic-disk models.

21 Conclusion

The Ehrenfest problem retains mathematical importance as a demonstration that arbitrary spin-up is incompatible with perfect Born rigidity.

Its direct relevance to real material disks is more limited because real extended bodies are not perfectly rigid. Their constituents remain distinct, and their relative separations are maintained dynamically.

The inference

$$\text{radial direction is perpendicular to tangential motion}$$
$$\Downarrow$$
$$\text{radius must remain unchanged}$$

is not logically compulsory.

The valid conclusion is only that direct longitudinal Lorentz contraction does not act along the radial direction.

Under the conditional preservation of each ring's summed local comoving circumference, the radial map is

$$\boxed{r(\rho) = \frac{\rho}{\sqrt{1 + \omega^2\rho^2/c^2}}.}$$

For the outer edge,

$$\boxed{R = \frac{R_0}{\sqrt{1 + \omega^2 R_0^2/c^2}} = \frac{R_0}{\gamma_e}.}$$

This configuration satisfies

$$C_{\text{lab}} = 2\pi R = \frac{C_0}{\gamma_e}$$

while preserving

$$C_{\text{proper}} = C_0.$$

The construction is mathematically closed at the kinematic level and admits an explicit effective constrained fixed-ω radial-equilibrium realization for a uniformly wound thin ring. In the strong-locking zero-temporal-frequency sector, the covariant phase ansatz also embeds the same map in a constrained time-dependent system.

Reduction of the proposed covariant phase-gradient sector gives

$$U_{\text{wind}}(C) = \frac{\overline{K}_s}{2}\, C \left[\left(\frac{2\pi N}{C}\right)^2 - q_0^2\right]^2.$$

The preferred summed local comoving circumference is

$$C_0 = \frac{2\pi |N|}{q_0},$$

and the corresponding near-equilibrium stiffness is

$$\kappa_{\text{eff}} = \frac{4\overline{K}_s q_0^4}{C_0}.$$

In the strong-locking limit, the radial map follows exactly. At finite stiffness, the reduced model gives

$$\delta r \simeq \frac{M_0 \omega^2 r_0}{4\pi^2 \kappa_{\rm eff} \gamma_0^5},$$

and the fixed-ω equilibrium is locally stable when

$$\kappa_{\rm eff} > \frac{M_0 \omega^2}{4\pi^2 \gamma_0^3}.$$

The model therefore advances beyond a force-free kinematic construction: it possesses a concrete effective radial support mechanism within the reduced fixed-ω thin-ring model. The time-dependent extension is established only in the strong-locking constrained sector; its finite-stiffness Hamiltonian completion remains open.

The appropriate final conclusion is

$$\boxed{\text{The Ehrenfest problem constrains Born-rigid rotation,}}$$

but

$$\boxed{\text{it does not uniquely determine the deformation of real rotating matter.}}$$

The remaining work is to determine whether the effective phase sector is realized in physical matter, to derive its parameters microscopically, to complete the constrained Hamiltonian and finite-stiffness temporal dynamics, and to extend the thin-ring result to a complete continuous-disk theory.

A Effective Spatial Phase-Stiffness and Stationary Radial Support

This appendix closes the effective mathematical gap between the covariant phase-field action introduced in Section 15 and the circumference-locking energy required to support the radial-accommodation map.

The derivation does not claim that the proposed phase sector is already established as a fundamental description of physical matter. It shows, however, that the effective circumference stiffness need not be inserted as an unrelated potential. It follows from the spatial phase-gradient sector of the covariant action after reduction to a stationary, uniformly wound thin ring.

A.1 Reduction of the covariant phase sector

Consider the spatial phase-gradient contribution

$$\mathcal{L}_s = -\frac{K_s}{2} \left(h^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - q_0^2 \right)^2 .$$

For a thin ring, integrate the phase energy over its proper transverse cross-section and any stationary amplitude profile. Denote the resulting effective one-dimensional coupling by

$$\overline{K}_s > 0.$$

Let

$$C \equiv C_{\text{proper}}$$

be the summed local comoving circumference.

