

A Note on the Algebraic and Combinatorial Properties of $\{-1, 0, +1\}^3$

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Abstract

This note documents basic properties of the set $L_3 = \{-1, 0, +1\}^3$, a finite set of 27 integer lattice points. We describe:

- Two binary operations: saturated addition \oplus (non-associative) and balanced product \otimes (associative, commutative), with \otimes distributing over \oplus ;
- The graph automorphism group $\text{Aut}(L_3) \cong S_3 \times (\mathbb{Z}_2)^3$ of order 48, with orbit decomposition $1 + 6 + 12 + 8$;
- 49 collinear triples summing to zero; the 49×27 incidence matrix over $\text{GF}(3)$ has rank 23 and a 4-dimensional null space spanned by $\{1, x, y, z\}$;
- The adjacency graph has 27 vertices, 54 edges, first Betti number $\beta_1 = 28$, and Euler characteristic $\chi = -27$;
- Componentwise order yields a distributive lattice with rank distribution 1-3-6-7-6-3-1.

All results presented are elementary and can be obtained by direct computation or are standard corollaries of general theorems. No claim of original mathematical discovery is made. This note serves purely as a convenient compilation for reference purposes.

1 Definition and Notation

Definition 1.1. *Let*

$$L_3 = \{-1, 0, +1\}^3 = \{(x, y, z) \in \mathbb{Z}^3 : x, y, z \in \{-1, 0, 1\}\}.$$

The set L_3 contains $|L_3| = 3^3 = 27$ points.

Remark 1.2. *The name “Luo-Shu cell” is a convenient label chosen by the author; this set is standard in combinatorics as the 3-ary 3-dimensional hypercube or the set of lattice points in a $3 \times 3 \times 3$ cube centered at the origin. No structural connection to the classical magic square is implied beyond the coincidental number of magic-sum lines (Section 4) which matches the row/column count of the 3×3 Luo-Shu square.*

For $\phi = (x, y, z) \in L_3$, we call x, y, z its coordinates.

2 Algebraic Operations

Definition 2.1 (Saturation function). Define $\text{sat} : \mathbb{Z} \rightarrow \{-1, 0, +1\}$ by

$$\text{sat}(t) = \begin{cases} -1 & \text{if } t \leq -1, \\ 0 & \text{if } t = 0, \\ +1 & \text{if } t \geq 1. \end{cases}$$

Definition 2.2 (Saturated addition). Define $\oplus : L_3 \times L_3 \rightarrow L_3$ componentwise:

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (\text{sat}(x_1 + x_2), \text{sat}(y_1 + y_2), \text{sat}(z_1 + z_2)).$$

The operation table for a single coordinate is:

\oplus	-1	0	+1
-1	-1	-1	0
0	-1	0	+1
+1	0	+1	+1

Proposition 2.3 (Properties of \oplus). 1. **Commutativity:** $\phi \oplus \psi = \psi \oplus \phi$.

2. **Identity:** $\phi \oplus (0, 0, 0) = \phi$.

3. **Non-associativity:** $((1, 0, 0) \oplus (1, 0, 0)) \oplus (-1, 0, 0) = (0, 0, 0) \neq (1, 0, 0) = (1, 0, 0) \oplus ((1, 0, 0) \oplus (-1, 0, 0))$.

4. **Non-associativity rate:** Among $27^3 = 19683$ triples in L_3 , exactly 7516 are non-associative. The proportion is $7516/19683 \approx 38.19\%$.

Proof. For (4): On a single coordinate, 4 of 27 triples are non-associative: $(-1, -1, +1)$, $(-1, +1, +1)$, $(+1, -1, -1)$, $(+1, +1, -1)$. Since \oplus is defined componentwise, a triple is associative in L_3 if and only if it is associative in each coordinate. Thus the proportion of associative triples in L_3 is $(23/27)^3 = 12167/19683$, and the non-associative proportion is $1 - (23/27)^3 = 7516/19683$. \square

Definition 2.4 (Balanced product). Define $\otimes : L_3 \times L_3 \rightarrow L_3$ componentwise by integer multiplication:

$$(x_1, y_1, z_1) \otimes (x_2, y_2, z_2) = (x_1x_2, y_1y_2, z_1z_2).$$

The operation table for a single coordinate is:

\otimes	-1	0	+1
-1	+1	0	-1
0	0	0	0
+1	-1	0	+1

Proposition 2.5 (Properties of \otimes). 1. **Commutativity:** $\phi \otimes \psi = \psi \otimes \phi$.

2. **Associativity:** $(\phi \otimes \psi) \otimes \chi = \phi \otimes (\psi \otimes \chi)$.

3. **Identity:** $\phi \otimes (1, 1, 1) = \phi$.

4. **Absorbing element:** $\phi \otimes (0, 0, 0) = (0, 0, 0)$.

