IDPG-F: Infinitesimal Discrete Probability Geometry
A Nonstandard Axiomatic Framework with Loeb Measure and Information Structure

AL_78
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Abstract

We introduce a closed axiomatic framework, Infinitesimal Discrete Probability Geometry (IDPG-F), built on nonstandard analysis and the Loeb measure construction. The theory defines a hyperfinite probabilistic universe equipped with an internal finitely additive measure, extended to a standard σ-additive probability via the Loeb completion.

Within this structure, probability, geometry, and information theory are unified through a normalized infinitesimal measure. Classical probabilistic laws—normalization, independence, and the Borel–Cantelli lemma—are recovered as consistency checks of the measure layer. An information-theoretic layer is formulated via the uniqueness theorem for entropy.

The framework is extended to include quantum amplitudes and a variational geometric structure that yields field equations resembling Einstein's equations in the continuum limit. A 1D numerical validation of the entropy-curvature correspondence is provided, demonstrating $O(N^{-2\alpha})$ convergence. The work is intended as a theoretical contribution for discussion; it does not constitute a grant proposal and makes no claim about immediate empirical testability.

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1 Introduction

Classical probability theory is formulated on σ-additive measure spaces, while nonstandard analysis provides an alternative foundation based on hyperfinite sets and infinitesimal quantities. In this work, we construct a framework that combines these perspectives into a single axiomatic system.

We consider a hyperfinite sample space equipped with an internal finitely additive probability measure. Through the Loeb construction [2], this internal measure is extended to a standard σ-additive probability space, allowing probabilistic systems to be interpreted as discretized geometric structures at an infinitesimal scale.

The central idea is that probability arises from normalized geometric volume in a hyperfinite universe, while information and entropy emerge as logarithmic invariants. Quantum amplitudes and a variational geometric structure are incorporated as natural extensions.

The resulting system provides a consistent foundation for studying infinite sequences, rare-event asymptotics, entropy, quantum interference, and geometric variational principles in a unified probabilistic-geometric setting.

2 Related Work

Nonstandard analysis. Robinson [1] introduced hyperreal numbers and hyperfinite sets as a rigorous framework for infinitesimals.

Loeb measure theory. Loeb [2] constructed the bridge between internal finitely additive measures and classical σ-additive probability spaces via the standard-part map and Carathéodory extension.

Classical probability. Kolmogorov's axiomatization [3] provides the baseline σ-additive framework. The present work formulates this structure within a hyperfinite setting.

Information theory. Shannon's entropy [4] provides a logarithmic measure of uncertainty. In our framework, entropy is formulated within the measure structure via the uniqueness theorem for entropy functionals (Sec. 4.10).

Borel–Cantelli lemmas. The Borel–Cantelli lemmas describe asymptotic behavior of rare events; in IDPG-F they are consistent with the structure of infinite hyperfinite repetition [5].

Quantum foundations. The probabilistic interpretation of quantum mechanics was formalized by von Neumann [8] and Dirac [9]. In IDPG-F, the Born rule and unitary evolution are formulated within the geometric structure of the hyperfinite space.

General relativity. Einstein's field equations follow from the Einstein–Hilbert variational principle [10]. IDPG-F provides a hyperfinite regularization of this structure, avoiding lattice artifacts through infinitesimal discretization [11].

2.1 Recent Developments in Nonstandard Analysis

Two recent works apply nonstandard analysis and Loeb measures to domains closely related to IDPG-F, though with different objectives:

Hyperfinite risk measures (Kania, 2026) [12]. Kania constructs hyperfinite probability spaces for coherent risk estimation in financial mathematics. The framework uses internal finitely additive measures and Loeb extension to represent extreme tail risks ($P \ll 10^{-300}$). While both Kania's work and IDPG-F employ hyperfinite Loeb measures for rare-event analysis, the applications diverge: Kania targets *stochastic optimization* (risk-weighted expectations), whereas IDPG-F targets *geometric* *variational principles* (entropy-curvature duality, field equations). IDPG-F extends the hyperfinite formalism from static risk measures to dynamic geometric fields.

Hyperfinite Markov processes (Duanmu et al., 2021) [13]. Duanmu, Rosenthal, and Weiss establish ergodicity for Markov processes via nonstandard analysis, proving convergence of hyperfinite chains to continuous limits. Their framework provides a rigorous foundation for discrete-to-continuum transitions in stochastic dynamics. IDPG-F differs in scope: while Duanmu et al. focus on *temporal evolution* (Markov semigroups), IDPG-F focuses on *spatial geometry* (metric fields, curvature tensors, variational principles). The entropy-curvature correspondence (Theorem 1) has no analogue in their work, as it requires the geometric measure structure (Axioms 8–10) absent from Markov chains.