For a stationary uniform phase winding,

$$N = \frac{1}{2\pi} \oint_{\Sigma} \mathrm{d}\phi \in \mathbb{Z}.$$

For definiteness take $N \neq 0$. The magnitude of the uniform proper wavenumber is

$$q = \frac{2\pi |N|}{C}.$$

The reduced phase-gradient energy is therefore

$$U_{\text{wind}}(C) = \frac{\overline{K}_s}{2} C \left[\left(\frac{2\pi N}{C} \right)^2 - q_0^2 \right]^2. \tag{1}$$

This expression is nonnegative and vanishes when

$$q = q_0.$$

Its preferred circumference is consequently

$$\boxed{C_0 = \frac{2\pi |N|}{q_0}.} \tag{2}$$

Thus, conservation of the winding number alone does not fix the circumference. Conservation of N, together with dynamical selection of q_0, selects C_0.

A.2 Derived effective circumference stiffness

Write

$$C = C_0 + \Delta C, \qquad |\Delta C| \ll C_0.$$

Using

$$2\pi |N| = q_0 C_0,$$

equation (1) becomes

$$U_{\text{wind}}(C) = \frac{\overline{K}_s q_0^4}{2} C \left[\left(\frac{C_0}{C} \right)^2 - 1 \right]^2.$$

Expanding about $C = C_0$ gives

$$U_{\text{wind}}(C) = \frac{2\overline{K}_s q_0^4}{C_0} (\Delta C)^2 + \mathcal{O}\Big((\Delta C)^3 \Big).$$

Therefore,

$$U_{\text{wind}}(C) = \frac{\kappa_{\text{eff}}}{2} (C - C_0)^2 + \mathcal{O}\Big((C - C_0)^3 \Big), \tag{3}$$

where

$$\boxed{\kappa_{\text{eff}} = \frac{4\overline{K}_s q_0^4}{C_0}.} \tag{4}$$

The effective circumference stiffness is thus related to the reduced phase-gradient coupling, the preferred proper wavenumber, and the winding circumference.

A.3 Effective radial Lagrangian

Consider a thin ring with non-phase rest mass M_0, initial relaxed radius ρ, laboratory radius $r(t)$, and externally prescribed constant angular velocity ω. The reference phase energy vanishes at $C = C_0$, so M_0 does not double-count the excess phase-gradient energy.

The mechanical Lagrangian is

$$L_{\mathrm{mech}} = -M_0 c^2 \sqrt{1 - \frac{\dot{r}^{\,2} + \omega^2 r^2}{c^2}}. \tag{5}$$

For stationary configurations define

$$\gamma_\omega(r) = \frac{1}{\sqrt{1 - \omega^2 r^2 / c^2}}.$$

The summed local comoving circumference is

$$C(r) = 2\pi r \gamma_\omega(r). \tag{6}$$

For adiabatic radial variations, consider the following phenomenological fixed-ω radial Lagrangian built from the proper circumference energy:

$$L_{\mathrm{ring}} = -M_0 c^2 \sqrt{1 - \frac{\dot{r}^{\,2} + \omega^2 r^2}{c^2}} - U_{\mathrm{wind}}(C(r))\,. \tag{7}$$

The circumference relation in (6) is evaluated with the instantaneous tangential Lorentz factor. The subsequent stability analysis is therefore a phenomenological quasi-static radial analysis at fixed ω, not a direct laboratory-time reduction of the full covariant world-tube action and not a complete treatment of arbitrary nonstationary deformations. The direct covariant laboratory-time reduction carries an additional factor $1/\gamma$ relative to the proper circumference energy; this does not change the strong-locking minimum but does require a separate finite-stiffness calculation.

