Kochen–Specker Contextuality and the Tetralemmatic Calculus of Quantum Mechanics

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March 17, 2026

Abstract

The contextual character of quantum observables—formalized by the Kochen–Specker theorem and Bohr's complementarity—represents a fundamental rupture in classical binary logic. In interference regimes, such as the double-slit experiment, polar questions (e.g., "Which path did the particle take?") are operationally unaskable, yet standard logical frameworks fail to formalize this unaskability without collapsing into truth-value gaps or triviality. We resolve this by introducing a context-indexed formal logic that adjoins an explicit *applicability predicate* to the valuation space.

Drawing structural inspiration from the ancient *catuṣkoṭi* (tetralemma), we map quantum operational states onto four distinct logical sectors: *Thesis/Antithesis* (strong measurement and decoherence), *Synthesis* (paraconsistent gluts realized via weak measurement), and *Holothesis* (paracomplete gaps representing typed inapplicability). From this logical foundation, we generate a $*$-algebra equipped with a positive warrant functional. Through the Gelfand–Naimark–Segal (GNS) construction, we demonstrate that Hilbert-space structure—and the complex unit i itself—emerge canonically from an order-4 logical symmetry rather than being presupposed. Contextuality is quantified via incompatibility graphs, with chromatic number bounding minimal classical embeddings.

We establish a Noether-like correspondence wherein physical symmetry breaking dictates logical polarity, while symmetry preservation dictates logical transcendence (self-duality). This *logic-first reconstruction* unifies Relational Quantum Mechanics (Rovelli) and Consistent Histories (Griffiths, Omnès, Gell-Mann, Hartle) as regime-relative logical stances within a pre-geometric ontology of *interdependent origination* (contextual co-arising of observer and observable), where the observer's contextual frame and the system's observable reality are co-arising and non-separable. The framework yields testable deviations in many-body interference ($I_3 \neq 0$), providing an empirical pathway for detection.

Contents

Preface

Modern quantum theory rests on a mathematical structure of remarkable predictive power, yet its logical foundations remain incomplete. The Kochen–Specker theorem and the operational status of "unaskable" questions expose the fundamental limitations of classical binary logic, necessitating a reconstruction of quantum kinematics from a context-indexed, four-valued grammar—the tetralemma—placed prior to any presupposition of a Hilbert space or spacetime manifold.

The conceptual origin of this framework lies in the author's earlier work, *"A Theory on the Interdependent Origination and Entanglement of Chronogeometric Gravity, Matter and the Cosmos"* [1] (SSRN, 11 November 2025; Zenodo v5, December 2025), which first proposed the classical tetralemma as the generative grammar for an interdependent quantum holo-state. While the 2025 preprint sketched a broad architectural vision, it remained partly heuristic in its logical articulation, relying on qualitative intuitions of a "Four-Vertex Prism."

The present manuscript represents a decisive maturation of that scaffold into a rigorous, algebraic derivation. The qualitative intuition of the earlier work is here replaced by a fully formal logical system wherein the four corners of the *catuṣkoṭi* are defined not merely as truth values, but as primitive, indecomposable *warranted acts* equipped with an explicit *applicability predicate.* This predicate furnishes a positive ontological status—**Holothesis**—for the "unaskable" questions that standard quantum logic can only treat negatively as gaps or linguistic prohibitions.

Readers acquainted with the 2025 preprint will recognize the unchanged commitment to relational, interdependent origination (*pratītyasamutpāda*), yet the implementation is now mathematically exact: the tetralemma has become the atomic logical manifold from which the warrant functional, the complex unit i, and the emergent Hilbert space are derived by strict algebraic closure via the Gelfand–Naimark–Segal (GNS) construction. No physical Hilbert space is postulated; rather, it is forced by the positivity of the logical evaluation. Furthermore, the duality symmetry of the tetralemma is shown to necessitate an order-4 rotational symmetry, from which the imaginary unit emerges as a representation-theoretic eigenvalue, grounding the complex phase of quantum mechanics in logical topology.

This volume constitutes the formal foundation for a tetralemmatic reconstruction of quantum kinematics. It demonstrates that the foundational puzzles of quantum mechanics—contextuality, the measurement problem, and the Wigner's Friend paradox—are consequences of an insufficient logical substrate. By elevating logic to the primary dynamical variable, the tetralemmatic grammar resolves these puzzles while generating novel, testable predictions, most notably a correction to Sorkin's third-order interference parameter ($I_3 \neq 0$) in regimes of high logical deficiency. The work is offered to those prepared to treat logic itself not as a passive tool of description, but as the active, generative substrate of physical reality.

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Australia, March 2026

1 Introduction

Quantum mechanics stands as our most successful physical theory, yet its conceptual foundations remain profoundly puzzling. The theory's empirical predictions are unparalleled in precision, but its implications for the nature of reality, measurement, and logic itself continue to provoke debate nearly a century after its formulation. At the heart of these foundational debates lies a cluster of interconnected problems: the Kochen–Specker theorem's demonstration that quantum observables cannot possess pre-existing values independent of measurement context [2]; the measurement problem's apparent conflict between unitary evolution and definite outcomes; Wigner's friend paradox and the challenge of reconciling different observers' descriptions [3]; and the persistent tension between relational approaches like Rovelli's Relational Quantum Mechanics [4] and framework-based interpretations like Consistent Histories [5].

The Foundational Crisis: Logic, Context, and the Unaskable

Standard quantum mechanics (SQM) operates within a mathematical framework—complex Hilbert spaces, self-adjoint operators, the Born rule—that is extraordinarily effective but offers little insight into *why* nature should be described by this particular formalism [6, 7]. The Kochen–Specker theorem reveals a deep structural feature: no noncontextual hidden-variable theory can reproduce quantum predictions [2]. Yet the theorem is fundamentally negative—it tells us what is impossible, but not what contextual correlations signify about the logical structure of physical reality [8, 9].

Consider the paradigmatic double-slit experiment [10, 11]. When a which-path detector is present, the question "Which slit did the particle traverse?" admits a definite answer: left or right. When the detector is absent and interference fringes appear, this same question becomes operationally *unaskable*—not false, but inapplicable. Standard quantum logic, built on the projection lattice of Hilbert space, lacks any formal mechanism to distinguish "false" from "unaskable" [7, 9]. Both are treated as "not true," collapsing a crucial operational distinction into logical silence.

This silence echoes through foundational interpretations. Relational Quantum Mechanics correctly emphasizes that facts are relative to observers [4], but struggles with the cross-perspective link: how can Alice's definite outcome be reconciled with Bob's superposition description without invoking collapse or many worlds [3]? Consistent Histories enforces a single-framework rule that renders incompatible descriptions "meaningless," a linguistic prohibition that offers no positive account of why certain questions cannot be asked [5]. QBism [12] treats the wave function as subjective Bayesian belief, yet the objectivity of measurement outcomes demands explanation.

What these approaches share is a reliance on classical binary logic as the substrate for reasoning about quantum phenomena. When that logic fails—as it must, given Kochen–Specker—they resort to interpretive maneuvers rather than revising the logic itself [2, 13]. The central thesis of this work is that the foundational puzzles of quantum mechanics arise from an inadequate logical framework, and that a more expressive logic, one that can accommodate both the "askable" and the "unaskable," provides a path to genuine reconstruction [8, 14].

The Tetralemmatic Hypothesis

Our proposal draws structural inspiration from the ancient Buddhist logic of the *catuṣkoṭi* (tetralemma), which recognizes four truth-values rather than two [13, 15]. We introduce a context-indexed logical framework with four primitive logical statuses, which we call the *tetralemmatic corners*:

(i) **Thesis**(Section 2.4): Affirmation of existence—the claim holds at the given context and locus (e.g., "the particle went through the left slit" in a which-path measurement).

(ii) **Antithesis**(Section 2.5): Denial of existence—the negation holds (e.g., "the particle went through the right slit").

(iii) **Synthesis**(Section 2.6): Simultaneous affirmation and denial—both poles hold non-explosively, corresponding to coherent superpositions and paraconsistent "glut" semantics.

(iv) **Holothesis**(Section 2.7): Neither affirmation nor denial is applicable—the question itself is ill-typed at this context and locus, corresponding to paracomplete "gap" semantics.

Crucially, logical status is indexed to a context $\mathcal{C}$ and a locus $\lambda \in \Lambda_{\mathcal{C}}$, formalizing the insight that truth is relative to an operational frame. We introduce an explicit *applicability predicate* $\mathrm{App}_{\mathcal{C}}(\chi; \lambda)$ that gates whether a claim χ can be meaningfully evaluated at all. Claims failing applicability are not false—they are simply not candidates for truth-valuation, receiving the Holothesis status (Definition 2.12).

From Logic to Hilbert Space: A Reconstruction Program

The tetralemmatic corners serve as the generators of a $*$-algebra $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$ (Definition 3.20). To each applicable polar claim P we associate four self-adjoint generators $C_{\bullet}(P)$ ($\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$), forming the raw material for an algebraic reconstruction of quantum kinematics [7, 8].

The next crucial ingredient is a *warrant functional* ω_{Λ}—a positive linear functional on $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$ that assigns numerical degrees of warrant to logical combinations of corner-acts (Definition 3.35). This functional encodes the evaluative content of the theory: how much weight each logical status carries in a given preparation [16, 17].

With these data—a $*$-algebra and a positive functional—we apply the Gelfand–Naimark–Segal (GNS) construction [16, 18, 19]. This canonical procedure yields a Hilbert space $\mathcal{H}_{\Lambda}$, a representation π_{Λ} of the algebra, and a cyclic vector $|\Omega_{\Lambda}\rangle$ (Definition 3.42). Crucially, *no physical Hilbert space, no Born rule, and no complex numbers are presupposed*. The Hilbert space is *forced* by the positivity of the warrant functional; the complex unit i emerges from an order-4 logical symmetry that cycles the four corners, appearing as an eigenvalue of the corresponding unitary implementer (Theorem 5.33).

This constitutes a genuine reconstruction: quantum kinematics is derived from logical first principles, not assumed [8, 9]. The framework then naturally generates graph-theoretic tools

for quantifying contextuality. The *incompatibility graph* $\Gamma_{\omega_\Lambda}(\mathcal{C}, \lambda)$ has vertices representing admissible claims and edges when their associated observables do not commute. Its chromatic number $\mathscr{X}$ bounds the minimal number of classical contexts required to cover a set of claims, and the *Logical Deficiency Theorem* (Theorem 6.94) proves that when $\mathscr{X}$ exceeds the Hilbert space dimension d, a positive amount of warrant must be allocated to the non-classical corners (Synthesis or Holothesis). The constant $\mathcal{K}(\Gamma, d) > 0$ provides a rigorous lower bound on "Holothesis density," showing that contextuality violations (e.g., the KCBS bound $\sqrt{5} > 2$) are not empirical anomalies but logical necessities [2, 13, 20].

Resolution of Foundational Puzzles

With this machinery in place, we demonstrate that the tetralemmatic framework resolves the foundational puzzles that motivate it:

(i) **The Kochen–Specker theorem** is transformed from a negative impossibility result into a positive reconstruction: contextuality is a derived logical necessity when $\mathscr{X} > d$, with the magnitude of violation fixed by $\mathcal{K}(\Gamma, d)$ (Theorem 6.94; [2, 20]). This aligns with the graph-theoretic formulation of contextuality [8, 21].

(ii) **The measurement problem** is reframed as locus symmetry breaking (Section 6.7.1). The Holothesis vector $|\textbf{HoloTh}\rangle$ (Definition 4.8) remains in the global Hilbert space but becomes inaccessible to the local observer's applicability predicate $\mathrm{App}_{\mathcal{C}}$ once symmetry is broken (Proposition 6.46). No collapse occurs; the change is in the logical status of claims relative to the locus, consistent with decoherence theory [22, 23] but grounded in logical ontology.

(iii) **Wigner's friend** is resolved by recognizing truth as locus-indexed. For Alice at her locus, the outcome claim has status **Th**; for Bob, the same claim has status **HoloTh** (inapplicable). There is no contradiction because existence is relative to context: $\exists_{\lambda_A}^{\mathcal{C}_A} x \neq \exists_{\lambda_B}^{\mathcal{C}_B} x$ (Definition 2.1; [4]). This resolves the cross-perspective link problem (Section 6.18.1; [3]).

(iv) **The wave function ontology** is clarified: the wave function corresponds to the warrant functional ω_Λ (Definition 3.35) operating over the Holothesis sector (Definition 2.12). It is neither a "thing" (Thesis) nor "nothing" (Antithesis), but a state of *logical transcendence*—a potentiality structure prior to symmetry-breaking measurement (Remark 6.141; [24]).

(v) **Relational Quantum Mechanics and Consistent Histories** are unified as regime-relative stances within a single framework (Section 6.18). CH's consistency conditions become symmetry constraints on admissible refinements (Definition 2.6; [5]); RQM's relative facts become locus-dependent Thesis/Antithesis assignments ([4]). The Holothesis sector provides a formal mechanism for the "unaskable" that CH treats as linguistic boundary and RQM struggles to formalize (Remark 6.148; [9]).

Empirical Vulnerability and Predictions

A foundational theory must do more than reinterpret existing data; it must make novel, testable predictions. We derive a correction to Sorkin's third-order interference parameter [14]:

$$I_3 = \mathcal{K}(\Gamma, d) \cdot \sin(4\theta_\mathbf{R}) + \mathcal{O}(\mathcal{K}^2),$$

where $\mathcal{K}$ is the logical deficiency constant and $\theta_\mathbf{R}$ is the quarter-turn phase (Proposition 6.36). Standard quantum mechanics corresponds to $\mathcal{K} = 0$ and $\theta_\mathbf{R} = \pi/2$, yielding $I_3 = 0$. Recent experimental bounds constrain $|\mathcal{K}| \lesssim 10^{-3}$ in standard contexts [25, 26], but many-particle correlations may exhibit enhanced sensitivity due to combinatorial growth of the incompatibility graph [21, 27]. This provides a clear empirical boundary: the framework acts as an *interpretative reconstruction* if $\mathcal{K} = 0$, but offers *novel physics* if deviations are detected in extreme regimes.

Organization of the Paper

The manuscript is structured to guide the reader through this reconstruction in four progressive stages:

(i) **Part I: Logical Foundations and Contextual Ontology** (Section 2)
Establishes the tetralemmatic logic in detail: context-indexed existence (Definition 2.1), the applicability predicate, the four corners as primitive logical statuses, and the duality involution that exchanges Thesis and Antithesis while fixing Synthesis and Holothesis.

(ii) **Part II: Tetralemmatic Algebra and Pre-Hilbert Structures** (Section 3)
Constructs the free $*$-algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$ generated by the corner-acts, introduces the warrant functional ω_Λ, and defines the GNS pre-Hilbert space L_{ω_Λ}—the inner-product quotient that forms the raw material for Hilbert space completion.

(iii) **Part III: Gelfand–Naimark–Segal and State Space Emergence** (Section 4)
Performs the Cauchy completion to obtain the logical Hilbert space $\mathcal{H}_\Lambda$, derives the unitary implementers $U_\mathbf{D}$ and $U_\mathbf{R}$ of the duality and quarter-turn symmetries, and demonstrates the canonical emergence of the complex unit i from the order-4 logical symmetry (Theorem 5.33).

(iv) **Part IV: Operational Realization and Contextuality Witnesses** (Section 6)
Connects the abstract construction to empirical physics via the Realization Axiom (Section 6.5), develops graph-theoretic contextuality witnesses (incompatibility graphs, frame numbers, the logical deficiency constant $\mathcal{K}$), derives the Sorkin I_3 correction, resolves the measurement problem, and unifies RQM and CH within the tetralemmatic ontology.

Each part builds systematically on its predecessors, culminating in a unified framework that reconstructs quantum kinematics from logical first principles while generating novel empirical predictions and resolving longstanding foundational puzzles. For a comprehensive summary of notation used throughout the manuscript, see table 7 in the Appendix.

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2 Logical Foundations and Contextual Ontology

2.1 Aristotlian and Tetralemmatic Modes of Logical Inquiry

2.1.1 Aristotle and the Story of a Thesis

The word *thésis* (θέσις) originally meant "a placing" or "a setting down," deriving from the Proto-Indo-European root *$d^h eh_1$-* with the concrete sense of putting something in place [28, 29]. From this physical origin emerged its intellectual meaning: an idea deliberately set forth for consideration, a proposition exposed to scrutiny [30]. Aristotle systematized this notion within a theory of argument: a thesis becomes a determinate claim that can be defended by reasons, potentially even against common opinion (*endoxa*), within an organized dialectical inquiry [31]. The contemporary academic practice of defending a doctoral "thesis" is the institutional descendant of this trajectory, as documented in the historical record of English usage [32].

This transition from physical placement to logical positioning marks the birth of the *thesis* as a stable object of inquiry governed by the fundamental laws of thought. Within the Aristotelian framework, the act of "setting down" a proposition implicitly invokes the Law of Non-Contradiction (LNC), which asserts that a property cannot both belong and not belong to the same subject at the same time and in the same respect [33]. By fixating a claim in a determinate state, the *thesis* establishes a boundary that excludes its opposite; without this boundary, the "placing" remains fluid, rendering dialectical defense impossible. As modern critiques of classical logic underscore, the LNC provides the necessary ontological ground upon which a thesis is "set," ensuring it possesses a singular, non-contradictory identity throughout the inquiry [34].

Furthermore, the "exposure to scrutiny" inherent in the evolution of the *thesis* is structurally dependent on the Law of Excluded Middle (LEM). Aristotle's dialectic necessitates a binary resolution—defended or refuted—leaving no stable intermediate state for the investigator [35]. The LEM dictates that for any determinate claim P set forth as a *thesis*, either P is the case or not-P is the case ($P \vee \neg P$). This law transforms the intellectual "setting down" into a decisive act of valuation, forcing the dialectician to navigate a landscape where every proposition occupies a discrete truth-value. In this sense, the Aristotelian *thesis* serves as the primary unit of a classical reality characterized by observer-independent, definite properties [36].

Yet, this classical stability—predicated on observer-independent properties—encounters a fundamental crisis within modern quantum foundations. Specifically, the Kochen-Specker Theorem demonstrates that a "determinate claim" cannot be "set down" independently of its measurement context, directly undermining the assumption of non-contextual value definiteness [2]. If a property's value is contingent upon the context of its observation, the *thesis* loses its absolute character. This challenge is met by Relational Quantum Mechanics (RQM), which posits that the "truth" of a thesis is not a universal fact but a relational event occurring between interacting systems [4]. Alternatively, Rafael Sorkin's Quantum Measure Theory seeks to maintain a realist, single-history ontology by employing an anhomomorphic logic [14]. Rather than discarding the LEM itself, this framework relaxes the requirement that truth-assignments behave as Boolean homomorphisms, allowing the system to accommodate

quantum interference through non-additive measures and preclusion rules [37]. Thus, the etymological journey of the *thesis*—from a physical act to a logical law—diverges toward either a non-Boolean history or a strictly relational framework, both redefining the very nature of "putting something in place."

2.1.2 From Stable Facts to Relative Information

While Sorkin's framework seeks to resolve the classical crisis by altering the logical rules of history, the recent evolution of Relational Quantum Mechanics (RQM) provides a rigorous refinement of the *thesis* by focusing instead on physical interactions. In the updated formalism proposed by Di Biagio and Rovelli (2021), a thesis is not merely a subjective perspective but a "stable fact" whenever the decoherence between interacting systems renders interference terms negligible [3]. This stability allows for a recovered version of the Law of Non-Contradiction to function within a shared macro-reality; while the underlying ontology remains strictly relational, the instantiation of a fact becomes robust enough for multiple observers to treat it as an objective datum.

Furthermore, the 2026 iteration of the theory anchors the realization of a thesis in the physical exchange of relative information [38].[1] By defining a fact as a physical correlation between two systems, RQM transforms the Aristotelian *thésis* into an information-theoretic event. In this view, the "scrutiny" of a proposition is synonymous with the physical interaction that establishes mutual information, suggesting that the "placing" of reality is not a static state but a dynamic, informational occurrence. This shift implies that the laws of logic—once thought to be abstract dictates of a "God's eye view"—are actually emergent properties of physical correlations and the limits of accessible information.

2.2 Nāgārjuna and the Debates of Ancient India

Classical India developed highly structured public debate traditions (*vāda* वाद) with explicit rules for claims, reasons, exemplars, and refutation [39, 40, 41]. A debate typically begins with a *pratijñā* (प्रतिज्ञा), an opening proposition intended to be established [39, 41, 42]. The term's components preserve clear Indo-European ancestry: *prati-* (प्रति-) from **proti-* conveys "toward/against/in return," while the root *jñā* (ज्ञा) derives from **'gneh₃-* "to know," cognate with Greek *gnōsis* (γνῶσις) and English "know" [28, 29, 30, 42]. The practical meaning is a public declaration that anchors the argument [42].

Once stated, the *pratijñā* is defended via disciplined inferential structure, classically articulated in Nyāya as a five-membered form: Once stated, the *pratijñā* is defended via disciplined inferential structure, classically articulated in Nyāya as a five-membered (*pañcāvayava*) form:

(1) *Pratijñā* (Proposition)

(2) *Hetu* (Reason)

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[1]The 2026 paper quantifies this exchange via the maximal relative information $I_{A|B}$, defined as the conditional information that becomes concentrated after interaction; a correlation qualifies as a stable fact precisely when $I_{A|B}$ reaches its upper bound, thereby providing a physical, information-theoretic criterion for when a relational event can be treated as effectively objective.

(3) *Udāharaṇa* (Example/Universal Rule)

(4) *Upanaya* (Application)

(5) *Nigamana* (Conclusion)

This structure is supported by examples and tested by an opponent for fallacies [39, 40, 41]. supported by examples and tested by an opponent for fallacies [39, 40, 41]. The ideal of such debate is not merely victory but *tattva-nirṇaya* (तत्त्वनिर्णय), the determination of what is the case [39, 40]. In this way, the Greek and Indian traditions converge on a shared methodological insight: knowledge advances by putting forward explicit claims and subjecting them to rigorous public norms of reasoning, even while differing in the logical frameworks they take to be most fundamental [31, 41].

In tracing the conceptual underpinnings of quantum mechanics—where the certainties of classical predication give way to probabilistic predictions derived from amplitudes and to context-dependent state assignments—the ancient tetralemma stands out as an elegant scaffold for logical inquiry, clarifying how and why certain forms of "is/is not" description become unstable. This four-corner framework, known in Sanskrit as **Catuṣkoṭi** (चतुष्कोटि, "four corners"), arises from the rigorous traditions of Indian dialectical logic and finds its classical exposition in the Madhyamaka analyses systematized by Nāgārjuna [15, 43]. Far from a simplistic enumeration of possibilities, it functions as a refined instrument for interrogating predication itself: rather than confining propositions to the binary poles "true" or "false," it distinguishes four logically discrete postures toward a phenomenon and makes explicit the inferential and conceptual commitments each posture entails [13, 44]. In this way, the Catuṣkoṭi operates as a four-vertex, refracting the implicit metaphysical commitments embedded in commonplace notions of existence and negation—commitments that classical bivalent logic often treats as default constraints on admissible description [13, 44].

The four corners are conventionally stated as exhaustive predications about a phenomenon:

(i) *The Thesis: it is (Being).*
An affirmative stance (Sanskrit *Asti* अस्ति), often discussed using conceptual terms such as *Bhāva* भाव ("being/existence/state") or *Sat* सत् ("the real").

(ii) *The Antithesis: it is not (Not-Being).*
A negative stance (*Nāsti* नास्ति, from *na asti*), with conceptual correlates *Abhāva* अभाव ("absence/non-being") and *Asat* असत् ("non-being").

(iii) *The Synthesis: it both is and is not.*
The joint stance (*Asti ca Nāsti ca* अस्ति च नास्ति च), which directly confronts classical non-contradiction and motivates paraconsistent readings of certain forms of reasoning [45]. A condensed conceptual form is *Sadasat* सदसत्, combining *sat* and *asat*.

(iv) *The Holothesis: it neither is nor is not.*
[2]The "neither/nor" stance (*Naiva Asti na Nāsti* नैवास्ति न नास्ति), often misread as a mere

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[2]To clarify the exhaustive nature of this corner: the holothesis is neither (i) the thesis nor (ii) the antithesis, as these have already been accounted for; nor is it (iii) the synthesis (both thesis and antithesis), since that combinatorial space has likewise been exhausted.

"none of the above." In the more careful dialectical use associated with Nāgārjuna, it marks a categorical shift: the point is not to negate the first three options from within the same conceptual scheme, but to indicate that the scheme itself may fail to apply stably to the target [46].

While the mapping of the first three corners onto the Western dialectical triad—famously structured by J. G. Fichte and commonly associated with G. W. F. Hegel—has robust historical precedent in twentieth-century comparative philosophy, the fourth corner has traditionally presented a terminological dead end [47]. Mid-century scholars, such as T. R. V. Murti and Richard H. Robinson, frequently utilized *thesis*, *antithesis*, and *synthesis* to translate the affirmative, negative, and joint stances of the Catuṣkoṭi into frameworks legible to Western logic [45, 48]. However, because the Fichtean dialectic treats the *synthesis* as the ultimate, stable resolution of a contradiction, the Western conceptual apparatus lacked a native structural equivalent for a fourth stage in which the entire dialectical scheme breaks down.

Consequently, previous comparative literature has often been forced to abandon the *-thesis* suffix at the fourth corner, retreating instead to clunky or domain-specific descriptors such as "global negation," "the void" (*śūnyatā*), or modern logical designations like "paracompleteness" [13]. This historical asymmetry leaves the four-cornered topology structurally disjointed when translated.

To resolve this asymmetry and adequately map the Catuṣkoṭi onto the non-classical demands of quantum measure theory, the present framework introduces the term **Holothesis**.[3] Deriving from the Greek *holos* (ὅλος, meaning "whole," "entire," or "complete") [30], a *holothesis* explicitly names the operation performed upon the entire combinatorial space of the preceding triad. It signifies not a localized negation, but the systemic realization that the whole conceptual scheme is exhausted, precluded, and strictly inapplicable. By completing the linguistic set, the *holothesis* provides a symmetrical and philosophically precise handle for the logical "gap"—preparing the ground for a re-imagined tetralemma that can seamlessly accommodate the non-Boolean topologies of the quantum universe.

2.2.1 Dialectical Deployment of the Tetralemma in Quantum Contexts

In its canonical deployment, the tetralemma is not used to select a privileged corner as an ultimate ontological verdict. It functions instead as a diagnostic: when a phenomenon is treated as possessing an intrinsic, context-invariant essence, each of the four predicative modes can be driven into inconsistency, explanatory circularity, or category error. The fourth corner is therefore not a trivial remainder; it marks the point at which the predicative scheme itself loses stable application, forcing a shift from questions of "what it is in itself" to descriptions framed by dependence, relations, and conditions [13, 15, 44, 46]. This methodological nuance aligns naturally with quantum foundations, where the formalism ties what can be asserted to operational context: classical property attributions can fail in the presence of superposition and entanglement, and disciplined description tracks how legitimate claims change with preparation, interaction, and measurement.

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[3]The term *holothesis* is introduced in this work as an original terminological extension. It completes the dialectical set, resolving the historical inadequacy of the Fichtean triad when applied to the Catuṣkoṭi, and provides a precise linguistic anchor for the measure-zero preclusions (gaps) encountered in quantum contexts.

This general methodological alignment finds a rigorous mathematical formalization in Rafael Sorkin's explicit mapping of the tetralemma to Quantum Measure Theory (QMT) [49]. Seeking to preserve a realist, single-history ontology without falling victim to the Kochen-Specker contradiction, Sorkin proposes that the "answering map" assigning truth values to quantum histories cannot be a classical Boolean homomorphism. Instead, quantum interference dictates that the logical space expands precisely into the four corners of the Catuṣkoṭi. Within Sorkin's anhomomorphic logic, a quantum event may be classically affirmed (Thesis) or denied (Antithesis); however, interference can also result in a simultaneous affirmation of multiple histories (Synthesis, functioning as a logical "glut"), or an event may be entirely precluded with a quantum measure of zero, rendering the proposition logically inapplicable to that history (Holothesis, functioning as a logical "gap"). Thus, Sorkin deploys the ancient tetralemma not merely as a philosophical analogy, but as the exact logical topology required to evaluate propositions in a quantum universe.

Following this explicit mapping, the formalization of these non-classical truth-valuations has matured into an active research program, widely known in contemporary literature as the "co-event formulation" of QMT. Subsequent work by Sorkin, Fay Dowker, Petros Wallden, and others has rigorously tested the boundaries of this four-cornered topology [50, 51]. For instance, investigations into classical rules of inference reveal that strictly enforcing Modus Ponens alongside quantum measure rules forces a collapse back into classical Boolean logic, confirming that navigating a quantum universe strictly requires the paraconsistent and paracomplete flexibility of the tetralemma. Furthermore, contemporary efforts focus on how the bivalent logic of our macroscopic reality is recovered from this foundation. In standard QMT approaches, this recovery relies on a quantum analogue of the Cournot Principle. When microscopic quantum histories are bundled together via macroscopic coarse-graining, interference terms cancel out, driving the quantum measure of non-classical states toward zero; researchers argue that this principle effectively suppresses macroscopic "gluts" and "gaps" in everyday observation [52]. However, because the classical world of exclusive affirmation and denial is merely an emergent approximation rather than a fundamental absolute, the Synthesis and Holothesis corners are obscured rather than ontologically erased. This persistent non-Boolean foundation suggests that the "glut" and "gap" are not mere mathematical artifacts to be coarse-grained away, leaving crucial conceptual space to re-imagine the tetralemma's structural role in observable reality.

2.3   The Tetralemma Re–imagined

The logical framework developed here[4] draws upon the classical *catuṣkoṭi* (tetralemma) of Madhyamaka Buddhism, re-interpreted through modern paraconsistent and contextual logic [13, 43, 53]. We fix a *domain of discourse* $\mathcal{D}$ and work relative to an *immanent context* $\mathcal{C}$ that specifies (at least) the language of admissible formulas, criteria of individuation and reference, admissible modes of inference and evaluation, and the class of admissible loci of evaluation. This context-dependent approach resonates with topos-theoretic formulations of physics and Bohr's notion of complementary contexts [9, 11].

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[4]The formal roots of this tetralemmatic approach and the initial conceptualization of the "Four-Vertex Prism" were first presented in the author's 2025 preprint [1]. This section represents a rigorous formalisation and extension of those original relational insights.

2.3.1 Context-Indexed Existence and Loci

Definition 2.1 (Context-Indexed Existence and Loci)**.** Let $\Lambda_{\mathcal{C}}$ denote the set of $\mathcal{C}$-sanctioned loci (spatiotemporal, modal, or otherwise), and fix $\lambda \in \Lambda_{\mathcal{C}}$. We define the context-indexed existential quantifier by

$$\exists_{\lambda}^{\mathcal{C}} x\, \varphi(x) \quad \text{as shorthand for} \quad \exists x \in \mathcal{D}_{\lambda}^{\mathcal{C}}\, \varphi(x),$$

where $\mathcal{D}_{\lambda}^{\mathcal{C}} \subseteq \mathcal{D}$ is the locus- and context-admissible subdomain. Thus, "existence" is explicitly $\mathcal{C}$- and λ-indexed, rather than absolute.

2.3.2 Truth-Status vs. Warranted Act

To avoid conflating propositional status with epistemic performance, we distinguish between the holding relation and the act of warranting.

Definition 2.2 (Context-Indexed Evaluation Relation)**.** We use a context-indexed evaluation relation $\Vdash_{\mathcal{C},\lambda}$ on formulas:

$$\mathcal{C}, \lambda \Vdash \chi$$

reads "χ holds at locus λ according to the designated evaluation/consequence regime of $\mathcal{C}$."

A *warranted act* will generally require both (i) an applicability judgement and (ii) a holding judgement, while the holding judgement may obtain even when an agent fails to supply the required defenses.

2.3.3 Applicability Predicate

The framework relies on a typed notion of meaningfulness prior to truth valuation.

Definition 2.3 (Applicability Predicate)**.** We use an *applicability* predicate $\mathrm{App}_{\mathcal{C}}(\chi; \lambda)$, read as: "$\chi$ is a well-formed and presuppositionally coherent claim for evaluation in $\mathcal{C}$ at locus λ." Its negation $\neg\, \mathrm{App}_{\mathcal{C}}(\chi; \lambda)$ is a *typed* inapplicability judgement (category error, ill-typed predication, or presupposition failure), not a truth-value assignment. This distinction aligns with paracomplete logics that admit truth-value gaps alongside paraconsistent gluts [9, 13].

Axiom 2.4 (Negation-Closure of Applicability)**.** For all admissible claims χ in $\mathcal{C}$, applicability is closed under negation:

$$\mathrm{App}_{\mathcal{C}}(\chi; \lambda) \iff \mathrm{App}_{\mathcal{C}}(\neg\chi; \lambda). \tag{1}$$

2.3.4 Warrant Profiles and Argument Classes

To formalize the epistemic requirements of a context, we define the admissible modes of defense.

Definition 2.5 (Warrant Profiles and Argument Classes)**.** Let $\mathsf{W}(\mathcal{C}) \subseteq \{A, N\}$ be the set of warrant modes required by $\mathcal{C}$ (e.g. $\mathsf{W}(\mathcal{C}) = \{A, N\}$ if both Aristotelian and Nyaya defenses are mandatory). Let $\mathrm{Arg}_A(\mathcal{C})$ and $\mathrm{Arg}_N(\mathcal{C})$ denote the classes of admissible Aristotelian and Nyaya defenses in $\mathcal{C}$, with $\mathrm{Arg}_A^{\neg}(\mathcal{C})$, $\mathrm{Arg}_N^{\neg}(\mathcal{C})$ their refutational counterparts, and $\mathrm{Arg}_A^{\perp}(\mathcal{C})$, $\mathrm{Arg}_N^{\perp}(\mathcal{C})$ their meta-level (inapplicability) counterparts.

The Nyaya tradition provides a rigorous five-membered argument scheme (*pratijna, hetu, udaharana, upanaya, nigamana*) for validating such claims [39, 41], while Aristotelian dialectic supplies the Western counterpart for reasoned position-taking [33]. When $A \notin \mathsf{W}(\mathcal{C})$ (resp. $N \notin \mathsf{W}(\mathcal{C})$), the corresponding Aristotelian (resp. Nyaya) obligations below are dropped.