Proposition 2.6 (Distributivity). *For all $\phi, \psi, \chi \in L_3$,*

$$\phi \otimes (\psi \oplus \chi) = (\phi \otimes \psi) \oplus (\phi \otimes \chi).$$

Proof. Check componentwise. For $a, b, c \in \{-1, 0, +1\}$:

- If $a = 0$: both sides are 0.
- If $a = 1$: both sides equal $\text{sat}(b + c)$.
- If $a = -1$: left-hand side = $-\text{sat}(b + c)$, right-hand side = $\text{sat}(-b - c) = -\text{sat}(b + c)$.

All cases agree. □

3 Symmetry Structure

Definition 3.1 (Adjacency graph). *The graph $\Gamma(L_3)$ has vertex set L_3 and edges between points at Manhattan distance 1:*

$$E = \{\{\phi, \psi\} : d_M(\phi, \psi) = 1\},$$

where

$$d_M((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

Equivalently, $\Gamma(L_3) = P_3 \square P_3 \square P_3$ (Cartesian product of three path graphs on 3 vertices).

Proposition 3.2 (Graph automorphism group). *The automorphism group of $\Gamma(L_3)$ is*

$$\text{Aut}(L_3) \cong S_3 \times (\mathbb{Z}_2)^3,$$

a semidirect product of order $6 \cdot 8 = 48$, where S_3 permutes coordinates and $(\mathbb{Z}_2)^3$ flips signs independently.

Proof. For Cartesian products of relatively prime graphs, $\text{Aut}(G \square H) = \text{Aut}(G) \wr \text{Aut}(H)$ [1, ?]. Since P_3 is prime and $\text{Aut}(P_3) \cong \mathbb{Z}_2$, we have

$$\text{Aut}(P_3 \square P_3 \square P_3) = \mathbb{Z}_2 \wr S_3 \cong S_3 \times (\mathbb{Z}_2)^3. \quad \square$$

Proposition 3.3 (Orbit decomposition). *Under the action of $\text{Aut}(L_3)$, L_3 decomposes into four orbits determined by the number of non-zero coordinates. These are exactly the sets of points of equal Hamming weight:*

Orbit	Representative	Size	Description	Hamming weight
O_1	(0, 0, 0)	1	center	0
O_2	(1, 0, 0)	6	face centers	1
O_3	(1, 1, 0)	12	edge centers	2
O_4	(1, 1, 1)	8	corners	3

The degrees in $\Gamma(L_3)$ are $\deg(O_1) = 6$, $\deg(O_2) = 5$, $\deg(O_3) = 4$, $\deg(O_4) = 3$.

4 Magic-Sum Constraints

Definition 4.1 (Magic-sum line). *A magic-sum line in L_3 is a set of three collinear points $\{\phi_1, \phi_2, \phi_3\}$ such that $\phi_1 + \phi_2 + \phi_3 = (0, 0, 0)$, where the sum is computed in \mathbb{Z}^3 (without saturation).*

Proposition 4.2 (Number of magic lines). *There are exactly 49 magic-sum lines in L_3 :*

<i>Axial (parallel to x-, y-, or z-axis)</i>	: 27
<i>Face-diagonal (diagonals of coordinate planes)</i>	: 18
<i>Space-diagonal (body diagonals through origin)</i>	: 4
<hr style="width: 100%; border: 0.5px solid black;"/>	
<i>Total</i>	: 49

Definition 4.3 (Incidence matrix). *Let M be the 49×27 matrix over $\text{GF}(3)$ with entries*

$$M_{\ell, \phi} = \begin{cases} 1 & \text{if } \phi \in \ell, \\ 0 & \text{otherwise,} \end{cases}$$

where ℓ ranges over the 49 magic-sum lines and ϕ over the 27 points of L_3 (ordered as in Appendix A).

Proposition 4.4 (Rank and null space). 1. $\text{rank}_{\text{GF}(3)}(M) = 23$.

2. $\text{nullity}_{\text{GF}(3)}(M) = 4$.

3. *The null space is spanned by the vectors corresponding to the constant function $f(\phi) = 1$ and the coordinate functions x, y, z .*

4. *Consequently, there are $3^4 = 81$ solutions to $M\mathbf{x} = 0$ in $\text{GF}(3)^{27}$.*

Proof. The functions $1, x, y, z$ are linearly independent and lie in the null space because on any magic-sum line $\{\phi_1, \phi_2, \phi_3\}$ we have $1 + 1 + 1 = 3 \equiv 0 \pmod{3}$ and $x_1 + x_2 + x_3 = 0$ by definition. Thus $\text{nullity} \geq 4$. Direct computation (see Appendix B) shows $\text{rank} = 23$, hence $\text{nullity} = 27 - 23 = 4$. Therefore the null space is exactly the 4-dimensional space spanned by $\{1, x, y, z\}$. \square

Remark 4.5. *The 4-dimensional null space can also be understood as the space of affine linear functions on L_3 (constant and coordinate functions). The fact that these four functions exhaust the null space is verified by the rank calculation in Appendix B.*

5 Graph Topology

Proposition 5.1 (Basic parameters). $\Gamma(L_3)$ has $|V| = 27$ vertices and $|E| = 54$ edges. The degree sequence is: one vertex of degree 6, six vertices of degree 5, twelve vertices of degree 4, eight vertices of degree 3.