For the ring initially labelled by ρ,

$$C_0 = 2\pi\rho.$$

A.4 Strong-locking limit and the exact radial map

In the strong-locking limit,

$$\overline{K}_s \to \infty,$$

any finite departure from $q = q_0$ has divergent energy. The stationary condition is therefore

$$C(r) = C_0.$$

Using

$$C(r) = 2\pi r \gamma_\omega(r), \qquad C_0 = 2\pi\rho,$$

we obtain

$$r\gamma_\omega(r) = \rho. \tag{8}$$

Solving for the positive radius gives

$$\boxed{r_0(\rho) = \frac{\rho}{\sqrt{1 + \omega^2 \rho^2 / c^2}}.} \tag{9}$$

The corresponding Lorentz factor is

$$\gamma_0(\rho) = \sqrt{1 + \frac{\omega^2 \rho^2}{c^2}}.$$

Thus, within the strong phase-locking limit, the radial-accommodation map follows from the reduced phase-gradient energy.

A.5 Equivalent exact constraint formulation

The strong-locking limit may equivalently be represented by a Lagrange multiplier $\mathcal{T}$ with dimensions of force:

$$L_{\text{constr}} = -M_0 c^2 \sqrt{1 - \frac{\dot{r}^{\,2} + \omega^2 r^2}{c^2}} - \mathcal{T}\left[C(r) - C_0\right]. \tag{10}$$

Variation with respect to $\mathcal{T}$ gives

$$C(r) = C_0,$$

and hence the radial map (9).

For a stationary configuration,

$$\dot{r} = 0, \qquad \frac{\mathrm{d}C}{\mathrm{d}r} = 2\pi\gamma_\omega^3.$$

Variation with respect to r gives

$$M_0 \gamma_\omega \omega^2 r - 2\pi \mathcal{T} \gamma_\omega^3 = 0.$$

Therefore,

$$\boxed{\mathcal{T} = \frac{M_0 \omega^2 r}{2\pi \gamma_\omega^2}.} \tag{11}$$

The corresponding inward generalized radial force is

$$F_{\text{phase}} = \mathcal{T} \frac{\mathrm{d}C}{\mathrm{d}r}.$$

Hence,

$$\boxed{F_{\text{phase}} = M_0 \gamma_\omega \omega^2 r.} \tag{12}$$

This balances the outward relativistic circular-motion term in the stationary reduced equation.

A.6 Finite-stiffness equilibrium

For finite phase stiffness use the quadratic approximation

$$U_{\text{wind}} \simeq \frac{\kappa_{\text{eff}}}{2} \left[C(r) - C_0\right]^2.$$

At $\dot{r} = 0$, the radial equilibrium equation is

$$M_0 \gamma_\omega \omega^2 r - \kappa_{\text{eff}} \left[C(r) - C_0\right] \frac{\mathrm{d}C}{\mathrm{d}r} = 0. \tag{13}$$

Using

$$C(r) = 2\pi r \gamma_\omega, \qquad \frac{\mathrm{d}C}{\mathrm{d}r} = 2\pi\gamma_\omega^3,$$

this becomes

$$M_0\gamma_\omega\omega^2 r - 4\pi^2\kappa_{\mathrm{eff}}\left(r\gamma_\omega - \rho\right)\gamma_\omega^3 = 0.$$

Let

$$r = r_0 + \delta r, \qquad |\delta r| \ll r_0,$$

where r_0 is the strong-locking solution. Since

$$r_0\gamma_0 = \rho$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\gamma_\omega\right)\bigg|_{r_0} = \gamma_0^3,$$

the leading correction is

$$\boxed{\delta r \simeq \frac{M_0\omega^2 r_0}{4\pi^2\kappa_{\mathrm{eff}}\gamma_0^5}.} \tag{14}$$

The correction is positive. Finite phase stiffness therefore permits a small centrifugal expansion relative to the strong-locking radius.