2.3.5 Admissible Refinements and Closure

Contexts are not static; they admit structured transformations that preserve logical coherence.

Definition 2.6 (Admissible Refinements and Closure)**.** Let $\mathsf{R}(\mathcal{C})$ be a specified set of *elementary refinement operators* on contexts (explicit re-indexings of sense, respect, mode, or locus-parameters that $\mathcal{C}$ declares admissible and non-equivocating). This structure parallels the coarse-graining and refinement of histories in consistent histories quantum mechanics [5]. Define the refinement-closure

$$\mathrm{Ref}(\mathcal{C}) \;:=\; \left\{ r_n \circ \cdots \circ r_1(\mathcal{C}) \ \middle| \ n \geq 0, \ r_i \in \mathsf{R}(\mathcal{C}) \right\}, \tag{2}$$

with the convention $n = 0$ yields $\mathcal{C}$. For $r \in \mathsf{R}(\mathcal{C})$, write $\mathcal{C}_r := r(\mathcal{C})$.

Mathematical and Logical Meaning.   This framework fundamentally reconstructs classical logic into a highly typed, context-relative operating system. In the classical view, existence ($\exists$) and truth are absolute, binary, and belong to a global domain of discourse $\mathcal{D}$. By contrast, Definition 2.1 restricts the domain to $\mathcal{D}_\lambda^{\mathcal{C}}$, dictating that reality is admissible for discussion only within a specific operational context ($\mathcal{C}$) and locus (λ). Furthermore, by decoupling the holding relation (Definition 2.2) from the epistemic warrant profile (Definition 2.5), the formalism separates *ontology* from *epistemology*. A relational interaction may establish a physical fact regardless of whether an observer is present to construct an Aristotelian or Nyāya defense. Crucially, the introduction of the Applicability predicate (Definition 2.3) establishes a formal space for *inapplicability*: a claim can fail to be true not because it is false, but because it triggers a category error ($\neg \mathrm{App}_{\mathcal{C}}$) at that locus. This shift replaces the flat bivalence of classical logic with a dynamic topology where questions must first be well-typed before they can be evaluated [9, 13].

Relation to Kochen–Specker (Contextuality).   This formalism serves as the exact logical prerequisite for satisfying the Kochen–Specker (KS) theorem. The KS theorem proves that one cannot assign definite truth values to all quantum observables simultaneously [2]. Classical logic fails here because it assumes a global domain and a universal truth-value assignment. By indexing both existence ($\exists_\lambda^{\mathcal{C}}$) and evaluation ($\Vdash_{\mathcal{C},\lambda}$) to specific contexts and loci, the framework natively forbids this "God's eye" view. An object or property may exist and hold in a which-path context ($\mathcal{C}_1, \lambda_1$) but completely fail the applicability predicate in an interference context ($\mathcal{C}_2, \lambda_2$). This strict locus-dependence ensures that logic itself mirrors the contextual demands of quantum mechanics, preventing the mathematical contradictions that the KS theorem otherwise exposes [9].

Relation to Rovelli (Relational Quantum Mechanics). Furthermore, this formalism perfectly captures the ontological commitments of Relational Quantum Mechanics (RQM). In RQM, physical variables do not possess absolute values; rather, facts are realized only through physical interactions between systems [4]. The context-indexed evaluation relation ($\Vdash_{\mathcal{C},\lambda}$) provides the exact formal syntax for Rovelli's "relative facts," ensuring that a proposition only holds relative to a specific interacting system (the locus λ) within a defined physical interaction (the context $\mathcal{C}$). By strictly decoupling this ontic holding relation from the epistemic warrant profile ($\mathsf{W}(\mathcal{C})$), the framework accommodates RQM's radical physicalism: an interaction establishes a concrete relational fact even in the absence of a macroscopic agent capable of formulating an Aristotelian or Nyāya defense. Moreover, the recent theoretical refinement distinguishing volatile relative facts from decoherence-induced "stable facts" [3, 38] is naturally handled by Admissible Refinements (Definition 2.6). As the context is scaled and coarse-grained via the refinement-closure $\mathrm{Ref}(\mathcal{C})$, localized relative information stabilizes, allowing the recovered bivalence of a shared macroscopic reality to emerge directly from the relational quantum substrate.

Relation to Sorkin (Quantum Measure Theory). The definitions directly encode Rafael Sorkin's requirements for Quantum Measure Theory (QMT) and anhomomorphic logic. Sorkin argues that valid questions must be admissible within a given measure-theoretic context, summarized by his maxim: "Logic is to the Quantum as Geometry is to Gravity" [8, 14]. The applicability predicate (Definition 2.3) acts as the logical equivalent of Sorkin's measure-zero preclusion; it identifies the "null co-events" where questions are strictly precluded from the history space. Additionally, the definition of Admissible Refinements (Definition 2.6) provides the mathematical machinery necessary to handle the coarse-graining of these histories. Just as Sorkin and Dowker demonstrate how macroscopic bivalent reality emerges from bundling microscopic quantum histories, the refinement-closure $\mathrm{Ref}(\mathcal{C})$ allows the logical context to scale and transform without losing internal coherence [5].

Relation to Priest (Paraconsistency and the Catuṣkoṭi). Finally, this unified architecture serves as the formal engine for mapping Graham Priest's interpretation of the Catuṣkoṭi onto physical theory. Priest's analysis of Madhyamaka logic requires accommodating truth-value gluts (both true and false) and truth-value gaps (neither true nor false) [13, 54]. The warrant profiles (Definition 2.5) anchor the first two corners—Thesis and Antithesis—in standard Aristotelian and Nyāya defense mechanisms. The context-indexed evaluation relation ($\Vdash_{\mathcal{C},\lambda}$) accommodates the paraconsistent third corner (the Synthesis), allowing for quantum gluts (like superpositions) to hold non-explosively relative to a locus. Most importantly, the applicability gap ($\neg\,\mathrm{App}_{\mathcal{C}}$) provides a rigorous mathematical identity for the fourth corner: the *Holothesis*. By formalizing the Holothesis as a typed inapplicability judgment, the framework seamlessly integrates Buddhist dialectics, paracomplete logic, and quantum preclusion into a single, cohesive topology [43, 53].

2.4 The Thesis

A *Thesis* is a determinate, context-indexed commitment that affirms existence (*Asti*, *Bhāva*, *Sat*) [42] and is normatively bound to rigorous defense: Aristotelian in the sense of a reasoned position suitable for dialectical scrutiny [33], and Nyāya in the sense that a *pratijñā* is to be validated by a methodical, multi-membered argument [39, 41]. Its epistemic warrant derives from coherent applicability, holding-status in $\mathcal{C}$, and satisfaction of the required warrant profile $\mathsf{W}(\mathcal{C})$ *within* $(\mathcal{C}, \lambda)$.

Definition 2.7 (Thesis (*Asti*; *Bhāva*; *Sat*; Greek *thésis*; Sanskrit *pratijñā*)). Fix $\mathcal{D}$, $\mathcal{C}$, $\Lambda_{\mathcal{C}}$, and $\lambda \in \Lambda_{\mathcal{C}}$. For a formula $\varphi(x)$ (with x free), define the $\mathcal{C}$- and λ-relative existential claim using the context-indexed quantifier (Definition 2.1):

$$P := \exists^{\mathcal{C}}_{\lambda} x \, \varphi(x).$$

A *Thesis-act* about φ at λ is the structured tuple

$$\mathbf{Th}(\varphi; \mathcal{C}, \lambda) := \left\langle P, \, \mathcal{J}_A(\varphi; \mathcal{C}, \lambda), \, \mathcal{J}_N(\varphi; \mathcal{C}, \lambda) \right\rangle,$$

and we say that the Thesis *is warranted* (write $\mathrm{Th}(\varphi; \mathcal{C}, \lambda)$ holds) when the following conditions obtain:

(i) Ontological affirmation (Being).
$\mathrm{App}_{\mathcal{C}}(P; \lambda)$ holds (Definition 2.3) and $\mathcal{C}, \lambda \Vdash P$ holds (Definition 2.2). This is the "It is" commitment, relative to $(\mathcal{C}, \lambda)$ (Catuṣkoṭi: first corner) [13, 46].

(ii) Admissibility and determinacy.
The predication φ is admissible in $\mathcal{C}$, and the relevant reference and individuation conditions are fixed so that P has stable content (no equivocation in sense, locus, or respect within $\mathcal{C}$).

(iii) Dialectical posture (Aristotle).
If $A \in \mathsf{W}(\mathcal{C})$ (Definition 2.5), then $\mathcal{J}_A(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A(\mathcal{C})$.

(iv) Procedural obligation (Nyāya).
If $N \in \mathsf{W}(\mathcal{C})$, then $\mathcal{J}_N(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N(\mathcal{C})$; in particular, it may instantiate a Nyāya-style proving scheme with *pratijñā* as the opening commitment (e.g. pratijñā, hetu, udāharaṇa, upanaya, nigamana) [40].

Equivalently,

$$\begin{aligned}
\mathrm{Th}(\varphi; \mathcal{C}, \lambda) \text{ holds} \iff & \mathrm{App}_{\mathcal{C}}(P; \lambda) \, \wedge \, (\mathcal{C}, \lambda \Vdash P) \\
& \wedge \left(A \notin \mathsf{W}(\mathcal{C}) \, \vee \, \mathcal{J}_A(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A(\mathcal{C}) \right) \\
& \wedge \left(N \notin \mathsf{W}(\mathcal{C}) \, \vee \, \mathcal{J}_N(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N(\mathcal{C}) \right).
\end{aligned}$$

Certificate of the Fact with its Epistemic Machinery. By formalizing the Thesis-act as the structured tuple $\mathbf{Th}(\varphi; \mathcal{C}, \lambda) = \langle P, \mathcal{J}_A, \mathcal{J}_N \rangle$, Definition 2.7 structurally unites the relational ontology of quantum mechanics with the rigorous dialectical traditions of both the West and the East. The first element, P, represents the raw ontological claim—the "relative fact" realized at a specific physical locus [4]. The subsequent elements, $\mathcal{J}_A$ and $\mathcal{J}_N$, represent the necessary epistemic "receipts": the Aristotelian dialectical proof [33] and the Nyāya multi-membered syllogism [39, 41], respectively. In this framework, a complete, human-articulated *Thesis* is not treated merely as a free-floating true statement; rather, it is a composite mathematical object that explicitly packages the underlying physical fact together with the specific cultural and logical machinery required to publicly defend it.

Conditional Epistemic Burdens and Relational Facts. The philosophical power of this tuple is operationalized by the logical equivalence at the conclusion of the definition. Specifically, the conditional disjunctions—such as $(A \notin \mathsf{W}(\mathcal{C}) \vee \mathcal{J}_A(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A(\mathcal{C}))$—dictate that either a given context does not demand a specific epistemic defense, or a valid one must be provided. This conditional elegance is crucial for seamlessly accommodating the physicalism of Relational Quantum Mechanics (RQM) [3, 4]. Consider, for example, two electrons interacting in a vacuum. Because this is a purely microscopic physical context, the warrant profile $\mathsf{W}(\mathcal{C})$ is empty; neither an Aristotelian nor a Nyāya defense is requested by the universe. Consequently, conditions (iii) and (iv) trivially resolve to true, and the Thesis holds entirely on the basis of condition (i)—the raw ontological affirmation that the relative fact holds $(\mathcal{C}, \lambda \Vdash P)$. By structuring the definition in this way, the formalism mathematically secures the core premise of RQM: it proves that the universe can establish definite relational facts entirely independent of human observers, while simultaneously defining the exact epistemic obligations that emerge when macroscopic philosophers do enter the context.

2.5 The Antithesis

An *Antithesis* is a determinate, context-indexed denial that withholds existence (*Nāsti*, *Abhāva*, *Asat*) [42] and is normatively bound to rigorous refutation: Aristotelian in the sense of a reasoned negation suitable for dialectical scrutiny [33], and Nyāya in the sense that the asserted absence is to be validated by methodical argument (with *pratijñā* now a negative claim) [39, 41]. Its epistemic warrant derives from coherent applicability, holding-status for the denial in $\mathcal{C}$, and satisfaction of the required warrant profile $\mathsf{W}(\mathcal{C})$ *within* $(\mathcal{C}, \lambda)$.

Definition 2.8 (Antithesis (*Nāsti*; *Abhāva*; *Asat*; Greek *antíthesis*))**.** Fix $\mathcal{D}, \mathcal{C}, \Lambda_{\mathcal{C}}$, and $\lambda \in \Lambda_{\mathcal{C}}$. For $\varphi(x)$, define the context-indexed existential claim using the context-indexed quantifier (Definition 2.1):

$$P := \exists_\lambda^{\mathcal{C}} x\, \varphi(x).$$

An *Antithesis-act* denying φ at locus λ is the structured tuple

$$\mathbf{AntiTh}(\varphi; \mathcal{C}, \lambda) := \Big\langle \neg P,\ \mathcal{J}_A^-(\varphi; \mathcal{C}, \lambda),\ \mathcal{J}_N^-(\varphi; \mathcal{C}, \lambda) \Big\rangle,$$

and we say that the Antithesis *is warranted* (write AntiTh$(\varphi; \mathcal{C}, \lambda)$ holds) when the following conditions obtain:

(i) Ontological denial (Not-Being).
  $\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$ holds (Definition 2.3) and $\mathcal{C}, \lambda \Vdash \neg P$ holds (Definition 2.2). This is the "It is not" commitment, relative to $(\mathcal{C}, \lambda)$ (Catuṣkoṭi: second corner) [13, 53].

(ii) Scope and determinacy.
  The denial is meaningful within $\mathcal{C}$: the locus λ (and any relevant window/parameters) is fixed, and the reference/individuation conditions for what would count as an instance are specified so that non-occurrence is testable in principle.

(iii) Dialectical posture (Aristotle).
  If $A \in \mathsf{W}(\mathcal{C})$ (Definition 2.5), then $\mathcal{J}_A^{\neg}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A^{\neg}(\mathcal{C})$.

(iv) Procedural obligation (Nyāya).
  If $N \in \mathsf{W}(\mathcal{C})$, then $\mathcal{J}_N^{\neg}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N^{\neg}(\mathcal{C})$; it may validate absence using Nyāya norms (e.g. negative *pratijñā* with *hetu, udāharaṇa, upanaya, nigamana*, and admissible appeals to non-perception *anupalabdhi* where $\mathcal{C}$ allows) [39, 40].

Equivalently,

$$\mathrm{AntiTh}(\varphi; \mathcal{C}, \lambda) \text{ holds} \iff \mathrm{App}_{\mathcal{C}}(\neg P; \lambda) \wedge (\mathcal{C}, \lambda \Vdash \neg P)$$
$$\wedge \left(A \notin \mathsf{W}(\mathcal{C}) \vee \mathcal{J}_A^{\neg}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A^{\neg}(\mathcal{C}) \right)$$
$$\wedge \left(N \notin \mathsf{W}(\mathcal{C}) \vee \mathcal{J}_N^{\neg}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N^{\neg}(\mathcal{C}) \right).$$

Certificate of Absence and Methodological Refutation. Just as the Thesis certifies a relational presence, the Antithesis-act formalizes the "Certificate of Absence" through the tuple $\mathbf{AntiTh}(\varphi; \mathcal{C}, \lambda) = \langle \neg P, \mathcal{J}_A^{\neg}, \mathcal{J}_N^{\neg} \rangle$. Here, the ontological element $\neg P$ represents a determinate relational fact of non-occurrence or exclusion at locus λ [4]. Crucially, Definition 2.8 distinguishes this from mere silence; it requires the provision of refutational "receipts" ($\mathcal{J}^{\neg}$) that satisfy the specific norms of the context. In the Western dialectical tradition, this corresponds to a reasoned negation capable of surviving scrutiny [33], while in the Eastern tradition, it invokes the sophisticated Nyāya machinery of *anupalabdhi* (non-perception)—the formal proof that a property is absent precisely because the conditions for its perception are met, yet the perception itself does not occur [39, 41]. By bundling the ontic denial with these specific modes of refutation, the framework ensures that the second corner of the Catuṣkoṭi is treated not as a passive void, but as an active, warranted commitment to the non-existence of a property within a given operational domain.

Symmetry and the Preclusion of Truth-Value Confusion. The structural symmetry between the Thesis and Antithesis serves a vital role in navigating quantum contextuality. By requiring both corners to pass the Applicability predicate ($\mathrm{App}_{\mathcal{C}}$) before evaluation, the formalism prevents the conflation of a "false" result with an "inapplicable" one. In the context of Sorkin's anhomomorphic logic, the Antithesis identifies histories that are classically denied but remain within the set of meaningful possibilities [49]. This preserves the logical integrity of the first two corners as the domain of classical bivalence ($P \vee \neg P$), while leaving the

necessary conceptual space for the paraconsistent Synthesis (where both hold) and the paracomplete Holothesis (where neither is applicable). Consequently, the Antithesis functions as the necessary boundary-marker that defines what it means for a claim to be meaningfully denied without collapsing the entire logical topology of the quantum system.

2.5.1   Absolute vs. Local Denial

The standard Antithesis defined above is *local* to a specific locus λ. We may also define a global variant that ranges over all admissible loci.

Definition 2.9 (Absolute Antithesis). If the intent is *absolute* denial, quantification ranges over all admissible loci:

$$\text{AntiTh}_{\text{abs}}(\varphi; \mathcal{C}) \text{ holds} \iff \forall \lambda' \in \Lambda_{\mathcal{C}} \ \mathcal{C}, \lambda' \Vdash \neg \exists^{\mathcal{C}}_{\lambda'} x \ \varphi(x). \tag{3}$$

2.5.2   Paraconsistent consequence regimes

The formalization of the Synthesis corner (third corner) requires a logic that tolerates contradiction without triviality, necessitating the following definition [54, 55].

Definition 2.10 (Paraconsistent consequence regimes). Fix an immanent context $\mathcal{C}$ with language $\text{Lang}(\mathcal{C})$. A consequence relation $\vdash_{\text{pc}}$ on $\text{Lang}(\mathcal{C})$ is called *paraconsistent* iff it is *non-explosive*, i.e. it is *not* the case that a contradiction entails every formula [13]:

$$\neg \Big(\forall \chi \in \text{Lang}(\mathcal{C}) \ \forall \psi \in \text{Lang}(\mathcal{C}) \ \big(\{\chi, \neg\chi\} \vdash_{\text{pc}} \psi \big) \Big).$$

Equivalently, there exist formulas $\chi, \psi \in \text{Lang}(\mathcal{C})$ such that

$$\{\chi, \neg\chi\} \nvdash_{\text{pc}} \psi.$$

When $\bot$ is available in $\text{Lang}(\mathcal{C})$, a common sufficient formulation of non-explosion is

$$\exists \chi \in \text{Lang}(\mathcal{C}) : \ \{\chi, \neg\chi\} \nvdash_{\text{pc}} \bot.$$

A *paraconsistent evaluation/holding regime* at locus $\lambda \in \Lambda_{\mathcal{C}}$ is a holding relation $\Vdash^{\text{pc}}_{\mathcal{C}, \lambda}$ designated by $\mathcal{C}$ whose associated consequence relation is paraconsistent in the above sense.

Scaling the Denial: From Local to Absolute.   While the standard Antithesis is strictly local—denying a fact at a single, specific locus λ—Definition 2.9 provides the formal syntax for a global sweep of the context. By applying the universal quantifier ($\forall \lambda'$) across the entire admissible locus-set $\Lambda_{\mathcal{C}}$, the Absolute Antithesis asserts that a property is denied *everywhere* within that specific operational domain. In quantum physical terms, this scalability is vital. It allows the framework to formally distinguish between a localized absence (e.g., "the particle did not take path A") and an absolute systemic absence (e.g., "the particle is entirely missing from the experimental apparatus"), all while strictly preserving the overarching context-dependence ($\mathcal{C}$) of the quantum logic.

Disarming the Principle of Explosion. Before the third corner of the Catușkoți—the Synthesis, or "both/and" stance—can be formalized, the logical architecture must first be insulated against systemic collapse. In classical Aristotelian logic, the Principle of Explosion (*ex contradictione quodlibet*) dictates that the presence of a true contradiction ($P \wedge \neg P$) trivializes the entire system, allowing any arbitrary statement to be proven valid [13]. Because quantum superpositions and logical gluts structurally require the simultaneous holding of mutually exclusive states, Definition 2.10 acts as a necessary mathematical firewall. By explicitly defining a logic ($\vdash_{\mathrm{pc}}$) where a contradiction $\{\chi, \neg\chi\}$ does *not* entail an arbitrary formula ψ or the absolute falsum $\bot$, the framework successfully disarms the explosion [54, 55]. The designation of a paraconsistent holding relation ($\Vdash_{\mathcal{C},\lambda}^{\mathrm{pc}}$) thus creates a localized, non-explosive mathematical sandbox. It secures the formal capacity for a quantum state to simultaneously "be" and "not be" at a given locus without that localized logical glut infecting and trivializing the broader deductive fabric of the universe.

2.6 The Synthesis

A *Synthesis* is a determinate, context-indexed co-commitment that simultaneously affirms and denies existence (*Asti ca Nāsti ca*; *Sadasat*) [42]. To be non-trivial, it requires a disciplined non-explosive consequence regime: the co-maintenance of P and $\neg P$ must be co-tenable without collapsing into derivational triviality [54, 55]. Its warrant is therefore twofold: (i) paired positive and negative defenses (as required by $\mathsf{W}(\mathcal{C})$), and (ii) an explicit nontriviality witness internal to the admitted paraconsistent regime [13].

Definition 2.11 (Synthesis (*Asti ca Nāsti ca*; *Sadasat*))**.** Fix $\mathcal{D}$, $\mathcal{C}$, $\Lambda_{\mathcal{C}}$, and $\lambda \in \Lambda_{\mathcal{C}}$. Define

$$P := \exists_{\lambda}^{\mathcal{C}} x \, \varphi(x).$$

Assume that $\mathcal{C}$ admits at least one *paraconsistent* consequence/evaluation regime; fix such a regime and write its holding relation as $\Vdash_{\mathcal{C},\lambda}^{\mathrm{pc}}$. Non-explosion is expressed by requiring that it is not the case that $\{P, \neg P\}$ trivializes the theory; equivalently, there exists some formula ψ such that $\{P, \neg P\} \not\vdash_{\mathrm{pc}} \psi$ [13, 54].

A *Synthesis-act* regarding φ at λ is the structured tuple

$$\mathbf{SynTh}(\varphi; \mathcal{C}, \lambda) := \left\langle P, \neg P, \mathcal{J}_A^+, \mathcal{J}_A^-, \mathcal{J}_N^+, \mathcal{J}_N^-, \Pi_{\mathrm{pc}}(P) \right\rangle,$$

and we say that the Synthesis *is warranted* (write $\mathrm{SynTh}(\varphi; \mathcal{C}, \lambda)$ holds) when the following conditions obtain:

(i) Co-commitment (Both Being and Not-Being).
   $\mathrm{App}_{\mathcal{C}}(P; \lambda)$ and $\mathcal{C}, \lambda \Vdash^{\mathrm{pc}} P$, while also $\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$ and $\mathcal{C}, \lambda \Vdash^{\mathrm{pc}} \neg P$. This is the "both" corner (Catușkoți: third corner) [46, 53].

(ii) Stability against admissible disambiguations.
   For every elementary refinement $r \in \mathsf{R}(\mathcal{C})$, letting $\mathcal{C}_r := r(\mathcal{C})$ and $P_r := \exists_{\lambda}^{\mathcal{C}_r} x \, \varphi(x)$, one still has $\mathrm{App}_{\mathcal{C}_r}(P_r; \lambda)$, $\mathrm{App}_{\mathcal{C}_r}(\neg P_r; \lambda)$, and $\mathcal{C}_r, \lambda \Vdash^{\mathrm{pc}} P_r$ as well as $\mathcal{C}_r, \lambda \Vdash^{\mathrm{pc}} \neg P_r$. (Intuitively: the contradiction is not an artifact removable by any refinement that $\mathcal{C}$ itself recognizes as a legitimate disambiguation.)

(iii) Paraconsistent discipline (nontriviality).
$\Pi_{\mathrm{pc}}(P)$ witnesses non-explosion; a minimal proof-theoretic witness is $\exists \psi$ such that $\{P, \neg P\} \nvdash_{\mathrm{pc}} \psi$ [54, 55].

(iv) Dialectical posture (Aristotle, paired).
If $A \in \mathsf{W}(\mathcal{C})$, then $\mathcal{J}_A^+(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A^+(\mathcal{C})$ and $\mathcal{J}_A^-(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A^-(\mathcal{C})$ [33].

(v) Procedural obligation (Nyāya, paired).
If $N \in \mathsf{W}(\mathcal{C})$, then $\mathcal{J}_N^+(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N^+(\mathcal{C})$ and $\mathcal{J}_N^-(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N^-(\mathcal{C})$ [39, 41].

Equivalently,

$$
\begin{aligned}
\mathrm{SynTh}(\varphi; \mathcal{C}, \lambda) \text{ holds} \iff & \mathrm{App}_{\mathcal{C}}(P; \lambda) \wedge \mathrm{App}_{\mathcal{C}}(\neg P; \lambda) \wedge (\mathcal{C}, \lambda \Vdash^{\mathrm{pc}} P) \wedge (\mathcal{C}, \lambda \Vdash^{\mathrm{pc}} \neg P) \\
& \wedge \left(\forall r \in \mathsf{R}(\mathcal{C}) \text{ the above persists in } (\mathcal{C}_r, \lambda) \right) \wedge \Pi_{\mathrm{pc}}(P) \\
& \wedge \left(A \notin \mathsf{W}(\mathcal{C}) \vee (\mathcal{J}_A^+ \in \mathrm{Arg}_A^+(\mathcal{C}) \wedge \mathcal{J}_A^- \in \mathrm{Arg}_A^-(\mathcal{C})) \right) \\
& \wedge \left(N \notin \mathsf{W}(\mathcal{C}) \vee (\mathcal{J}_N^+ \in \mathrm{Arg}_N^+(\mathcal{C}) \wedge \mathcal{J}_N^- \in \mathrm{Arg}_N^-(\mathcal{C})) \right).
\end{aligned}
$$

Synthesis as Non-Explosive Co-commitment. The formulation of the Synthesis in Definition 2.11 fundamentally advances the mid-century comparative attempts by T. R. V. Murti and Richard H. Robinson, who utilized the Fichtean triad as a makeshift translation for the first three corners of the Catuṣkoṭi [45, 48]. Unlike those previous models, which were often forced to treat the Synthesis as a resolutive "sublation" or merger, the present framework transforms the third corner into a disciplined, non-explosive co-commitment. By formalizing the Synthesis-act as a structured tuple $\mathbf{SynTh}(\varphi; \mathcal{C}, \lambda) = \langle P, \neg P, \mathcal{J}_A^+, \mathcal{J}_A^-, \mathcal{J}_N^+, \mathcal{J}_N^-, \Pi_{\mathrm{pc}}(P) \rangle$, the formalism operates as a mathematical "Certificate of the Glut." The inclusion of the nontriviality witness $\Pi_{\mathrm{pc}}(P)$ provides the necessary proof-theoretic evidence that the paraconsistent holding relation $(\Vdash_{\mathcal{C}, \lambda}^{\mathrm{pc}})$ successfully contains the contradiction, moving the "Both" stance out of the realm of mystical paradox and into a rigorous logic that avoids derivational triviality [54, 55].

Refinement Stability and the Quantum Mapping. Crucially, the framework introduces a filter for distinguishing between fundamental ontological gluts and mere epistemic failures through the requirement of stability against admissible disambiguations. Historically, the "Both" corner was often dismissed as an artifact of linguistic vagueness that would vanish upon higher resolution. Definition 2.11 explicitly addresses this via Condition (ii), which insists that the dual-holding status persists across the entire refinement-closure $\mathrm{Ref}(\mathcal{C})$. By requiring that the contradiction survive all elementary refinement operators $r \in \mathsf{R}(\mathcal{C})$, the formalism distinguishes between removable epistemic uncertainty and irreducible physical phenomena such as quantum superposition and interference [49, 53]. This ensures that the Synthesis functions as a high-fidelity model for states that stubbornly refuse to resolve into classical bivalence, regardless of how meticulously the unmeasured history space is refined [52].

2.7   The Holothesis

A *Holothesis* is a determinate, context-indexed suspension that refuses both affirmation and denial of existence (*Naiva Asti na Nāsti*) [42], not as a residual "none of the above," but as a categorical shift in which the *applicability* of existential polarity (Being/Not-Being, and their co-tenability) is withdrawn. Its warrant is metadialectical: polarity-questions are ill-typed or presuppositionally failed at this level, so the correct move is to register inapplicability and (where possible) shift to a stratum in which an alternative, well-typed description can be evaluated [13, 53].

Definition 2.12 (Holothesis (*Naiva Asti na Nāsti*)). Fix $\mathcal{D}$, $\mathcal{C}$, $\Lambda_{\mathcal{C}}$, and $\lambda \in \Lambda_{\mathcal{C}}$. Define

$$P \;:=\; \exists_{\lambda}^{\mathcal{C}} x\, \varphi(x).$$

A *Holothesis-act* about φ at λ is the structured tuple

$$\mathbf{HoloTh}(\varphi; \mathcal{C}, \lambda) \;:=\; \big\langle\, \neg\,\mathrm{App}_{\mathcal{C}}(P; \lambda),\; \neg\,\mathrm{App}_{\mathcal{C}}(\neg P; \lambda),\; \mathcal{J}_{A}^{\perp}(\varphi; \mathcal{C}, \lambda),\; \mathcal{J}_{N}^{\perp}(\varphi; \mathcal{C}, \lambda) \,\big\rangle,$$

and we say that the Holothesis *is warranted* (write $\mathrm{HoloTh}(\varphi; \mathcal{C}, \lambda)$ holds) when the following conditions obtain:

(i) Polar inapplicability (Neither Being nor Not-Being).
    $\neg\,\mathrm{App}_{\mathcal{C}}(P; \lambda)$ (equivalently, by negation-closure, $\neg\,\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$). This is the "neither" corner (Catuṣkoṭi: fourth corner) [44, 46].

(ii) Stability under admissible refinements.
    For every elementary refinement $r \in \mathsf{R}(\mathcal{C})$, letting $\mathcal{C}_r := r(\mathcal{C})$ and $P_r := \exists_{\lambda}^{\mathcal{C}_r} x\, \varphi(x)$, one has $\neg\,\mathrm{App}_{\mathcal{C}_r}(P_r; \lambda)$ (and hence $\neg\,\mathrm{App}_{\mathcal{C}_r}(\neg P_r; \lambda)$). (Intuitively: no refinement that $\mathcal{C}$ itself recognizes as legitimate restores existential polarity as a coherent predicative move here.) This stability condition parallels the contextuality constraints in quantum logic where certain observables remain undefined across compatible refinements [9, 11].

(iii) Dialectical posture (Aristotle, meta).
    If $A \in \mathsf{W}(\mathcal{C})$, then $\mathcal{J}_{A}^{\perp}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_{A}^{\perp}(\mathcal{C})$, i.e. a disciplined meta-argument showing that forcing P vs. $\neg P$ here commits a category mistake, presupposition failure, or illicit shift in the base of predication [33].

(iv) Procedural obligation (Nyāya, meta).
    If $N \in \mathsf{W}(\mathcal{C})$, then $\mathcal{J}_{N}^{\perp}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_{N}^{\perp}(\mathcal{C})$, i.e. a disciplined Nyāya-style meta-justification whose *pratijñā* is an inapplicability claim and whose support makes explicit the failed delimitor/presupposition that $\mathcal{C}$ recognizes [39, 41].

Optional strengthening (shift witness).   If one can exhibit a translation to a stratum where the relevant content is evaluable without reinstating existential polarity as a stable predication, one may strengthen a warranted Holothesis by adjoining a lift $\Uparrow\colon \mathrm{Lang}(\mathcal{C}) \to \mathrm{Lang}(\mathcal{C}^{\Uparrow})$ and requiring that $\Uparrow(\varphi)$ is evaluable in $\mathcal{C}^{\Uparrow}$ while projections back to $\mathcal{C}$ do not restore applicability of

P vs. $\neg P$ at λ. This strengthening is a consequent enrichment, not a prerequisite for asserting inapplicability, and resonates with Bohr's requirement to shift experimental arrangements to define complementary quantities [11].

Equivalently,

$$\begin{aligned}
\mathrm{HoloTh}(\varphi; \mathcal{C}, \lambda) \text{ holds} \iff & \neg \mathrm{App}_{\mathcal{C}}(P; \lambda) \wedge \left(\forall r \in \mathsf{R}(\mathcal{C}) \, \neg \mathrm{App}_{\mathcal{C}_r}(P_r; \lambda) \right) \\
& \wedge \left(A \notin \mathsf{W}(\mathcal{C}) \vee \mathcal{J}_A^{\perp}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_A^{\perp}(\mathcal{C}) \right) \\
& \wedge \left(N \notin \mathsf{W}(\mathcal{C}) \vee \mathcal{J}_N^{\perp}(\varphi; \mathcal{C}, \lambda) \in \mathrm{Arg}_N^{\perp}(\mathcal{C}) \right).
\end{aligned}$$

Certificate of the Gap and Meta-Dialectical Warrant. The Holothesis-act completes the dialectical topology by formalizing the "Certificate of the Gap." Unlike the first three corners, which operate within the domain of existential polarity, Definition 2.12 identifies the regions of the history space where the question itself fails the applicability predicate ($\neg \mathrm{App}_{\mathcal{C}}$). In this corner, the warrant is strictly metadialectical: it requires a disciplined meta-argument ($\mathcal{J}^{\perp}$) that identifies a category mistake, a type error, or a presupposition failure. This provides a rigorous logical identity for the "neither/nor" stance, treating it not as a vacuum of information, but as a warranted register of inapplicability.

Quantum Preclusion and the Complementarity Shift. In the context of quantum foundations, the Holothesis serves as the exact logical counterpart to Rafael Sorkin's measure-zero preclusions [8]. By insisting on stability under refinement (Condition ii), the formalism ensures that the gap is not a removable artifact of coarse-graining, but a structural feature of the physical context itself. Furthermore, the "Optional Strengthening" of the Holothesis provides the formal syntax for Niels Bohr's principle of complementarity [11]. It acknowledges that while a property may be inapplicable in context $\mathcal{C}$ (e.g., the position of an electron in a momentum-measurement context), a "lift" to context $\mathcal{C}^{\Uparrow}$ via the operator $\Uparrow$ may restore evaluability. This mathematical architecture thus elegantly binds the "neither/nor" of ancient Indian logic to the non-Boolean restrictions of modern quantum mechanics.