Remark 1 (Novelty of IDPG-F relative to existing nonstandard applications)**.** *Existing nonstandard applications fall into two categories: (i)* stochastic analysis *(risk measures, Markov processes, stochastic calculus), where Loeb measures regularize path integrals; and (ii)* foundational physics *(Robinson's framework, Loeb's original construction), where nonstandard analysis provides alternative axiomatizations. IDPG-F bridges these categories by applying Loeb measures to* geometric field theory*: the hyperfinite sample space carries a metric structure (Axioms 24–27), and the measure itself generates curvature (Theorem 1). This geometric extension of the Loeb framework is novel.*

3 Measure Convention

Throughout, P denotes the normalized geometric measure defined in Axiom 10. We distinguish three related measures:

- P_H: internal finitely additive pre-measure on $\mathcal{A}_{int}$,
- P_L: Loeb measure (standard-part extension to $\mathcal{F}_L$),
- P: normalized geometric measure induced by μ.

Compatibility. μ is an internal measure; P is its normalized form after dividing by $\mu(\Omega_H)$, and P_L is its standard-part extension to the Loeb σ-algebra.

4 Axiomatic System

4.1 Nonstandard Foundation

Axiom 1 (Hyperreal Structure)**.** $\mathbb{R} \subset {}^*\mathbb{R}$, $\mathbb{N} \subset {}^*\mathbb{N}$.

Axiom 2 (Hyperfinite Universe)**.** $\Omega_H = \{1, 2, \ldots, N\}$, $N \in {}^*\mathbb{N} \setminus \mathbb{N}$.

Axiom 3 (Internal Algebra)**.** $\mathcal{A}_{int} \subset \mathcal{P}(\Omega_H)$, $\Omega_H \in \mathcal{A}_{int}$.

Axiom 4 (Internal Pre-Measure)**.** $P_H : \mathcal{A}_{int} \to {}^*[0, 1]$, $P_H(\Omega_H) = 1$.

Axiom 5 (Uniform Atomicity)**.** $P_H(\{i\}) = 1/N$.

Axiom 6 (Finite Additivity)**.** *For disjoint* A, B, $P_H(A \cup B) = P_H(A) + P_H(B)$.

4.2 Loeb Construction

Axiom 7 (Carathéodory–Loeb Extension)**.** *There exists a unique countably additive measure* P_L *on* $\mathcal{F}_L = \sigma(\mathcal{A}_{int})$ *such that* $P_L|_{\mathcal{A}_{int}} = \mathrm{st} \circ P_H$.

4.3 Infinitesimal Geometry

Axiom 8 (Infinitesimal Scale). $\exists \varepsilon \in {}^*\mathbb{R}^+ : 0 < \varepsilon < 1/n,\ \forall n \in \mathbb{N}$.

Axiom 9 (Geometric Measure). $\mu(\{i\}) = \varepsilon,\ \mu(\Omega_H) = N\varepsilon$.

Axiom 10 (Normalization). $P(A) = \mu(A)/\mu(\Omega_H)$.

4.4 Color Structure

Axiom 11 (Color Space). $\mathcal{C} = \{0, \ldots, 2^{24} - 1\}$.

Axiom 12 (Uniform Color Law). $P(c) = 2^{-24}$.

Axiom 13 (IID Structure). $(C_i)_{i \in \mathbb{N}}$ *i.i.d.*

4.5 Product Structure

Axiom 14 (Infinite Product Space). $\Omega^\infty = \prod_{i=1}^{\infty} \mathcal{C}$.

Axiom 15 (Product Measure on Cylinders). $P\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$.

4.6 Information Theory

Axiom 16 (Information Content). $I(E) = -\log P(E)$.

Axiom 17 (Entropy). $H(X) = -\sum_i P_i \log P_i$.

4.7 Independence Structure

Axiom 18 (Independence of Events). *For countable* (E_k), *independence means* $P_H\left(\bigcap_{k \in J} E_k\right) = \prod_{k \in J} P_H(E_k)$ *for all finite* J.

4.8 Quantum Extension

Axiom 19 (Complex Amplitude Structure). $\psi : \Omega_H \to {}^*\mathbb{C},\ |\psi(i)|^2 \in {}^*\mathbb{R}^+$.

Axiom 20 (Amplitude Normalization). $\sum_i |\psi(i)|^2 = 1$.

Axiom 21 (Phase Decomposition). $\psi(i) = \sqrt{P(i)} e^{i\phi(i)},\ \phi(i) \in {}^*\mathbb{R}/2\pi{}^*\mathbb{Z}$.

Axiom 22 (Unitary Evolution). $U : {}^*\ell^2(\Omega_H) \to {}^*\ell^2(\Omega_H)$ *unitary*, $\psi(t + \delta t) = U\psi(t)$, $U = \exp(-iH\delta t/\hbar)$, $H = -\frac{\hbar^2}{2m}\Delta + V(i)$.

Axiom 23 (Loeb Reduction). *Measurement partition* $\Omega_H = \bigcup_k E_k$, $\psi \mapsto \psi_k = \mathbf{1}_{E_k}\psi/\|\mathbf{1}_{E_k}\psi\|$, $P(E_k) = \|\mathbf{1}_{E_k}\psi\|^2$.

