

# The Whole Is a Function, the Parts Are Subfunctions

—A Rigorous Proof in the Framework of Category Theory

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## Abstract

We present a rigorous proof, entirely within the framework of category theory, of the proposition that “the whole is a function, the parts are subfunctions”. Using the Yoneda embedding, any object  $X$  (the “whole”) is identified with its representable functor  $h_X = \text{Hom}(X, -)$ , which can be viewed as a “function”. For a subobject  $i : A \hookrightarrow X$  (the “part”), the embedding induces a natural transformation  $\text{res} : h_X \rightarrow h_A$  whose components are restriction maps  $f \mapsto f \circ i$ . Thus the part corresponds to the functor  $h_A$  and the restriction natural transformation precisely expresses that the part is a subfunction of the whole. This result is independent of any specific mathematical structure and reveals the fundamental functorial nature of the whole–part relation.

**Keywords:** whole and part; Yoneda lemma; subobject; natural transformation; category theory

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# 1 Introduction

The relationship between a whole and its parts has been a central theme in philosophy and science for millennia. Reductionism attempts to explain the whole as a mere sum of its parts, while holism insists that the whole possesses emergent properties not reducible to its parts. Both views, however, often presuppose that parts can exist independently of the whole. In this note we adopt a purely categorical perspective and show that, in any category, the whole is naturally represented as a *function* (a representable functor) and any part appears as a *subfunction* (a natural transformation induced by a monomorphism). The proof is elementary, relying only on the Yoneda lemma and basic properties of subobjects. This categorical formulation provides a rigorous foundation for the philosophical thesis and opens the way to quantitative refinements in more concrete settings such as the Zhu-Liang universal recursive element system.

## 2 Preliminaries from Category Theory

We work in an arbitrary category  $\mathcal{C}$ . Objects are denoted by  $X, Y, Z, \dots$ . For any two objects  $X, Y$ , the set of morphisms from  $X$  to  $Y$  is written  $\text{Hom}(X, Y)$ . Composition is associative and identity morphisms exist.

**Definition 2.1** (Subobject). *A subobject of an object  $X$  is an equivalence class of monomorphisms  $i : A \hookrightarrow X$ , where two monomorphisms  $i : A \hookrightarrow X$  and  $i' : A' \hookrightarrow X$  are equivalent if there exists an isomorphism  $\phi : A \rightarrow A'$  such that  $i' \circ \phi = i$ . A monomorphism  $i$  is a morphism satisfying left cancellation:  $i \circ g = i \circ h \Rightarrow g = h$ .*

### 2.1 Yoneda Embedding

For each object  $X$  of  $\mathcal{C}$ , define the functor

$$h_X = \text{Hom}(X, -) : \mathcal{C} \rightarrow \mathbf{Set}, \quad Y \mapsto \text{Hom}(X, Y).$$

For a morphism  $g : Y \rightarrow Z$ ,  $h_X(g) : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$  is given by composition  $f \mapsto g \circ f$ . The assignment  $X \mapsto h_X$  extends to a functor  $y : \mathcal{C} \rightarrow [\mathcal{C}, \mathbf{Set}]$ , the Yoneda embedding. The Yoneda lemma states that for any functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ ,

$$\text{Nat}(h_X, F) \cong F(X),$$

naturally in  $X$  and  $F$ . In particular, the Yoneda embedding is fully faithful:  $\text{Hom}(X, Y) \cong \text{Nat}(h_X, h_Y)$ . Hence the object  $X$  is determined (up to isomorphism) by the functor  $h_X$ .

## 3 The Whole as a Function

By the Yoneda embedding, we may identify the “whole” object  $X$  with the functor  $h_X$ . This functor takes any object  $Y$  to the set of all morphisms from  $X$  to  $Y$ , which can be thought of as the “functions” from the whole to  $Y$ . Thus the whole is represented as a function (in fact a functor) that encodes the entire relational network of  $X$  with all other objects.

**Proposition 3.1** (Whole as a function). *In any category  $\mathcal{C}$ , the object  $X$  is isomorphic to the functor  $h_X = \text{Hom}(X, -)$  via the Yoneda embedding. Hence the whole is equivalently a functor (a “function” in the categorical sense).*

## 4 Parts as Subfunctions

Let  $i : A \hookrightarrow X$  be a monomorphism representing a subobject of  $X$  (a “part”). For each object  $Y$ , define a restriction map

$$\text{res}_Y : \text{Hom}(X, Y) \longrightarrow \text{Hom}(A, Y), \quad f \longmapsto f \circ i.$$

These maps are natural in  $Y$ : for any morphism  $g : Y \rightarrow Z$ , the diagram

$$\begin{array}{ccc} \text{Hom}(X, Y) & \xrightarrow{\text{res}_Y} & \text{Hom}(A, Y) \\ g \circ - \downarrow & & \downarrow g \circ - \\ \text{Hom}(X, Z) & \xrightarrow{\text{res}_Z} & \text{Hom}(A, Z) \end{array}$$

commutes because  $(g \circ f) \circ i = g \circ (f \circ i)$ . Hence  $\text{res} : h_X \rightarrow h_A$  is a natural transformation between the representable functors  $h_X$  and  $h_A$ .