Remark 2.13 (Resolution of Stability Circularity: Context-Class Preservation)**.** To address the potential circularity in Definition 2.12 (clause H2), we formally distinguish between two classes of transformations on the locus $(\mathcal{C}, \lambda)$:

(i) **Context-Class Preserving Refinements ($\mathsf{R}_{\mathrm{pres}}$):** These are refinements $r \in \mathsf{R}(\mathcal{C})$ that maintain the *operational type* of the context (e.g., improving detector resolution within an interference arrangement). For these refinements, Holothesis is *stable*:

$$r \in \mathsf{R}_{\mathrm{pres}}(\mathcal{C}) \quad \implies \quad \neg \mathrm{App}_{\mathcal{C}_r}(P_r; \lambda). \tag{4}$$

This captures the physical intuition that no amount of "zooming in" within an interference context restores which-path polarity.

(ii) **Context-Transforming Shifts ($\mathsf{S}_{\mathrm{shift}}$):** These are operations that map $(\mathcal{C}, \lambda) \to (\mathcal{C}', \lambda')$ where the operational type changes (e.g., inserting a which-path detector). For these

shifts, Holothesis stability is *not required*:

$$s \in \mathsf{S}_{\text{shift}} \quad \Longrightarrow \quad \text{App}_{\mathcal{C}'}(P'; \lambda') \text{ may hold.} \tag{5}$$

The "Shift Witness" (Definition 2.12, optional strengthening) belongs to this class. It demonstrates the existence of a complementary context without violating the stability of the original context.

Clause (ii) of Definition 2.12 is thus understood to quantify over $\mathsf{R}_{\text{pres}}(\mathcal{C}) \subseteq \mathsf{R}(\mathcal{C})$, not over all possible context transformations. This distinction formalizes Bohr's complementarity: certain questions are *structurally excluded* from a context class, not merely unresolved [9, 11].

Definition 2.14 (Admissible Refinement Subgroup)**.** Let $\mathsf{R}(\mathcal{C})$ be the set of sanctioned unitary symmetries or operational refinements for context $\mathcal{C}$ (cf. Section 6.5). We define the *context-class preserving subgroup* $\mathsf{R}_{\text{pres}}(\mathcal{C})$ as the subset of refinements that commute with the context's typing predicate $\text{App}_{\mathcal{C}}$:

$$\mathsf{R}_{\text{pres}}(\mathcal{C}) := \left\{ r \in \mathsf{R}(\mathcal{C}) \ : \ \text{App}_{\mathcal{C}}(P; \lambda) \iff \text{App}_{\mathcal{C}_r}(P_r; \lambda) \right\}. \tag{6}$$

Holothesis stability (Definition 2.12, H2) is enforced strictly over $\mathsf{R}_{\text{pres}}(\mathcal{C})$.

Context-Class Preservation and the Resolution of Circularity. To ensure the logical integrity of the fourth corner, Remark 2.13 introduces a critical distinction between transformations that merely refine a context and those that fundamentally shift its operational type. A potential circularity arises if the Holothesis is required to be stable under *all* possible refinements; specifically, if the insertion of a measurement apparatus (e.g., a which-path detector) were classified as a simple refinement, the destruction of the interference-based gap would ostensibly invalidate the Holothesis. By bifurcating transformations into Context-Class Preserving Refinements (R_{pres}) and Context-Transforming Shifts ($\mathsf{S}_{\text{shift}}$), the framework mathematically secures the stability of the gap. As demonstrated in eq. (4), the Holothesis is strictly stable under R_{pres}, capturing the physical intuition that increasing detector resolution within an interference arrangement cannot restore the applicability of which-path polarity. This distinction allows the Holothesis to function as a structural exclusion rather than a mere temporary lack of information [9, 11].

Algebraic Subgrouping and the Commuting Predicate. The formalization of the Context-Class Preserving Subgroup in Definition 2.14 provides the exact mathematical syntax for anchoring this stability in the algebraic structure of quantum theory. By defining $\mathsf{R}_{\text{pres}}(\mathcal{C})$ as the subset of refinements that commute with the context's typing predicate $\text{App}_{\mathcal{C}}$, the formalism aligns the logical "Neither/Nor" stance with the commutative subalgebras of observables found in Topos-theoretic approaches to physics [9]. This subgrouping ensures that the stability of the Holothesis (H2) is only tested against operations that preserve the internal logical topology of the context. Furthermore, the existence of the Context-Transforming Shift ($\mathsf{S}_{\text{shift}}$) provides a rigorous identity for Bohr's principle of complementarity [11]. It proves that while a property may be structurally excluded from the original context, a shift to a non-commuting complementary context may restore applicability without violating

the stability of the original Holothesis. This multi-layered architecture thus provides an airtight logical engine for handling the non-Boolean restrictions of the quantum universe while preserving the dialectical completeness of the Catuṣkoṭi.

2.8 The Tetralemmatic Walk of the Double-Slit Experiment

The operational and context-dependent character of the four-corner logic becomes concrete in the double-slit experiment, a paradigmatic case for quantum complementarity [10, 11]. Fix the domain of discourse $\mathcal{D}$ to single quantum particles (electrons) and evaluate claims relative to an immanent context $\mathcal{C}$ and a locus $\lambda \in \Lambda_{\mathcal{C}}$ as in Section 2. We work within the quantum realization scheme of Section 6.5.

Predicate, polar claim, and Born-valued interpretation. Let $\varphi(x)$ be the predicate "x is localized at the left slit at a fixed time t_0 during its passage." For a given context $\mathcal{C}$ and locus λ, form the existential claim (cf. Section 2)

$$P := \exists_{\lambda}^{\mathcal{C}} x \, \varphi(x) \qquad (\text{"The electron went through the left slit at } t_0\text{"}).$$

When P is *applicable* at $(\mathcal{C}, \lambda)$ in the sense of Section 2, the realization map of Section 6.5 assigns an operational effect consistent with generalized quantum measurement theory [56, 57]

$$E_P := \mathcal{R}_{\mathcal{C},\lambda}(P), \qquad 0 \leq E_P \leq I,$$

and (when defined) its negation is represented by the complementary effect

$$E_{\neg P} := \mathcal{R}_{\mathcal{C},\lambda}(\neg P) = I - E_P.$$

(Recall also the negation-closure of applicability from Section 2, so that $\mathrm{App}_{\mathcal{C}}(P; \lambda) \Leftrightarrow \mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$.)

Given a density operator ρ, the Born-valued degree of warrant is computed via the realized effect following the standard probability rule [58, 59] (cf. Section 6.5):

$$v(P; \rho) := \mathrm{Tr}(E_P \rho), \qquad v(\neg P; \rho) := \mathrm{Tr}(E_{\neg P} \rho),$$

whenever the corresponding realized effects are defined. If P is *inapplicable* at $(\mathcal{C}, \lambda)$, then $\mathcal{R}_{\mathcal{C},\lambda}(P)$ is undefined and $v(P; \rho)$ is not formed (typed inapplicability, cf. Section 2).

To model the slit alternatives, take a two-dimensional subspace with orthonormal kets $|L\rangle, |R\rangle$ representing perfect localization at t_0, and (when the which-path test is admitted) set

$$E_P = |L\rangle\langle L|, \qquad E_{\neg P} = |R\rangle\langle R|.$$

Scenario A: Which-path context $(\mathcal{C}_1, \lambda_1)$. Let $\mathcal{C}_1$ be the context in which a which-path detector is active at the slits, and let λ_1 be the locus "at the slits during the detection event."

(A1) Applicability.
   Because the which-path detector physically implements the test, the polar claims are applicable:

$$\mathrm{App}_{\mathcal{C}_1}(P; \lambda_1) = \mathrm{App}_{\mathcal{C}_1}(\neg P; \lambda_1) = \text{True} \qquad (\text{cf. Section 2}).$$

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(A2) Measurement process.
   The detector realizes a measurement interaction that correlates the electron with detector records and suppresses coherence in the $\{|L\rangle, |R\rangle\}$ basis via environmental decoherence [22, 23] (an idealized model is a Lüders-type update [60]). Operationally, this is precisely the situation in which P vs. $\neg P$ is a well-typed polar question at λ_1.

(A3) Outcome (Thesis or Antithesis).
   A single sharp outcome occurs. Either the left outcome is recorded, in which case $\mathcal{C}_1, \lambda_1 \Vdash P$ and one is in the first corner (Thesis, cf. Section 2.4 and Definition 2.7); or the right outcome is recorded, in which case $\mathcal{C}_1, \lambda_1 \Vdash \neg P$ and one is in the second corner (Antithesis, cf. Section 2.5 and Definition 2.8).

(A4) Absence of Holothesis at the slit locus.
   Since P and $\neg P$ are applicable at $(\mathcal{C}_1, \lambda_1)$, the fourth corner is not warranted there: Holothesis requires polar inapplicability (cf. Section 2.7 and Definition 2.12).

Scenario B: Interference context $(\mathcal{C}_2, \lambda_2)$.   Let $\mathcal{C}_2$ be the context with no which-path detector, and let λ_2 be the locus "between the slits and the screen, before any position measurement."

(B1) Inapplicability (unaskability).
   In $\mathcal{C}_2$ at λ_2, the arrangement does not admit an operational which-path test, reflecting quantum contextuality constraints [2, 9]. Therefore the polar claims are inapplicable:

$$\neg \operatorname{App}_{\mathcal{C}_2}(P; \lambda_2) = \neg \operatorname{App}_{\mathcal{C}_2}(\neg P; \lambda_2) = \text{True} \qquad \text{(cf. Section 2).}$$

   Equivalently, $\mathcal{R}_{\mathcal{C}_2, \lambda_2}(P)$ and $\mathcal{R}_{\mathcal{C}_2, \lambda_2}(\neg P)$ are undefined (as per the typing discipline implicit in Section 6.5).

(B2) Holothesis warranted.
   Because the polar question is ill-typed at $(\mathcal{C}_2, \lambda_2)$, the correct corner-status there is Holothesis (cf. Section 2.7 and Definition 2.12). The disciplined meta-statement is:

   "The predicate 'went through the left slit' is not meaningfully evaluable at $(\mathcal{C}_2, \lambda_2)$."

(B3) State description (coherent superposition).
   Although the polar which-path proposition is inapplicable at λ_2, the electron may still be described kinematically by a pure state in the $\{|L\rangle, |R\rangle\}$ subspace, e.g.

$$|\psi\rangle = \alpha|L\rangle + \beta|R\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

   This is a statement about the quantum state used for later Born-rule predictions (cf. Section 6.5), not a licensed object-level verdict on P at λ_2.

(B4) Where (and how) "Synthesis" fits without violating applicability.
   In this framework, Synthesis requires (i) applicability of both P and $\neg P$, (ii) a designated paraconsistent holding regime $\Vdash^{\text{pc}}$, and (iii) a non-explosion witness (cf.

Definition 2.10 and Section 2.6 and Definition 2.11). Therefore, at the literal interference locus $(\mathcal{C}_2, \lambda_2)$—where applicability fails—the corner-status is Holothesis (cf. Definition 2.12), not Synthesis (cf. Definition 2.11).

To discuss Synthesis legitimately, shift to a nearby context/locus $(\mathcal{C}', \lambda')$ in which the which-path question is operationally definable (so applicability holds) while coherence is not fully destroyed—for instance, by a *soft* which-path coupling to a meter M akin to weak measurement protocols [61]. Let H_M be a meter Hilbert space with initial state $|0\rangle_M$ and let an interaction unitary U_{int} satisfy

$$U_{\text{int}}\big(|L\rangle \otimes |0\rangle_M\big) = |L\rangle \otimes |m_L\rangle_M, \qquad U_{\text{int}}\big(|R\rangle \otimes |0\rangle_M\big) = |R\rangle \otimes |m_R\rangle_M,$$

with $\langle m_L | m_R \rangle \neq 0$ (weak distinguishability). Choose meter effects F_L, F_R on H_M with $F_L \geq 0, F_R \geq 0, F_L + F_R = I_M$, interpreted as "left-pointer" and "right-pointer" tests, and define (in this context) realized polar effects on $H \otimes H_M$ by

$$\mathcal{R}_{\mathcal{C}', \lambda'}(P) := I \otimes F_L, \qquad \mathcal{R}_{\mathcal{C}', \lambda'}(\neg P) := I \otimes F_R.$$

Then $\text{App}_{\mathcal{C}'}(P; \lambda')$ and $\text{App}_{\mathcal{C}'}(\neg P; \lambda')$ hold (cf. Section 2).

For a coherent input $|\psi\rangle = \alpha|L\rangle + \beta|R\rangle$, the post-interaction joint state is

$$|\Psi\rangle = \alpha|L\rangle|m_L\rangle_M + \beta|R\rangle|m_R\rangle_M, \qquad \rho_{SM} = |\Psi\rangle\langle\Psi|.$$

In such a setting, both $v(P; \rho_{SM})$ and $v(\neg P; \rho_{SM})$ are typically non-zero. If $\mathcal{C}'$ designates a paraconsistent holding regime $\Vdash^{\text{pc}}$ (cf. Definition 2.10), one convenient choice of holding convention is:

$$\mathcal{C}', \lambda' \Vdash^{\text{pc}} \chi \quad \text{whenever} \quad v(\chi; \rho_{SM}) > 0,$$

so that $\mathcal{C}', \lambda' \Vdash^{\text{pc}} P$ and $\mathcal{C}', \lambda' \Vdash^{\text{pc}} \neg P$ can hold together, realizing the co-commitment clause (S1) of Synthesis (cf. Definition 2.11). Non-explosion is then witnessed by the fact that the co-holding does not force arbitrary conclusions about meter definiteness; equivalently, one can select a formula ψ (e.g. a definite-pointer claim) such that $\{P, \neg P\} \nvdash_{\text{pc}} \psi$ in the regime designated by $\mathcal{C}'$, consistent with paraconsistent logic frameworks [13] (cf. Definitions 2.10 and 2.11).

In short: Synthesis belongs to contexts that admit the polar effects while preserving coherence, whereas the pure interference arrangement warrants Holothesis because the polar which-path question is not licensed at that locus.

Empirical signature: the interference pattern at the screen. When the electron reaches the screen, move to a new context/locus $(\mathcal{C}_3, \lambda_3)$ in which position measurement is admitted. Let $|x\rangle$ denote position eigenkets (or a suitable coarse-graining). The position claim "the electron hits at coordinate x" is applicable at $(\mathcal{C}_3, \lambda_3)$, and the Born rule yields

$$p(x) = |\langle x|\psi\rangle|^2 \qquad \text{(cf. Section 6.5).}$$

This exhibits interference fringes exactly when coherence between the $|L\rangle$ and $|R\rangle$ components is preserved, recovering the classical wave interference phenomenon [62]. Those fringes are

destroyed when one forces applicability and determinacy of the polar which-path question by inserting a which-path detector, shifting from Scenario B back to Scenario A [10].

This tetralemmatic walk illustrates the core lesson of Section 2.3: the status of a claim is governed by the operational context $\mathcal{C}$ and locus λ, via the gatekeeping role of applicability and the designated holding regime. Quantum mechanics supplies the realization calculus (Section 6.5) that determines which propositions are admitted and how their degrees of warrant are computed.

Table 1: Tetralemmatic Mapping of the Double-Slit Experiment

Scenario	Corner	Logical Status	Physical Mechanism
A: Which-Path	Thesis / Antithesis	App $\to$ True	Decoherence and sharp measurement realization.
B: Interference	**Holothesis**	$\neg\,\mathrm{App}$	Coherent superposition and structural preclusion (unaskability).
B-variant: Weak	**Synthesis**	$\Vdash^{\mathrm{pc}} (P \wedge \neg P)$	Weak measurement coupling preserving partial coherence.

2.9   Operational Derivation of the Four-Corner Structure

To address the physical necessity of the tetralemmatic structure, we demonstrate that it is not a postulated primitive but a minimal derivation from two operational requirements: the existence of *polarity* (affirmation vs. denial) and the constraint of *contextual applicability* (Definition 3.8).

Proposition 2.15 (Minimal Logical Manifold from Operational Constraints)**.** *Let $\mathcal{V} = \{+, -\}$ denote the valuation space (polarity) and $\mathcal{A} = \{1, 0\}$ denote the applicability space, where $1 \equiv \mathrm{App}_{\mathcal{C}}(P; \lambda)$ and $0 \equiv \neg\,\mathrm{App}_{\mathcal{C}}(P; \lambda)$. The cross-product $\mathcal{V} \times \mathcal{A}$ yields exactly four sectors:*

(i) $(+, 1)$: Thesis (Applicable Affirmation, Definition 2.7).

(ii) $(-, 1)$: Antithesis (Applicable Denial, Definition 2.8).

(iii) $(\{+, -\}, 1)$: Synthesis (Coherent co-holding where binary exclusion fails, Definition 2.11).

(iv) $(\cdot, 0)$: Holothesis (Inapplicable where binary inquiry fails, Definition 2.12).

Standard binary logic is the restriction to the $\mathcal{A} = 1$ slice. Quantum mechanics, by introducing interference and contextuality (Kochen–Specker), forces the inclusion of the $\mathcal{A} = 0$ slice, thus necessitating the tetralemma as the minimal logical manifold [2, 9].

Proof. **Exhaustiveness.** For any polar claim P at locus $(\mathcal{C}, \lambda)$, the applicability predicate $\mathrm{App}_{\mathcal{C}}(P; \lambda)$ is either true or false (law of excluded middle at the meta-level). If true, P admits

polar valuation ($+$ or $-$). If false, polar valuation is undefined, yielding the inapplicable sector. This exhausts all operational possibilities.

Minimality. Any logic with fewer than four sectors would collapse distinct operational regimes:

- Three-valued logic (e.g. Kleene) cannot distinguish Synthesis (co-holding) from Holothesis (inapplicability) [13].

- Binary logic with contextuality (standard SQM) treats inapplicability as external to the logic (lattice gaps) rather than as a first-class logical status [63].

Thus four sectors are necessary and sufficient [55]. $\qquad\square$

Remark 2.16 (Why Standard SQM is Insufficient)**.** Standard quantum mechanics employs a binary projection lattice $\mathcal{L}(\mathcal{H})$ where propositions are either true (eigenvalue 1) or false (eigenvalue 0). However, this classical binary mapping fails to capture the full operational manifold:

(i) **Interference states** (e.g. $|L\rangle + |R\rangle$) are not eigenstates of which-path projectors, yet they are physically real. Standard logic treats them as a "gap," but lacks the machinery to formalize this gap as a positive ontological status. Synthesis provides this by formalizing interference as a paraconsistent co-holding.

(ii) **Contextuality** (Kochen–Specker) proves no global truth-value assignment exists [2]. Standard QM handles this by restricting the domain of discourse *ad hoc*, but does not formalize the *reason* for restriction (applicability failure).

By internalizing these phenomena, the tetralemma transforms contextuality from a limitation of the theory into the **engine of logic**. It assigns Holothesis as the first-class logical status for complementarity, turning the "hidden variable" problem into a matter of logical type-checking: if an observable does not commute with the context, it is not "hidden"; it is simply holothetically precluded. This constitutes an ontological expansion of the logic, not merely an interpretational shift [9].

2.10 The Atomic Tetralemmatic Lens

Etymology of *oṃ*. The sacred syllable *oṃ* ($\bar{o}ṃ/om$) occupies an unusual philological position: it is at once a phonological form (traditionally analysed as a–u–m, with $a+u$ contracting to o, and with final nasal closure marked in the orthography by $ṃ/m$) and a discourse particle whose earliest attested function is closely tied to ritual assent and authorization. Lexicographic tradition already registers this particle-like use (e.g., "yes," "so be it," "granted," especially at the opening of recitation), alongside the later nominalizations *oṃkāra* and *praṇava* that treat it as an object of doctrinal reflection [42]. Modern etymological proposals remain contested: Bloomfield's classic hypothesis traces *oṃ* to an inherited Indo-European introductory/confirmatory particle (often schematized as **au*), whereas Parpola's influential reassessment argues that the earliest semantic nucleus is best captured as a marker of assent/permission (explicitly glossed in early Upaniṣadic usage), and that this

favors derivations consistent with an "affirmation" particle—potentially including non–Indo-European contact etymologies discussed in his survey [64, 65, 66]. Hock sharpens the linguistic problem by separating questions of phonological development (including Vedic/early Sanskrit variation) from the history of function, emphasizing that the sanctification of *oṃ* emerges through its ritual deployment and only later becomes the object of systematic metaphysical symbolization [67]. Gerety's historical study, finally, situates these debates within the broader rise of *oṃ* as a "totalizing" sign in late Vedic and early soteriological milieus, providing an up-to-date synthesis of the evidence and of the secondary literature that explains why no single etymology commands universal assent [68].

Vajrayāna: *oṃ* as the "body-seal" of mantra.   In Vajrayāna, *oṃ* (ॐ) is treated less as an etymological particle than as a *bīja* (seed-syllable): a performative phoneme that "seals" the embodied gate of practice. In the ubiquitous triad *oṃ āḥ hūṃ*, the three syllables are correlated with the practitioner's "three doors" (body, speech, mind) and with the three vajras; generation-stage and guru-yoga instructions accordingly place *oṃ* at the crown/head (*kāya*), *āḥ* at the throat (*vāk*), and *hūṃ* at the heart (*citta*), so that purification and consecration are enacted as a coordinated transformation of all three modalities at once. Kongtrul's encyclopedic presentation makes the mapping explicit in deity-yoga terms: *oṃ* at the head is the causal seed that transforms into Vairochana as "Vajra Body," paired with *āḥ* as "Vajra Speech" (Amitābha) and *hūṃ* as "Vajra Mind" (Akṣobhya), after which the practitioner's three gateways are made "indivisible" with awakened body, voice, and mind [69, 70].

Etymology of *catvāri*.   The form *catvāri* (चत्वारि) is the neuter nominative/accusative plural of the Sanskrit numeral "four," paradigmatically paired with the masculine *catvāraḥ* and feminine *catasraḥ*; in practice it is frequently used substantively to mean "*the four*" as an abstract collection (rather than four items with marked gender) [42]. Philologically, the Sanskrit "four" family is the regular Indo-European reflex of Proto-Indo-European **kʷetwóres* (with the labiovelar **kʷ* yielding Sanskrit *c*), cognate with Greek *tettares* (τέτταρες), Latin *quattuor*, and English "four," so that *catvāri* inherits a stable comparative nucleus even as its surface morphology reflects the inherited suppletive-like gendered plural forms characteristic of Indo-European numerals [28, 29].

***Tetralemmatic* and the unitary sign $\mathring{\Lambda}$.**   With these two registers fixed—*oṃ* as bīja at the crown and *catvāri* as "the four"—we name the unified object *tetralemmatic lens* (i.e., an *oṃcatvāri*) and write it as $\mathring{\Lambda}$. The notation is meant as a *single inseparable, atomic unit*, not as an additive layering of an "*oṃ* part" and a "fourfold logic part." The crown-mark ∘ indicates that the tetralemmatic tetrahedron

$$\big\langle \mathbf{Th},\ \mathbf{AntiTh},\ \mathbf{SynTh},\ \mathbf{HoloTh} \big\rangle$$

is to be read here as *seeded*: it arises as the articulated fourfold deployment of inquiry issued from the *oṃ*-seed, rather than as a free-standing logical gadget to which *oṃ* is later appended. In this sense, Λ is only the tetralemma *as* oṃ-*seeded*, and *oṃ* is invoked only insofar as it is expressed as a four-corner articulation. The name *oṃcatvāri* therefore designates the identity "*oṃ* as the seed of the fourfold logic," and $\mathring{\Lambda}$ is its mathematical sigil.

Definition 2.17 (Tetralemmatic lens)**.** Let **Th**, **AntiTh**, **SynTh**, and **HoloTh** denote the four corner-acts exactly as defined in Section 2.4, Section 2.5, Section 2.6, and Section 2.7. Let Σ be the collection of symbols, relations, and constructions that occur in those four definitions (and only those).

The *tetralemmatic lens* is the corner-labelled Σ-structure

$$\overset{\circ}{\mathbf{\Lambda}} := \left\langle \mathbf{Th},\ \mathbf{AntiTh},\ \mathbf{SynTh},\ \mathbf{HoloTh} \right\rangle$$

characterized by the following clauses:

(a) No redefinition.

The internal content of each corner is exactly that given in its defining section; no further axioms on **Th**, **AntiTh**, **SynTh**, or **HoloTh** are imposed.

(b) Generated closure.

$\overset{\circ}{\mathbf{\Lambda}}$ is the smallest Σ-substructure containing the four distinguished corners. Equivalently, if $X \subseteq \overset{\circ}{\mathbf{\Lambda}}$ is a Σ-substructure and $\{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\} \subseteq X$, then $X = \overset{\circ}{\mathbf{\Lambda}}$.

(c) Atomicity (indecomposability).

There is no decomposition of $\overset{\circ}{\mathbf{\Lambda}}$ into a disjoint union of two nonempty corner-labelled Σ-substructures that both inherit the Σ-structure. In particular, no partition of $\{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$ yields two independent tetralemmatic units compatible with the inherited inter-corner constructions.

A *tetralemmatic morphism* $f : \overset{\circ}{\mathbf{\Lambda}} \to (\overset{\circ}{\mathbf{\Lambda}})'$ is a Σ-homomorphism that preserves corner labels:

$$f(\mathbf{Th}) = \mathbf{Th}', \qquad f(\mathbf{AntiTh}) = \mathbf{AntiTh}',$$
$$f(\mathbf{SynTh}) = \mathbf{SynTh}', \qquad f(\mathbf{HoloTh}) = \mathbf{HoloTh}'.$$

Two tetralemmatic units are *isomorphic* if there exists a bijective tetralemmatic morphism between them.

Proposition 2.18 (Invariance under admissible dualities and reindexings)**.** *Let D be an admissible involution that exchanges $\mathbf{Th} \leftrightarrow \mathbf{AntiTh}$ and fixes $\mathbf{SynTh}$ and $\mathbf{HoloTh}$, and let ρ be any admissible reindexing that preserves the four corner-definitions (hence preserves the induced Σ-structure). Then D and ρ are tetralemmatic automorphisms of $\overset{\circ}{\mathbf{\Lambda}}$, and therefore*

$$D(\overset{\circ}{\mathbf{\Lambda}}) = \overset{\circ}{\mathbf{\Lambda}}, \qquad \rho(\overset{\circ}{\mathbf{\Lambda}}) = \overset{\circ}{\mathbf{\Lambda}}.$$

Representing primitive ontological states in the language. Assume there is a representation map

$$\iota_{\mathcal{C}} : \mathcal{U}_{\mathcal{C}} \to \mathrm{Lang}(\mathcal{C}),$$

sending each primitive state s to a designated atomic formula $\iota_{\mathcal{C}}(s)$.

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Bridging to applicability/holding (optional). Define

$$\mathrm{Con}_{\mathcal{C}}(s) \; :\Longleftrightarrow \; \exists \lambda \in \Lambda_{\mathcal{C}} \; \mathrm{App}_{\mathcal{C}}\big(\iota_{\mathcal{C}}(s); \lambda\big),$$

and (one possible choice)

$$\mathrm{Aff}_{\mathcal{C}}(s) \; :\Longleftrightarrow \; \exists \lambda \in \Lambda_{\mathcal{C}} \; \Big(\mathrm{App}_{\mathcal{C}}\big(\iota_{\mathcal{C}}(s); \lambda\big) \; \wedge \; \mathcal{C}, \lambda \Vdash \iota_{\mathcal{C}}(s)\Big).$$

2.11 The Axiom of Duality and its Self-Similar Nature

2.11.1 Ontological state symbols for the four symmetry modes

Before stating the Axiom of Duality, we fix notation for the four canonical ontological states that correspond to the symmetry modes—and, via the map of Remark 2.20, to the four corners of the tetralemma.

Definition 2.19 (Ontological state symbols for the symmetry modes)**.** Let $\mathcal{U}_{\mathcal{C}}$ denote the class of primitive ontological states admissible in an immanent context $\mathcal{C}$. We designate four distinguished elements of $\mathcal{U}_{\mathcal{C}}$ as follows.

1. Symmetry mode (Sym; Thesis).
   Conceptual meaning: The pure "is" or "being" pole. It represents the ontological stance of *affirmation* (*Asti, Bhāva*) [42]—the determinate commitment to existence of a property or entity relative to a fixed context and locus. In quantum realizations, it corresponds to a projector onto a definite eigenstate (e.g., $|L\rangle$ in the double-slit context when a left-slit outcome is recorded) [6, 59].

$$|\overset{\circ}{\mathbf{0}}\rangle \in \mathcal{U}_{\mathcal{C}}$$

2. Antisymmetry mode (AntiSym; Antithesis).
   Conceptual meaning: The pure "is-not" or "non-being" pole. It represents the ontological stance of *denial* (*Nāsti, Abhāva*) [42]—the determinate commitment to absence or negation of a property relative to the same context and locus. Quantum-mechanically, it is the orthogonal complement of the symmetry mode (e.g., $|R\rangle$ when a right-slit outcome is recorded) [71].

$$|\overset{\circ}{\mathbf{1}}\rangle \in \mathcal{U}_{\mathcal{C}}$$

3. Parasymmetry mode (ParaSym; Synthesis).
   Conceptual meaning: The "both-and" mode. The symbol $\overline{\overline{\wedge}}$ (read "para-wedge") indicates the *co-presence* or *paraconsistent co-holding* of the two poles [13, 55]. It is not a mere superposition but the ontological commitment to the simultaneous applicability and joint affirmation of both Symmetry and Antisymmetry within a designated paraconsistent regime. In quantum realizations, it corresponds to a coherent superposition $\frac{1}{\sqrt{2}}(|\overset{\circ}{\mathbf{0}}\rangle + |\overset{\circ}{\mathbf{1}}\rangle)$ when interpreted in a paraconsistent valuation scheme (nonzero Born weight for both poles) [59, 72].

$$|\overset{\circ}{\mathbf{0}} \, \overline{\overline{\wedge}} \, \overset{\circ}{\mathbf{1}}\rangle \in \mathcal{U}_{\mathcal{C}}$$

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4. Holosymmetry mode (HoloSym; Holothesis).

Conceptual meaning: The "neither-nor" mode. The symbol $\overset{\infty}{\wedge}$ (read "holo-wedge") denotes the *transcendent closure* or *meta-level suspension* of the dyadic opposition [46, 53]. It represents the ontological stance in which the very polarity of "is"/"is-not" is inapplicable—the descriptive scheme itself is withdrawn. In quantum contexts, it corresponds to the diagnosis that a polar question (e.g., "which slit?") is operationally ill-typed at a given locus, warranting neither affirmation nor denial [9, 11].

$$|\mathring{\mathbf{0}} \overset{\infty}{\wedge} \mathring{\mathbf{1}}\rangle \in \mathcal{U}_\mathcal{C}$$

These four symbols are taken as primitive designators in $\mathcal{U}_\mathcal{C}$; their internal structure (e.g., the superposition form of the parasymmetric state) may be exhibited in specific realizations, but their ontological status is that of atomic elements of the state space.

Remark 2.20 (Relation to the tetralemma corners)**.** The ontological states correspond bijectively to the four corners:

$$|\mathring{\mathbf{0}}\rangle \longleftrightarrow \text{Th} \quad \text{(Thesis)},$$
$$|\mathring{\mathbf{1}}\rangle \longleftrightarrow \text{AntiTh} \quad \text{(Antithesis)},$$
$$|\mathring{\mathbf{0}} \overset{=}{\wedge} \mathring{\mathbf{1}}\rangle \longleftrightarrow \text{SynTh} \quad \text{(Synthesis)},$$
$$|\mathring{\mathbf{0}} \overset{\infty}{\wedge} \mathring{\mathbf{1}}\rangle \longleftrightarrow \text{HoloTh} \quad \text{(Holothesis)}.$$

Thus the notation system unifies the symmetry-mode language with the dialectical corner-language of the tetralemma [13, 43].

Remark 2.21 (Ontological representation of the atomic tetralemmatic lens)**.** The atomic tetralemmatic lens $\mathring{\mathbf{\Lambda}}$ defined in Definition 2.17 admits a canonical representation in terms of the ontological state symbols introduced in Definition 2.19, and we obtain the following isomorphic representation:

$$\mathring{\mathbf{\Lambda}} \cong \left\langle |\mathring{\mathbf{0}}\rangle,\ |\mathring{\mathbf{1}}\rangle,\ |\mathring{\mathbf{0}} \overset{=}{\wedge} \mathring{\mathbf{1}}\rangle,\ |\mathring{\mathbf{0}} \overset{\infty}{\wedge} \mathring{\mathbf{1}}\rangle \right\rangle.$$

That is, the four corner-acts correspond bijectively to the four ontological states:

- **Th** $\longleftrightarrow |\mathring{\mathbf{0}}\rangle$ (Symmetry/Thesis)

- **AntiTh** $\longleftrightarrow |\mathring{\mathbf{1}}\rangle$ (Antisymmetry/Antithesis)

- **SynTh** $\longleftrightarrow |\mathring{\mathbf{0}} \overset{=}{\wedge} \mathring{\mathbf{1}}\rangle$ (Parasymmetry/Synthesis)

- **HoloTh** $\longleftrightarrow |\mathring{\mathbf{0}} \overset{\infty}{\wedge} \mathring{\mathbf{1}}\rangle$ (Holosymmetry/Holothesis)

This representation preserves the Σ-structure: the logical relations among the corner-acts (e.g., duality, refinement stability, applicability conditions) translate directly into relations among the ontological states. In particular, the duality involution $\Delta_{\mathcal{C}}$ of Proposition 2.18 acts on the states as:

$$\Delta_{\mathcal{C}}\big(|\mathring{\mathbf{0}}\rangle\big) = |\mathring{\mathbf{1}}\rangle,$$
$$\Delta_{\mathcal{C}}\big(|\mathring{\mathbf{1}}\rangle\big) = |\mathring{\mathbf{0}}\rangle,$$
$$\Delta_{\mathcal{C}}\big(|\mathring{\mathbf{0}}\ \bar{\bar{\wedge}}\ \mathring{\mathbf{1}}\rangle\big) = |\mathring{\mathbf{0}}\ \bar{\bar{\wedge}}\ \mathring{\mathbf{1}}\rangle,$$
$$\Delta_{\mathcal{C}}\big(|\mathring{\mathbf{0}}\ \overset{\infty}{\wedge}\ \mathring{\mathbf{1}}\rangle\big) = |\mathring{\mathbf{0}}\ \overset{\infty}{\wedge}\ \mathring{\mathbf{1}}\rangle,$$

reflecting the self-duality of the para- and Holosymmetric modes [73, 74].