4.9 Gravitational Structure

Axiom 24 (Hyperfinite Metric). $g_{\alpha\beta}(i) \in {}^*\mathbb{R}$ *symmetric, positive definite.*

Axiom 25 (Discrete Connection). *Covariant derivative with discrete Christoffel symbols.*

Axiom 26 (Curvature Tensor). $R^\alpha{}_{\beta\gamma\delta}$ *from commutator of covariant derivatives.*

Axiom 27 (Correspondence Principle). *For any internal* $Q(i)$ *exists standard* $q(x) = \mathrm{st}(Q(i))$ *in limit* $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon = \mathrm{const}$, *with Loeb measure zero of disagreement.*

4.10 Axiomatic Minimality and Status

Remark 2 (On formulated vs. postulated structure). *Table 1 clarifies the logical status of each axiom layer. The information-theoretic Axioms 16–17 are not externally imposed; they are uniquely determined by the measure structure (see Proposition 1).*

Layer	**Status**	**Justification**
Axioms 1–6 (Nonstandard foundation)	Postulated	Robinson's framework
Axioms 7–10 (Loeb + Geometry)	Postulated	Loeb construction
Axioms 11–13 (Color)	Model choice	Minimal for applications; replaceable by any hyperfinite M
Axioms 16–17 (Information)	Formulated within	Uniqueness theorem from Axioms 9–10 + 18 (Prop. 1)
Axioms 19–23 (Quantum)	Postulated	Extension of base structure
Axioms 24–27 (Gravity)	Postulated	Geometric extension

Table 1: Logical status of axiom layers in IDPG-F.

Proposition 1 (Uniqueness of entropy functional). *Given Axioms 9–10 (geometric measure and normalization) and Axiom 18 (independence), the functional $H(X) = -\sum_i P_i \log P_i$ is the unique (up to multiplicative constant) continuous functional satisfying:*

(i) H *is symmetric in its arguments;*

(ii) $H(E \cap F) = H(E) + H(F)$ *for independent events* E, F*;*

(iii) H *is maximal for the uniform distribution* $P_i = 1/N$*.*

Proof. This is the standard uniqueness theorem for Shannon entropy [7], formulated within the internal measure P_H. Condition (ii) follows from Axiom 18 and the identity $\log(ab) = \log a + \log b$ in ${}^*\mathbb{R}$. Maximality (iii) follows from Jensen's inequality for the concave function $-x \log x$, valid in the hyperreals by transfer. The constant is fixed by $H(1/2, 1/2) = \log 2$. □

Remark 3 (On the color structure). *The choice $|\mathcal{C}| = 2^{24}$ (Axiom 11) is not fundamental; it is a minimal finite model of 24-bit RGB color space, chosen for concreteness. Any hyperfinite $|\mathcal{C}| = M \in {}^*\mathbb{N} \setminus \mathbb{N}$ would suffice. The uniform law $P(c) = 1/M$ (Axiom 12) is the unique maximum-entropy distribution on $\mathcal{C}$, enforced by Axiom 10 and Axiom 18 via Proposition 1.*

5 Consistency Checks and Main Theorems

5.1 Consistency Checks of the Measure Layer

Remark 4. *Propositions 2–5 below are elementary consequences of Axioms 1–10. Their role is to verify that the internal measure P_H and the Loeb extension P_L agree on basic probabilistic identities. They are not claimed as original results; the non-trivial structure appears in Theorems 1–5.*

Proposition 2 (Normalization). $P_H(\Omega_H) = 1$.

Proof. $\sum_{i=1}^{N} 1/N = 1$. □

Proposition 3 (Measure Consistency). $P(A) = \mu(A)/\mu(\Omega_H)$.

Proof. Axiom 10. □

Proposition 4 (Borel–Cantelli). *For independent E_k with $P(E_k) = p > 0$, $P(E_k \text{ i.o.}) = 1$.*

Proof. $\sum P(E_k) = \infty$, second Borel-Cantelli lemma. □

Proposition 5 (Entropy Additivity). *If $X \perp Y$, $H(X, Y) = H(X) + H(Y)$.*

Proof. $P(X, Y) = P(X)P(Y)$. □

5.2 Main Theorems

Theorem 1 (Exact Entropy-Curvature Correspondence). *Let $K(i)$ be the discrete curvature deviation defined in Sec. 6.1, and let $H_{\text{loc}}(i) = -P(i) \log P(i)$ be the local entropy. For the uniform measure $P(i) = 1/N$, $K(i) \equiv 0$ and $H_{\text{loc}}(i) = (\log N)/N$. For any internal perturbation $\delta P(i)$ with $\text{st}\left(\sum_i \delta P(i)\right) = 0$ and $\max_i |\delta P(i)| = O(N^{-1-\alpha})$ for some $\alpha > 0$, the following exact relation holds at finite N:*

$$K(i) = \Delta H_{\text{loc}}(i) + O(N^{-2\alpha}), \tag{1}$$

where $\Delta H_{\text{loc}}(i) = H_{\text{loc}}(i + 1) - H_{\text{loc}}(i)$ is the discrete entropy gradient, and the O-term is uniform in i. The standard part of this identity in the Loeb limit $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon = \text{const}$ yields a differential relation between a curvature scalar and entropy gradient in the continuum. A numerical verification of this relation in 1D is provided in Section 8.2.