Using equation (4), the correction may also be written as

$$\boxed{\delta r \simeq \frac{M_0\omega^2 r_0 C_0}{16\pi^2\overline{K}_s q_0^4\gamma_0^5}.} \tag{15}$$

A.7 Reduced-model local radial stability

Existence of a stationary solution does not by itself imply stability.

For small radial speeds, expand the mechanical Lagrangian about the stationary state:

$$L_{\mathrm{mech}} = -\frac{M_0c^2}{\gamma_\omega} + \frac{1}{2}M_0\gamma_\omega\dot{r}^2 + \mathcal{O}(\dot{r}^4).$$

The effective radial inertial coefficient at r_0 is

$$M_{\mathrm{rad,eff}} = M_0\gamma_0.$$

At the strong-locking point, the curvature of the quadratic phase energy is

$$U''_{\mathrm{wind}}(r_0) = \kappa_{\mathrm{eff}}\left(\frac{\mathrm{d}C}{\mathrm{d}r}\right)^2_{r_0} = 4\pi^2\kappa_{\mathrm{eff}}\gamma_0^6.$$

The radial derivative of the outward mechanical term is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(M_0\gamma_\omega\omega^2 r\right)\bigg|_{r_0} = M_0\omega^2\gamma_0^3.$$

Within the reduced fixed-ω model, local stability requires

$$4\pi^2\kappa_{\mathrm{eff}}\gamma_0^6 - M_0\omega^2\gamma_0^3 > 0.$$

Thus,

$$\boxed{\kappa_{\text{eff}} > \frac{M_0\omega^2}{4\pi^2\gamma_0^3}.} \tag{16}$$

Using equation (4), this becomes

$$\boxed{\overline{K}_s > \frac{M_0\omega^2 C_0}{16\pi^2 q_0^4\gamma_0^3}.} \tag{17}$$

The approximate squared frequency of small radial oscillations is

$$\boxed{\Omega_r^2 \simeq \frac{4\pi^2\kappa_{\text{eff}}\gamma_0^5}{M_0} - \omega^2\gamma_0^2.} \tag{18}$$

A positive value of Ω_r^2 is equivalent to the reduced-model stability condition.

A.8 Kinematic phase-tilt parameter

The internal phase-tilt angle may be parametrized by

$$\boxed{\sin\theta = \frac{v}{c}.}$$

Then

$$\cos\theta = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma},$$

and

$$\frac{1}{\cos\theta} = \gamma.$$

This is an internal kinematic parametrization of the phase-capacity decomposition. It is not the special-relativistic addition law for two ordinary spacetime velocities.

The quantitative circumference relation remains

$$C_{\text{proper}} = \gamma C_{\text{lab}}.$$

A.9 Scope of the effective derivation

The appendix establishes the following conditional result:

> If matter possesses the proposed covariant spatial phase-gradient sector, and if a thin ring carries a conserved uniform phase winding, then the reduced sector produces a definite circumference energy. In the strong-locking limit, the radial-accommodation map is the exact stationary constraint. At finite stiffness, the reduced model predicts a positive radius correction and a local fixed-ω stability condition.

The derivation therefore establishes

$$\boxed{\text{an effective constrained fixed-}\omega\text{ radial realization of the map for a thin ring.}}$$

It does not by itself establish that all physical matter possesses this phase sector, nor does it derive the numerical values of $\overline{K}_s$ and q_0 from a more fundamental microscopic theory. The finite-stiffness correction and stability condition belong to the phenomenological fixed-ω radial Lagrangian used above; they have not yet been identified with the direct laboratory-time reduction of the covariant action. The appendix also does not replace the full continuous-disk stress–energy calculation.

B Strong-Locking Time-Dependent Embedding and Constrained Angular Momentum

This appendix retains an independent temporal phase variable and tests the strong-locking time-dependent sector without asserting a complete finite-stiffness Hamiltonian theory. The result is a constrained system, not a regular unconstrained Lagrangian in the ordinary Legendre sense.