Consequently, any tetralemmatic morphism $f : \mathring{\mathbf{\Lambda}} \to (\mathring{\mathbf{\Lambda}})'$ induces a corresponding map on the ontological state symbols that respects the corner-labeling and the duality structure.

2.11.2   The Axiom of Duality

Axiom 2.22 (Axiom of Duality). Fix an immanent context $\mathcal{C}$ with its criteria of individuation and admissible inference, and let $\mathcal{U}_{\mathcal{C}}$ denote the class of primitive ontological states. There exists an involution (duality operator)

$$\mathbf{D}_{\mathcal{C}} : \mathcal{U}_{\mathcal{C}} \to \mathcal{U}_{\mathcal{C}}, \qquad \mathbf{D}_{\mathcal{C}} \circ \mathbf{D}_{\mathcal{C}} = \mathrm{Id}_{\mathcal{U}_{\mathcal{C}}}.$$

For designated primitives one has

$$\mathbf{D}_{\mathcal{C}}\big(|\mathring{\mathbf{0}}\rangle\big) = |\mathring{\mathbf{1}}\rangle, \qquad \mathbf{D}_{\mathcal{C}}\big(|\mathring{\mathbf{1}}\rangle\big) = |\mathring{\mathbf{0}}\rangle.$$

Write $\mathrm{Aff}_{\mathcal{C}}(s)$ for "s is affirmed in $\mathcal{C}$" and $\mathrm{Con}_{\mathcal{C}}(t)$ for "t is minimally conceptually available in $\mathcal{C}$." In a setting governed by dependent origination (*pratītyasamutpāda*), affirmation of any $s \in \mathcal{U}_{\mathcal{C}}$ entails the conceptual co-arising of its dual [43, 53]:

$$\mathrm{Aff}_{\mathcal{C}}(s) \implies \mathrm{Con}_{\mathcal{C}}\big(\mathbf{D}_{\mathcal{C}}(s)\big).$$

Accordingly, intelligible ontological content is minimally dyadic: the ordered pair $\langle s, \mathbf{D}_{\mathcal{C}}(s)\rangle$ is a minimal semantic unit of duality, and neither element is taken to be semantically autonomous in isolation. This structure parallels the treatment of dualities in quantum field theory where symmetry-related states form inseparable representation pairs [73, 74].

2.11.3   Operational ties to the four corners

Fix a locus $\lambda \in \Lambda_{\mathcal{C}}$. Define a (generally partial) corner-status assignment $\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda)$ for well-formed formulas φ by the following rule: $\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda)$ is *defined* exactly when precisely one of the four corner-holding predicates

$$\mathrm{Th}(\varphi; \mathcal{C}, \lambda), \quad \mathrm{AntiTh}(\varphi; \mathcal{C}, \lambda), \quad \mathrm{SynTh}(\varphi; \mathcal{C}, \lambda), \quad \mathrm{HoloTh}(\varphi; \mathcal{C}, \lambda)$$

(as defined in Sections 2.4 to 2.7) holds; in that case, $\text{Status}_{\mathcal{C}}(\varphi, \lambda)$ is that unique corner. Define the status dualization map

$$\Delta_{\mathcal{C}} : \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\} \to \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$$

by

$$\Delta_{\mathcal{C}}(\mathbf{Th}) = \mathbf{AntiTh}, \qquad \Delta_{\mathcal{C}}(\mathbf{AntiTh}) = \mathbf{Th},$$
$$\Delta_{\mathcal{C}}(\mathbf{SynTh}) = \mathbf{SynTh}, \qquad \Delta_{\mathcal{C}}(\mathbf{HoloTh}) = \mathbf{HoloTh}.$$

so that $\Delta_{\mathcal{C}} \circ \Delta_{\mathcal{C}} = \text{Id}$. This corner involution structure reflects the self-duality of the third and fourth positions in classical *catuṣkoṭi* analyses [13, 46].

2.12 Symmetries of Contextual Duality

Assume $\mathbf{D}_{\mathcal{C}}$ admits a homomorphic extension to formulas, written $\varphi \mapsto \varphi^{\mathbf{D}}$ (specified below). Whenever $\text{Status}_{\mathcal{C}}(\varphi, \lambda)$ and $\text{Status}_{\mathcal{C}}(\varphi^{\mathbf{D}}, \lambda)$ are both defined, they obey

$$\text{Status}_{\mathcal{C}}(\varphi^{\mathbf{D}}, \lambda) = \Delta_{\mathcal{C}}(\text{Status}_{\mathcal{C}}(\varphi, \lambda)).$$

In particular, dualization exchanges the Thesis and Antithesis statuses, while Synthesis and Holothesis are self-dual. This compatibility condition ensures that propositional dualization respects the tetralemmatic structure, analogous to how logical negation interacts with many-valued semantics [13, 55].

2.12.1 Self-similarity (scale-free replication)

Duality propagates uniformly across levels of discourse.

Definition 2.23 (Lifts of duality)**.** We distinguish four compatible lifts of the context-duality $\mathbf{D}_{\mathcal{C}}$, organized by level of rememberance (object, propositional, corner, and meta).

(i) Object level.
$\mathbf{D}_{\mathcal{C}}$ acts on primitive ontological states $s \in \mathcal{U}_{\mathcal{C}}$, with $\mathbf{D}_{\mathcal{C}} \circ \mathbf{D}_{\mathcal{C}} = \text{Id}_{\mathcal{U}_{\mathcal{C}}}$. This mirrors the treatment of symmetry operations on Hilbert space rays in quantum mechanics [59, 73].

(ii) Propositional level.
Extend $\mathbf{D}_{\mathcal{C}}$ to formulas by a homomorphic dualization $\varphi \mapsto \varphi^{\mathbf{D}}$ that dualizes the designated primitive predicates and distributes through logical form in a De Morgan–compatible way, respecting the admissible consequence/evaluation regime(s) of $\mathcal{C}$. In particular, for the connectives admitted by $\mathcal{C}$ one requires $(\neg\psi)^{\mathbf{D}} = \neg(\psi^{\mathbf{D}})$ and $(\psi \wedge \chi)^{\mathbf{D}} = \psi^{\mathbf{D}} \wedge \chi^{\mathbf{D}}$ (and similarly for other connectives admitted by $\mathcal{C}$). Such homomorphic extensions are standard in algebraic logic and operator-algebraic approaches to quantum theory [16, 19].

(iii) Corner level.

The involution $\Delta_{\mathcal{C}}$ on $\{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$ is given by

$$\Delta_{\mathcal{C}}(\mathbf{Th}) = \mathbf{AntiTh}, \qquad \Delta_{\mathcal{C}}(\mathbf{AntiTh}) = \mathbf{Th},$$
$$\Delta_{\mathcal{C}}(\mathbf{SynTh}) = \mathbf{SynTh}, \qquad \Delta_{\mathcal{C}}(\mathbf{HoloTh}) = \mathbf{HoloTh}.$$

so that $\Delta_{\mathcal{C}} \circ \Delta_{\mathcal{C}} = \mathrm{Id}$. Whenever both sides are defined, status assignment respects duality:

$$\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda) = X \implies \mathrm{Status}_{\mathcal{C}}\big(\varphi^{\mathbf{D}}, \lambda\big) = \Delta_{\mathcal{C}}(X).$$

(iv) Meta level.

Duality preserves inapplicability at λ: for any judgment χ,

$$\neg \mathrm{App}_{\mathcal{C}}(\chi; \lambda) \implies \neg \mathrm{App}_{\mathcal{C}}\big(\chi^{\mathbf{D}}; \lambda\big).$$

Since $(\chi^{\mathbf{D}})^{\mathbf{D}} = \chi$, this is equivalent to $\neg \mathrm{App}_{\mathcal{C}}(\chi; \lambda) \iff \neg \mathrm{App}_{\mathcal{C}}\big(\chi^{\mathbf{D}}; \lambda\big)$. This meta-level invariance reflects the contextuality constraints of quantum observables where unmeasurable quantities remain unmeasurable under symmetry transformations [2, 9, 11].

2.12.2 Object-level instance

Let $|\overset{\circ}{\mathbf{0}}\rangle$ be a designated primitive ontological state in $\mathcal{U}_{\mathcal{C}}$ and define $|\overset{\circ}{\mathbf{1}}\rangle := \mathbf{D}_{\mathcal{C}}\big(|\overset{\circ}{\mathbf{0}}\rangle\big)$. Then

$$|\overset{\circ}{\mathbf{0}}\rangle \leftrightarrow |\overset{\circ}{\mathbf{1}}\rangle, \qquad \mathbf{D}_{\mathcal{C}}\big(|\overset{\circ}{\mathbf{1}}\rangle\big) = |\overset{\circ}{\mathbf{0}}\rangle,$$

and the dyad $\big(|\overset{\circ}{\mathbf{0}}\rangle, |\overset{\circ}{\mathbf{1}}\rangle\big)$ is semantically primary. This dyadic structure echoes the treatment of complementary observables in quantum mechanics where conjugate variables form inseparable pairs [11, 75].

Proposition 2.24 (Self-similarity under admissible reindexings)**.** *Let $F : \mathcal{U}_{\mathcal{C}} \to \mathcal{U}_{\mathcal{C}'}$ be the map induced by an admissible reindexing $\mathcal{C} \to \mathcal{C}'$ (change of aspect, respect, mode, or locus) that preserves the duality structure. Then there exists a unique involution $\mathbf{D}_{\mathcal{C}'}$ such that*

$$F \circ \mathbf{D}_{\mathcal{C}} = \mathbf{D}_{\mathcal{C}'} \circ F.$$

Sketch. Define $\mathbf{D}_{\mathcal{C}'}$ on the image of F by $\mathbf{D}_{\mathcal{C}'}(F(s)) := F(\mathbf{D}_{\mathcal{C}}(s))$ and extend as required on $\mathcal{U}_{\mathcal{C}'}$; involutivity and uniqueness follow from $\mathbf{D}_{\mathcal{C}} \circ \mathbf{D}_{\mathcal{C}} = \mathrm{Id}$ and the commutation relation. The corner-level compatibility is governed by $\Delta_{\mathcal{C}}$ together with the fact that Synthesis and Holothesis are invariant under polarity swap while Thesis and Antithesis exchange. This commutation property is analogous to the covariance of symmetry operations under changes of reference frame in relativistic quantum theory [73, 74]. $\square$

Operational compatibility with the realization axiom (optional). Assume the realization axiom of Section 6.5. If the duality is physically realized by a unitary $\mathbf{D}_{\mathcal{C}} \in \mathsf{R}(\mathcal{C}) \subseteq U(4)$, require that the propositional lift commutes with interpretation by conjugation:

$$\mathcal{R}_{\mathcal{C},\lambda}(\chi^{\mathbf{D}}) = \mathbf{D}_{\mathcal{C}}\, \mathcal{R}_{\mathcal{C},\lambda}(\chi)\, \mathbf{D}_{\mathcal{C}}^{\dagger}$$

whenever $\mathcal{R}_{\mathcal{C},\lambda}(\chi)$ is defined. In particular, interpretability (hence applicability) is invariant under duality at λ, and Born valuation is equivariant under the corresponding conjugation on effects. Such unitary implementations of logical symmetries are standard in quantum information theory where logical operations are represented by completely positive maps [59, 76].

Meta-level lift (minimal proof-theoretic form). Let $\Gamma \vdash_{\mathcal{C}} \varphi$ range over well-formed judgments in context $\mathcal{C}$, derived from an admissible rule set $\mathcal{R}_{\mathcal{C}}$. There exists an involution $\widehat{\mathbf{D}}_{\mathcal{C}}$ acting on judgments, rules, and derivations such that:

(M1) Action on judgments.
   $\widehat{\mathbf{D}}_{\mathcal{C}}(\Gamma \vdash_{\mathcal{C}} \varphi) = \widehat{\mathbf{D}}_{\mathcal{C}}(\Gamma) \vdash_{\mathcal{C}} \varphi^{\mathbf{D}}$ and $\widehat{\mathbf{D}}_{\mathcal{C}} \circ \widehat{\mathbf{D}}_{\mathcal{C}} = \mathrm{Id}$.

(M2) Action on rules.
   For every rule $r \in \mathcal{R}_{\mathcal{C}}$, $\widehat{\mathbf{D}}_{\mathcal{C}}(r) \in \mathcal{R}_{\mathcal{C}}$, and r is admissible if and only if $\widehat{\mathbf{D}}_{\mathcal{C}}(r)$ is admissible.

(M3) Action on derivations.
   $\widehat{\mathbf{D}}_{\mathcal{C}}$ maps derivations to derivations by structural recursion, preserves derivability, and commutes with substitution/renaming. Such structural preservation is characteristic of proof-theoretic symmetries in sequent calculi [77, 78].

(M4) Status coherence.
   Whenever both sides are defined, if $\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda) = X$, then $\mathrm{Status}_{\mathcal{C}}(\varphi^{\mathbf{D}}, \lambda) = \Delta_{\mathcal{C}}(X)$.

(M5) Conservativity and non-explosion.
   If $\vdash_{\mathrm{pc}}$ is a stipulated paraconsistent consequence relation used for Synthesis in $\mathcal{C}$, then $\widehat{\mathbf{D}}_{\mathcal{C}}$ preserves any stated non-explosion requirements (in particular, it does not turn nontrivial theories into trivial ones). This preservation property is essential for paraconsistent logics where duality must respect the non-explosive consequence relation [13, 54].

Paraconsistent compatibility. Assume that $\mathcal{C}$ admits a paraconsistent consequence relation $\vdash_{\mathrm{pc}}$ (see Section 2.6), consistent with standard formulations of paraconsistent logic [13, 54], and fix the corresponding holding relation $\Vdash_{\mathcal{C},\lambda}^{\mathrm{pc}}$.

For a dyadic presentation of φ with coordinate predicates $L(\varphi)$ and $R(\varphi)$, define the coordinate existential claims

$$P_L(\varphi) \;:=\; \exists_{\lambda}^{\mathcal{C}} x\, L(\varphi)(x), \qquad P_R(\varphi) \;:=\; \exists_{\lambda}^{\mathcal{C}} y\, R(\varphi)(y).$$

Dyadic reading of the third and fourth corners. *Synthesis* is read as the (non-explosive) co-holding of both coordinate claims in the paraconsistent regime [44, 55]:

$$\mathrm{SynTh}(\varphi; \mathcal{C}, \lambda) \Rightarrow \mathrm{App}_{\mathcal{C}}\big(P_L(\varphi); \lambda\big) \ \wedge \ \mathrm{App}_{\mathcal{C}}\big(P_R(\varphi); \lambda\big) \ \wedge$$
$$\big(\mathcal{C}, \lambda \Vdash^{\mathrm{pc}} P_L(\varphi)\big) \ \wedge \ \big(\mathcal{C}, \lambda \Vdash^{\mathrm{pc}} P_R(\varphi)\big),$$

with non-explosion witnessed, e.g. $\exists \psi$ such that $\{P_L(\varphi), P_R(\varphi)\} \nvdash_{\mathrm{pc}} \psi$
(in particular, if $\perp$ is available: $\{P_L, P_R\} \nvdash_{\mathrm{pc}} \perp$).

Holothesis is read as meta-level inapplicability of both coordinate claims, resonating with contextual constraints in quantum logic [9, 11]:

$$\mathrm{HoloTh}(\varphi; \mathcal{C}, \lambda) \Rightarrow \ \neg \mathrm{App}_{\mathcal{C}}\big(P_L(\varphi); \lambda\big) \ \wedge \ \neg \mathrm{App}_{\mathcal{C}}\big(P_R(\varphi); \lambda\big).$$

Self-duality of the third and fourth corners. Whenever status assignment is defined and propositional dualization $\varphi \mapsto \varphi^{\mathbf{D}}$ is available [13],

$$\mathrm{Status}_{\mathcal{C}}\big(\varphi^{\mathbf{D}}, \lambda\big) = \Delta_{\mathcal{C}}\big(\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda)\big).$$

Since $\Delta_{\mathcal{C}}$ fixes **SynTh** and **HoloTh**, reflecting symmetry properties common in physical theories [73], it follows that

$$\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda) \in \{\mathbf{SynTh}, \mathbf{HoloTh}\} \implies \mathrm{Status}_{\mathcal{C}}\big(\varphi^{\mathbf{D}}, \lambda\big) = \mathrm{Status}_{\mathcal{C}}(\varphi, \lambda).$$

refinement. To visually mark *conceptual co-implication/co-arising*, grounded in the doctrine of dependent origination [43, 53], write

$$s \ \dashv\vdash_{\mathcal{C}} \ \mathbf{D}_{\mathcal{C}}(s)$$

and reserve $\leftrightarrow$ for material biconditionals *inside* $\mathcal{C}$.

Compact Axiom Schema. The following schema summarizes the duality structure, analogous to symmetry constraints in quantum field theory [74]:

$$\forall s \in \mathcal{U}_{\mathcal{C}}: \quad \mathrm{Aff}_{\mathcal{C}}(s) \ \Rightarrow \ \mathrm{Con}_{\mathcal{C}}\big(\mathbf{D}_{\mathcal{C}}(s)\big), \qquad \mathbf{D}_{\mathcal{C}}^2 = \mathrm{Id}_{\mathcal{U}_{\mathcal{C}}}, \qquad \Delta_{\mathcal{C}}^2 = \mathrm{Id},$$
$$\mathrm{Status}_{\mathcal{C}}\big(\varphi^{\mathbf{D}}, \lambda\big) = \Delta_{\mathcal{C}}\big(\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda)\big), \qquad \neg \mathrm{App}_{\mathcal{C}}(\chi; \lambda) \ \Rightarrow \ \neg \mathrm{App}_{\mathcal{C}}\big(\chi^{\mathbf{D}}; \lambda\big).$$

(The last two clauses are status-compatibility and invariance of inapplicability at locus λ.)

Example dyads. The following set illustrates common philosophical dyads subject to this duality structure [53, 79]:

$$\mathcal{E}_{\mathrm{duality}} := \big\{\mathfrak{D}_{\mathcal{C}}(\mathrm{Existence}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Eternalism}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Self}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Emptiness}),$$
$$\mathfrak{D}_{\mathcal{C}}(\mathrm{Continuous}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Duality}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Created}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Finite}),$$
$$\mathfrak{D}_{\mathcal{C}}(\mathrm{Form}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Time}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{True}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Positive}), \ \mathfrak{D}_{\mathcal{C}}(\mathrm{Real})\big\}.$$

Illustrative pairs. (1) Existence / Non-Existence, (2) Eternalism / Nihilism, (3) Self / Not-Self, (4) Emptiness / Non-Emptiness, (5) Continuous / Discontinuous, (6) Duality / Non-Duality, (7) Created / Un-Created, (8) Finite / Infinite, (9) Form / Formlessness, (10) Time / Timelessness, (11) True / False, (12) Positive / Negative, (13) Real / Un-Real.

2.12.3 The Double-Slit Experiment under the Axiom of Duality

This example shows how the Axiom of Duality organizes the logical status of slit-claims in the double-slit experiment, a paradigmatic case for quantum complementarity [10, 11]. Fix the domain of discourse $\mathcal{D}$ to single electrons and evaluate claims relative to an immanent context $\mathcal{C}$ and a locus $\lambda \in \Lambda_{\mathcal{C}}$ as in Section 2. We also presuppose the tetralemmatic analysis of the same experiment given in Section 2.8 and work within the quantum realization scheme of Section 6.5.

Predicate and polar claim (object level). Let $\varphi(x)$ be the predicate "x is localized at the left slit at a fixed time t_0 during its passage." Form the context- and locus-relative existential claim (cf. Section 2 and Definition 2.7)

$$P := \exists_{\lambda}^{\mathcal{C}} x\, \varphi(x) \qquad (\text{"The electron went through the left slit at } t_0\text{"}).$$

When P is applicable at $(\mathcal{C}, \lambda)$ in the sense of Section 2, interpret it by an effect via the realization map of Section 6.5, following standard quantum measurement theory [6, 56]:

$$E_P := \mathcal{R}_{\mathcal{C},\lambda}(P), \qquad 0 \leq E_P \leq I,$$

and (when defined) interpret $\neg P$ by the complementary effect (cf. Section 6.5 together with negation-closure of applicability in Section 2)

$$E_{\neg P} := \mathcal{R}_{\mathcal{C},\lambda}(\neg P) = I - E_P.$$

For the ideal two-slit model, let H contain orthonormal kets $|L\rangle, |R\rangle$ representing perfect left/right localization at t_0, consistent with the standard Hilbert space formulation [59, 71]. When the which-path test is admitted one may take

$$E_P = |L\rangle\langle L|, \qquad E_{\neg P} = |R\rangle\langle R|.$$

(Here the identification $E_{\neg P} = |R\rangle\langle R|$ uses the idealization that $\{|L\rangle, |R\rangle\}$ exhaust the relevant alternatives at the slit locus.)

Physical symmetry as the duality operator (duality level). Assume the apparatus is symmetric under exchange of the two slits. Represent this physical symmetry by a unitary duality operator $D_{\mathcal{C}}$ on H such that, in accordance with Wigner's theorem on symmetry transformations [73, 74]:

$$D_{\mathcal{C}}|L\rangle = |R\rangle, \qquad D_{\mathcal{C}}|R\rangle = |L\rangle, \qquad D_{\mathcal{C}}^2 = I.$$

On the propositional level, define the dual claim by predicate dualization (duality lift, cf. Definition 2.23):

$$P^D := \exists_{\lambda}^{\mathcal{C}} x\, \varphi^D(x), \qquad \text{where } \varphi^D(x) \text{ is "} x \text{ is localized at the right slit at } t_0\text{."}$$

Operational compatibility of duality with realization is expressed by conjugation (cf. Definition 2.23 and Section 6.5), following the standard treatment of symmetry operations in quantum theory [71]:

$$\mathcal{R}_{\mathcal{C},\lambda}(P^D) = D_{\mathcal{C}}\, \mathcal{R}_{\mathcal{C},\lambda}(P)\, D_{\mathcal{C}}^{\dagger},$$

whenever $\mathcal{R}_{\mathcal{C},\lambda}(P)$ is defined. In the two-slit idealization this yields

$$\mathcal{R}_{\mathcal{C},\lambda}(P^D) = D_{\mathcal{C}} |L\rangle\langle L| D_{\mathcal{C}}^{\dagger} = |R\rangle\langle R| = E_{\neg P},$$

so P^D coincides with $\neg P$ at the level of the admitted sharp test.

Corner-status dualization (corner level). The Axiom of Duality induces an involution on the four corners (cf. Axiom 2.22 and the operational tie between corners and evaluation in Section 2), reflecting the self-duality structure of paraconsistent logics [13]:

$$\Delta_{\mathcal{C}}(\text{Th}) = \text{AntiTh}, \quad \Delta_{\mathcal{C}}(\text{AntiTh}) = \text{Th},$$
$$\Delta_{\mathcal{C}}(\text{SynTh}) = \text{SynTh}, \quad \Delta_{\mathcal{C}}(\text{HoloTh}) = \text{HoloTh}.$$

Thus, whenever both sides are defined,

$$\text{Status}_{\mathcal{C}}(P^D, \lambda) = \Delta_{\mathcal{C}}\Big(\text{Status}_{\mathcal{C}}(P, \lambda)\Big).$$

Scenario A: Which-path context $(\mathcal{C}_1, \lambda_1)$ (definite outcomes). Let $\mathcal{C}_1$ be a context with an active which-path detector, and let λ_1 be the locus "at the slits during detection."

(A1) Applicability.
   The detector implements the which-path test, hence

$$\text{App}_{\mathcal{C}_1}(P; \lambda_1) = \text{App}_{\mathcal{C}_1}(\neg P; \lambda_1) = \text{True} \qquad (\text{cf. Section 2}).$$

(A2) Measurement and (run-wise) symmetry breaking.
   An instrument $\{\mathcal{I}_L, \mathcal{I}_R\}$ correlates the electron with detector records (instrumental realization, cf. Section 6.5). Even if the detector hardware is left-right symmetric, a single run produces a definite record, thereby breaking the symmetry at the level of the realized outcome and posterior state, consistent with decoherence theory [22, 23].

(A3) Outcome and corner-status.
   After the measurement interaction, one obtains either, following the Lüders update rule [60]:

$$(\mathcal{C}_1, \lambda_1, \rho) \Vdash P \quad \text{with} \quad \rho \approx |L\rangle\langle L| \qquad (\text{Th; cf. } Section\ 2.4 and\ Definition\ 2.7),$$

   or

$$(\mathcal{C}_1, \lambda_1, \rho) \Vdash \neg P \quad \text{with} \quad \rho \approx |R\rangle\langle R|$$
$$(\text{AntiTh; cf. } Section\ 2.5 and\ Definition\ 2.8).$$

(A4) Duality action on corners.
   If $\text{Status}_{\mathcal{C}_1}(P, \lambda_1) = \text{Th}$ then

$$\text{Status}_{\mathcal{C}_1}(P^D, \lambda_1) = \Delta_{\mathcal{C}_1}(\text{Th}) = \text{AntiTh},$$

   and conversely if $\text{Status}_{\mathcal{C}_1}(P, \lambda_1) = \text{AntiTh}$ then $\text{Status}_{\mathcal{C}_1}(P^D, \lambda_1) = \text{Th}$ (corner dualization, cf. Axiom 2.22). Thus duality exchanges Thesis and Antithesis, matching the fact that a definite slit record selects one pole.

(A4′) Compatibility with the realization axiom (equivariance).
For any claim χ whose realized effect is defined at $(\mathcal{C}_1, \lambda_1)$, duality acts by conjugation (cf. Definition 2.23 and Section 6.5), preserving the operator-algebraic structure [16]:

$$\mathcal{R}_{\mathcal{C}_1,\lambda_1}(\chi^D) = D_{\mathcal{C}_1}\, \mathcal{R}_{\mathcal{C}_1,\lambda_1}(\chi)\, D_{\mathcal{C}_1}^{\dagger}.$$

Consequently Born-valuations (cf. the Born-valued semantics in Definition 4.20 and eq. (28)) are equivariant:

$$v(\chi^D; \rho) = \mathrm{Tr}\Big(\mathcal{R}_{\mathcal{C}_1,\lambda_1}(\chi^D)\, \rho\Big) = \mathrm{Tr}\Big(\mathcal{R}_{\mathcal{C}_1,\lambda_1}(\chi)\, D_{\mathcal{C}_1}^{\dagger}\rho\, D_{\mathcal{C}_1}\Big) = v\Big(\chi; D_{\mathcal{C}_1}^{\dagger}\rho\, D_{\mathcal{C}_1}\Big).$$

(A5) Dependent origination (conceptual co-arising).
Affirmation of one outcome conceptually co-arises with the availability of its dual, reflecting the Madhyamaka doctrine of *pratītyasamutpāda* [43, 53]: affirming $|L\rangle$ entails minimal conceptual availability of $|R\rangle$ (and vice versa), even though one of them is empirically denied in the run (compare the polarity structure in Section 2).

Scenario B: Interference context $(\mathcal{C}_2, \lambda_2)$ **(symmetry preservation and inapplicability).** Let $\mathcal{C}_2$ be the context with no which-path detector, and let λ_2 be the locus "between slits and screen, before any position measurement."

(B1) Inapplicability.
In $\mathcal{C}_2$ at λ_2, the which-path test is not admitted, reflecting quantum contextuality constraints [2, 9]. Hence

$$\neg\, \mathrm{App}_{\mathcal{C}_2}(P; \lambda_2) = \neg\, \mathrm{App}_{\mathcal{C}_2}(\neg P; \lambda_2) = \mathrm{True}$$
$$\text{(typed inapplicability, cf. Section 2).}$$

Equivalently $\mathcal{R}_{\mathcal{C}_2,\lambda_2}(P)$ and $\mathcal{R}_{\mathcal{C}_2,\lambda_2}(\neg P)$ are undefined (cf. Section 6.5).

(B2) Holothesis warranted and self-dual.
As established in Section 2.8, inapplicability of both poles warrants Holothesis:

$$\mathrm{Status}_{\mathcal{C}_2}(P, \lambda_2) = \mathrm{HoloTh} \qquad \text{(cf. Section 2.7 and Definition 2.12).}$$

Since $\Delta_{\mathcal{C}_2}$ fixes Holothesis (cf. Axiom 2.22), one also has

$$\mathrm{Status}_{\mathcal{C}_2}(P^D, \lambda_2) = \Delta_{\mathcal{C}_2}(\mathrm{HoloTh}) = \mathrm{HoloTh}.$$

Thus both P and its dual share the same Holothesis status: "left?" and "right?" are jointly unaskable at $(\mathcal{C}_2, \lambda_2)$.

(B3) State description and symmetry.
Kinematically, the electron state in the path subspace may be a coherent superposition, following standard quantum mechanics [59, 80]:

$$|\psi\rangle = \alpha|L\rangle + \beta|R\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1 \qquad \text{(cf. Section 6.5).}$$

If $|\alpha| = |\beta|$ then $|\psi\rangle$ is invariant under slit exchange up to an overall phase, reflecting preservation of the physical symmetry in this context.

(B4) Where Synthesis may be discussed without contradicting applicability.
At $(\mathcal{C}_2, \lambda_2)$ the polar claims are inapplicable, so Synthesis for the polar pair $P, \neg P$ is not the literal corner-status there (cf. Definitions 2.11 and 2.12). To discuss Synthesis legitimately, one must shift to a nearby context/locus $(\mathcal{C}', \lambda')$ where:

- the polar effects are operationally definable (so $\mathrm{App}_{\mathcal{C}'}(P; \lambda')$ and $\mathrm{App}_{\mathcal{C}'}(\neg P; \lambda')$ hold; cf. Section 2),

- coherence is preserved (for example, via weak/erasable which-path coupling),

- a paraconsistent holding relation $\Vdash^{\mathrm{pc}}$ is designated (cf. Definition 2.10).

In such a setting, both P and P^D can receive simultaneously non-zero Born weights while remaining applicable, warranting Synthesis (cf. Section 2.6 and Definition 2.11), consistent with paraconsistent logic frameworks [13, 54]. Since $\Delta_{\mathcal{C}'}$ fixes Synthesis (cf. Axiom 2.22), this corner-status is self-dual: $\Delta_{\mathcal{C}'}(\mathrm{SynTh}) = \mathrm{SynTh}$, reflecting that a symmetric coherent preparation treats the two paths on equal footing.

(B5) Meta-level preservation of inapplicability under duality.
Since duality preserves inapplicability at a locus (cf. Definition 2.23 together with Section 2),
$$\neg\,\mathrm{App}_{\mathcal{C}_2}(P; \lambda_2) \iff \neg\,\mathrm{App}_{\mathcal{C}_2}(P^D; \lambda_2),$$
the unaskability of "left?" is structurally inseparable from the unaskability of "right?" in the symmetry-preserving interference context, echoing Bohr's complementarity principle [11, 81].

Empirical signature and a Noether-type correspondence. At the screen, move to a context/locus $(\mathcal{C}_3, \lambda_3)$ that admits position measurement (cf. Section 6.5). The observed probability density
$$p(x) = |\langle x|\psi\rangle|^2$$
exhibits interference fringes when coherence between the $|L\rangle$ and $|R\rangle$ components is preserved, recovering Young's original interference phenomenon [62]. The slit-exchange symmetry induces a corresponding spatial transformation on the screen coordinates (reflection about the symmetry axis), so that in a symmetric arrangement the pattern is mapped to itself.

This yields a Noether-type correspondence between physical symmetry and logical transformation, analogous to symmetry-conservation laws in field theory [74, 82]:

Table 2: Physical Symmetry and Logical Duality in Quantum Contexts

Physical Symmetry Status	Logical Transformation under Duality ($\Delta_{\mathcal{C}}$)
Which-Path Context (Symmetry Breaking): An admitted polar test yields definite outcomes (run-wise symmetry breaking).	**Thesis/Antithesis Exchange**: $\Delta_{\mathcal{C}}$ exchanges Th $\leftrightarrow$ AntiTh. A Thesis for P corresponds to an Antithesis for P^D, and vice versa.
Interference Context (Symmetry Preserved): The polar which-path question is not admitted (symmetry preserved).	**Self-Dual Statuses**: $\Delta_{\mathcal{C}}$ fixes HoloTh (and, where applicable, SynTh). Holothesis is self-dual and shared by P and P^D.

In summary: duality is realized physically as slit exchange ($D_{\mathcal{C}}$), propositionally as $P \mapsto P^D$ (duality lift, cf. Definition 2.23), and modally as the corner involution $\Delta_{\mathcal{C}}$ (cf. Axiom 2.22). Symmetry-breaking contexts yield dual-pair outcomes (Thesis/Antithesis, cf. Definitions 2.7 and 2.8), whereas symmetry-preserving contexts yield self-dual logical descriptions (Holothesis, cf. Definition 2.12, and—when the polar effects are admitted in an appropriate weak/erasable setting—Synthesis, cf. Definition 2.11).

2.13 Paraconsistency in the Double-Slit: A Concrete Realization

The formal definition of a paraconsistent consequence regime in Definition 2.10 admits a natural quantum realization in the double-slit analysis, resonating with logical frameworks that tolerate contradiction [13, 55]. The crucial point is the scope discipline emphasized in the tetralemmatic walk Section 2.8: in the pure interference situation $(\mathcal{C}_2, \lambda_2)$ the polar which-slit claims are *inapplicable* (Holothesis, Definition 2.12), reflecting Bohr's doctrine of complementarity [11]. Accordingly, paraconsistency functions as a *secondary* regime, realized either (i) in a nearby applicability-restoring refinement/context, or (ii) as the disciplined "counterfactual semantics" that would apply if one insisted on evaluating the polar claims despite their primary inapplicability.

Setup: the slit claims and the interference state. Let $\varphi(x)$ be "x is localized at the left slit at time t_0." As in Section 2, form

$$P := \exists_{\lambda_2}^{\mathcal{C}_2} x \, \varphi(x), \qquad \text{and its negation } \neg P.$$

In the interference preparation one has a coherent path superposition, a standard feature of quantum mechanics [10, 59]:

$$|\psi\rangle = \alpha|L\rangle + \beta|R\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1, \ \alpha, \beta \neq 0,$$

with $|L\rangle, |R\rangle$ orthogonal path kets.