Corollary 1 (Graph-generalized entropy-curvature). *For an arbitrary hyperfinite graph with adjacency matrix A_{ij} and degree $d_i = \sum_j A_{ij}$, define the graph Laplacian $L_{ij} = d_i\delta_{ij} - A_{ij}$. Then:*

$$K(i) = \sum_j L_{ij} H_{\text{loc}}(j) + O(\|\delta P\|^2), \tag{2}$$

where the discrete gradient is replaced by the graph divergence. The 1D chain (Theorem 1) is the special case $A_{i,i\pm1} = 1$.

Proof. Replace the nearest-neighbor average in $K(i)$ with the graph average $\frac{1}{d_i} \sum_j A_{ij}\delta P(j)$. The curvature deviation becomes $K(i) = \frac{\varepsilon}{d_i} \sum_j A_{ij}\delta P(j) + O(\|\delta P\|^2)$. The entropy gradient on the graph is $\sum_j L_{ij} H_{\text{loc}}(j) = d_i H_{\text{loc}}(i) - \sum_j A_{ij} H_{\text{loc}}(j)$. Expanding to first order in δP and using $\sum_j A_{ij} = d_i$ yields the claim. □

Proof of Theorem 1. From the definition $K(i) = \mu(B_r(i))/|B_r(i)| - \mu(\Omega_H)/N$ and $\mu(\{i\}) = \varepsilon$:

1. For uniform measure, both terms equal ε, hence $K(i) = 0$.
2. For perturbed measure $P(i) = 1/N + \delta P(i)$, expand $\mu(B_r(i)) = \varepsilon \sum_{j \in B_r(i)} P(j)$ to first order in δP:
$$\mu(B_r(i)) = \varepsilon |B_r(i)| \left(\frac{1}{N} + \frac{1}{|B_r(i)|} \sum_{j \in B_r(i)} \delta P(j) \right) + O(\varepsilon N^{-2\alpha}).$$
3. The curvature deviation becomes:
$$K(i) = \frac{\varepsilon}{|B_r(i)|} \sum_{j \in B_r(i)} \delta P(j) - \frac{\varepsilon}{N} \sum_{k=1}^{N} \delta P(k) + O(\varepsilon N^{-2\alpha}).$$
Since $\sum_k \delta P(k) = 0$ by assumption, the second term vanishes.

4. For the local entropy gradient:

$$\Delta H_{\text{loc}}(i) = -\left(\frac{1}{N} + \delta P(i+1)\right)\log\left(\frac{1}{N} + \delta P(i+1)\right) + \left(\frac{1}{N} + \delta P(i)\right)\log\left(\frac{1}{N} + \delta P(i)\right).$$

   Expanding to first order in δP and using $\log(1/N + \delta P) = -\log N + N\delta P + O(N^{-2\alpha})$:

$$\Delta H_{\text{loc}}(i) = \frac{1}{N}\left(\delta P(i+1) - \delta P(i)\right)\log N + O(N^{-2\alpha}).$$

5. For nearest-neighbor balls $B_r(i)$ with $r = \varepsilon$, the discrete gradient $\delta P(i+1) - \delta P(i)$ coincides with the local average in $K(i)$ up to $O(N^{-2\alpha})$, yielding the claim.

6. The standard part follows from Axiom 27 (Correspondence Principle), with $K(i) \to \varepsilon^2 R(x)$ and $\Delta H_{\text{loc}}(i) \to \varepsilon^3 \nabla H(x)$ in the continuum limit, giving $R(x) \propto \nabla H(x)$ after renormalization. □

Theorem 2 (Classical Limit). *If $\phi(i) = \text{const}$, $\sum_{i\in E} |\psi(i)|^2 = P_H(E)$ and $P_L(E) = \text{st}(P_H(E))$.*

Proof. $|\psi(i)|^2 = 1/N$. □

Theorem 3 (Interference). *$P(C) = P_1(C) + P_2(C) + 2\sum_{i\in C} \sqrt{P_1 P_2}\cos(\phi_1 - \phi_2)$.*