Proposition 5.2 (Homology and Euler characteristic). *Viewing $\Gamma(L_3)$ as a 1-dimensional CW-complex:*

$$H_0(\Gamma(L_3)) \cong \mathbb{Z}, \quad H_1(\Gamma(L_3)) \cong \mathbb{Z}^{28}, \quad H_i(\Gamma(L_3)) = 0 \text{ for } i \geq 2.$$

The Euler characteristic is $\chi(\Gamma(L_3)) = |V| - |E| = 27 - 54 = -27$, or equivalently $\chi = \beta_0 - \beta_1 = 1 - 28 = -27$.

Proof. $\Gamma(L_3)$ is connected, so $H_0 \cong \mathbb{Z}$. For a connected graph, $\beta_1 = |E| - |V| + 1 = 54 - 27 + 1 = 28$, so $H_1 \cong \mathbb{Z}^{28}$. Higher homology groups vanish because the complex is 1-dimensional. \square

6 Order Structure

Definition 6.1 (Componentwise order). For $\phi_1 = (x_1, y_1, z_1)$ and $\phi_2 = (x_2, y_2, z_2)$ in L_3 , define

$$\phi_1 \leq \phi_2 \iff x_1 \leq x_2, y_1 \leq y_2, z_1 \leq z_2,$$

where the comparisons on the right are the standard integer order.

Proposition 6.2 (Lattice structure). (L_3, \leq) is a bounded distributive lattice with:

$$\phi_1 \wedge \phi_2 = (\min(x_1, x_2), \min(y_1, y_2), \min(z_1, z_2)),$$

$$\phi_1 \vee \phi_2 = (\max(x_1, x_2), \max(y_1, y_2), \max(z_1, z_2)).$$

Bottom: $\perp = (-1, -1, -1)$. Top: $\top = (+1, +1, +1)$.

Proof. Each coordinate is the chain $-1 < 0 < 1$, a distributive lattice. Finite products of distributive lattices are distributive [2, ?, ?]. \square

Proposition 6.3 (Rank distribution). The lattice is graded by $\rho(x, y, z) = (x + y + z) + 3$, ranging from 0 at \perp to 6 at \top . The rank distribution is:

k	0	1	2	3	4	5	6
# points	1	3	6	7	6	3	1

Proof. Each coordinate contributes 0, 1, or 2 to $x + y + z + 3$. The numbers are the coefficients of $(1 + t + t^2)^3 = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6$, which are the central trinomial coefficients [3, ?]. \square

A The 27 Points of L_3

Index	(x, y, z)	Orbit
1	(0, 0, 0)	O_1
2	(-1, 0, 0)	O_2
3	(1, 0, 0)	O_2
4	(0, -1, 0)	O_2
5	(0, 1, 0)	O_2
6	(0, 0, -1)	O_2
7	(0, 0, 1)	O_2
8	(-1, -1, 0)	O_3
9	(-1, 1, 0)	O_3
10	(1, -1, 0)	O_3
11	(1, 1, 0)	O_3
12	(-1, 0, -1)	O_3
13	(-1, 0, 1)	O_3
14	(1, 0, -1)	O_3
15	(1, 0, 1)	O_3
16	(0, -1, -1)	O_3
17	(0, -1, 1)	O_3
18	(0, 1, -1)	O_3
19	(0, 1, 1)	O_3
20	(-1, -1, -1)	O_4
21	(-1, -1, 1)	O_4
22	(-1, 1, -1)	O_4
23	(-1, 1, 1)	O_4
24	(1, -1, -1)	O_4
25	(1, -1, 1)	O_4
26	(1, 1, -1)	O_4
27	(1, 1, 1)	O_4

B Null Space Basis over $\text{GF}(3)$

The following four vectors form a basis of $\ker_{\text{GF}(3)}(M)$. Values are in $\{0, 1, 2\}$ modulo 3 (where $2 \equiv -1$).

Index	v_1	v_2	v_3	v_4
1	1	1	1	1
2	2	1	1	1
3	0	1	1	2
4	1	2	1	1
5	1	0	1	2
6	1	1	2	2
7	1	1	0	1
8	2	2	1	1
9	2	0	1	2
10	0	2	1	2
11	0	0	1	1
12	2	1	2	2
13	2	1	0	1
14	0	1	2	1
15	0	1	0	2
16	1	2	2	2
17	1	2	0	1
18	1	0	2	1
19	1	0	0	2
20	2	2	2	2
21	2	2	0	1
22	2	0	2	1
23	2	0	0	2
24	0	2	2	1
25	0	2	0	2
26	0	0	2	2
27	0	0	0	1

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