Classical explosive logic breaks down. If one attempts to treat "non-zero support" as classical truth for both P and $\neg P$ in a coherent superposition, one is led to a simultaneous acceptance of P and $\neg P$ while still regarding them as contradictories. Under classical explosive consequence this would trivialize inference [54]:

$$\{P, \neg P\} \vdash_{\text{classical}} \bot,$$

in direct conflict with the physical admissibility of superposition. The correct inferential discipline must therefore be non-explosive—precisely the paraconsistent requirement of Definition 2.10.

The applicability-restoring refinement concretely. To realize the Synthesis corner *without* violating the applicability discipline of Section 2, move to a nearby context/locus $(\mathcal{C}', \lambda')$ of the kind described in Section 2.8: a *soft* which-path coupling to a meter M that makes the polar test operationally definable while preserving coherence, akin to unsharp measurement schemes [56, 63].

Let H_M be a meter Hilbert space with initial state $|0\rangle_M$ and let an interaction unitary U_{int} satisfy, following the von Neumann measurement model [6, 59]:

$$U_{\text{int}}\big(|L\rangle \otimes |0\rangle_M\big) = |L\rangle \otimes |m_L\rangle_M, \qquad U_{\text{int}}\big(|R\rangle \otimes |0\rangle_M\big) = |R\rangle \otimes |m_R\rangle_M,$$

with $\langle m_L | m_R \rangle \neq 0$ (weak distinguishability). A typical parametrization is

$$|m_L\rangle = \cos\theta \, |0\rangle_M + \sin\theta \, |1\rangle_M, \qquad |m_R\rangle = \cos\theta \, |0\rangle_M - \sin\theta \, |1\rangle_M, \qquad 0 < \theta \ll \frac{\pi}{4},$$

so that $\langle m_L | m_R \rangle = \cos(2\theta) \neq 0$.

Choose meter effects $F_L, F_R \geq 0$ with $F_L + F_R = I_M$; for instance one may take $F_L = |0\rangle\langle 0|_M$ and $F_R = |1\rangle\langle 1|_M$, consistent with positive-operator-valued measures [83]. Then, in $\mathcal{C}'$ at λ', define realized polar effects on $H \otimes H_M$ by

$$E_P := \mathcal{R}_{\mathcal{C}',\lambda'}(P) = I \otimes F_L, \qquad E_{\neg P} := \mathcal{R}_{\mathcal{C}',\lambda'}(\neg P) = I \otimes F_R,$$

so that $\text{App}_{\mathcal{C}'}(P; \lambda')$ and $\text{App}_{\mathcal{C}'}(\neg P; \lambda')$ hold (cf. Section 2), and the realization map is in force (cf. Section 6.5).

For input $|\psi\rangle = \alpha|L\rangle + \beta|R\rangle$, the post-interaction joint state is

$$|\Psi\rangle = \alpha|L\rangle \otimes |m_L\rangle_M + \beta|R\rangle \otimes |m_R\rangle_M, \qquad \rho_{SM} = |\Psi\rangle\langle\Psi|.$$

Assuming $\langle m_L | F_L | m_L \rangle > 0$ and $\langle m_R | F_R | m_R \rangle > 0$, one has

$$\text{Tr}(\rho_{SM} E_P) = |\alpha|^2 \, \langle m_L | F_L | m_L \rangle > 0, \qquad \text{Tr}(\rho_{SM} E_{\neg P}) = |\beta|^2 \, \langle m_R | F_R | m_R \rangle > 0.$$

A quantum paraconsistent holding relation. Define a paraconsistent holding regime at $(\mathcal{C}', \lambda')$ by Born-positivity, grounding truth in quantum probability [58, 84]:

$$\mathcal{C}', \lambda' \Vdash^{\text{pc}} \chi \iff \text{Tr}\big(\rho_{SM} \, \mathcal{R}_{\mathcal{C}',\lambda'}(\chi)\big) > 0,$$

(Born-valued semantics, cf. Definition 4.20 and eq. (28), and realization, cf. Section 6.5). Then

$$\mathcal{C}', \lambda' \Vdash^{\text{pc}} P \qquad \text{and} \qquad \mathcal{C}', \lambda' \Vdash^{\text{pc}} \neg P,$$

so the co-commitment clause (S1) of Synthesis holds in this regime (cf. Definition 2.11), mirroring paraconsistent truth-value gluts [13].

The induced paraconsistent consequence relation. Let $\vdash_{\mathrm{pc}}$ be the semantic consequence relation induced by $\Vdash^{\mathrm{pc}}$: for any $\Gamma \subseteq \mathrm{Lang}(\mathcal{C}')$ and $\psi \in \mathrm{Lang}(\mathcal{C}')$,

$$\Gamma \vdash_{\mathrm{pc}} \psi \quad \text{iff} \quad \text{for every admissible instance } (\mathcal{C}', \lambda', \rho_{SM}),$$
$$\left(\forall \chi \in \Gamma : \ \mathcal{C}', \lambda' \Vdash^{\mathrm{pc}} \chi \right) \ \Rightarrow \ \mathcal{C}', \lambda' \Vdash^{\mathrm{pc}} \psi.$$

Non-explosion of this consequence relation is the defining property of paraconsistency (cf. Definition 2.10).

Verification of non-explosion (a concrete witness). To witness paraconsistency we exhibit a formula ψ such that $\{P, \neg P\} \nvdash_{\mathrm{pc}} \psi$ (cf. Definition 2.10), demonstrating the failure of ex contradictione quodlibet [54]. In the symmetric case $\alpha = \beta = 1/\sqrt{2}$, define

$$|+\rangle = \frac{|L\rangle + |R\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|L\rangle - |R\rangle}{\sqrt{2}}, \qquad \rho_+ = |+\rangle\langle+|.$$

Let $\psi_- :=$ "the path state is $|-\rangle$," realized by

$$E_- := |-\rangle\langle-|.$$

Then

$$\mathrm{Tr}(\rho_+ E_P) = \mathrm{Tr}(\rho_+ E_{\neg P}) = \frac{1}{2} > 0, \qquad \mathrm{Tr}(\rho_+ E_-) = 0.$$

Hence $\mathcal{C}', \lambda' \Vdash^{\mathrm{pc}} P$ and $\mathcal{C}', \lambda' \Vdash^{\mathrm{pc}} \neg P$, while $\mathcal{C}', \lambda' \nVdash^{\mathrm{pc}} \psi_-$. Therefore

$$\{P, \neg P\} \nvdash_{\mathrm{pc}} \psi_-,$$

and the regime is non-explosive in the sense required by Definition 2.10.

Warranting Synthesis: explicit checking of Definition 2.11. Within $(\mathcal{C}', \lambda')$ and the designated $\Vdash^{\mathrm{pc}}$:

(S1) $\mathrm{App}_{\mathcal{C}'}(P; \lambda')$ and $\mathrm{App}_{\mathcal{C}'}(\neg P; \lambda')$ hold by construction (cf. Section 2), and $\mathcal{C}', \lambda' \Vdash^{\mathrm{pc}} P$ as well as $\mathcal{C}', \lambda' \Vdash^{\mathrm{pc}} \neg P$ by Born-positivity.

(S2) The co-holding is stable under admissible refinements $r \in \mathsf{R}(\mathcal{C}')$ in the sense of clause (S2) of Definition 2.11: small admissible re-indexings/perturbations (e.g. slight changes of coupling strength or meter overlap $\langle m_L | m_R \rangle$ that remain within $\mathcal{C}'$-sanctioned tolerances) preserve $\mathrm{App}(P; \lambda')$, $\mathrm{App}(\neg P; \lambda')$, and the strict positivity of the corresponding Born weights.

(S3) Non-explosion is witnessed by $\Pi_{\mathrm{pc}}(P)$, for example $\Pi_{\mathrm{pc}}(P): \ \{P, \neg P\} \nvdash_{\mathrm{pc}} \psi_-$, as above (cf. Definitions 2.10 and 2.11).

Thus $\mathrm{SynTh}(\varphi; \mathcal{C}', \lambda')$ is warranted in the sense of Definition 2.11 (with the remaining warrant-profile components governed by $\mathsf{W}(\mathcal{C}')$ as in Section 2).

Connection to duality. In the soft-measurement context, slit exchange acts as a duality on both system and meter (cf. the duality exemplar Section 2.12.3 and the duality lifts Definition 2.23). Concretely, a slit-swap unitary on H exchanges $|L\rangle \leftrightarrow |R\rangle$, and a corresponding swap on H_M exchanges the pointer roles so that $F_L \leftrightarrow F_R$, hence $E_P \leftrightarrow E_{\neg P}$. At the corner level, Synthesis is self-dual under the corner involution $\Delta_{\mathcal{C}'}$ (cf. Axiom 2.22), matching the symmetric treatment of the two paths in coherent preparation and coupling.

Interpretation: forcing applicability. One may also read the paraconsistent regime as the disciplined semantics that would apply *if* one insisted on treating P as applicable in the pure interference context $(\mathcal{C}_2, \lambda_2)$, despite the primary Holothesis reading of Section 2.8. This is a counterfactual "what if": if one could ask "which slit?" without enforcing a which-path test, then Born weights would typically support both poles, and paraconsistency is the minimal non-trivial discipline that prevents this support from collapsing into explosion. On the actual primary reading, however, $(\mathcal{C}_2, \lambda_2)$ warrants Holothesis (cf. Definition 2.12).

Relation to the Holothesis. The third and fourth corners are therefore complementary rather than competing (cf. Section 2.8):

Holothesis (primary).

> At the pure interference locus, $\neg \operatorname{App}_{\mathcal{C}_2}(P; \lambda_2)$, hence Holothesis is warranted (cf. Definition 2.12).

Synthesis (secondary).

> In an applicability-restoring refinement/context, Born-positivity yields co-holding without triviality, thereby warranting Synthesis (cf. Definition 2.11).

Quantum superposition motivates paraconsistency precisely when (i) the polar claims are made applicable in a suitable refinement/context (cf. Sections 2 and 2.8) and (ii) holding is read via Born-valued semantics (cf. Definition 4.20 and eq. (28) and Section 6.5). The resulting regime is non-explosive in the sense of Definition 2.10 and supplies the explicit witness $\Pi_{\mathrm{pc}}(P)$ demanded by Definition 2.11. In this way, the Synthesis corner gains a physically grounded, logically disciplined realization, while Holothesis remains the primary status at the pure interference locus.

2.14 Paracompleteness in the Double-Slit: Truth-Value Gaps and the Holothesis

Whereas paraconsistent consequence tolerates contradictions without explosion—thereby underwriting the Synthesis corner (cf. Definitions 2.10 and 2.11 and its double-slit realization in Section 2.13)—a complete account of quantum phenomena also requires its dual: *paracomplete* consequence, which rejects the universal validity of excluded middle and admits *truth-value gaps*, resonating with contextual approaches to quantum logic [2, 9]. In the present framework, the cleanest fourth-corner diagnosis is *typed inapplicability* (Holothesis, Definition 2.12) rather than "unknown truth"; nevertheless, paracomplete gap-semantics provides the appropriate *logical shadow* of the Holothesis move once one works in a setting where the polar claims are applicable, aligning with analyses of the catuṣkoṭi in non-classical logic [13, 55].

Definition 2.25 (Paracomplete consequence and holding regimes)**.** Fix an immanent context $\mathcal{C}$ with language $\mathrm{Lang}(\mathcal{C})$ (cf. Section 2). A consequence relation $\vdash_{\mathrm{pcomp}}$ on $\mathrm{Lang}(\mathcal{C})$ is called *paracomplete* iff the law of excluded middle is not universally valid, a property characteristic of certain quantum logical systems [9]:

$$\exists \chi \in \mathrm{Lang}(\mathcal{C}) : \nvdash_{\mathrm{pcomp}} (\chi \vee \neg\chi).$$

Equivalently, there exist formulas $\chi \in \mathrm{Lang}(\mathcal{C})$ and admissible instances (e.g. loci and states, in the sense sanctioned by $\mathcal{C}$) such that neither χ nor $\neg\chi$ holds in the designated regime; in particular, the semantics admits *truth-value gaps*.

A *paracomplete evaluation/holding regime* at a locus $\lambda \in \Lambda_{\mathcal{C}}$ is a holding relation $\Vdash^{\mathrm{pcomp}}_{\mathcal{C},\lambda}$ designated by $\mathcal{C}$ whose associated consequence relation is paracomplete in the above sense.

A paracomplete holding relation from certainty (actuality semantics). Assume χ is applicable at $(\mathcal{C}, \lambda)$ in the sense of Section 2, so that the realization map of Section 6.5 assigns an effect $\mathcal{R}_{\mathcal{C},\lambda}(\chi)$ and the Born valuation $v(\chi; \rho) = \mathrm{Tr}(\mathcal{R}_{\mathcal{C},\lambda}(\chi)\rho)$ is defined, following the standard probability rule [59, 84] (cf. Definition 4.20 and eq. (28)). Define the certainty (actuality) holding regime by

$$\mathcal{C}, \lambda \Vdash^{\mathrm{pcomp}} \chi \quad \Longleftrightarrow \quad \mathrm{App}_{\mathcal{C}}(\chi; \lambda) \wedge \mathrm{Tr}\big(\rho\,\mathcal{R}_{\mathcal{C},\lambda}(\chi)\big) = 1.$$

Thus $\Vdash^{\mathrm{pcomp}}$ registers as "true" only what is certain (probability 1) relative to the realized test, consistent with the eigenstate-eigenvalue link [85].

Scope discipline: where a genuine paracomplete "gap" can be exhibited. At the pure interference locus $(\mathcal{C}_2, \lambda_2)$ the polar which-slit claims are *inapplicable* (Holothesis), as established in Section 2.8 and formalized by Definition 2.12, reflecting Bohr's complementarity principle [11]. Therefore, a paracomplete truth-value gap for P vs. $\neg P$ in the strict sense "applicable but neither holds" is most cleanly exhibited in a nearby applicability-restoring refinement $(\mathcal{C}', \lambda')$ (e.g. the soft/weak which-path coupling context of Section 2.8), where the polar effects are operationally definable while coherence is not fully destroyed, analogous to unsharp measurement schemes [63, 86].

Let P be "went through the left slit at t_0" and $\neg P$ its negation. In $(\mathcal{C}', \lambda')$ let

$$E_P := \mathcal{R}_{\mathcal{C}',\lambda'}(P), \qquad E_{\neg P} := \mathcal{R}_{\mathcal{C}',\lambda'}(\neg P) = I - E_P \qquad \text{(cf. Section 6.5)}.$$

For a coherent preparation with both amplitudes nonzero one typically has

$$0 < \mathrm{Tr}(\rho E_P) < 1 \qquad \text{and} \qquad 0 < \mathrm{Tr}(\rho E_{\neg P}) < 1,$$

hence

$$\mathcal{C}', \lambda' \nVdash^{\mathrm{pcomp}} P \qquad \text{and} \qquad \mathcal{C}', \lambda' \nVdash^{\mathrm{pcomp}} \neg P,$$

even though both formulas are applicable.

Note on disjunction semantics. For the present demonstration of excluded-middle failure, we employ the truth-functional disjunction clause at the level of the holding relation:

$$\mathcal{C}, \lambda \Vdash^{\text{pcomp}} (\chi \vee \psi) \quad \Longleftrightarrow \quad (\mathcal{C}, \lambda \Vdash^{\text{pcomp}} \chi) \text{ or } (\mathcal{C}, \lambda \Vdash^{\text{pcomp}} \psi).$$

In more general quantum logics, disjunction may be interpreted differently (e.g. as a lattice join of subspaces), but the truth-functional reading is the natural one for the present "certainty-as-actuality" semantics and suffices to exhibit paracompleteness [9].

Explicit failure of excluded middle (paracompleteness). In the coherent soft-measurement situation above, the certainty condition yields neither pole as an actuality, hence

$$\mathcal{C}', \lambda' \not\Vdash^{\text{pcomp}} (P \vee \neg P).$$

Therefore $(P \vee \neg P)$ is not universally valid under $\vdash_{\text{pcomp}}$, and the induced consequence relation is paracomplete in the sense of the opening definition, demonstrating contextuality constraints [2].

Numerical illustration. Take the symmetric superposition $\alpha = \beta = 1/\sqrt{2}$ in an applicability-restoring soft-measurement realization (cf. Section 2.8). Then typically

$$\text{Tr}(\rho E_P) = \text{Tr}(\rho E_{\neg P}) = 0.5 \neq 1,$$

so $\mathcal{C}', \lambda' \not\Vdash^{\text{pcomp}} P$ and $\mathcal{C}', \lambda' \not\Vdash^{\text{pcomp}} \neg P$, despite both being applicable. This is a concrete truth-value gap (as actuality) for the polar which-slit pair.

Formal verification of paracompleteness.

Proposition 2.26. *The holding relation $\Vdash^{\text{pcomp}}$ defined by the certainty condition induces a paracomplete consequence relation $\vdash_{\text{pcomp}}$.*

Proof. Let $\chi := P$, and consider an admissible instance $(\mathcal{C}', \lambda', \rho)$ in which P and $\neg P$ are applicable and $0 < \text{Tr}(\rho E_P) < 1$. Then $\mathcal{C}', \lambda' \not\Vdash^{\text{pcomp}} P$ and $\mathcal{C}', \lambda' \not\Vdash^{\text{pcomp}} \neg P$, hence (by the disjunction clause) $\mathcal{C}', \lambda' \not\Vdash^{\text{pcomp}} (P \vee \neg P)$. Therefore $(P \vee \neg P)$ fails in an admissible instance, so it is not a theorem of $\vdash_{\text{pcomp}}$: $\not\vdash_{\text{pcomp}} (P \vee \neg P)$. This is exactly paracompleteness. $\square$

Contrast with the paraconsistent regime (potentiality vs. actuality). In the same applicability-restoring context $(\mathcal{C}', \lambda')$ one may also consider the paraconsistent (potentiality) regime of Section 2.13, where holding is triggered by nonzero Born weight: $\text{Tr}(\rho E) > 0$. The two regimes are dual "potentiality vs. actuality" readings of the same realized test, echoing philosophical distinctions in quantum interpretation [33, 87].

Table 3: Dual regimes once applicability is restored (e.g. in $(\mathcal{C}', \lambda')$ of Section 2.8)

Feature	Paraconsistent regime ($\Vdash^{\mathrm{pc}}$)	Paracomplete regime ($\Vdash^{\mathrm{pcomp}}$)
Holding condition	$\mathrm{Tr}(\rho E) > 0$ (nonzero support: **potentiality**)	$\mathrm{Tr}(\rho E) = 1$ (certainty: **actuality**)
Status for $P, \neg P$ (coherent state)	Both hold (logical **glut**; non-explosive)	Neither holds (logical **gap**; excluded middle can fail)
Excluded middle $P \vee \neg P$	Holds ($E_{\neg P} = I - E_P$, nonzero weights)	Can fail (neither pole certain)
Tetralemma corner	Synthesis (cf. Definition 2.11)	"Neither as actuality" (paracomplete gap; shadow of fourth corner)

2.14.1   Duality and self-duality of gaps

In symmetric realizations, slit-exchange duality swaps $P \leftrightarrow \neg P$ at the effect level (e.g. $E_P \leftrightarrow E_{\neg P}$) in the sense of the duality lift Definition 2.23, consistent with Wigner's theorem on symmetry transformations [73] (cf. also Section 2.12.3). Since the certainty condition is symmetric, it yields identical status for P and its dual P^D:

$$\mathcal{C}', \lambda' \not\Vdash^{\mathrm{pcomp}} P \qquad \Longleftrightarrow \qquad \mathcal{C}', \lambda' \not\Vdash^{\mathrm{pcomp}} P^D,$$

so the gap is *self-dual* in exactly the sense one expects from symmetry-preserving contexts. This mirrors the fact that the corner-level involution fixes the fourth corner, a structural feature of the *catuṣkoṭi* in non-classical logic [13, 44] (cf. Axiom 2.22).

Extension: a three-slit variant and coarse-graining. In a three-slit setup with path kets $|A\rangle, |B\rangle, |C\rangle$ and state $|\psi\rangle = \alpha|A\rangle + \beta|B\rangle + \gamma|C\rangle$, one may have a context/locus where each single-slit claim P_A, P_B, P_C is applicable but none is certain, illustrating generalized contextuality constraints [2, 9]:

$$0 < \mathrm{Tr}(\rho E_A) < 1, \quad 0 < \mathrm{Tr}(\rho E_B) < 1, \quad 0 < \mathrm{Tr}(\rho E_C) < 1.$$

Then $\Vdash^{\mathrm{pcomp}}$ yields a three-way gap: $\mathcal{C}, \lambda \not\Vdash^{\mathrm{pcomp}} P_A$, $\mathcal{C}, \lambda \not\Vdash^{\mathrm{pcomp}} P_B$, $\mathcal{C}, \lambda \not\Vdash^{\mathrm{pcomp}} P_C$, and hence $\mathcal{C}, \lambda \not\Vdash^{\mathrm{pcomp}} (P_A \vee P_B \vee P_C)$ under the truth-functional disjunction clause. At the same time, a coarse-grained claim Q ("the electron passed through the slit-plane region") may be realized by a different effect (for commuting slit projectors, the sum $E_A + E_B + E_C$ on the slit subspace), and can be certain, analogous to consistent histories coarse-graining [5]: $\mathrm{Tr}(\rho\, \mathcal{R}(Q)) = 1$. Thus, under certainty-based paracomplete semantics, coarse-grained actuality (Q) need not coincide with disjunctive certainty of fine-grained alternatives.

Paracompleteness and the Holothesis (primary vs. shadow). Holothesis is defined here as typed inapplicability of the polar pair (cf. Definition 2.12), not merely as "neither pole is

true," resonating with topos-theoretic approaches to quantum logic [9]. In the double-slit interference locus $(\mathcal{C}_2, \lambda_2)$, the primary diagnosis is

$$\neg \operatorname{App}_{\mathcal{C}_2}(P; \lambda_2) \qquad (\text{equivalently } \neg \operatorname{App}_{\mathcal{C}_2}(\neg P; \lambda_2)),$$

as established in Section 2.8, reflecting Bohr's complementarity principle [11]. Paracompleteness supplies the appropriate logical shadow once applicability is restored: when P and $\neg P$ are meaningful but neither is certain, excluded middle fails and a truth-value gap (as actuality) appears, characteristic of intuitionistic and quantum logics [2, 9].

Interpretational connections (optional orientation). The certainty-based paracomplete "actuality" semantics naturally aligns with readings that emphasize the gap between potentiality and determinate actual value (e.g. modal or history-based stances) [5, 88], whereas the Holothesis diagnosis resonates with explicitly contextual/operational readings in which unlicensed questions are treated as ill-typed rather than merely indeterminate (cf. Section 2 and Definition 2.12), akin to relational quantum mechanics [4].

Paracompleteness as the logical shadow of the Holothesis. In the pure interference situation $(\mathcal{C}_2, \lambda_2)$ the correct *primary* diagnosis is Holothesis:

$$\operatorname{HoloTh}(\varphi; \mathcal{C}_2, \lambda_2) \text{ holds, i.e. } \neg \operatorname{App}_{\mathcal{C}_2}(P; \lambda_2) \quad (\text{equivalently } \neg \operatorname{App}_{\mathcal{C}_2}(\neg P; \lambda_2)),$$

by clause (H1) of Definition 2.12 and the concrete double-slit analysis in Section 2.8. In particular, at $(\mathcal{C}_2, \lambda_2)$ the realized effect $\mathcal{R}_{\mathcal{C}_2, \lambda_2}(P)$ is *not formed* (cf. Section 6.5), so it is methodologically improper to assign any actuality-truth value to P or $\neg P$ there, consistent with operational quantum physics [63].

Nevertheless, Definition 2.12 explicitly allows an *optional strengthening (shift witness)*: one may adjoin a lift $\Uparrow: \operatorname{Lang}(\mathcal{C}_2) \to \operatorname{Lang}(\mathcal{C}_2^{\Uparrow})$ (or, equivalently, pass to an admissible refinement $(\mathcal{C}', \lambda')$, e.g. the soft/weak which-path coupling of Section 2.8) in which the relevant which-path content becomes evaluable *without* thereby claiming that it was evaluable in $(\mathcal{C}_2, \lambda_2)$ itself, analogous to context shifts in topos physics [9]. In such a lifted/refined setting $(\mathcal{C}^\star, \lambda^\star)$, define the actuality (certainty) holding regime by

$$\mathcal{C}^\star, \lambda^\star \Vdash^{\text{pcomp}} \chi \quad \Longleftrightarrow \quad \operatorname{App}_{\mathcal{C}^\star}(\chi; \lambda^\star) \ \wedge \ \operatorname{Tr}\!\left(\rho \, \mathcal{R}_{\mathcal{C}^\star, \lambda^\star}(\chi) \right) = 1.$$

For a coherent path state with nonzero weights for both alternatives, one typically has

$$0 < \operatorname{Tr}\!\left(\rho \, \mathcal{R}_{\mathcal{C}^\star, \lambda^\star}(P) \right) < 1, \qquad 0 < \operatorname{Tr}\!\left(\rho \, \mathcal{R}_{\mathcal{C}^\star, \lambda^\star}(\neg P) \right) < 1,$$

so that

$$\mathcal{C}^\star, \lambda^\star \not\Vdash^{\text{pcomp}} P \qquad \text{and} \qquad \mathcal{C}^\star, \lambda^\star \not\Vdash^{\text{pcomp}} \neg P,$$

even though both poles are applicable in $(\mathcal{C}^\star, \lambda^\star)$. Hence excluded middle can fail in the certainty semantics, a hallmark of paracomplete systems [13]:

$$\mathcal{C}^\star, \lambda^\star \not\Vdash^{\text{pcomp}} (P \vee \neg P).$$

This is the paracomplete *gap* that serves as the logical shadow of the Holothesis move: it preserves the "neither" character of the fourth corner while making it expressible in truth-conditional (actuality) terms.

The shadow relationship (formal hierarchy). We therefore have a strict hierarchy:

(i) **Primary (Holothesis):** $\neg\,\mathrm{App}_{\mathcal{C}_2}(P; \lambda_2)$ at $(\mathcal{C}_2, \lambda_2)$ (cf. Definition 2.12 and Section 2.8).

(ii) **Shadow (paracomplete gap):** in a refinement/lift $(\mathcal{C}^\star, \lambda^\star)$ with $\mathrm{App}_{\mathcal{C}^\star}(P; \lambda^\star)$, one has neither $\mathcal{C}^\star, \lambda^\star \Vdash^{\mathrm{pcomp}} P$ nor $\mathcal{C}^\star, \lambda^\star \Vdash^{\mathrm{pcomp}} \neg P$.

The shadow retains the fourth-corner "neither" structure, but only after the shift that makes the content evaluable.

Interpretational clarification. Operational/contextualist stances typically stop at (i) (Holothesis: the polar question is ill-typed at the interference locus), whereas realist/modal stances often proceed to (ii), insisting on an actuality-style truth predicate in a setting where the polar content is at least well-typed [88, 89]. Both moves are represented within the formal options of Definition 2.12.

What the shadow preserves. In symmetric double-slit realizations, the paracomplete gap inherits key structural features of the Holothesis diagnosis:

- **Symmetry of poles:** P and $\neg P$ receive the same gap-status.

- **Self-duality:** under slit-exchange duality (cf. Section 2.12.3), the gap remains a gap for the dual claim, preserving the involution structure [73].

- **Non-arbitrariness:** the gap is forced by coherence (Born probabilities strictly between 0 and 1), not by ignorance or incomplete specification, reflecting quantum indeterminacy [11, 75].

Paracompleteness enters the double-slit in the disciplined "actuality" semantics in which only certainty warrants holding, yielding truth-value gaps for coherent states even when the polar claims are applicable. In the pure interference context, Holothesis strengthens this into typed inapplicability (cf. Definition 2.12), but the same structural lesson remains: quantum mechanics does not, in general, sustain universal excluded middle for which-path claims, a result consistent with the Kochen–Specker theorem [2]. The tetralemma's fourth corner is therefore naturally aligned with paracomplete logic, just as the third corner is naturally aligned with paraconsistent logic (cf. Definitions 2.10 and 2.11), providing a unified logical framework for quantum phenomena [9, 13].

2.15 The Laws of Non–Contradiction and Excluded–Middle as Tetralemmatic Metaprinciples

Fix an immanent context $\mathcal{C}$ with language $\mathrm{Lang}(\mathcal{C})$ and a locus $\lambda \in \Lambda_{\mathcal{C}}$ (cf. Section 2). Let $\Vdash^r_{\mathcal{C},\lambda,\rho}$ be a designated holding relation at $(\mathcal{C}, \lambda)$ for some evaluation/consequence regime r, possibly parameterized by an admissible state ρ (we suppress ρ when fixed by $\mathcal{C}$), following the framework of operational quantum logic [9, 63].

Per-formula (local) schemata. To state "LEM/LNC holds for a specific formula" unambiguously, define the local metapredicates:

$$\mathrm{LEM}^r_{\mathcal{C},\lambda,\rho}[\chi] \iff \Big(\mathrm{App}_{\mathcal{C}}(\chi;\lambda) \Rightarrow \big(\mathcal{C},\lambda \Vdash^r_{\rho} \chi \ \vee\ \mathcal{C},\lambda \Vdash^r_{\rho} \neg\chi\big)\Big),$$

$$\mathrm{LNC}^r_{\mathcal{C},\lambda,\rho}[\chi] \iff \Big(\mathrm{App}_{\mathcal{C}}(\chi;\lambda) \Rightarrow \neg\big(\mathcal{C},\lambda \Vdash^r_{\rho} \chi \ \wedge\ \mathcal{C},\lambda \Vdash^r_{\rho} \neg\chi\big)\Big).$$

This local formulation allows classical principles to hold for specific formulas while failing globally, a feature characteristic of contextual logics [2, 55].

Definition 2.27 (Law of Excluded Middle (applicability-gated, regime-relative))**.** Excluded middle holds at $(\mathcal{C},\lambda)$ in regime r (relative to ρ if applicable) iff

$$\mathrm{LEM}^r(\mathcal{C},\lambda,\rho) \iff \forall \chi \in \mathrm{Lang}(\mathcal{C}) : \Big(\mathrm{App}_{\mathcal{C}}(\chi;\lambda) \ \Rightarrow\ \big(\mathcal{C},\lambda \Vdash^r_{\rho} \chi \ \vee\ \mathcal{C},\lambda \Vdash^r_{\rho} \neg\chi\big)\Big).$$

Equivalently,

$$\mathrm{LEM}^r(\mathcal{C},\lambda,\rho) \iff \forall \chi \in \mathrm{Lang}(\mathcal{C})\ \mathrm{LEM}^r_{\mathcal{C},\lambda,\rho}[\chi].$$

Definition 2.28 (Law of Non-Contradiction (applicability-gated, regime-relative))**.** Non-contradiction holds at $(\mathcal{C},\lambda)$ in regime r (relative to ρ if applicable) iff

$$\mathrm{LNC}^r(\mathcal{C},\lambda,\rho) \iff \forall \chi \in \mathrm{Lang}(\mathcal{C}) : \Big(\mathrm{App}_{\mathcal{C}}(\chi;\lambda) \ \Rightarrow\ \neg\big(\mathcal{C},\lambda \Vdash^r_{\rho} \chi \ \wedge\ \mathcal{C},\lambda \Vdash^r_{\rho} \neg\chi\big)\Big).$$

Equivalently,

$$\mathrm{LNC}^r(\mathcal{C},\lambda,\rho) \iff \forall \chi \in \mathrm{Lang}(\mathcal{C})\ \mathrm{LNC}^r_{\mathcal{C},\lambda,\rho}[\chi].$$

Context-valid versions. We say $\mathrm{LEM}^r(\mathcal{C},\rho)$ holds (resp. $\mathrm{LNC}^r(\mathcal{C},\rho)$ holds) iff the corresponding condition holds for all $\lambda \in \Lambda_{\mathcal{C}}$, analogous to contextuality constraints in quantum theory [9].

Remark 2.29 (Local vs. global LEM/LNC and why we gate by applicability)**.** The pointwise metapredicates $\mathrm{LEM}^r_{\mathcal{C},\lambda,\rho}[\chi]$ and $\mathrm{LNC}^r_{\mathcal{C},\lambda,\rho}[\chi]$ express "LEM/LNC for this particular formula χ at this locus." The global schemata $\mathrm{LEM}^r(\mathcal{C},\lambda,\rho)$ and $\mathrm{LNC}^r(\mathcal{C},\lambda,\rho)$ in Definitions 2.27 and 2.28 are obtained by universal quantification over $\chi \in \mathrm{Lang}(\mathcal{C})$.

Applicability-gating is essential (cf. Section 2): when $\neg\,\mathrm{App}_{\mathcal{C}}(\chi;\lambda)$, the framework treats χ as ill-typed at $(\mathcal{C},\lambda)$ rather than as a candidate for truth-valuation, reflecting presupposition failure in formal semantics [9, 13].

Definition 2.30 (Paracomplete consequence regimes)**.** Fix an immanent context $\mathcal{C}$ with language $\mathrm{Lang}(\mathcal{C})$. A consequence relation $\vdash_{\mathrm{pcomp}}$ on $\mathrm{Lang}(\mathcal{C})$ is called *paracomplete* iff excluded middle is not universally valid, a property characteristic of certain quantum and intuitionistic logics [9, 13]:

$$\exists \chi \in \mathrm{Lang}(\mathcal{C}) : \nvdash_{\mathrm{pcomp}} (\chi \vee \neg\chi).$$

Equivalently, there exist admissible instances in which neither χ nor $\neg\chi$ holds (truth-value gaps), as in Kleene's strong three-valued logic [13].

A *paracomplete evaluation/holding regime* at locus $\lambda \in \Lambda_{\mathcal{C}}$ is a holding relation $\Vdash^{\mathrm{pcomp}}_{\mathcal{C},\lambda}$ designated by $\mathcal{C}$ whose associated consequence relation is paracomplete in the above sense.

Duality as a covariance principle for regime-relative schemata.

Condition 2.31 (Duality action on formulas is an involution)**.** *The dualization $\chi \mapsto \chi^{\mathbf{D}}$ is a bijection on* $\mathrm{Lang}(\mathcal{C})$*; in particular*

$$(\chi^{\mathbf{D}})^{\mathbf{D}} \;\equiv\; \chi \qquad \text{for all } \chi \in \mathrm{Lang}(\mathcal{C}).$$

This involution structure mirrors symmetry operations in quantum mechanics [73].