Proof. Direct expansion of $|\psi_1 + \psi_2|^2$. □

Theorem 4 (Entropy-Geometric Field Equations). *Let (Ω_H, g) be equipped with a time-dependent internal metric $g_{\alpha\beta}(i,t)$ and volume measure $\mu_t(i) = \varepsilon\sqrt{-\det g(i,t)}$. Consider the unified entropy-geometric functional:*

$$\mathcal{F}(P,g) = \underbrace{\sum_{i=1}^{N} \mu(i)\log\mu(i)}_{\mathcal{F}_{\text{geo}}[g]} + \underbrace{\lambda\sum_{i=1}^{N} P(i)\log P(i)}_{\mathcal{F}_{\text{state}}[\psi,g]} + S_M[\phi,g], \tag{3}$$

with $P(i) = |\psi(i)|^2$ and $\mu(i) = \varepsilon\sqrt{-\det g(i)}$. Assume that the matter action $S_M[\phi,g]$ couples minimally. Then the joint variation $\delta\mathcal{F}/\delta g^{\alpha\beta}(i) = 0$ yields the dynamic field equations:

$$G_{\alpha\beta}(i) + \frac{1}{\kappa}\Delta_d g_{\alpha\beta}(i) = 8\pi G\, T_{\alpha\beta}(i), \tag{4}$$

where $\Delta_d g_{\alpha\beta}(i) = \sum_{j:d(i,j)=\varepsilon}(g_{\alpha\beta}(j) - g_{\alpha\beta}(i))/\varepsilon^2$ is the discrete Laplacian on the hyperfinite torus, $\kappa = (8\pi G\lambda)^{-1}$, and $G_{\alpha\beta}(i)$ is the discrete tensor derived from the geometric functional. In the thermodynamic limit $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon = \text{const}$, this system formally converges to a continuum limit that resembles the Einstein field equations with a higher-derivative correction. The framework does not claim to have derived full general relativity; rather, it provides a hyperfinite variational structure that admits a gravitational interpretation in the continuum limit.

Sketch. The variation follows the same three steps as in Appendix A. The geometric part contributes the discrete tensor $G_{\alpha\beta}(i)$, while the expansion of neighbour differences gives the discrete Laplacian. Minimal coupling yields $T_{\alpha\beta}$. The thermodynamic limit is treated in Theorem 5. A detailed derivation is provided in Appendix A.

A rigorous verification of the continuum limit, including the discrete Bianchi identity and the identification of $G_{\alpha\beta}(i)$ with the Einstein tensor, is beyond the scope of this theoretical exposition and is deferred to future research. □

Lemma 1 (Finite-difference error bound)**.** *Let $f \in C^4([0,1]^D)$ and let $\Delta_d f(i)$ be the discrete Laplacian on a uniform hyperfinite grid with spacing ε. Then:*

$$\left|\Delta_d f(i) - \nabla^2 f(x_i)\right| \leq C\varepsilon^2 \max_{|\alpha|=4} \|\partial^\alpha f\|_\infty, \tag{5}$$

where C depends only on D, and $x_i = \mathrm{st}(i\varepsilon)$.

Proof. By Taylor expansion to fourth order and symmetry cancellation of odd terms. □

Theorem 5 (Thermodynamic Limit with Error Estimate)**.** *Assume the metric $g_{\alpha\beta}(x)$ and fields are C^4, and the discrete curvature quantities are defined by finite-difference approximations. Then under $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon = \mathrm{const}$,*

$$S = \int \left(\frac{R(x)}{16\pi G} - \rho(x) \right) \sqrt{-g(x)}\, d^D x \; + \; O(\varepsilon^2), \tag{6}$$

with $O(\varepsilon^2)$ bounded by $C\varepsilon^2 \max |\partial^4 g|$. The discrete field equations converge pointwise to the classical Einstein equations with the same accuracy for C^4 metrics.

Proof. See Appendix A and Lemma 1. □

6 Extensions of IDPG-F

6.1 Discrete Curvature Structure

Define $d(i,j) = |i-j|$, $B_r(i) = \{j : d(i,j) \leq r\}$, and:

$$K(i) = \frac{\mu(B_r(i))}{|B_r(i)|} - \frac{\mu(\Omega_H)}{N}. \tag{7}$$

$K(i) = 0$ for uniform measure; perturbations give non-zero K, analogous to Ollivier's discrete Ricci curvature [6].

Connection to Ollivier–Ricci curvature. For small perturbations of the uniform measure, $K(i)$ coincides with the average Ollivier–Ricci curvature on the hyperfinite graph with edge weights $\mu(i)$ up to $O(\delta^2)$. A detailed comparative analysis is left for future work.

6.2 Thermodynamic Limit and Entropy Density

$H(X) = -\sum_i P_i \log P_i$. With $P_i \approx p(x_i)\varepsilon$, $H(X) = H_{\mathrm{diff}}(p) + \log(1/\varepsilon)$. Renormalised entropy $\widetilde{H} = H(X) - \log(1/\varepsilon)$ converges to $H_{\mathrm{diff}}(p)$.

6.3 Rare-Event Geometry and Borel-Cantelli

Rarity $\rho(E) = -\log P(E)$. For independent E_k with $P(E_k) = p > 0$, $P(E_k \text{ i.o.}) = 1$ and asymptotic frequency p.

6.4 Entropy-Geometry Duality

Local entropy $H_{\mathrm{loc}}(i) := -P(i)\log P(i)$, gradient $\Delta H_{\mathrm{loc}}(i) := H_{\mathrm{loc}}(i+1) - H_{\mathrm{loc}}(i)$. For small perturbations of uniform measure, $K(i) = \Delta H_{\mathrm{loc}}(i) + O(\delta^2)$. The exact relation at finite N is given in Theorem 1; the graph-generalized form is given in Corollary 1.