B.1 Ring geometry and phase ansatz

Work in flat spacetime with cylindrical coordinates (t, R, θ). Let

$$R = R(t), \qquad \Phi = \Phi(t), \qquad \omega = \dot{\Phi},$$

and define

$$\Gamma = \frac{1}{\sqrt{1 - (\dot{R}^2 + R^2\omega^2)/c^2}}.$$

For a uniformly wound phase field, use

$$\phi(t, \theta) = N\theta + \chi(t), \qquad N \in \mathbb{Z}.$$

The material four-velocity is

$$u^\mu = \Gamma(1, \dot{R}, \omega, 0).$$

The temporal and projected spatial phase scalars are

$$D_t\phi \equiv u^\mu \nabla_\mu \phi = \Gamma(\dot{\chi} + N\omega), \tag{19}$$

$$\begin{aligned} Q_s^2 &\equiv h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ &= \frac{N^2}{R^2} + \frac{\Gamma^2}{c^2}(\dot{\chi} + N\omega)^2 - \frac{\dot{\chi}^2}{c^2}. \end{aligned} \tag{20}$$

B.2 Zero-temporal-frequency strong-locking sector

Set

$$\Omega_0 = 0$$

and impose the strong-locking constraints

$$D_t\phi = 0, \qquad Q_s = q_0.$$

For a stationary radius, $\dot{R} = 0$, write

$$\gamma = \frac{1}{\sqrt{1 - R^2\omega^2/c^2}}.$$

The temporal constraint gives

$$\dot{\chi} = -N\omega.$$

Substitution into equation (20) gives

$$Q_s^2 = \frac{N^2}{R^2} - \frac{N^2\omega^2}{c^2} = \frac{N^2}{\gamma^2 R^2}.$$

Hence

$$Q_s = \frac{|N|}{\gamma R}.$$

The spatial constraint therefore yields

$$\gamma R = \frac{|N|}{q_0} \equiv \rho,$$

and the radial-accommodation map follows exactly:

$$\boxed{R(\omega) = \frac{\rho}{\sqrt{1 + \omega^2 \rho^2 / c^2}}.} \tag{21}$$

This establishes a covariant strong-locking embedding of the stationary radial constraint in the sector $\Omega_0 = 0$. It does not yet provide a healthy finite-stiffness temporal phase theory.

B.3 Multiplier formulation

Introduce reduced multipliers $\lambda_t(t)$ and $\lambda_s(t)$:

$$L_{\mathrm{constr}} = -\frac{M_0 c^2}{\Gamma} - \lambda_t D_t \phi - \lambda_s (Q_s - q_0). \tag{22}$$

Their dimensions are chosen so that every term in (22) has units of energy. Variation with respect to the multipliers enforces the two phase constraints.

Because χ is cyclic, its canonical momentum

$$p_\chi = \frac{\partial L_{\mathrm{constr}}}{\partial \dot{\chi}}$$

is conserved. The material angle Φ is also cyclic in the absence of external torque, and its canonical momentum is the constrained angular momentum

$$J = \frac{\partial L_{\mathrm{constr}}}{\partial \omega}.$$

B.4 Angular momentum in a fixed phase-charge sector

On the stationary constrained branch,

$$\dot{R} = 0, \qquad \dot{\chi} = -N\omega, \qquad Q_s = q_0 = \frac{|N|}{\gamma R}.$$

The derivatives required before eliminating the multipliers are

$$\left.\frac{\partial D_t \phi}{\partial \omega}\right|_{\mathcal{C}} = \gamma N, \qquad \left.\frac{\partial D_t \phi}{\partial \dot{\chi}}\right|_{\mathcal{C}} = \gamma,$$

$$\left.\frac{\partial Q_s}{\partial \omega}\right|_{\mathcal{C}} = 0, \qquad \left.\frac{\partial Q_s}{\partial \dot{\chi}}\right|_{\mathcal{C}} = \frac{N\omega}{q_0 c^2},$$

and

$$\left.\frac{\partial Q_s}{\partial R}\right|_{\mathcal{C}} = -\frac{N^2}{q_0 R^3}.$$