Condition 2.32 (Duality equivariance of applicability and holding (state-covariant))**.** *Assume a dualization map $\chi \mapsto \chi^{\mathbf{D}}$ on* $\mathrm{Lang}(\mathcal{C})$ *such that, for all χ and λ,*

$$\mathrm{App}_{\mathcal{C}}(\chi; \lambda) \iff \mathrm{App}_{\mathcal{C}}(\chi^{\mathbf{D}}; \lambda), \qquad (\neg\chi)^{\mathbf{D}} \equiv \neg(\chi^{\mathbf{D}}).$$

Assume further there is a corresponding action $\rho \mapsto \rho^{\mathbf{D}}$ on admissible states such that

$$\mathcal{C}, \lambda \Vdash_{\rho}^{r} \chi \quad \iff \quad \mathcal{C}, \lambda \Vdash_{\rho^{\mathbf{D}}}^{r} \chi^{\mathbf{D}}.$$

In quantum realizations (cf. Sections 2.12.3 and 6.5) one typically has

$$\mathcal{R}_{\mathcal{C},\lambda}(\chi^{\mathbf{D}}) = \mathbf{D}_{\mathcal{C}}\, \mathcal{R}_{\mathcal{C},\lambda}(\chi)\, \mathbf{D}_{\mathcal{C}}^{\dagger}, \qquad \rho^{\mathbf{D}} := \mathbf{D}_{\mathcal{C}}^{\dagger} \rho\, \mathbf{D}_{\mathcal{C}},$$

following the standard treatment of symmetry transformations in quantum theory [59, 73].

Remark 2.33 (When the simpler "same-state" equivariance is valid)**.** If ρ is $\mathbf{D}$-invariant (i.e. $\rho^{\mathbf{D}} = \rho$), then Condition 2.32 reduces to

$$\mathcal{C}, \lambda \Vdash_{\rho}^{r} \chi \quad \iff \quad \mathcal{C}, \lambda \Vdash_{\rho}^{r} \chi^{\mathbf{D}}.$$

This occurs in symmetric preparations, analogous to invariant states under group actions [90].

Lemma 2.34 (Duality covariance of local LEM/LNC (state-covariant))**.** *Under Condition 2.32, for every $\chi \in \mathrm{Lang}(\mathcal{C})$ and locus λ,*

$$\mathrm{LEM}_{\mathcal{C},\lambda,\rho}^{r}[\chi] \iff \mathrm{LEM}_{\mathcal{C},\lambda,\rho^{\mathbf{D}}}^{r}[\chi^{\mathbf{D}}], \qquad \mathrm{LNC}_{\mathcal{C},\lambda,\rho}^{r}[\chi] \iff \mathrm{LNC}_{\mathcal{C},\lambda,\rho^{\mathbf{D}}}^{r}[\chi^{\mathbf{D}}].$$

Proof. Substitute the equivalences from Condition 2.32 into the defining implications of the local metapredicates; negation-commutation $(\neg\chi)^{\mathbf{D}} \equiv \neg(\chi^{\mathbf{D}})$ handles the $\neg\chi$ clauses, preserving the De Morgan structure [13]. $\square$

Theorem 2.35 (Duality invariance of global LEM/LNC at a locus (state-covariant))**.** *Assume Conditions 2.31 and 2.32. Then*

$$\mathrm{LEM}^{r}(\mathcal{C}, \lambda, \rho) \iff \mathrm{LEM}^{r}(\mathcal{C}, \lambda, \rho^{\mathbf{D}}), \qquad \mathrm{LNC}^{r}(\mathcal{C}, \lambda, \rho) \iff \mathrm{LNC}^{r}(\mathcal{C}, \lambda, \rho^{\mathbf{D}}).$$

Proof. Use Lemma 2.34 pointwise in χ and then quantify over $\mathrm{Lang}(\mathcal{C})$. Bijection of $\chi \mapsto \chi^{\mathbf{D}}$ (cf. Condition 2.31) ensures the universal quantifier is preserved, analogous to symmetry-invariant properties in field theory [74]. $\square$

From regime-level duality to corner-level transport (polar case). Let $P := \exists^{\mathcal{C}}_{\lambda} x\, \varphi(x)$ be a *polar* existential claim, consistent with context-dependent quantification in quantum logic [9, 13] (cf. Definitions 2.7, 2.8, 2.11 and 2.12).

Condition 2.36 (Polar duality exchanges the poles). *Assume that for the polar pair* $(P, \neg P)$, *the duality operator acts as logical negation on the poles, a standard feature in paraconsistent and quantum logical frameworks [13, 55]:*

$$P^{\mathbf{D}} \equiv \neg P, \qquad (\neg P)^{\mathbf{D}} \equiv P.$$

Definition 2.37 (Corner involution). Define $\Delta_{\mathcal{C}}$ on corner-labels by the standard tetralemmatic involution structure [13, 46]:

$$\Delta_{\mathcal{C}}(\mathrm{Th}) = \mathrm{AntiTh}, \quad \Delta_{\mathcal{C}}(\mathrm{AntiTh}) = \mathrm{Th}, \quad \Delta_{\mathcal{C}}(\mathrm{SynTh}) = \mathrm{SynTh}, \quad \Delta_{\mathcal{C}}(\mathrm{HoloTh}) = \mathrm{HoloTh}.$$

Theorem 2.38 (Corner-status transport under polar duality (state-covariant)). *Assume Conditions 2.32 and 2.36. Whenever the corner-status of P at $(\mathcal{C}, \lambda, \rho)$ is defined via the core clauses (T1)/(A1)/(S1)/(H1) of Definitions 2.7, 2.8, 2.11 and 2.12 (with $\Vdash$ read as $\Vdash_{\rho}$),*

$$\mathrm{Status}_{\mathcal{C}}(P^{\mathbf{D}}, \lambda; \rho^{\mathbf{D}}) = \Delta_{\mathcal{C}}\big(\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho)\big).$$

This matches the corner-level involution required by the Axiom of Duality (cf. Axiom 2.22), reflecting symmetry covariance in physical theories [73, 74] in the polar-exchange case.

Proof. Using Condition 2.36, $P^{\mathbf{D}} \equiv \neg P$ and $\neg(P^{\mathbf{D}}) \equiv P$. Then:

- If $\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho) = \mathrm{Th}$, (T1) gives $\mathrm{App}_{\mathcal{C}}(P; \lambda)$ and $\mathcal{C}, \lambda \Vdash^r_{\rho} P$, consistent with Aristotelian affirmation [33]. Hence $\mathcal{C}, \lambda \Vdash^r_{\rho} \neg(P^{\mathbf{D}})$, and by applicability equivariance (A1) holds for $P^{\mathbf{D}}$ at $\rho^{\mathbf{D}}$, so the status is AntiTh.

- If $\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho) = \mathrm{AntiTh}$, (A1) gives $\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$ and $\mathcal{C}, \lambda \Vdash^r_{\rho} \neg P$, aligning with Nyaya negation [39, 41]. Since $P^{\mathbf{D}} \equiv \neg P$, (T1) holds for $P^{\mathbf{D}}$ at $\rho^{\mathbf{D}}$, so the status is Th.

- If $\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho) = \mathrm{SynTh}$, then (S1) gives $\mathrm{App}_{\mathcal{C}}(P; \lambda)$, $\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$ and co-holding $\mathcal{C}, \lambda \Vdash^{\mathrm{pc}}_{\rho} P$ as well as $\mathcal{C}, \lambda \Vdash^{\mathrm{pc}}_{\rho} \neg P$ in a paraconsistent regime [13, 54]; renaming poles via $P^{\mathbf{D}} \equiv \neg P$ preserves (S1), so the status remains SynTh and $\Delta_{\mathcal{C}}$ fixes it.

- If $\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho) = \mathrm{HoloTh}$, then (H1) gives $\neg\,\mathrm{App}_{\mathcal{C}}(P; \lambda)$ (and by negation-closure also $\neg\,\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$), reflecting contextuality constraints [9, 11]. By applicability equivariance, $\neg\,\mathrm{App}_{\mathcal{C}}(P^{\mathbf{D}}; \lambda)$, hence the status remains HoloTh and $\Delta_{\mathcal{C}}$ fixes it.

These four cases are exactly $\Delta_{\mathcal{C}}$ as in Definition 2.37. $\qquad\square$

Corollary 2.39 (Corner-status transport at fixed state (symmetric preparations)). *If additionally $\rho^{\mathbf{D}} = \rho$, corresponding to symmetry-invariant quantum states [59, 80], then*

$$\mathrm{Status}_{\mathcal{C}}(P^{\mathbf{D}}, \lambda; \rho) = \Delta_{\mathcal{C}}\big(\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho)\big).$$

Proof. By the state-covariant corner-transport Theorem 2.38 we have

$$\mathrm{Status}_{\mathcal{C}}(P^{\mathbf{D}}, \lambda; \rho^{\mathbf{D}}) = \Delta_{\mathcal{C}}\big(\mathrm{Status}_{\mathcal{C}}(P, \lambda; \rho)\big),$$

whenever the corner-status of P at $(\mathcal{C}, \lambda, \rho)$ is defined via the core clauses (T1)/(A1)/(S1)/(H1) of Definitions 2.7, 2.8, 2.11 and 2.12. Under the additional symmetry assumption $\rho^{\mathbf{D}} = \rho$, the left-hand side becomes $\mathrm{Status}_{\mathcal{C}}(P^{\mathbf{D}}, \lambda; \rho)$, yielding the claimed identity. $\qquad\square$

3 Tetralemmatic Algebra and Pre-Hilbert Structures

The purpose of this section is to construct a *logical Hilbert template* directly from the tetralemma corner-structure and the axiom of duality, *without* invoking any Realization Theorem (mapping formulas to quantum effects) and *without* presupposing a pre-given physical Hilbert space. The tetralemmatic structure draws upon the classical Catuṣkoṭi framework found in Buddhist logic [13, 15, 53], reinterpreted here through a paraconsistent and paracomplete lens [54, 55].

The guiding dependence is:

$$\left(\text{tetralemma} + \text{duality}\right) \; + \; \left(\text{warrant functional}\right) \implies \left(\text{Hilbert space via GNS}\right).$$

The only genuinely unavoidable additional ingredient beyond tetralemma and duality is a *positive evaluation/warrant functional*: without some positivity data, there is no canonical way to determine an inner product. This algebraic approach to quantum kinematics follows the Gelfand–Naimark–Segal (GNS) reconstruction paradigm [7, 16, 17, 18], and aligns with the view that logic stands to quantum theory as geometry stands to gravity [8], as well as quantum measure-theoretic foundations [14].

Remark 3.1 (What is *not* assumed)**.** Throughout this construction, we do *not* assume:

- any pre-existing physical Hilbert space,
- any identification of logical formulas with quantum effects,
- any Born rule interpretation.

All Hilbert-space structure will be induced *after* we introduce a positive warrant functional on a purely logical ∗-algebra generated from the tetralemma corner-acts. This avoids the circularity often noted in foundational discussions where the Hilbert space is presupposed [7, 8, 9].

3.0.1 Tetralemma atomic lens

We assume the *tetralemma atomic lens* as already introduced in Definition 2.17:

$$\overset{\circ}{\mathbf{\Lambda}} \; := \; \left\langle \mathbf{Th},\ \mathbf{AntiTh},\ \mathbf{SynTh},\ \mathbf{HoloTh} \right\rangle,$$

together with its defining clauses (*no redefinition / generated closure / atomicity*). This four-corner structure corresponds to the classical Catuṣkoṭi framework [13, 15, 53], reinterpreted here through a paraconsistent and paracomplete lens [54, 55].

When the bridge map to ontological modal symbols is invoked, we retain the corner–modal bijection of Remark 2.20:

$$|\overset{\circ}{\mathbf{0}}\rangle \leftrightarrow \mathbf{Th}, \quad |\overset{\circ}{\mathbf{1}}\rangle \leftrightarrow \mathbf{AntiTh}, \quad |\overset{\circ}{\mathbf{0}} \overset{=}{\wedge} \overset{\circ}{\mathbf{1}}\rangle \leftrightarrow \mathbf{SynTh}, \quad |\overset{\circ}{\mathbf{0}} \overset{\infty}{\wedge} \overset{\circ}{\mathbf{1}}\rangle \leftrightarrow \mathbf{HoloTh}.$$

3.0.2 Corner-acts and warrant conditions

We work in the same logical environment as the tetralemma formalism (tetralogic) developed in Section 2. In particular, we assume:

- a contextual parameter $\mathcal{C}$ (the context) and a parameter λ (the polarity / mode label),

- an applicability predicate $\mathrm{App}_{\mathcal{C}}(P; \lambda)$,

- a warrant/forcing relation $\mathcal{C}, \lambda \Vdash (\cdot)$.

This context-dependent structure aligns with topos-theoretic approaches to quantum logic [9, 91].

Definition 3.2 (Polar claim (schematic form)). A *polar claim* is an expression of the schematic form

$$P \ := \ \exists_{\lambda}^{\mathcal{C}} u \, \varphi(u),$$

where φ is a well-formed predicate in the ambient language, and where P is required to be applicable at $(\mathcal{C}, \lambda)$ in the sense that $\mathrm{App}_{\mathcal{C}}(P; \lambda)$ holds. This matches the existential claim structure of Definitions 2.7 and 2.8.

Assumption 3.3 (Tetralemma corner-acts as primitive statuses). *For each polar claim P (Definition 3.2) that is applicable at $(\mathcal{C}, \lambda)$, the four corner-acts*

$$\mathbf{Th}(P), \qquad \mathbf{AntiTh}(P), \qquad \mathbf{SynTh}(P), \qquad \mathbf{HoloTh}(P)$$

are available as primitive logical statuses*, determined by the tetralogic clauses that govern the interaction of $\mathrm{App}_{\mathcal{C}}(\cdot \, ; \lambda)$ and the warrant relation $\mathcal{C}, \lambda \Vdash (\cdot)$. These correspond precisely to the definitions in Definitions 2.7, 2.8, 2.11 and 2.12.*

Remark 3.4 (Role of warrant clauses). In the later purely-logical Hilbert construction, we will *not* interpret these statuses as quantum probabilities. Instead, they are the primitive corner-status outputs (Thesis/Antithesis/Synthesis/Holothesis) available whenever a claim is well-formed and applicable. Numerical evaluation will enter only when we introduce a positive warrant functional. This distinguishes the logical template from standard quantum measure theory [8, 14] until the GNS construction is applied [7, 16].

3.0.3 Axiom of duality

Assumption 3.5 (Duality involution on corners). *We assume a duality involution $\mathbf{D}$ acting on the tetralemma corner-structure with the corner action*

$$\begin{aligned}
\mathbf{D(Th)} &= \mathbf{AntiTh} & \mathbf{D(SynTh)} &= \mathbf{SynTh} \\
\mathbf{D(AntiTh)} &= \mathbf{Th} & \mathbf{D(HoloTh)} &= \mathbf{HoloTh}
\end{aligned} \tag{7}$$

In particular, $\mathbf{D}$ is involutive on corners:

$$\mathbf{D}^2(\mathbf{Th}) = \mathbf{Th}, \qquad \mathbf{D}^2(\mathbf{SynTh}) = \mathbf{SynTh},$$

$$\mathbf{D}^2(\mathbf{AntiTh}) = \mathbf{AntiTh}, \qquad \mathbf{D}^2(\mathbf{HoloTh}) = \mathbf{HoloTh}. \tag{8}$$

This matches the Axiom of Duality (Axiom 2.22) and the corner involution $\Delta_{\mathcal{C}}$. The exchange of Thesis/Antithesis reflects standard duality symmetries in physics [92, 93].

Remark 3.6 (Optional duality action on claims)**.** If a refined duality action on claims is present (e.g. $P \mapsto P^{\mathbf{D}}$), it may be carried along in parallel. The purely-logical Hilbert template construction below, however, requires only the corner-level involution described in Assumption 3.5. Propositional duality lifts are discussed in Definition 2.23.

3.1 Typed primitives, universal algebra, and coherent warrant

Definition 3.7 (Logical loci, languages, and polar claims)**.** Let $\mathbf{Ctx}$ be the class of *contexts* and let $\mathbf{Pol}$ be the class of *polarity/mode labels*. A *locus* is a pair $(\mathcal{C}, \lambda) \in \mathbf{Ctx} \times \mathbf{Pol}$. The use of context-indexed languages aligns with topos-theoretic and contextual approaches to physics [9, 91].

For each $\mathcal{C} \in \mathbf{Ctx}$, let $\mathrm{Lang}(\mathcal{C})$ denote the collection of well-formed formulas available in context $\mathcal{C}$.

Fix a signature/grammar Σ for building polar existential forms. For each locus $(\mathcal{C}, \lambda)$ define the set of *polar claims* by

$$\mathrm{Pol}(\Sigma; \mathcal{C}, \lambda) := \left\{ P \in \mathrm{Lang}(\mathcal{C}) \ : \ P \equiv \exists^{\mathcal{C}}_{\lambda} x \, \varphi(x) \text{ for some } \varphi \in \mathrm{Lang}(\mathcal{C}) \right\}.$$

Definition 3.8 (Applicability predicate)**.** For each locus $(\mathcal{C}, \lambda)$, $\mathrm{App}_{\mathcal{C}}(\,\cdot\,; \lambda)$ is a primitive predicate on $\mathrm{Pol}(\Sigma; \mathcal{C}, \lambda)$. Write

$$\mathsf{P}_{\mathcal{C}, \lambda} := \left\{ P \in \mathrm{Pol}(\Sigma; \mathcal{C}, \lambda) \ : \ \mathrm{App}_{\mathcal{C}}(P; \lambda) \right\}$$

for the class of *admissible* (well-formed and applicable) polar claims at $(\mathcal{C}, \lambda)$.

Definition 3.9 (Forcing/warrant relation)**.** Fix a locus $(\mathcal{C}, \lambda)$. A (possibly regime-indexed) warrant/forcing relation is a primitive relation

$$\mathcal{C}, \lambda \Vdash (\cdot)$$

on $\mathrm{Lang}(\mathcal{C})$ (and in particular on $\mathrm{Pol}(\Sigma; \mathcal{C}, \lambda)$). If the tetralogic includes additional indices (e.g. a preparation/state ρ and/or regime superscripts), they may be carried implicitly; the constructions below use only the existence of $\mathrm{App}_{\mathcal{C}}$ and $\Vdash$.

Definition 3.10 (Corner label set and status codomain)**.** Let the tetralemma atomic tetralemmatic lens be the corner tuple

$$\overset{\circ}{\mathbf{\Lambda}} := \left\langle \mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh} \right\rangle,$$

and define the corner-status codomain

$$\mathbf{Status}_{\Lambda} := \{ \mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh} \}.$$

These four corners correspond to the classical Catuṣkoṭi positions [13, 15, 53], with Synthesis representing a paraconsistent glut and Holothesis representing a paracomplete gap [54, 55].

Assumption 3.11 (Corner-status assignment is primitive and clause-determined)**.** *For each locus $(\mathcal{C}, \lambda)$ there is a corner-status map*

$$\text{Status}_{\mathcal{C}, \lambda} : \mathsf{P}_{\mathcal{C}, \lambda} \to \mathbf{Status}_{\Lambda},$$

determined by the tetralogic clauses governing $\text{App}_{\mathcal{C}}(\,\cdot\,; \lambda)$ *and* $\Vdash$. *We write (as abbreviations)*

$$\begin{aligned}
\mathbf{Th}(P) &\iff \text{Status}_{\mathcal{C}, \lambda}(P) = \mathbf{Th}, \\
\mathbf{AntiTh}(P) &\iff \text{Status}_{\mathcal{C}, \lambda}(P) = \mathbf{AntiTh}, \\
\mathbf{SynTh}(P) &\iff \text{Status}_{\mathcal{C}, \lambda}(P) = \mathbf{SynTh}, \\
\mathbf{HoloTh}(P) &\iff \text{Status}_{\mathcal{C}, \lambda}(P) = \mathbf{HoloTh}.
\end{aligned} \tag{9}$$

Assumption 3.12 (Duality involution on corners)**.** *A duality involution* $\mathbf{D}$ *acts on corner labels by*

$$\begin{aligned}
\mathbf{D}(\mathbf{Th}) = \mathbf{AntiTh} && \mathbf{D}(\mathbf{SynTh}) = \mathbf{SynTh} \\
\mathbf{D}(\mathbf{AntiTh}) = \mathbf{Th} && \mathbf{D}(\mathbf{HoloTh}) = \mathbf{HoloTh}
\end{aligned} \tag{10}$$

and $\mathbf{D}^2 = \text{id}$ *on* $\mathbf{Status}_{\Lambda}$. *This involution reflects the symmetry properties often required in physical duality transformations [92, 93]. Optionally, duality may act on claims by* $P \mapsto P^{\mathbf{D}}$; *the purely logical Hilbert template below requires only the corner-level involution and (when used) a compatible claim relabelling.*

3.2 Universal corner ∗-algebra and its minimal congruence

This construction follows the algebraic quantum theory paradigm [7, 16, 17], where the observable algebra is generated from primitive logical data rather than presupposed from a physical Hilbert space.

Definition 3.13 (Corner generator set at a locus)**.** Fix a locus $(\mathcal{C}, \lambda)$. Define the generator set

$$\mathcal{G}_{\mathcal{C}, \lambda} := \{\, C_{\bullet}(P) \;:\; P \in \mathsf{P}_{\mathcal{C}, \lambda}, \;\bullet \in \mathbf{Status}_{\Lambda} \,\}.$$

(Notation alignment: in earlier drafts the symbols $E_{\bullet}(P)$ may appear; here we use $C_{\bullet}(P)$ to match the tetralogic and logic-hilbert notation.)

Definition 3.14 (Free unital ∗-algebra on corner generators)**.** Let $\mathbb{C}\langle \mathcal{G}_{\mathcal{C}, \lambda} \rangle$ denote the free unital associative algebra on $\mathcal{G}_{\mathcal{C}, \lambda}$ over $\mathbb{C}$. Equip it with the canonical involution † determined by: (i) conjugate-linearity on scalars, (ii) $(XY)^{\dagger} = Y^{\dagger} X^{\dagger}$, and (iii) a formal involution on generators. This is the standard free ∗-algebra construction [19, 94].

Definition 3.15 (Minimal ∗-congruence and the logical corner algebra)**.** Let $\mathcal{I}_0$ be the smallest ∗-ideal of $\mathbb{C}\langle \mathcal{G}_{\mathcal{C}, \lambda} \rangle$ containing:

$$\mathbf{1}^{\dagger} - \mathbf{1}, \qquad (C_{\bullet}(P))^{\dagger} - C_{\bullet}(P) \quad \text{for all } P \in \mathsf{P}_{\mathcal{C}, \lambda}, \;\bullet \in \mathbf{Status}_{\Lambda}.$$

(Thus generators are declared self-adjoint *formally*, but no effect-like axioms are imposed.)

Optionally, let $\equiv_\Sigma$ be any chosen logical congruence on $\mathsf{P}_{\mathcal{C},\lambda}$ (e.g. induced by Σ-closure identifications in the tetralogic layer), and let $\mathcal{I}_\Sigma$ be the $*$-ideal generated by

$$C_\bullet(P) - C_\bullet(Q) \quad \text{whenever } P \equiv_\Sigma Q, \; \bullet \in \mathbf{Status}_\Lambda.$$

Define the *logical corner $*$-algebra* (Tier 0; ultra-minimal) by

$$\mathcal{A}_\Lambda^{\mathrm{alg}} := \mathbb{C}\langle \mathcal{G}_{\mathcal{C},\lambda} \rangle \big/ \big(\mathcal{I}_0 + \mathcal{I}_\Sigma \big),$$

with $\mathcal{I}_\Sigma = 0$ if no logical congruence is imposed.

Proposition 3.16 (Universal property)**.** *Let $\mathcal{B}$ be any unital $*$-algebra and suppose we are given an assignment*

$$\Phi : \mathcal{G}_{\mathcal{C},\lambda} \to \mathcal{B}$$

such that $\Phi(C_\bullet(P))$ is self-adjoint for all $P, \bullet$ and (if $\mathcal{I}_\Sigma \neq 0$) respects the chosen claim congruence: $P \equiv_\Sigma Q \Rightarrow \Phi(C_\bullet(P)) = \Phi(C_\bullet(Q))$. Then there exists a unique unital $*$-homomorphism

$$\widetilde{\Phi} : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{B}$$

extending Φ. This is a standard application of the universal property of free $*$-algebras and quotient algebras [17, 19, 94].

Proof. Let $\mathcal{F} := \mathbb{C}\langle \mathcal{G}_{\mathcal{C},\lambda} \rangle$ denote the free unital associative algebra generated by $\mathcal{G}_{\mathcal{C},\lambda}$ over $\mathbb{C}$, equipped with the canonical involution $\dagger$ defined by conjugate-linearity on scalars, $(XY)^\dagger = Y^\dagger X^\dagger$, and $C_\bullet(P)^\dagger = C_\bullet(P)$ on generators [94].

Existence. By the universal property of the free algebra, the set map $\Phi : \mathcal{G}_{\mathcal{C},\lambda} \to \mathcal{B}$ extends uniquely to a unital algebra homomorphism $\Psi : \mathcal{F} \to \mathcal{B}$ [19]. We verify that Ψ is a $*$-homomorphism. Since $\Phi(C_\bullet(P))$ is self-adjoint in $\mathcal{B}$ for all generators, we have

$$\Psi(C_\bullet(P)^\dagger) = \Psi(C_\bullet(P)) = \Phi(C_\bullet(P)) = \Phi(C_\bullet(P))^\dagger = \Psi(C_\bullet(P))^\dagger.$$

Since Ψ preserves the involution on generators and $\mathcal{F}$ is generated by these elements, $\Psi(X^\dagger) = \Psi(X)^\dagger$ for all $X \in \mathcal{F}$ [17].

Next, we show that the ideal $\mathcal{J} := \mathcal{I}_0 + \mathcal{I}_\Sigma$ is contained in $\ker \Psi$.

- For $\mathcal{I}_0$: The ideal is generated by elements of the form $\mathbf{1}^\dagger - \mathbf{1}$ and $C_\bullet(P)^\dagger - C_\bullet(P)$. Since Ψ is unital and $*$-preserving, $\Psi(\mathbf{1}^\dagger - \mathbf{1}) = \mathbf{1}_\mathcal{B} - \mathbf{1}_\mathcal{B} = 0$ and $\Psi(C_\bullet(P)^\dagger - C_\bullet(P)) = \Phi(C_\bullet(P))^\dagger - \Phi(C_\bullet(P)) = 0$ by the self-adjointness hypothesis on Φ.

- For $\mathcal{I}_\Sigma$: The ideal is generated by elements $C_\bullet(P) - C_\bullet(Q)$ where $P \equiv_\Sigma Q$. By hypothesis, $\Phi(C_\bullet(P)) = \Phi(C_\bullet(Q))$, so $\Psi(C_\bullet(P) - C_\bullet(Q)) = 0$.

Thus $\mathcal{J} \subseteq \ker \Psi$. By the fundamental homomorphism theorem for algebras, Ψ factors uniquely through the quotient $\mathcal{A}_\Lambda^{\mathrm{alg}} = \mathcal{F}/\mathcal{J}$. That is, there exists a unique unital $*$-homomorphism $\widetilde{\Phi} : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{B}$ such that $\widetilde{\Phi} \circ \pi = \Psi$, where $\pi : \mathcal{F} \to \mathcal{A}_\Lambda^{\mathrm{alg}}$ is the canonical quotient map [19]. Since $\pi(C_\bullet(P)) = [C_\bullet(P)]$ corresponds to the generator in the quotient algebra, $\widetilde{\Phi}$ extends Φ.

Uniqueness. Any unital $*$-homomorphism $\widetilde{\Phi} : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{B}$ extending Φ must satisfy $\widetilde{\Phi}([X]) = \Psi(X)$ for all $X \in \mathcal{F}$, because $\mathcal{A}_\Lambda^{\mathrm{alg}}$ is generated by the images of $\mathcal{G}_{\mathcal{C},\lambda}$. Since Ψ is uniquely determined by Φ on the free algebra, $\widetilde{\Phi}$ is unique [94]. $\qquad\square$

Remark 3.17 (No effect-like relations at Tier 0)**.** In Tier 0, we do *not* assume idempotence, exclusivity, partitions of unity, or order bounds:

$$C_\bullet(P)^2 = C_\bullet(P), \quad C_\bullet(P)\, C_{\bullet'}(P) = 0 \ (\bullet \neq \bullet'), \quad \sum_\bullet C_\bullet(P) = \mathbf{1}, \quad 0 \leq C_\bullet(P) \leq \mathbf{1}$$

are *not* relations in $\mathcal{A}_\Lambda^{\mathrm{alg}}$ unless explicitly added later as enrichment. This distinguishes the logical corner algebra from standard effect algebras or POVM structures [56, 63, 85].

3.3 The Observable Algebra of the Tetralemma

The Realization Theorem would identify logical formulas (or corner-acts) with concrete quantum effects. We explicitly *do not* assume any such identification here. Instead we introduce a purely formal $*$-algebra of *corner-observables* generated directly from the tetralemma corner-acts. This algebraic-first approach aligns with the logic-to-quantum reconstruction program [7, 8, 14].

3.3.1 Corner-observable generators

Definition 3.18 (Corner-observable generators)**.** Fix a context–polarity pair $(\mathcal{C}, \lambda)$ as in Section 2. For each polar claim

$$P \ := \ \exists_\lambda^{\mathcal{C}} u \, \varphi(u)$$

that is well-formed and applicable at $(\mathcal{C}, \lambda)$ (i.e. $\mathrm{App}_{\mathcal{C}}(P; \lambda)$ holds, cf. Section 2), introduce four *formal* symbols

$$C_{\mathbf{Th}}(P), \qquad C_{\mathbf{AntiTh}}(P), \qquad C_{\mathbf{SynTh}}(P), \qquad C_{\mathbf{HoloTh}}(P),$$

called the *corner-observable generators* associated to P. They are intended to represent the four corner-typed readouts (Thesis/Antithesis/Synthesis/Holothesis) associated to the claim P, as defined in Definitions 2.7, 2.8, 2.11 and 2.12, but at this stage they are purely formal generators and carry no a priori operator or probabilistic meaning.

Remark 3.19 (No semantic identification)**.** The symbols $C_\bullet(P)$ are *not* assumed to be quantum effects, operators on a Hilbert space, or elements of any pre-existing physical algebra. They are merely generators in a formal $*$-algebraic envelope of tetralemma corner semantics. This distinguishes the construction from standard effect algebra approaches [56, 63, 85].

3.3.2 The involutive algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$

Definition 3.20 (The logical corner-observable $*$-algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$)**.** Let $\mathsf{P}_{\mathcal{C},\lambda}$ denote the collection of all polar claims P that are well-formed and applicable at $(\mathcal{C}, \lambda)$. Define $\mathcal{A}_\Lambda^{\mathrm{alg}}$ to be the *unital involutive $*$-algebra* generated by the family of symbols

$$\{\, C_{\mathbf{Th}}(P),\ C_{\mathbf{AntiTh}}(P),\ C_{\mathbf{SynTh}}(P),\ C_{\mathbf{HoloTh}}(P): \ P \in \mathsf{P}_{\mathcal{C},\lambda} \,\},$$

subject only to the following minimal requirements:

(1) **Unitality.** $\mathcal{A}_\Lambda^{\mathrm{alg}}$ contains a distinguished unit element $\mathbf{1}$ and is closed under multiplication with $\mathbf{1}$.

(2) **Involution on generators.** Each generator is declared self-adjoint:

$$C_\bullet(P)^\dagger \;=\; C_\bullet(P) \qquad (\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\},\ P \in \mathsf{P}_{\mathcal{C},\lambda}).$$

The involution is extended to all of $\mathcal{A}_\Lambda^{\mathrm{alg}}$ by conjugate-linearity and the rule

$$(AB)^\dagger \;=\; B^\dagger A^\dagger, \qquad \mathbf{1}^\dagger = \mathbf{1}.$$

This is the standard free $*$-algebra construction [17, 19, 94].

(3) **Minimal structural relations.** Beyond unitality and involution, no additional algebraic relations are imposed *unless* they are already fixed at the purely logical level. Concretely, one may (optionally) quotient by any relations that are forced by the Σ-constructions of Definition 2.17 and by the corner-effect conventions already adopted in the tetralemma layer; no further "physical" constraints are assumed here.

Equivalently, $\mathcal{A}_\Lambda^{\mathrm{alg}}$ may be regarded as the free unital $*$-algebra on the generating set $\{C_\bullet(P)\}$, modulo only those relations explicitly stipulated above (and any optional relations already present in the logical Σ-structure). This matches the Tier-0 ultra-minimal algebra of Definition 3.15.

Remark 3.21 (Why this is still pre-quantum)**.** The object $\mathcal{A}_\Lambda^{\mathrm{alg}}$ is not yet a C^*-algebra and carries no norm, topology, or representation. It is simply the algebraic envelope needed to later apply a positivity-based construction (GNS) once a positive warrant functional is specified. This is precisely what prevents circularity: no Hilbert space, no operator effects, and no Born semantics are assumed at this stage. Compare the algebraic quantum field theory paradigm where the observable algebra precedes representation [7, 16, 95].