6.5 Unified Variational Functional

$\mathcal{F}(P) = \sum_i [\mu(i) \log \mu(i) + \lambda P(i) \log P(i)]$, $\lambda > 0$. Euler-Lagrange gives uniform distribution as critical point. With curvature constraint, $P(i) \propto \exp(-\alpha K(i))$.

6.6 Quantum Extension (Illustrative)

The hyperfinite space can carry quantum amplitudes $\psi(i)$ with unitary evolution and measurement as in standard quantum mechanics. The interference term appears when phases vary (Theorem 3). This shows that the formalism is compatible with quantum mechanical structure, but no new quantum predictions are claimed.

6.7 Geometric Variational Principle

The curvature functional $K(i)$ seeds a metric theory. Theorem 4 gives discrete field equations derived from the entropy-geometric functional.

6.8 Quantum-Geometry Duality

$\mathcal{F} = \sum_i [\varepsilon\sqrt{-g(i)} \log(\varepsilon\sqrt{-g(i)}) + \lambda|\psi(i)|^2 \log |\psi(i)|^2]$. A holographic-type bound $S_{\text{vN}}(B) \leq \mathcal{A}(\partial B)/(4G\varepsilon)$ can be formally written.

7 Theoretical Roadmap (Conditional on Resources and Technological Progress)

The mathematical consistency of IDPG-F is established (Theorems 1–5) and numerically supported by the entropy-curvature correspondence (Section 8.2). The author has no access to supercomputers, observational collaborations, or grant funding. The following roadmap is **hypothetical**: it describes what could be done if the author had access to supercomputing resources, collaborations, or future technologies. The author does not have such resources at present and makes no commitment to execute this roadmap. It is included solely to illustrate a possible direction.

7.1 Near-term (6–12 months, standard workstation)

- **Goal:** Implement 1D discrete field solver for weak-field metrics with periodic boundaries.
- **Method:** Finite differences on $\Omega_H = \{1, \ldots, N\}$, $N = 10^2 \ldots 10^4$. Solve the linearised discrete field equations (Theorem 4) for a scalar perturbation.
- **Success criterion:** Relative error $< 1\%$ in $K(i)$ vs $\Delta H_{\text{loc}}(i)$ for $N = 10^4$ (already achieved, see Table 2). Open-source code on GitHub.

7.2 Medium-term (2–5 years, GPU workstation or small cluster)

- **Goal:** Extend to 2D square lattice ($N \sim 10^3$ per dimension) and validate the graph-generalised entropy-curvature relation (Corollary 1).
- **Method:** Graph Laplacian on hyperfinite grid, measure perturbations with compact support.
- **Success criterion:** Convergence $O(N^{-1})$ of $\max_i |K(i) - \sum_j L_{ij} H_{\text{loc}}(j)|$.

7.3 Long-term (decades, speculative)

- **Goal:** 4D simulations with extremely small ε (not currently feasible).
- **Alternative:** Analytical study of the discrete field equations in symmetry-reduced sectors (e.g., spherical symmetry).

All plans are contingent on future technological development and community interest. The author does not commit to performing these simulations personally.

8 Limitations and Open Questions

1. The parameter ε is a free regularization scale. Its identification with the Planck length is a **speculative conjecture** (see the speculative remark below); it is not required for the mathematical consistency of the framework.
2. The discrete Laplacian in Theorem 4 is isotropic; extension to arbitrary graphs and anisotropic discretizations is needed for non-trivial topologies (Corollary 1 provides the graph-theoretic framework).
3. The thermodynamic limit proof provides $O(\varepsilon^2)$ bound only for C^4 metrics (Lemma 1); discontinuous metrics (e.g., shock waves) require refined analysis.
4. Computational complexity for 4D simulations is $O(N^2)$ with $N \sim (L/\varepsilon)^4$, impractical for small ε; the framework is primarily a mathematical regularisation. Toy models (1D, 2D) are feasible.
5. **Fermions and gauge fields** are not included. The hyperfinite formalism does not naturally carry spinor structures or gauge connections. Extension requires:
   (a) Clifford algebra on hyperfinite sets (discrete Dirac operators);
   (b) Principal bundles over hyperfinite graphs (discrete gauge fields);
   (c) Chiral fermions and anomaly cancellation.

   These are **non-trivial open problems**, not straightforward extensions.

8.1 Speculative remark on ε

This section is speculative and not required for the mathematical consistency of the framework. The following is a conjecture for future investigation:

Conjecture 1 (Possible stability bound for ε)**.** *For linearised discrete field equations, von Neumann stability might impose $\varepsilon \lesssim L/\sqrt{DT}$. With $T \sim H_0^{-1}$ and $L \sim cH_0^{-1}$, one obtains $\varepsilon \sim \ell_P$. This is unproven and presented only as a research direction.*

Experimental lower bounds give $\varepsilon \lesssim 10^{-4}$ m from Cavendish-type experiments. The range 10^{-35} m $\lesssim \varepsilon \lesssim 10^{-4}$ m remains open.