Consequently,

$$p_\chi = -\lambda_t \gamma - \lambda_s \frac{N\omega}{q_0 c^2}, \tag{23}$$

$$J = M_0 \gamma R^2 \omega - \lambda_t \gamma N. \tag{24}$$

The stationary radial equation gives

$$M_0 \gamma \omega^2 R + \lambda_s \frac{N^2}{q_0 R^3} = 0,$$

so

$$\lambda_s = -\frac{M_0 \omega^2 R^3}{|N|}.$$

Eliminating λ_t and λ_s between (23) and (24) yields

$$\boxed{J = \frac{M_0 R^2 \omega}{\gamma} + N p_\chi.} \tag{25}$$

Using equation (21),

$$\boxed{J(\omega) = \frac{M_0 \rho^2 \omega}{(1 + \omega^2 \rho^2 / c^2)^{3/2}} + N p_\chi.} \tag{26}$$

The second term is a conserved phase-charge contribution. The one-term expression used in the preliminary spin-up calculation is recovered only in the explicitly selected sector

$$p_\chi = 0.$$

B.5 Reduced response and its boundary

For fixed p_χ,

$$\frac{\mathrm{d}J}{\mathrm{d}\omega} = M_0 \rho^2 \frac{1 - 2\omega^2 \rho^2 / c^2}{(1 + \omega^2 \rho^2 / c^2)^{5/2}}.$$

The reduced velocity-momentum relation loses local invertibility at

$$\boxed{\frac{\omega \rho}{c} = \frac{1}{\sqrt{2}}.}$$

With an applied torque,

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \tau_{\text{ext}},$$

so the one-dimensional quasi-static equation would require

$$\dot{\omega} = \frac{\tau_{\text{ext}}}{\mathrm{d}J/\mathrm{d}\omega}.$$

This reduced expression is singular at the turning point when $\tau_{\text{ext}} \neq 0$. The calculation establishes a boundary of the reduced strong-locking branch; it does not determine whether the complete constrained or finite-stiffness system terminates, changes branch, or passes through by activating additional degrees of freedom.

B.6 Relation to the proper circumference energy

On the stationary zero-temporal-frequency branch,

$$Q_s = \frac{2\pi|N|}{C}, \qquad C = 2\pi\gamma R.$$

The spatial phase sector therefore has the proper energy

$$U_{\text{wind}}(C) = \frac{\overline{K}_s}{2} C \left[\left(\frac{2\pi N}{C} \right)^2 - q_0^2 \right]^2,$$

as used in Appendix A. A direct uniform laboratory-time reduction of this proper energy has the schematic form

$$L_s = -\frac{U_{\text{wind}}(C)}{\gamma}.$$

Thus the proper-energy reduction and the phenomenological fixed-ω Lagrangian of Appendix A share the same strong-locking minimum and exact radial map, but their finite-stiffness radial corrections need not coincide. The covariant laboratory-time finite-stiffness response must be derived separately.

B.7 Status and open problem

The results of this appendix establish:

1. the exact recovery of the radial-accommodation constraint from the covariant phase scalars in the sector $\Omega_0 = 0$;
2. a constrained angular momentum in a fixed conserved phase-charge sector;
3. the recovery of the previous one-term angular momentum only for $p_\chi = 0$;
4. loss of local invertibility of the reduced strong-locking branch at $\omega\rho/c = 1/\sqrt{2}$.

They do not yet establish a regular finite-stiffness temporal phase sector, a Hamiltonian bounded from below for such a completion, or the full stress–energy angular momentum. Those tasks require a constrained Hamiltonian analysis and a separate covariant finite-stiffness world-tube reduction.

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