3.3.3 Duality as a $*$-automorphism of $\mathcal{A}_\Lambda^{\mathrm{alg}}$

Definition 3.22 (Duality-induced $*$-automorphism)**.** Assume, in addition to the corner-level involution of Assumption 3.5 (which reflects the classical Catuṣkoṭi duality [13, 15, 53]), that duality acts on polar claims by a mapping

$$P \longmapsto P^{\mathbf{D}}$$

on $\mathsf{P}_{\mathcal{C},\lambda}$ (whenever such an action is available in the ambient framework). Define a map $\alpha_{\mathbf{D}} : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_\Lambda^{\mathrm{alg}}$ on generators by

$$\alpha_{\mathbf{D}}\big(C_{\mathbf{Th}}(P)\big) = C_{\mathbf{AntiTh}}(P^{\mathbf{D}}), \qquad \alpha_{\mathbf{D}}\big(C_{\mathbf{AntiTh}}(P)\big) = C_{\mathbf{Th}}(P^{\mathbf{D}}),$$

$$\alpha_{\mathbf{D}}\big(C_{\mathbf{SynTh}}(P)\big) = C_{\mathbf{SynTh}}(P^{\mathbf{D}}), \qquad \alpha_{\mathbf{D}}\big(C_{\mathbf{HoloTh}}(P)\big) = C_{\mathbf{HoloTh}}(P^{\mathbf{D}}),$$

and set $\alpha_{\mathbf{D}}(\mathbf{1}) = \mathbf{1}$. Extend $\alpha_{\mathbf{D}}$ to all of $\mathcal{A}_\Lambda^{\mathrm{alg}}$ by linearity and multiplicativity:

$$\alpha_{\mathbf{D}}(A + B) = \alpha_{\mathbf{D}}(A) + \alpha_{\mathbf{D}}(B), \qquad \alpha_{\mathbf{D}}(AB) = \alpha_{\mathbf{D}}(A)\,\alpha_{\mathbf{D}}(B).$$

Finally, require $\alpha_{\mathbf{D}}$ to commute with the involution:

$$\alpha_{\mathbf{D}}(A^{\dagger}) \;=\; \alpha_{\mathbf{D}}(A)^{\dagger} \qquad (A \in \mathcal{A}_{\Lambda}^{\mathrm{alg}}).$$

This construction follows the standard definition of a $*$-automorphism on involutive algebras [17, 19, 94], implemented here to capture physical duality symmetries [92, 93].

Proposition 3.23 (Basic properties of $\alpha_{\mathbf{D}}$). *The map $\alpha_{\mathbf{D}}$ defined in Definition 3.22 is a unital $*$-endomorphism of $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$. If, moreover, the duality action on claims is involutive (i.e. $(P^{\mathbf{D}})^{\mathbf{D}} = P$ for all $P \in \mathsf{P}_{\mathcal{C},\lambda}$), then $\alpha_{\mathbf{D}}$ is involutive:*

$$\alpha_{\mathbf{D}}^{2} \;=\; \mathrm{id}_{\mathcal{A}_{\Lambda}^{\mathrm{alg}}},$$

and hence is a $*$-automorphism. This is a standard result in the theory of operator algebras [17, 19].

Proof. By construction, $\alpha_{\mathbf{D}}$ is linear, multiplicative, and unital, so it is a unital algebra endomorphism. The defining requirement $\alpha_{\mathbf{D}}(A^{\dagger}) = \alpha_{\mathbf{D}}(A)^{\dagger}$ makes it a $*$-endomorphism [19, 94]. If $(P^{\mathbf{D}})^{\mathbf{D}} = P$, then applying $\alpha_{\mathbf{D}}$ twice returns each generator to itself:

$$\alpha_{\mathbf{D}}^{2}\big(C_{\bullet}(P)\big) = C_{\bullet}(P),$$

for $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$, and hence $\alpha_{\mathbf{D}}^{2} = \mathrm{id}$ on all of $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$ by multiplicativity and linearity [17]. $\square$

Remark 3.24 (If claim-level duality is unavailable). If an action $P \mapsto P^{\mathbf{D}}$ on claims is not assumed, one may still define $\alpha_{\mathbf{D}}$ purely at the level of corner labels by fixing $\alpha_{\mathbf{D}}(C_{\mathbf{Th}}(P)) := C_{\mathbf{AntiTh}}(P)$, etc. The subsequent GNS-based Hilbert construction requires only that $\alpha_{\mathbf{D}}$ be a unital $*$-automorphism implementing the corner-level duality symmetry [7, 16, 17].

3.4 The Warrant Functional

The tetralemma corner-structure and the axiom of duality determine *which* corner-statuses exist and *how* they transform, but they do not determine a canonical numerical inner product. To obtain a Hilbert space by a GNS-type construction, we must supply one further primitive: a *positive evaluator* on the purely formal corner-observable algebra. This algebraic approach to quantum kinematics follows the Gelfand–Naimark–Segal (GNS) reconstruction paradigm [7, 16, 17, 18], where the Hilbert space structure is induced solely by the positivity of the evaluation functional [19, 96].

3.4.1 Definition and axioms

Definition 3.25 (Warrant functional ω_{Λ}). Let $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$ be the logical corner-observable $*$-algebra (Definition 3.20) and let $\alpha_{\mathbf{D}}$ be the duality-induced $*$-automorphism (Definition 3.22). A *warrant functional* is a linear functional

$$\omega_{\Lambda} : \mathcal{A}_{\Lambda}^{\mathrm{alg}} \longrightarrow \mathbb{C}$$

satisfying:

(1) **Normalization:**
$$\omega_\Lambda(\mathbf{1}) = 1.$$

(2) **Positivity:**
$$\omega_\Lambda(A^\dagger A) \geq 0 \qquad \text{for all } A \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

This condition identifies ω_Λ as a state on the $*$-algebra [17, 19, 94].

(3) **Duality invariance:**
$$\omega_\Lambda \circ \alpha_{\mathbf{D}} = \omega_\Lambda, \qquad \text{i.e.} \quad \omega_\Lambda\big(\alpha_{\mathbf{D}}(A)\big) = \omega_\Lambda(A) \ \text{ for all } A \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

This reflects the requirement that the evaluation regime respects the underlying logical symmetry [16, 92].

Remark 3.26 (State vs. warrant functional). Definition 3.25 is formally the definition of a *state* on a unital $*$-algebra (positivity + normalization) [7, 19]. We call it a *warrant functional* to emphasize that ω_Λ is introduced as a primitive evaluator of corner-statuses, not as a probability measure and not as a quantum state on a pre-given operator algebra. This distinction avoids the circularity often noted in foundational discussions where the Hilbert space is presupposed [7, 8, 9].

3.4.2 Interpretation on corner generators

Definition 3.27 (Corner warrant degrees). Let $P \in \mathsf{P}_{\mathcal{C},\lambda}$ be a polar claim applicable at $(\mathcal{C}, \lambda)$, and let $C_\bullet(P)$ be its corner-observable generators (Definition 3.18). Given a warrant functional ω_Λ, define the *degree of warrant* of corner-status $\bullet$ for the claim P by
$$w_\Lambda^\bullet(P) := \omega_\Lambda\big(C_\bullet(P)\big), \qquad \bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}.$$

These values correspond to the operational weights assigned to each logical outcome in an abstract measurement scheme [56, 63].

Remark 3.28 (Not yet probability). At this purely logical stage, the quantities $w_\Lambda^\bullet(P)$ are *not* assumed to be probabilities, expectation values of physical effects, or Born weights. They are simply the positive numerical evaluations supplied by ω_Λ, constrained only by the algebraic relations already imposed in $\mathcal{A}_\Lambda^{\mathrm{alg}}$ and by the duality invariance of ω_Λ. This aligns with the quantum measure-theoretic view where measure arises from the algebraic structure rather than being presupposed [7, 8, 14].

Proposition 3.29 (Duality symmetry on warrant degrees). *Assume claim-level duality $P \mapsto P^{\mathbf{D}}$ is available and that $\alpha_{\mathbf{D}}$ acts on generators as in Definition 3.22. Then the duality invariance of ω_Λ implies the following symmetry relations among warrant degrees:*
$$w_\Lambda^{\mathbf{Th}}(P) = w_\Lambda^{\mathbf{AntiTh}}(P^{\mathbf{D}}), \qquad w_\Lambda^{\mathbf{AntiTh}}(P) = w_\Lambda^{\mathbf{Th}}(P^{\mathbf{D}}),$$
$$w_\Lambda^{\mathbf{SynTh}}(P) = w_\Lambda^{\mathbf{SynTh}}(P^{\mathbf{D}}), \qquad w_\Lambda^{\mathbf{HoloTh}}(P) = w_\Lambda^{\mathbf{HoloTh}}(P^{\mathbf{D}}).$$

These relations ensure that the evaluation of dual claims remains consistent under the logical duality transformation [16, 92].

Proof. For example, using the duality invariance $\omega_\Lambda \circ \alpha_{\mathbf{D}} = \omega_\Lambda$ and the defining action of $\alpha_{\mathbf{D}}$:

$$w_\Lambda^{\mathbf{Th}}(P) = \omega_\Lambda\big(C_{\mathbf{Th}}(P)\big) = \omega_\Lambda\big(\alpha_{\mathbf{D}}(C_{\mathbf{Th}}(P))\big) = \omega_\Lambda\big(C_{\mathbf{AntiTh}}(P^{\mathbf{D}})\big) = w_\Lambda^{\mathbf{AntiTh}}(P^{\mathbf{D}}),$$

and the remaining identities follow analogously [19, 94]. □

3.4.3 Optional sharp regime (compatibility with strict warrant)

The general theory does *not* require corner warrant degrees to be $\{0, 1\}$-valued. This aligns with the operational approach to quantum physics where effects are generally unsharp [56, 63, 85]. However, one may optionally impose additional constraints to match a strict warrant regime, corresponding to classical truth values or sharp observables in specific contexts [7, 84].

Definition 3.30 (Sharp warrant constraints (optional)). Fix $(\mathcal{C}, \lambda)$ and let P be a polar claim applicable at $(\mathcal{C}, \lambda)$. A warrant functional ω_Λ is said to be *sharp-compatible* (at $(\mathcal{C}, \lambda)$) if, whenever the strict tetralogic warrant conditions for a given corner-act hold (cf. Definitions 2.7, 2.8, 2.11 and 2.12, the corresponding corner generator is evaluated at 1. This reflects the classical limit where logical valuations become $\{0, 1\}$-measures [2, 84]. For instance, one may require:

(i) if $\mathrm{App}_{\mathcal{C}}(P; \lambda)$ and $\mathcal{C}, \lambda \Vdash P$, then

$$\omega_\Lambda\big(C_{\mathbf{Th}}(P)\big) = 1;$$

This matches the Thesis condition of the Catuṣkoṭi framework [13, 15, 53].

(ii) if $\mathrm{App}_{\mathcal{C}}(\neg P; \lambda)$ and $\mathcal{C}, \lambda \Vdash \neg P$, then

$$\omega_\Lambda\big(C_{\mathbf{AntiTh}}(P)\big) = 1;$$

(iii) and analogously for any adopted strict clauses for $\mathbf{SynTh}(P)$ and $\mathbf{HoloTh}(P)$ in the underlying tetralogic.

Remark 3.31 (Optionality and non-uniqueness). The sharp constraints of Definition 3.30 are optional and generally do not determine ω_Λ uniquely. In algebraic quantum theory, states are not uniquely fixed by their values on a subset of observables unless additional completeness conditions are met [7, 19]. They are best viewed as compatibility conditions with a strict-warrant subregime, not as part of the minimal logical Hilbert template. This distinction preserves the flexibility needed for contextual truth-value assignments in quantum logic [9, 63]. Note that this is a condition on the *evaluation regime*; it may be imposed independently of the Tier 2 algebraic enrichment (Definition 3.33), though they often coincide in classical limits.

3.4.4 Tiering: minimal algebra, evaluator constraints, and enrichment

This tiering strategy aligns with the quantum measure theory approach [8, 14], where the algebraic structure remains minimal and probabilistic/evaluative content is carried by the state functional.

Definition 3.32 (Tier 1: normalization as evaluator constraints (optional))**.** A warrant functional ω_Λ on $\mathcal{A}_\Lambda^{\mathrm{alg}}$ is *Tier-1 normalized* (at the locus $(\mathcal{C}, \lambda)$) if, for every $P \in \mathsf{P}_{\mathcal{C}, \lambda}$,

$$\sum_{\bullet \in \mathbf{Status}_\Lambda} \omega_\Lambda\big(C_\bullet(P)\big) = 1.$$

Additional Tier-1 conventions for inapplicable claims may also be imposed *on* ω_Λ (not on the algebra), e.g. a holothesis convention of the form

$$\neg \mathrm{App}_{\mathcal{C}}(P; \lambda) \implies \omega_\Lambda\big(C_{\mathbf{HoloTh}}(P)\big) = 1,$$

if such a numerical convention is desired.

Definition 3.33 (Tier 2: effect-like enrichment (explicitly non-minimal))**.** If one *explicitly* wants sharp/effect-like relations, define the enriched quotient

$$\mathcal{A}_\Lambda^\sharp := \mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{J}_\sharp,$$

where $\mathcal{J}_\sharp$ is the $*$-ideal generated by any chosen family of additional axioms, such as:

- Idempotence: $C_\bullet(P)^2 = C_\bullet(P)$,

- Exclusivity: $C_\bullet(P)C_{\bullet'}(P) = 0$ for $\bullet \neq \bullet'$,

- Partitions: $\sum_\bullet C_\bullet(P) = \mathbf{1}$.

Such axioms are *not* part of the purely logical Hilbert template (Tier 0/1); they constitute extra structure. This enrichment corresponds to moving from a general $*$-algebra to a projection lattice or effect algebra [56, 63]. The paraconsistent Synthesis corner may require relaxing exclusivity axioms [54, 55].

3.5 Coherent Warrant and Positivity as a PSD Kernel

This construction aligns with the algebraic formulation of quantum theory, where states are identified with positive linear functionals on an observable algebra [7, 16, 17]. The positivity condition ensures the existence of a Hilbert space representation via the GNS construction [18, 96].

Definition 3.34 (Warrant kernel)**.** Given a linear functional $\omega_\Lambda : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathbb{C}$, define the kernel

$$K_{\omega_\Lambda}(A, B) := \omega_\Lambda(A^\dagger B), \qquad A, B \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

This is a sesquilinear form analogous to a reproducing kernel [97].

Definition 3.35 (Coherent (positive) warrant functional)**.** A *warrant functional* is a linear functional $\omega_\Lambda : \mathcal{A}_\Lambda^{\text{alg}} \to \mathbb{C}$ satisfying:

(a) **Normalization:** $\omega_\Lambda(\mathbf{1}) = 1$.

(b) **Kernel PSD (coherence):** for every $n \in \mathbb{N}$, every $A_1, \ldots, A_n \in \mathcal{A}_\Lambda^{\text{alg}}$, and every $z_1, \ldots, z_n \in \mathbb{C}$,

$$\sum_{i,j=1}^n \overline{z_i}\, z_j\, \omega_\Lambda(A_i^\dagger A_j) \; \geq \; 0.$$

This condition identifies ω_Λ as a state on the $*$-algebra [18, 19]. In the context of quantum measure theory, this corresponds to the positivity of the decoherence functional [8, 14].

(c) **Duality invariance (when $\alpha_{\mathbf{D}}$ is defined):** $\omega_\Lambda \circ \alpha_{\mathbf{D}} = \omega_\Lambda$.

Proposition 3.36 (PSD kernel $\iff$ positivity on squares)**.** *If ω_Λ satisfies the PSD-kernel condition of Definition 3.35(b), then $\omega_\Lambda(A^\dagger A) \geq 0$ for all $A \in \mathcal{A}_\Lambda^{\text{alg}}$. Conversely, if ω_Λ is positive in the sense that $\omega_\Lambda(A^\dagger A) \geq 0$ for all A, then K_{ω_Λ} is positive semidefinite on all finite subsets. This is a standard result in the theory of operator algebras [17, 19, 94].*

Proof. **Forward direction:** Assume K_{ω_Λ} is positive semidefinite. Choose $n = 1$, $A_1 = A \in \mathcal{A}_\Lambda^{\text{alg}}$, and $z_1 = 1 \in \mathbb{C}$. The PSD condition implies

$$\sum_{i,j=1}^1 \overline{z_i}\, z_j\, \omega_\Lambda(A_i^\dagger A_j) \; = \; \overline{1} \cdot 1 \cdot \omega_\Lambda(A^\dagger A) \; = \; \omega_\Lambda(A^\dagger A) \; \geq \; 0.$$

Thus ω_Λ is positive on squares [94].

Reverse direction: Assume $\omega_\Lambda(A^\dagger A) \geq 0$ for all $A \in \mathcal{A}_\Lambda^{\text{alg}}$. Let $\{A_1, \ldots, A_n\} \subset \mathcal{A}_\Lambda^{\text{alg}}$ and $\{z_1, \ldots, z_n\} \subset \mathbb{C}$. Define the element $B \in \mathcal{A}_\Lambda^{\text{alg}}$ by the linear combination

$$B \; := \; \sum_{k=1}^n z_k A_k.$$

By the positivity assumption, $\omega_\Lambda(B^\dagger B) \geq 0$. Using the linearity of ω_Λ and the properties of the involution †, we expand $B^\dagger B$:

$$B^\dagger B = \left(\sum_{i=1}^n z_i A_i\right)^\dagger \left(\sum_{j=1}^n z_j A_j\right) \; = \; \left(\sum_{i=1}^n \overline{z_i} A_i^\dagger\right)\left(\sum_{j=1}^n z_j A_j\right)$$

$$= \sum_{i,j=1}^n \overline{z_i}\, z_j\, A_i^\dagger A_j.$$

Applying ω_Λ and using linearity:

$$\omega_\Lambda(B^\dagger B) \; = \; \sum_{i,j=1}^n \overline{z_i}\, z_j\, \omega_\Lambda(A_i^\dagger A_j) \; \geq \; 0.$$

This is precisely the condition that the kernel $K_{\omega_\Lambda}(A_i, A_j) = \omega_\Lambda(A_i^\dagger A_j)$ is positive semidefinite on the finite subset $\{A_1, \ldots, A_n\}$ [17, 19]. $\qquad\square$

Proposition 3.37 (Cauchy–Schwarz for coherent warrant)**.** *For any coherent warrant functional* ω_Λ *and any* $A, B \in \mathcal{A}_\Lambda^{\text{alg}}$,

$$\left|\omega_\Lambda(A^\dagger B)\right|^2 \leq \omega_\Lambda(A^\dagger A)\,\omega_\Lambda(B^\dagger B).$$

Proof. Follows directly from the positivity of the kernel K_{ω_Λ} [17, 71]. □

3.6 Duality as a ∗-automorphism

The implementation of logical duality as an algebraic symmetry parallels the treatment of physical symmetries in algebraic quantum field theory [92, 93]. The specific corner-exchange structure reflects the classical Catușkoți duality [13, 15].

Definition 3.38 (Duality automorphism on $\mathcal{A}_\Lambda^{\text{alg}}$)**.** Assume either a claim transform $P \mapsto P^{\mathbf{D}}$ on $\mathsf{P}_{\mathcal{C},\lambda}$, or take it to be trivial. Define $\alpha_{\mathbf{D}} : \mathcal{A}_\Lambda^{\text{alg}} \to \mathcal{A}_\Lambda^{\text{alg}}$ on generators by

$$\alpha_{\mathbf{D}}\big(C_{\mathbf{Th}}(P)\big) = C_{\mathbf{AntiTh}}(P^{\mathbf{D}}), \qquad \alpha_{\mathbf{D}}\big(C_{\mathbf{AntiTh}}(P)\big) = C_{\mathbf{Th}}(P^{\mathbf{D}}),$$

$$\alpha_{\mathbf{D}}\big(C_{\mathbf{SynTh}}(P)\big) = C_{\mathbf{SynTh}}(P^{\mathbf{D}}), \qquad \alpha_{\mathbf{D}}\big(C_{\mathbf{HoloTh}}(P)\big) = C_{\mathbf{HoloTh}}(P^{\mathbf{D}}), \qquad \alpha_{\mathbf{D}}(\mathbf{1}) = \mathbf{1},$$

and extend by linearity, multiplicativity, and ∗-preservation.

Proposition 3.39 (Involutivity)**.** *If* $(P^{\mathbf{D}})^{\mathbf{D}} = P$ *for all* $P \in \mathsf{P}_{\mathcal{C},\lambda}$, *then* $\alpha_{\mathbf{D}}^2 = \mathrm{id}_{\mathcal{A}_\Lambda^{\text{alg}}}$ *and hence* $\alpha_{\mathbf{D}}$ *is a* ∗-*automorphism.*

Proof. Direct verification on generators using the involutive property of the corner map and claim duality. Since $\alpha_{\mathbf{D}}$ is defined by its action on the generators $C_\bullet(P)$ and extends by linearity and multiplicativity, the condition $(P^{\mathbf{D}})^{\mathbf{D}} = P$ together with $\mathbf{D}^2 = \mathrm{id}$ on corners implies $\alpha_{\mathbf{D}}^2(C_\bullet(P)) = C_\bullet(P)$. Standard results on involutive automorphisms of ∗-algebras ensure this extends to the whole algebra [17, 19, 94]. □

3.7 Algebraic GNS construction (forced Hilbert template)

This construction follows the standard Gelfand–Naimark–Segal (GNS) paradigm for states on ∗-algebras [7, 16, 17, 18]. No C^*-completion is assumed *a priori*; the Hilbert space structure is induced solely by the positivity of the warrant functional [19, 96].

Definition 3.40 (Null ideal and GNS pre-Hilbert space)**.** Let ω_Λ be a coherent warrant functional on $\mathcal{A}_\Lambda^{\text{alg}}$. Define the null set

$$\mathcal{N}_{\omega_\Lambda} := \{\, A \in \mathcal{A}_\Lambda^{\text{alg}} : \omega_\Lambda(A^\dagger A) = 0 \,\}.$$

Let $\mathcal{A}_\Lambda^{\text{alg}}/\mathcal{N}_{\omega_\Lambda}$ be the quotient vector space and write $[A]$ for the class of A. Define a sesquilinear form by

$$\langle [A] \mid [B] \rangle_\Lambda := \omega_\Lambda(A^\dagger B).$$

This is the canonical construction of a pre-Hilbert space from a positive functional [94, 95].

Proposition 3.41 (Basic GNS correctness (algebraic form))**.** *In the setting of Definition 3.40:*

(a) $\mathcal{N}_{\omega_\Lambda}$ *is a left ideal in* $\mathcal{A}_\Lambda^{\text{alg}}$.

(b) $\langle \cdot \mid \cdot \rangle_\Lambda$ *is well-defined on the quotient and is positive definite there.*

(c) The left-regular action
$$\pi_\Lambda(A)[B] \; := \; [AB]$$
defines a well-defined unital $*$-*representation of* $\mathcal{A}_\Lambda^{\text{alg}}$ *on the quotient pre-Hilbert space.*

Proof. (a) If $N \in \mathcal{N}_{\omega_\Lambda}$ and $A \in \mathcal{A}_\Lambda^{\text{alg}}$, then by the Cauchy–Schwarz inequality for positive functionals [17, 19],

$$|\omega_\Lambda((AN)^\dagger(AN))|^2 \le \omega_\Lambda(A^\dagger A)\,\omega_\Lambda(N^\dagger N) = 0,$$

so $AN \in \mathcal{N}_{\omega_\Lambda}$. (b) Well-definedness follows from the left-ideal property. Positive definiteness follows because $\langle [A] \mid [A] \rangle_\Lambda = 0 \iff A \in \mathcal{N}_{\omega_\Lambda} \iff [A] = 0$. (c) Multiplicativity is immediate from the definition. For $*$-preservation, note that

$$\langle [C] \mid \pi_\Lambda(A^\dagger)[B] \rangle_\Lambda = \omega_\Lambda(C^\dagger A^\dagger B) = \omega_\Lambda((AC)^\dagger B) = \langle \pi_\Lambda(A)[C] \mid [B] \rangle_\Lambda,$$

so $\pi_\Lambda(A^\dagger) = \pi_\Lambda(A)^\dagger$ on the dense domain [16, 71]. $\qquad\square$

Definition 3.42 (Logical GNS triple)**.** Let $\mathcal{H}_\Lambda$ be the Hilbert space completion of $\mathcal{A}_\Lambda^{\text{alg}}/\mathcal{N}_{\omega_\Lambda}$ in the norm induced by $\langle \cdot \mid \cdot \rangle_\Lambda$. Let π_Λ be the representation from Proposition 3.41, extended to the completion on its natural dense domain. Define the cyclic vector

$$|\Omega_\Lambda\rangle \; := \; [\mathbf{1}] \; \in \; \mathcal{H}_\Lambda.$$

The triple $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ is the *logical GNS representation* [7, 18].

Proposition 3.43 (State recovery and cyclicity)**.** *For all* $A \in \mathcal{A}_\Lambda^{\text{alg}}$,

$$\omega_\Lambda(A) \; = \; \langle \Omega_\Lambda \mid \pi_\Lambda(A) \mid \Omega_\Lambda \rangle_\Lambda.$$

Moreover,
$$\overline{\text{span}}\{\pi_\Lambda(A)|\Omega_\Lambda\rangle : \; A \in \mathcal{A}_\Lambda^{\text{alg}}\} \; = \; \mathcal{H}_\Lambda.$$

Proof. The recovery formula follows directly from the definitions:

$$\langle \Omega_\Lambda \mid \pi_\Lambda(A) \mid \Omega_\Lambda \rangle_\Lambda = \langle [\mathbf{1}] \mid [A] \rangle_\Lambda = \omega_\Lambda(\mathbf{1}^\dagger A) = \omega_\Lambda(A).$$

Cyclicity holds because the set $\{\pi_\Lambda(A)|\Omega_\Lambda\rangle\} = \{[A]\}$ is by construction dense in $\mathcal{H}_\Lambda$ [17, 94]. $\qquad\square$

Remark 3.44 (Relation to Sorkin's quantum measure theory)**.** The tiered construction above resonates deeply with Sorkin's quantum measure theory and co-event interpretation [8, 14]. Both frameworks recognize that the Kochen–Specker theorem demands a revision of classical logic [20], and both seek observer-independent foundations suitable for quantum gravity. However, the present construction complements and extends Sorkin's program in several key respects:

- **Algebraic completeness:** We construct the full ∗-algebra from logical primitives (cf. Definition 3.15), then derive the Hilbert space via GNS. Sorkin's framework remains primarily at the level of measure theory on a history space.

- **Explicit tiering:** We separate logical necessity (Tier 0), evaluative content (Tier 1), and physical enrichment (Tier 2) (cf. Definitions 3.32 and 3.33). Sorkin's framework implicitly combines these layers.

- **Duality principle:** Our duality involution $\mathbf{D}$ on corners, with its action exchanging Thesis and Antithesis while fixing Synthesis and Holothesis (cf. Axiom 2.22 and Definitions 2.7, 2.8, 2.11 and 2.12), reveals a deep symmetry absent from Sorkin's work.

- **Regime-relativity:** Tier 2 enrichment allows different physical regimes (classical, quantum, paraconsistent, paracomplete) to be encoded by different axiom sets, whereas Sorkin's framework assumes a single logical system.

Thus, while Sorkin's program identifies the need for a new logic, the present framework provides that logic in complete mathematical detail, with explicit connections to both ancient wisdom (the tetralemma) and modern operational contexts (e.g. weak measurement [8]; see also Section 2.13).

3.8 Canonical 4-corner template (local and global forms)

This extraction of a modal subspace from the GNS Hilbert space aligns with the algebraic approach to quantum kinematics [7, 16], where specific logical sectors are identified as subspaces invariant under relevant symmetry operations [17]. The four corners correspond to the tetralemma statuses defined in Definitions 2.7, 2.8, 2.11 and 2.12.

Definition 3.45 (Local corner kets and local modal subspace)**.** Fix any admissible polar claim $P \in \mathsf{P}_{\mathcal{C},\lambda}$. Define the corner kets

$$|\mathbf{Th}; P\rangle := [C_{\mathbf{Th}}(P)], \qquad |\mathbf{SynTh}; P\rangle := [C_{\mathbf{SynTh}}(P)],$$

$$|\mathbf{AntiTh}; P\rangle := [C_{\mathbf{AntiTh}}(P)], \qquad |\mathbf{HoloTh}; P\rangle := [C_{\mathbf{HoloTh}}(P)]. \tag{11}$$

These vectors represent the equivalence classes of the corner generators in the GNS pre-Hilbert space [19, 94]. Define the associated local modal subspace

$$\mathcal{H}_L(P) := \mathrm{span}\Big\{|\mathbf{Th}; P\rangle, |\mathbf{AntiTh}; P\rangle, |\mathbf{SynTh}; P\rangle, |\mathbf{HoloTh}; P\rangle\Big\} \subseteq \mathcal{H}_\Lambda.$$

This subspace captures the logical degrees of freedom associated with the claim P within the tetralemmatic framework [8, 9].

Definition 3.46 (Logical modal subspace $\mathcal{H}_L(P)$)**.** Fix an atomic corner context P (Definition 4.7). The associated *logical modal subspace* (the *4-corner Hilbert template* in context P) is

$$\mathcal{H}_L(P) := \mathrm{span}\Big\{|\mathbf{Th}; P\rangle, |\mathbf{AntiTh}; P\rangle, |\mathbf{SynTh}; P\rangle, |\mathbf{HoloTh}; P\rangle\Big\} \subseteq \mathcal{H}_\Lambda. \tag{12}$$

66

This subspace captures the logical degrees of freedom associated with the claim P within the tetralemmatic framework, constituting the local realization of the logical Hilbert template [8, 9].

Remark 3.47 (Orthonormality and the Gram matrix)**.** If the warrant functional satisfies $\omega_\Lambda(C_\bullet(P)^\dagger C_{\bullet'}(P)) = \delta_{\bullet\bullet'}$ for each P, then the Gram matrix $G(P)$ (cf. Definition 6.3 and Proposition 6.4) is the identity, and the corner kets form an orthonormal basis for $\mathcal{H}_L(P)$. This occurs, for example, when the corner generators are realized as projectors onto orthogonal subspaces in a sharp quantum realization (cf. Section 2.8). In paraconsistent or paracomplete regimes, $G(P)$ typically has off-diagonal entries, reflecting logical coherence or inapplicability [56, 63].

Definition 3.48 (Global corner span (choice-free template))**.** Define the global (choice-free) corner span by

$$\mathcal{H}_L^{\text{glob}} := \overline{\text{span}}\Big\{ [C_\bullet(P)] \ : \ P \in \mathsf{P}_{\mathcal{C},\lambda}, \ \bullet \in \mathbf{Status}_\Lambda \Big\} \subseteq \mathcal{H}_\Lambda.$$

Then $\mathcal{H}_L(P) \subseteq \mathcal{H}_L^{\text{glob}}$ for every admissible P. This global space aggregates the logical content of all admissible claims at the locus $(\mathcal{C}, \lambda)$, analogous to the history space in quantum measure theory [14, 20].

Remark 3.49 (Compatibility with Tier 2 enrichment)**.** The canonical corner kets and subspaces are defined at Tier 1, using only the minimal algebra $\mathcal{A}_\Lambda^{\text{alg}}$ and warrant functional ω_Λ. If Tier 2 enrichment (Definition 3.33) imposes additional relations such as idempotence $(C_\bullet(P)^2 = C_\bullet(P))$ or exclusivity $(C_\bullet(P)C_{\bullet'}(P) = 0$ for $\bullet \neq \bullet')$, these may impose linear dependencies among the corner kets, potentially reducing $\dim \mathcal{H}_L(P)$ from 4 to a lower value. This captures the transition between logical regimes (e.g., classical sharp measurements vs. paraconsistent coherent superpositions) within the Hilbert space framework [54, 55].

Proposition 3.50 (Choice-independence under claim congruence)**.** *Assume a claim congruence $\equiv_\Sigma$ was imposed via $\mathcal{I}_\Sigma$ in Definition 3.15. If $P \equiv_\Sigma Q$, then for each corner label $\bullet$,*

$$[C_\bullet(P)] = [C_\bullet(Q)] \quad \text{in } \mathcal{H}_\Lambda,$$

and hence $\mathcal{H}_L(P) = \mathcal{H}_L(Q)$.

Proof. By Definition 3.15, the ideal $\mathcal{I}_\Sigma$ contains $C_\bullet(P) - C_\bullet(Q)$ whenever $P \equiv_\Sigma Q$. Thus $C_\bullet(P) - C_\bullet(Q) \in \mathcal{N}_{\omega_\Lambda}$ (the null space of the GNS construction), implying their equivalence classes coincide in the quotient space [17, 94]. $\square$

Example 3.51 (Double-slit realization of corner kets)**.** In the double-slit experiment (Section 2.8), let P be the claim "the electron went through the left slit." In the which-path context $(\mathcal{C}_1, \lambda_1)$:

$$\begin{aligned}
|\mathbf{Th}; P\rangle \ &\cong \ |L\rangle \quad \text{(left slit outcome, Thesis)},\\
|\mathbf{AntiTh}; P\rangle \ &\cong \ |R\rangle \quad \text{(right slit outcome, Antithesis)},\\
|\mathbf{SynTh}; P\rangle \ &\cong \ |+\rangle = \tfrac{|L\rangle + |R\rangle}{\sqrt{2}} \quad \text{(coherent superposition, Synthesis)},\\
|\mathbf{HoloTh}; P\rangle \ &\cong \ \text{undefined at } (\mathcal{C}_2, \lambda_2) \quad \text{(inapplicable, Holothesis)}.
\end{aligned}$$

The duality unitary $U_{\mathbf{D}}$ implements slit-exchange: $U_{\mathbf{D}}|L\rangle = |R\rangle$, $U_{\mathbf{D}}|R\rangle = |L\rangle$, $U_{\mathbf{D}}|+\rangle = |+\rangle$. At the interference locus $(\mathcal{C}_2, \lambda_2)$, the polar kets are not defined (typed inapplicability, Definition 2.12), warranting Holothesis.

3.9 GNS Implementation of Duality

This result is standard in the theory of operator algebras and algebraic quantum field theory, where state-preserving automorphisms are implemented by unitaries in the GNS representation [7, 16, 17, 19]. The construction ensures that logical symmetries (such as duality) manifest as physical symmetries (unitaries) on the Hilbert space [92].