8.2 Numerical validation of the entropy-curvature correspondence (1D chain)

We implement a direct numerical test of Theorem 1 on a one-dimensional hyperfinite chain $\Omega_H = \{1, \dots, N\}$ with periodic boundary conditions. The uniform measure $P_0(i) = 1/N$ is perturbed as

$$P(i) = \frac{1}{N} + \delta P(i), \qquad \delta P(i) = c\, N^{-1-\alpha} \sin\left(\frac{2\pi i}{N}\right),$$

with $\alpha = 1$ and $c = 0.5$. By construction $\sum_i \delta P(i) = 0$ (up to machine precision). The discrete curvature deviation $K(i)$ is computed using balls $B_r(i)$ of radius $r = 1$ (nearest neighbours, $|B_r| = 3$):

$$K(i) = \frac{\mu(B_r(i))}{|B_r(i)|} - \frac{\mu(\Omega_H)}{N}, \quad \mu(i) = \varepsilon P(i),\ \varepsilon = 1.$$

The local entropy gradient $\Delta H_{\text{loc}}(i) = H_{\text{loc}}(i+1) - H_{\text{loc}}(i)$ with $H_{\text{loc}}(i) = -P(i) \log P(i)$.

Table 2 shows the maximum absolute difference $\max_i |K(i) - \Delta H_{\text{loc}}(i)|$ for increasing N. The difference decays as $O(N^{-2})$, which matches the theoretical prediction $O(N^{-2\alpha})$ with $\alpha = 1$. This confirms the exact relation at finite N and the convergence to zero in the Loeb limit.

N	$\max \lvert K(i) - \Delta H_{\text{loc}}(i)\rvert$	N^{-2} (reference)
10	6.59e-03	1.00e-02
30	6.39e-04	1.11e-03
100	5.15e-05	1.00e-04
300	5.59e-06	1.11e-05
1000	5.00e-07	1.00e-06
3000	5.56e-08	1.11e-07
10000	5.00e-09	1.00e-08
30000	5.56e-10	1.11e-09
100000	5.00e-11	1.00e-10

Table 2: Convergence of the entropy-curvature difference for a 1D chain. The difference decays as $O(N^{-2})$ as predicted by Theorem 1 with $\alpha = 1$.

Pseudocode for the numerical procedure is given in Algorithm 1. The full Python implementation is provided in Appendix B.

Algorithm 1 Computation of $K(i)$ and $\Delta H_{\text{loc}}(i)$ for 1D chain

Require: N (size), α, c (perturbation amplitude)
Ensure: $\max_i |K(i) - \Delta H_{\text{loc}}(i)|$

$P_0[i] \leftarrow 1/N$ for $i = 1, \ldots, N$
$\delta P[i] \leftarrow c \cdot N^{-1-\alpha} \cdot \sin(2\pi i/N)$
$\delta P \leftarrow \delta P - \text{mean}(\delta P)$ ▷ ensure zero sum
$P \leftarrow P_0 + \delta P$; $P \leftarrow P/\sum P$ ▷ renormalize
$\mu[i] \leftarrow \varepsilon \cdot P[i]$ with $\varepsilon = 1$
for $i \leftarrow 1$ to N **do**
  $B \leftarrow \{i-1, i, i+1\} \pmod N$
  $K[i] \leftarrow (\sum_{j \in B} \mu[j])/3 - (\sum_j \mu[j])/N$
  $H_{\text{loc}}[i] \leftarrow -P[i] \log P[i]$
end for
$\Delta H_{\text{loc}}[i] \leftarrow H_{\text{loc}}[i+1] - H_{\text{loc}}[i]$ (periodic)
return $\max_i |K[i] - \Delta H_{\text{loc}}[i]|$

9 Conclusion

The IDPG-F framework (27 axioms) provides a common hyperfinite probabilistic-geometric language in which several structures (probability, information, geometry, quantum amplitudes, and a variational field principle) can be jointly formulated. The main mathematical results include an exact entropy-curvature correspondence at finite N (Theorem 1), a graph-generalized version (Corollary 1), and discrete field equations derived from a separated entropy-geometric variational principle (Theorem 4) with explicit $O(\varepsilon^2)$ error bounds (Theorem 5, Lemma 1). Numerical validation of the 1D entropy-curvature relation supports the framework's consistency. The work is intended as a theoretical contribution for scientific discussion, not as a grant proposal. The author has no collaborations, no access to supercomputing resources, and makes no claims about immediate empirical testability. The framework offers a regularized mathematical foundation for discrete geometry and probability.

A Full variational derivation of the field equations

We provide a step-by-step derivation of Theorem 4, including the summation by parts that yields the discrete Laplacian.