**Definition 4.1** (Subfunction). *Given the whole  $X$  and a subobject  $i : A \hookrightarrow X$ , the natural transformation  $\text{res} : h_X \rightarrow h_A$  is called the subfunction induced by  $A$ .*

Thus the part  $A$  corresponds to the functor  $h_A$ , and the embedding  $i$  yields a restriction natural transformation that takes any morphism from the whole  $X$  to  $Y$  and sends it to its restriction to the part  $A$ . In this sense, the part is precisely a “subfunction” of the whole function  $h_X$ .

## 5 The Rigorous Proof

We now assemble the above observations into a concise proof.

**Theorem 5.1** (The Whole Is a Function, the Parts Are Subfunctions). *In any category  $\mathcal{C}$ :*

1. *The whole object  $X$  is represented by the functor  $h_X = \text{Hom}(X, -)$ , i.e., the whole is a function (a representable functor).*
2. *For any subobject  $i : A \hookrightarrow X$  (a part), there exists a natural transformation  $\text{res} : h_X \rightarrow h_A$  whose component at  $Y$  sends a morphism  $f : X \rightarrow Y$  to its restriction  $f \circ i$ .*
3. *Consequently, the part  $A$  corresponds to the functor  $h_A$ , and the natural transformation  $\text{res}$  expresses that the part is a subfunction of the whole.*

*Proof.* 1. By the Yoneda embedding, the functor  $y : \mathcal{C} \rightarrow [\mathcal{C}, \mathbf{Set}]$  is fully faithful, so  $X \cong h_X$  as objects in the functor category. Thus the whole is identified with the function  $h_X$ .

2. Given a monomorphism  $i : A \hookrightarrow X$ , define  $\text{res}_Y : \text{Hom}(X, Y) \rightarrow \text{Hom}(A, Y)$  by  $\text{res}_Y(f) = f \circ i$ . The family  $\{\text{res}_Y\}_{Y \in \mathcal{C}}$  is natural as shown above; hence  $\text{res} : h_X \rightarrow h_A$  is a natural transformation.

3. The functor  $h_A$  is exactly the image of the part  $A$  under the Yoneda embedding. The natural transformation  $\text{res}$  provides the restriction operation, so the part behaves as a subfunction of  $h_X$ .

□

## 6 Example: The Category of Sets

In the category **Set** of sets and functions, the theorem takes a familiar form:

- The whole is a set  $X$ .
- $h_X(Y) = \text{Hom}(X, Y)$  is the set of all functions from  $X$  to  $Y$ .
- A subobject  $i : A \hookrightarrow X$  is simply a subset  $A \subseteq X$  with the inclusion map.
- For any set  $Y$ ,  $\text{res}_Y(f) = f|_A$ , the restriction of the function  $f : X \rightarrow Y$  to  $A$ .

Thus the whole is the function  $h_X$ , and a part  $A$  gives the subfunction  $h_A$  together with the restriction natural transformation.

## 7 Discussion and Philosophical Implications

The proof is purely categorical and holds in *any* category, regardless of additional structure. It shows that the relation “whole – part” is fundamentally *functorial*: the whole is a representable functor, and a part is a natural transformation induced by a monomorphism. This shifts the ontological perspective from a static “element–subset” paradigm to a dynamic “function–subfunction” paradigm. The whole is not a container of elements but a relational network encoded by its morphisms; the part is not an independent atom but a restriction of that network to a subobject.

The categorical formulation provides a qualitative foundation that can be refined in more concrete settings. For instance, in the Zhu-Liang universal recursive element system [2, 3], this abstract framework is instantiated with the truth space  $\Omega$  (a terminal coalgebra) and truth functions  $h_A : A \rightarrow \Omega$ . The recursive nesting of projections  $x_n$  gives a quantitative measure of how parts are related to the whole via the hierarchical metric  $d_\Omega(x, y) = 2^{-k}$ , and the Calamity Axiom A5 introduces entropic weights and the notion of emergent depth.

Thus the categorical proof establishes the *form* of the whole–part relation, while the Zhu-Liang system provides its *substance*—making the abstract functorial relation computable and applicable to cognition, intelligence, and civilization theory.

## 8 Conclusion

We have given a rigorous proof, entirely within category theory, that the whole is a function (a representable functor) and its parts are subfunctions (natural transformations induced by subobjects). The proof is elementary, relying only on the Yoneda lemma and the definition of subobjects. It is valid in any category and captures the essential structural relation between a whole and its parts. This categorical foundation not only clarifies the philosophical debate but also serves as a solid basis for more refined theories, such as the Zhu-Liang universal recursive element system, where the abstract functions are concretized as truth functions and the natural transformations become measurable restrictions.

The whole is a function, the parts are subfunctions.

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## Conflict of Interest

The author declares no conflict of interest.

## Data Availability

This paper is a purely theoretical exposition and does not involve experimental data.

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