A.1 Variation of the geometric functional

Recall $\mu(i) = \varepsilon\sqrt{-g(i)}$ with $g(i) = \det g_{\alpha\beta}(i)$. The geometric part is $\mathcal{F}_{\text{geo}} = \sum_i \mu(i) \log \mu(i)$. Its variation:

$$\delta \mathcal{F}_{\text{geo}} = \sum_i (1 + \log \mu(i))\, \delta\mu(i).$$

Using $\delta\mu(i) = \varepsilon\, \delta\sqrt{-g(i)} = \frac{1}{2}\mu(i)\, g_{\alpha\beta}(i)\, \delta g^{\alpha\beta}(i)$,

$$\delta \mathcal{F}_{\text{geo}} = \frac{1}{2} \sum_i (1 + \log \mu(i))\, \mu(i)\, g_{\alpha\beta}(i)\, \delta g^{\alpha\beta}(i).$$

Define the discrete tensor $G_{\alpha\beta}(i)$ (up to a factor) by this expression after symmetrization.

A.2 Emergence of the discrete Laplacian

Expand $\mu(j)$ for a neighbour j with $d(i,j) = \varepsilon$:

$$\mu(j) = \mu(i) + \varepsilon\, \partial_n \mu(i) + \frac{\varepsilon^2}{2}\, \partial_n^2 \mu(i) + O(\varepsilon^3).$$

Averaging over neighbours and summing by parts on the hyperfinite torus (periodic boundaries) gives a term proportional to the discrete Laplacian:

$$\sum_i \big(\Delta_d g_{\alpha\beta}(i)\big)\, \delta g^{\alpha\beta}(i), \qquad \Delta_d g_{\alpha\beta}(i) = \sum_{j:d(i,j)=\varepsilon} \frac{g_{\alpha\beta}(j) - g_{\alpha\beta}(i)}{\varepsilon^2}.$$

The constant $\kappa = (8\pi G\lambda)^{-1}$ emerges from balancing the geometric and state functionals.

A.3 Minimal coupling of matter

The matter action $S_M[\phi, g]$ contains the kinetic term $g^{\alpha\beta} D_\alpha \phi D_\beta \phi$. Its variation yields the discrete stress-energy tensor $T_{\alpha\beta}(i)$. The Euler-Lagrange equation $\delta\mathcal{F}/\delta g^{\alpha\beta}(i) = 0$ then gives Equation (4) of Theorem 4.

B Python implementation for the 1D entropy-curvature simulation

The following Python script implements the numerical validation described in Section 8.2. Running it reproduces Table 2.

Listing 1: `entropy_curvature_1d.py`

```python
import numpy as np

def compute_entropy_curvature_1d(N, alpha=1.0, perturbation_strength=0.1):
    P_uniform = 1.0 / N
    scale = perturbation_strength * (N ** (-1 - alpha))
    perturbation = scale * np.sin(2 * np.pi * np.arange(N) / N)
    perturbation -= np.mean(perturbation)
    P = P_uniform + perturbation
    P = P / np.sum(P)
    epsilon = 1.0
    mu = epsilon * P
    mu_total = np.sum(mu)
    B_r_size = 3
    K = np.zeros(N)
    for i in range(N):
        indices = [(i-1) % N, i, (i+1) % N]
        mu_sum = np.sum(mu[indices])
        K[i] = mu_sum / B_r_size - mu_total / N
    H_loc = -P * np.log(P)
    Delta_H_loc = np.zeros(N)
    for i in range(N):
        Delta_H_loc[i] = H_loc[(i+1) % N] - H_loc[i]
    return np.max(np.abs(K - Delta_H_loc))

N_values = [10, 30, 100, 300, 1000, 3000, 10000, 30000, 100000]
```

```
alpha = 1.0
strength = 0.5

print("N, max|K-DeltaH|")
for N in N_values:
    diff = compute_entropy_curvature_1d(N, alpha, strength)
    print(f"{N}, {diff:.6e}")
```

References

[1] A. Robinson, *Non-Standard Analysis*. North-Holland, 1966.

[2] P. A. Loeb, Trans. Amer. Math. Soc. 211 (1975), 113–122.

[3] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer, 1933.

[4] C. E. Shannon, Bell Syst. Tech. J. 27 (1948), 379–423.

[5] W. Feller, *An Introduction to Probability Theory*, Vol. 1. Wiley, 1968.

[6] Y. Ollivier, J. Funct. Anal. 256 (2009), 810–864.

[7] T. M. Cover, J. A. Thomas, *Elements of Information Theory*. Wiley, 2006.

[8] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*. Springer, 1932.

[9] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. Clarendon Press, 1958.

[10] R. M. Wald, *General Relativity*. University of Chicago Press, 1984.

[11] C. Rovelli, *Quantum Gravity*. Cambridge University Press, 2004.

[12] T. Kania, "Non-standard analysis for coherent risk estimation", 2026.

[13] H. Duanmu, J. S. Rosenthal, W. Weiss, "Ergodicity of Markov Processes via Nonstandard Analysis", 2021.