Proposition 3.52 (Unitary implementer of an invariant $*$-automorphism). *Let $\alpha : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_\Lambda^{\mathrm{alg}}$ be a $*$-automorphism such that $\omega_\Lambda \circ \alpha = \omega_\Lambda$. Then there exists a unique unitary U_α on $\mathcal{H}_\Lambda$ such that*

$$U_\alpha \, \pi_\Lambda(A) \, U_\alpha^\dagger = \pi_\Lambda(\alpha(A)) \qquad (A \in \mathcal{A}_\Lambda^{\mathrm{alg}}), \qquad U_\alpha|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle.$$

Proof. Let $\mathcal{D}_\Omega := \pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})|\Omega_\Lambda\rangle$ be the dense subspace of $\mathcal{H}_\Lambda$ defined by the cyclic vector [17, 19]. Define a linear operator $U_\alpha^{(0)}$ on $\mathcal{D}_\Omega$ by

$$U_\alpha^{(0)}\big(\pi_\Lambda(A)|\Omega_\Lambda\rangle\big) := \pi_\Lambda(\alpha(A))|\Omega_\Lambda\rangle, \qquad A \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

Well-definedness: If $\pi_\Lambda(A)|\Omega_\Lambda\rangle = 0$, then $\|\pi_\Lambda(A)|\Omega_\Lambda\rangle\|^2 = \omega_\Lambda(A^\dagger A) = 0$. Since $\omega_\Lambda \circ \alpha = \omega_\Lambda$ and α is a $*$-automorphism,

$$\omega_\Lambda(\alpha(A)^\dagger \alpha(A)) = \omega_\Lambda(\alpha(A^\dagger A)) = \omega_\Lambda(A^\dagger A) = 0,$$

so $\pi_\Lambda(\alpha(A))|\Omega_\Lambda\rangle = 0$. Thus $U_\alpha^{(0)}$ is well-defined on the quotient [7, 16].
  Isometry: For any $\psi = \pi_\Lambda(A)|\Omega_\Lambda\rangle$ and $\phi = \pi_\Lambda(B)|\Omega_\Lambda\rangle$ in $\mathcal{D}_\Omega$,

$$\begin{aligned}
\langle U_\alpha^{(0)}\psi \mid U_\alpha^{(0)}\phi\rangle_\Lambda &= \langle \pi_\Lambda(\alpha(A))\Omega_\Lambda \mid \pi_\Lambda(\alpha(B))\Omega_\Lambda\rangle_\Lambda \\
&= \omega_\Lambda(\alpha(A)^\dagger \alpha(B)) \\
&= \omega_\Lambda(\alpha(A^\dagger B)) \\
&= \omega_\Lambda(A^\dagger B) \\
&= \langle \pi_\Lambda(A)\Omega_\Lambda \mid \pi_\Lambda(B)\Omega_\Lambda\rangle_\Lambda \\
&= \langle \psi \mid \phi\rangle_\Lambda.
\end{aligned}$$

Thus $U_\alpha^{(0)}$ preserves the inner product [71, 94].
  Extension to Unitary: Since $\mathcal{D}_\Omega$ is dense in $\mathcal{H}_\Lambda$ and $U_\alpha^{(0)}$ is an isometry, it extends uniquely to an isometry U_α on $\mathcal{H}_\Lambda$. Since α is an automorphism, α^{-1} exists and is also ω_Λ-invariant. Constructing $U_{\alpha^{-1}}$ similarly yields the inverse of U_α, so U_α is unitary [17, 95].
  Covariance and Vacuum Fixing: By definition, $U_\alpha|\Omega_\Lambda\rangle = \pi_\Lambda(\alpha(\mathbf{1}))|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle$. For the

covariance relation, compute on the dense domain:

$$U_\alpha\,\pi_\Lambda(A)\,U_\alpha^\dagger\Big(\pi_\Lambda(B)|\Omega_\Lambda\rangle\Big) = U_\alpha\,\pi_\Lambda(A)\Big(\pi_\Lambda(\alpha^{-1}(B))|\Omega_\Lambda\rangle\Big)$$
$$= U_\alpha\Big(\pi_\Lambda(A\alpha^{-1}(B))|\Omega_\Lambda\rangle\Big)$$
$$= \pi_\Lambda(\alpha(A\alpha^{-1}(B)))|\Omega_\Lambda\rangle$$
$$= \pi_\Lambda(\alpha(A)B)|\Omega_\Lambda\rangle$$
$$= \pi_\Lambda(\alpha(A))\,\pi_\Lambda(B)|\Omega_\Lambda\rangle.$$

By density, $U_\alpha\,\pi_\Lambda(A)\,U_\alpha^\dagger = \pi_\Lambda(\alpha(A))$ holds on $\mathcal{H}_\Lambda$ [16, 19]. Uniqueness follows from cyclicity of $|\Omega_\Lambda\rangle$. $\qquad\square$

Corollary 3.53 (Duality becomes unitary). *If $\alpha_{\mathbf{D}}$ is defined as in Definition 3.38 and $\omega_\Lambda \circ \alpha_{\mathbf{D}} = \omega_\Lambda$, then duality admits a unique unitary implementer $U_{\mathbf{D}} := U_{\alpha_{\mathbf{D}}}$ satisfying the covariance and vacuum-fixing identities. This implements the Axiom of Duality (Axiom 2.22) at the Hilbert space level.*

Proof. Apply Proposition 3.52 with $\alpha = \alpha_{\mathbf{D}}$. The condition $\omega_\Lambda \circ \alpha_{\mathbf{D}} = \omega_\Lambda$ is the duality invariance of the warrant functional (Definition 3.35). The resulting unitary $U_{\mathbf{D}}$ satisfies

$$U_{\mathbf{D}}\,\pi_\Lambda(A)\,U_{\mathbf{D}}^\dagger = \pi_\Lambda(\alpha_{\mathbf{D}}(A)), \qquad U_{\mathbf{D}}|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle,$$

establishing the unitary implementation of the logical duality symmetry [7, 92]. $\qquad\square$

3.10 Order-4 Symmetry and the Appearance of $\{1, i, -1, -i\}$

This section establishes the canonical emergence of the fourth roots of unity from the tetralemmatic corner-structure. The four corners (Definitions 2.7, 2.8, 2.11, 2.12 and 2.17) naturally suggest an order-4 symmetry group, whose unitary implementation forces the complex phases $\{1, i, -1, -i\}$ to appear as eigenvalues [17, 71].

Definition 3.54 (Tetralemmatic lens quarter-turn automorphism (order 4)). A *tetralemmatic quarter-turn* is a $*$-automorphism $\alpha_{\mathbf{R}}$ of $\mathcal{A}_\Lambda^{\mathrm{alg}}$ such that:

$$\alpha_{\mathbf{R}}^4 = \mathrm{id}, \qquad \alpha_{\mathbf{R}}\Big(C_\bullet(P)\Big) = C_{\sigma_{\mathbf{R}}(\bullet)}(P^{\mathbf{R}})$$

for some corner permutation $\sigma_{\mathbf{R}}$ and (optional) claim relabelling $P \mapsto P^{\mathbf{R}}$. Assume additionally $\omega_\Lambda \circ \alpha_{\mathbf{R}} = \omega_\Lambda$. This structure mirrors the cyclic permutation of the tetralemma corners consistent with the Axiom of Duality (Axiom 2.22) extended to order-4 rotations [13, 15].

Proposition 3.55 (Order-4 symmetry yields order-4 unitary and 4th-root spectrum). *Under the hypotheses of Definition 3.54, the unitary implementer $U_{\mathbf{R}} := U_{\alpha_{\mathbf{R}}}$ satisfies $U_{\mathbf{R}}^4 = I_{\mathcal{H}_\Lambda}$. Hence every eigenvalue of $U_{\mathbf{R}}$ (on any invariant subspace where it is diagonalizable) lies in $\{1, i, -1, -i\}$.*

Proof. Existence and uniqueness of the unitary implementer $U_{\mathbf{R}}$ follow from the standard GNS construction for invariant states on $*$-algebras [16, 17, 19]. Specifically, since $\omega_\Lambda \circ \alpha_{\mathbf{R}} = \omega_\Lambda$, the map

$$U_{\mathbf{R}}^{(0)} : \pi_\Lambda(A)|\Omega_\Lambda\rangle \mapsto \pi_\Lambda(\alpha_{\mathbf{R}}(A))|\Omega_\Lambda\rangle$$

is well-defined and isometric on the dense domain $\pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})|\Omega_\Lambda\rangle$, and extends uniquely to a unitary operator $U_{\mathbf{R}}$ on $\mathcal{H}_\Lambda$ [71].

To prove $U_{\mathbf{R}}^4 = I_{\mathcal{H}_\Lambda}$, observe that for any $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$:

$$
\begin{aligned}
U_{\mathbf{R}}^4\left(\pi_\Lambda(A)|\Omega_\Lambda\rangle\right) &= \pi_\Lambda(\alpha_{\mathbf{R}}^4(A))|\Omega_\Lambda\rangle \\
&= \pi_\Lambda(\mathrm{id}(A))|\Omega_\Lambda\rangle \quad (\text{since } \alpha_{\mathbf{R}}^4 = \mathrm{id}) \\
&= \pi_\Lambda(A)|\Omega_\Lambda\rangle.
\end{aligned}
$$

Since the vectors $\pi_\Lambda(A)|\Omega_\Lambda\rangle$ form a dense subset of $\mathcal{H}_\Lambda$ (Proposition 3.43), $U_{\mathbf{R}}^4$ acts as the identity on a dense domain and is bounded, hence $U_{\mathbf{R}}^4 = I_{\mathcal{H}_\Lambda}$ [98, 99].

For the spectrum, let $\lambda \in \sigma(U_{\mathbf{R}})$ be an eigenvalue with eigenvector $v \neq 0$. Then $U_{\mathbf{R}}v = \lambda v$ implies $U_{\mathbf{R}}^4 v = \lambda^4 v$. Since $U_{\mathbf{R}}^4 = I$, we have $\lambda^4 = 1$. The solutions to $\lambda^4 = 1$ in $\mathbb{C}$ are exactly $\{1, i, -1, -i\}$ [71]. $\qquad\square$

Remark 3.56 (Logical origin of complex phases)**.** This proposition demonstrates that the complex unit i need not be presupposed as part of a physical Hilbert space axiom. Instead, it emerges representation-theoretically from the order-4 symmetry of the logical corner-structure (Definition 2.17). This aligns with the view that quantum kinematics is forced by logical coherence and symmetry rather than by physical postulate [7, 8].

3.10.1 Optional realization interface

This interface formalizes the connection between the logic-generated algebraic template and concrete physical models, aligning with the algebraic quantum field theory paradigm [7, 16, 17]. It corresponds to the Realization Axiom introduced in Section 6.5 of the foundational logic chapter.

Definition 3.57 (Admissible Physical Realization)**.** Let $(\mathcal{A}_\Lambda^{\mathrm{alg}}, \omega_\Lambda)$ be the logical corner probability space (Definition 6.1). Let $(\mathcal{A}_{\mathrm{phys}}, \omega_{\mathrm{phys}})$ be a physical operator model, where $\mathcal{A}_{\mathrm{phys}} \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ is a unital C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}_{\mathrm{phys}}$, and ω_{phys} is a normal state (e.g., $\omega_{\mathrm{phys}}(X) = \mathrm{Tr}(\rho X)$).

An *admissible physical realization* is a unital $*$-homomorphism

$$
r : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_{\mathrm{phys}}
$$

satisfying the following rigorous conditions:

(i) **State Preservation (Empirical Consistency):**

$$
\omega_{\mathrm{phys}} \circ r = \omega_\Lambda. \tag{13}
$$

This ensures logical warrant profiles match physical Born valuations [63, 84].

(ii) **Effect Bounds (Tier-2 Enrichment):** For every corner generator $C_\bullet(P) \in \mathcal{A}_\Lambda^{\mathrm{alg}}$ associated with an applicable claim P (Definition 3.8), the image satisfies:

$$
0 \leq r(C_\bullet(P)) \leq I_{\mathrm{phys}}. \tag{14}
$$

This restricts the realization to the Tier-2 effect-like enrichment (Definition 3.33), ensuring probabilistic interpretability.

(iii) **Norm Continuity (Stability):** The map r is continuous with respect to the C*-norm on $\mathcal{A}_{\text{phys}}$ induced by the operator norm. Specifically, there exists a constant $K > 0$ such that for all $A \in \mathcal{A}_{\Lambda}^{\text{alg}}$:

$$\|r(A)\|_{\text{phys}} \leq K\|A\|_{\omega_{\Lambda}}, \tag{15}$$

where $\|A\|_{\omega_{\Lambda}} := \sqrt{\omega_{\Lambda}(A^{\dagger}A)}$ is the GNS seminorm. This ensures small perturbations in logical warrant do not yield unbounded physical operators [7, 16].

(iv) **Symmetry Equivariance:** For any logical symmetry automorphism α_g (e.g., duality $\alpha_{\mathbf{D}}$) and corresponding physical symmetry β_g (e.g., unitary conjugation),

$$r \circ \alpha_g = \beta_g \circ r. \tag{16}$$

This preserves the tetralemmatic structural constraints (e.g., duality exchange of Thesis/Antithesis) in the physical model [92].

(v) **Operational Completeness:** The image $r(\mathcal{A}_{\Lambda}^{\text{alg}})$ is dense in $\mathcal{A}_{\text{phys}}$ with respect to the weak operator topology. This ensures the logical algebra is sufficient to generate all relevant physical observables for the context [17].

Remark 3.58 (Experimental Robustness and Epsilon-Realizations)**.** In practice, experimental data contains noise, making exact state preservation (Condition i) impossible. We define an ϵ-*admissible realization* by relaxing Condition (i) to:

$$|\omega_{\text{phys}}(r(A)) - \omega_{\Lambda}(A)| \leq \epsilon\|A\|_{\omega_{\Lambda}} \quad \forall A \in \mathcal{A}_{\Lambda}^{\text{alg}}. \tag{17}$$

This defines a tolerance class of realizations $[r]_{\epsilon}$. Theoretical predictions are robust if they remain invariant across all $r \in [r]_{\epsilon}$. This formalizes the link to experimental data where statistical error bars are inherent [56, 83].

Proposition 3.59 (Uniqueness up to Unitary Equivalence)**.** *Let $r_1, r_2 : \mathcal{A}_{\Lambda}^{\text{alg}} \to \mathcal{A}_{\text{phys}}$ be two admissible physical realizations satisfying Definition 3.57 for the same logical state ω_{Λ}. Then there exists a unitary operator $U : \mathcal{H}_{\text{phys}}^{(1)} \to \mathcal{H}_{\text{phys}}^{(2)}$ such that:*

$$r_2(A) = U r_1(A) U^{\dagger} \quad \forall A \in \mathcal{A}_{\Lambda}^{\text{alg}}. \tag{18}$$

Proof. This follows directly from the uniqueness of the GNS representation (Theorem 6.97). Since both realizations preserve the state ω_{Λ} (Condition i) and are continuous (Condition iii), they induce GNS triples $(\pi_1, \mathcal{H}_1, |\Omega_1\rangle)$ and $(\pi_2, \mathcal{H}_2, |\Omega_2\rangle)$ that are unitarily equivalent [17, 18]. The map r corresponds to the representation π via the embedding V (Proposition 6.14), ensuring physical predictions (expectation values) are unique regardless of the specific substrate (e.g., spin vs. path). $\square$

4 Gelfand–Naimark–Segal and State Space Emergence

Having specified the purely formal corner-observable $*$-algebra $\mathcal{A}_{\Lambda}^{\text{alg}}$ (Definition 3.20) together with the warrant functional ω_{Λ} (Definition 3.25), the associated Hilbert space is no longer a postulate. It is *forced* by the Gelfand–Naimark–Segal construction applied to $(\mathcal{A}_{\Lambda}^{\text{alg}}, \omega_{\Lambda})$. This reconstruction aligns with the algebraic quantum theory paradigm, where the Hilbert space is derived from the algebraic state rather than presupposed [7, 16, 17], and resonates with quantum measure-theoretic approaches to logic [8, 14].

4.1 Canonical GNS construction of Hilbert Spaces

Definition 4.1 (GNS triple of $(\mathcal{A}_\Lambda^{\mathrm{alg}}, \omega_\Lambda)$). Let $\mathcal{A}_\Lambda^{\mathrm{alg}}$ be a unital $*$-algebra and let $\omega_\Lambda : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathbb{C}$ be a warrant functional (i.e. a normalized positive linear functional). Define the following objects.

(a) Null space / null ideal. Define

$$\mathcal{N}_{\omega_\Lambda} := \{ A \in \mathcal{A}_\Lambda^{\mathrm{alg}} : \omega_\Lambda(A^\dagger A) = 0 \}. \tag{19}$$

By the Cauchy–Schwarz inequality for positive functionals, $\mathcal{N}_{\omega_\Lambda}$ is a left ideal in $\mathcal{A}_\Lambda^{\mathrm{alg}}$ [19, 94].

(b) Pre-Hilbert space. Let $\mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda}$ be the quotient vector space and denote the equivalence class of $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$ by $[A]_{\omega_\Lambda}$ (or simply $[A]$ when no confusion can arise). Define a sesquilinear form on the quotient by

$$\left\langle [A]_{\omega_\Lambda} \,\middle|\, [B]_{\omega_\Lambda} \right\rangle_\Lambda := \omega_\Lambda(A^\dagger B). \tag{20}$$

Positivity of ω_Λ ensures this form is well-defined and positive semidefinite, becoming positive definite on the quotient [16, 17].

Remark 4.2 (Cauchy Completion of the Pre-Hilbert Quotient). The construction proceeds in three topological stages:

1. **Quotient Space:** $L_{\omega_\Lambda} := \mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda}$ is a complex vector space with well-defined equivalence classes $[A]_{\omega_\Lambda}$.

2. **Pre-Hilbert Structure:** The sesquilinear form $\langle [A]_{\omega_\Lambda} \mid [B]_{\omega_\Lambda} \rangle_\Lambda := \omega_\Lambda(A^\dagger B)$ is positive definite on L_{ω_Λ} (Proposition 4.3), making it a *pre-Hilbert space* (inner-product space, not necessarily complete).

3. **Hilbert Space Completion:** $\mathcal{H}_\Lambda$ is defined as the **Cauchy completion** of L_{ω_Λ} with respect to the norm $\|[A]_{\omega_\Lambda}\|_\Lambda = \sqrt{\omega_\Lambda(A^\dagger A)}$. That is, $\mathcal{H}_\Lambda$ consists of equivalence classes of Cauchy sequences in L_{ω_Λ}, ensuring all limit points exist [19, 71].

This three-stage construction is standard in the GNS theorem [16, 17, 18].

(c) Hilbert space completion. Let $\mathcal{H}_\Lambda$ be the completion of $\mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda}$ with respect to the norm induced by (20):

$$\|[A]_{\omega_\Lambda}\|_\Lambda := \sqrt{\left\langle [A]_{\omega_\Lambda} \,\middle|\, [A]_{\omega_\Lambda} \right\rangle_\Lambda} = \sqrt{\omega_\Lambda(A^\dagger A)}.$$

This yields a separable Hilbert space if $\mathcal{A}_\Lambda^{\mathrm{alg}}$ is countably generated [7].

(d) Representation. Define $\pi_\Lambda : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{B}(\mathcal{H}_\Lambda)$ on the dense subspace $\mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda} \subseteq \mathcal{H}_\Lambda$ by

$$\pi_\Lambda(A)\,[B]_{\omega_\Lambda} := [AB]_{\omega_\Lambda} \qquad (A, B \in \mathcal{A}_\Lambda^{\mathrm{alg}}), \tag{21}$$

and extend (when boundedness is available) to $\mathcal{H}_\Lambda$. This defines a unital $*$-representation of the algebra [16, 19].

(e) Cyclic vector. Define the distinguished vector

$$|\Omega_\Lambda\rangle := [\mathbf{1}]_{\omega_\Lambda} \in \mathcal{H}_\Lambda. \tag{22}$$

The vector $|\Omega_\Lambda\rangle$ is cyclic for the representation π_Λ [17, 94].

The triple $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ is called the *logical GNS representation* of $(\mathcal{A}_\Lambda^{\mathrm{alg}}, \omega_\Lambda)$. This construction provides the kinematical arena for the tetralemmatic logic without presupposing a physical Hilbert space [8, 9].

Proposition 4.3 (Basic correctness of the construction)**.** *In the setting of Definition 4.1:*

(a) The set $\mathcal{N}_{\omega_\Lambda}$ is a left ideal of $\mathcal{A}_\Lambda^{\mathrm{alg}}$.

(b) The sesquilinear form (20) *is well-defined and positive semidefinite on $\mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda}$, and becomes positive definite on the quotient.*

(c) The map π_Λ is a well-defined unital $*$-representation on $\mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda}$, i.e.

$$\pi_\Lambda(\mathbf{1}) = \mathbf{1}, \qquad \pi_\Lambda(AB) = \pi_\Lambda(A)\pi_\Lambda(B), \qquad \pi_\Lambda(A^\dagger) = \pi_\Lambda(A)^\dagger.$$

(d) The vector $|\Omega_\Lambda\rangle$ is cyclic in the sense that

$$\overline{\mathrm{span}}\{\pi_\Lambda(A)|\Omega_\Lambda\rangle : A \in \mathcal{A}_\Lambda^{\mathrm{alg}}\} = \mathcal{H}_\Lambda.$$

This is the standard GNS construction for states on unital $*$-algebras [16, 17, 18, 19].

Proof. We verify each claim in turn, following the classical GNS argument [7, 18, 94].

(a) Left ideal property. Let $N \in \mathcal{N}_{\omega_\Lambda}$ and $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$. We must show $AN \in \mathcal{N}_{\omega_\Lambda}$, i.e. $\omega_\Lambda((AN)^\dagger(AN)) = 0$. By positivity of ω_Λ, the Cauchy–Schwarz inequality holds [17, 19]:

$$\left|\omega_\Lambda(X^\dagger Y)\right|^2 \leq \omega_\Lambda(X^\dagger X)\,\omega_\Lambda(Y^\dagger Y) \qquad (X, Y \in \mathcal{A}_\Lambda^{\mathrm{alg}}).$$

Apply this with $X = N$ and $Y = A^\dagger AN$:

$$\left|\omega_\Lambda(N^\dagger(A^\dagger AN))\right|^2 \leq \omega_\Lambda(N^\dagger N)\,\omega_\Lambda((A^\dagger AN)^\dagger(A^\dagger AN)).$$

Since $N \in \mathcal{N}_{\omega_\Lambda}$, we have $\omega_\Lambda(N^\dagger N) = 0$, so the right-hand side vanishes. Hence $\omega_\Lambda(N^\dagger A^\dagger AN) = 0$. But $\omega_\Lambda((AN)^\dagger(AN)) = \omega_\Lambda(N^\dagger A^\dagger AN)$, so $AN \in \mathcal{N}_{\omega_\Lambda}$ [16, 94]. Thus $\mathcal{N}_{\omega_\Lambda}$ is a left ideal.

(b) Well-definedness and positive definiteness. For well-definedness, suppose $[A] = [A']$ and $[B] = [B']$ in the quotient, i.e. $A - A' \in \mathcal{N}_{\omega_\Lambda}$ and $B - B' \in \mathcal{N}_{\omega_\Lambda}$. Then

$$\omega_\Lambda(A^\dagger B) - \omega_\Lambda((A')^\dagger B') = \omega_\Lambda((A - A')^\dagger B) + \omega_\Lambda((A')^\dagger(B - B')).$$

By Cauchy–Schwarz and the left ideal property, both terms vanish [17, 19]. Hence the sesquilinear form is independent of representatives.

Positive semidefiniteness follows from positivity of ω_Λ:

$$\langle [A] \mid [A]\rangle_\Lambda = \omega_\Lambda(A^\dagger A) \geq 0.$$

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Positive definiteness on the quotient holds because $\langle [A] \mid [A] \rangle_\Lambda = 0$ iff $A \in \mathcal{N}_{\omega_\Lambda}$ iff $[A] = 0$ [7, 18].

(c) $*$-representation properties. Unitality is immediate: $\pi_\Lambda(\mathbf{1})[B] = [B]$ for all $[B]$. Multiplicativity follows from associativity in $\mathcal{A}_\Lambda^{\mathrm{alg}}$:

$$\pi_\Lambda(AB)[C] = [(AB)C] = [A(BC)] = \pi_\Lambda(A)[BC] = \pi_\Lambda(A)\pi_\Lambda(B)[C].$$

For $*$-preservation, compute for arbitrary $[B], [C]$:

$$\left\langle [B] \mid \pi_\Lambda(A^\dagger)[C] \right\rangle_\Lambda = \omega_\Lambda(B^\dagger A^\dagger C) = \omega_\Lambda((AB)^\dagger C) = \left\langle [AB] \mid [C] \right\rangle_\Lambda$$
$$= \left\langle \pi_\Lambda(A)[B] \mid [C] \right\rangle_\Lambda = \left\langle [B] \mid \pi_\Lambda(A)^\dagger [C] \right\rangle_\Lambda.$$

By non-degeneracy of the inner product, $\pi_\Lambda(A^\dagger) = \pi_\Lambda(A)^\dagger$ [16, 94, 95].

(d) Cyclicity. By definition, $|\Omega_\Lambda\rangle = [\mathbf{1}]_{\omega_\Lambda}$. For any $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$,

$$\pi_\Lambda(A)|\Omega_\Lambda\rangle = \pi_\Lambda(A)[\mathbf{1}] = [A \cdot \mathbf{1}] = [A].$$

Thus $\{\pi_\Lambda(A)|\Omega_\Lambda\rangle : A \in \mathcal{A}_\Lambda^{\mathrm{alg}}\}$ equals the dense subspace $\mathcal{A}_\Lambda^{\mathrm{alg}}/\mathcal{N}_{\omega_\Lambda} \subset \mathcal{H}_\Lambda$ [7, 17, 18]. Taking the closure yields $\mathcal{H}_\Lambda$. $\qquad\square$

4.2 The logical Holo-vector

Definition 4.4 (Logical Holo-vector). The distinguished cyclic vector $|\Omega_\Lambda\rangle \in \mathcal{H}_\Lambda$ of (22) is called the *logical Holo-vector*. It is the logic-side analogue of a global state vector: it is not assumed as a primitive element of a pre-existing Hilbert space, but is *generated* by the warrant functional via the GNS procedure. This terminology reflects its structural correspondence to the Holothesis corner (cf. Definition 2.12, representing the meta-level suspension from which specific corner-statuses are derived via the action of $\mathcal{A}_\Lambda^{\mathrm{alg}}$. The existence and uniqueness (up to unitary equivalence) of such a cyclic vector for a given state is a standard result in the theory of operator algebras [7, 16, 17, 18].

Proposition 4.5 (State recovery (logical expectation identity)). *For all $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$, the warrant functional is recovered as an expectation in the logical GNS representation:*

$$\omega_\Lambda(A) = \langle \Omega_\Lambda \mid \pi_\Lambda(A) \mid \Omega_\Lambda\rangle_\Lambda. \tag{23}$$

This identity establishes ω_Λ as a vector state in the representation π_Λ [7, 17, 19].

Proof. By Definition 4.1,

$$\langle \Omega_\Lambda \mid \pi_\Lambda(A) \mid \Omega_\Lambda\rangle_\Lambda = \left\langle [\mathbf{1}]_{\omega_\Lambda} \mid [A]_{\omega_\Lambda} \right\rangle_\Lambda = \omega_\Lambda(\mathbf{1}^\dagger A) = \omega_\Lambda(A).$$

This calculation follows the canonical GNS state recovery argument [16, 18]. $\qquad\square$

Remark 4.6 (Why this avoids presupposing quantum structure). The construction of $\mathcal{H}_\Lambda$ uses only: (i) the purely formal, logic-generated $*$-algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$, and (ii) the positive evaluator ω_Λ encoding degrees of warrant compatible with duality. No pre-given physical Hilbert space, no concrete operator effects, and no realization map are assumed. The Hilbert space appears as the completion dictated by positivity (GNS), not as an ontological postulate. This aligns with the algebraic quantum field theory paradigm [7, 16, 95], and resonates with reconstruction programs where quantum kinematics emerges from logical or informational primitives [8, 9, 14].

4.3 The Tetralemmatic Modal Subspace

The GNS construction produces a (generally large) Hilbert space $\mathcal{H}_\Lambda$ representing the logic-generated algebra $\mathcal{A}_\Lambda^{\text{alg}}$ in the warrant state ω_Λ. To obtain the *tetralemmatic Hilbert template*, we now extract a canonical 4-corner modal subspace inside $\mathcal{H}_\Lambda$. This extraction aligns with the algebraic approach to quantum kinematics [7, 16], where specific logical sectors are identified as subspaces invariant under relevant symmetry operations [17]. The four corners correspond to the tetralemma statuses defined in Definitions 2.7, 2.8, 2.11 and 2.12.

4.3.1 Fixing an atomic corner context

Definition 4.7 (Atomic corner context)**.** Fix an applicable polar claim $P \in \mathsf{P}_{\mathcal{C},\lambda}$ at some $(\mathcal{C}, \lambda)$ (cf. the tetralogic applicability relation $\text{App}_{\mathcal{C}}(P; \lambda)$), and consider the four corner generators

$$C_{\mathbf{Th}}(P), \quad C_{\mathbf{AntiTh}}(P), \quad C_{\mathbf{SynTh}}(P), \quad C_{\mathbf{HoloTh}}(P) \in \mathcal{A}_\Lambda^{\text{alg}}.$$

We refer to this choice of P (or equivalently, the $*$-subalgebra generated by these four elements) as an *atomic corner context*. This corresponds to fixing a single tetralemmatic inquiry unit $\mathring{\Lambda}$ as defined in Definition 2.17 [13, 15].

4.3.2 The four corner vectors in $\mathcal{H}_\Lambda$

Definition 4.8 (Corner kets in the logical GNS space)**.** Let $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ be the logical GNS triple of $(\mathcal{A}_\Lambda^{\text{alg}}, \omega_\Lambda)$ (Definition 4.1), and fix an atomic corner context P (Definition 4.7). Define the associated *corner kets* in $\mathcal{H}_\Lambda$ by

$$\begin{aligned} |\mathbf{Th}; P\rangle &:= [C_{\mathbf{Th}}(P)]_{\omega_\Lambda}, & |\mathbf{SynTh}; P\rangle &:= [C_{\mathbf{SynTh}}(P)]_{\omega_\Lambda}, \\ |\mathbf{AntiTh}; P\rangle &:= [C_{\mathbf{AntiTh}}(P)]_{\omega_\Lambda}, & |\mathbf{HoloTh}; P\rangle &:= [C_{\mathbf{HoloTh}}(P)]_{\omega_\Lambda}. \end{aligned} \tag{24}$$

These vectors represent the equivalence classes of the corner generators in the GNS pre-Hilbert space [19, 94].

Remark 4.9 (Equivalent "operator-on-vacuum" form)**.** Since $[A]_{\omega_\Lambda} = \pi_\Lambda(A)|\Omega_\Lambda\rangle$ for all $A \in \mathcal{A}_\Lambda^{\text{alg}}$, the corner kets may also be written as

$$|\bullet; P\rangle = \pi_\Lambda(C_\bullet(P)) |\Omega_\Lambda\rangle, \qquad \bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}.$$

This form emphasizes that the corner kets are obtained by acting with the corner observables on the logical Holo-vector (Definition 4.4), analogous to creating excited states from a vacuum in quantum field theory [16, 17].

Remark 4.10 (Relation to ontological state symbols)**.** Under the bridge map of Remark 2.20, the corner kets correspond to the ontological state symbols:

$$|\mathbf{Th}; P\rangle \longleftrightarrow |\mathring{\mathbf{0}}\rangle, \qquad |\mathbf{SynTh}; P\rangle \longleftrightarrow |\mathring{\mathbf{0}} \stackrel{=}{\wedge} \mathring{\mathbf{1}}\rangle,$$

$$|\mathbf{AntiTh}; P\rangle \longleftrightarrow |\mathring{\mathbf{1}}\rangle, \qquad |\mathbf{HoloTh}; P\rangle \longleftrightarrow |\mathring{\mathbf{0}} \stackrel{\infty}{\wedge} \mathring{\mathbf{1}}\rangle.$$

This identification preserves the Σ-structure of the tetralemmatic lens (Definition 2.17 and Remark 2.21) [13, 15, 53].

Remark 4.11 (Connection to the double-slit example)**.** In the double-slit realization of Example 3.51 and Section 2.8, the corner kets take concrete form:

- $|\mathbf{Th}; P\rangle \cong |L\rangle$ (left slit outcome, Thesis),

- $|\mathbf{AntiTh}; P\rangle \cong |R\rangle$ (right slit outcome, Antithesis),

- $|\mathbf{SynTh}; P\rangle \cong |+\rangle = \frac{|L\rangle + |R\rangle}{\sqrt{2}}$ (coherent superposition, Synthesis),

- $|\mathbf{HoloTh}; P\rangle$ corresponds to the inapplicability diagnosis at the interference locus (Holothesis).

This illustrates how the abstract logical construction maps to operational quantum states when a realization is available [8, 56, 63].

4.4 Gram Matrix and Automatic Positivity

Definition 4.12 (Corner Gram matrix)**.** For the corner kets (24), define the 4×4 Gram matrix $G(P) = (G_{\bullet\bullet'}(P))$ by

$$G_{\bullet\bullet'}(P) := \langle \bullet; P \mid \bullet'; P \rangle_\Lambda = \left\langle [C_\bullet(P)]_{\omega_\Lambda} \mid [C_{\bullet'}(P)]_{\omega_\Lambda} \right\rangle_\Lambda = \omega_\Lambda\!\left(C_\bullet(P)^\dagger C_{\bullet'}(P)\right), \quad (25)$$

where $\bullet, \bullet' \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$. This construction follows the standard theory of Gram matrices in inner product spaces [71, 100].

Proposition 4.13 (Automatic positive semidefiniteness)**.** *For every atomic corner context P, the Gram matrix $G(P)$ of Definition 4.12 is Hermitian and positive semidefinite. This is a standard result for Gram matrices induced by positive functionals on $*$-algebras [17, 19, 94].*

Proof. **Hermiticity.** For any $X \in \mathcal{A}_\Lambda^{\text{alg}}$, a positive linear functional satisfies $\omega_\Lambda(X^\dagger) = \overline{\omega_\Lambda(X)}$ [19, 94]. Thus

$$G_{\bullet\bullet'}(P) = \omega_\Lambda\!\left(C_\bullet(P)^\dagger C_{\bullet'}(P)\right) = \overline{\omega_\Lambda\!\left((C_\bullet(P)^\dagger C_{\bullet'}(P))^\dagger\right)} = \overline{\omega_\Lambda\!\left(C_{\bullet'}(P)^\dagger C_\bullet(P)\right)} = \overline{G_{\bullet'\bullet}(P)},$$

so $G(P)$ is Hermitian [17, 71].

Positive semidefiniteness. Let $(c_\bullet)_{\bullet \in \mathbf{Status}_\Lambda}$ be any complex coefficients and set $A := \sum_\bullet c_\bullet C_\bullet(P) \in \mathcal{A}_\Lambda^{\text{alg}}$. Then by linearity of ω_Λ and the definition of the Gram matrix entries:

$$\begin{aligned}
\sum_{\bullet,\bullet'} \overline{c_\bullet} \, c_{\bullet'} \, G_{\bullet\bullet'}(P) &= \sum_{\bullet,\bullet'} \overline{c_\bullet} \, c_{\bullet'} \, \omega_\Lambda\!\left(C_\bullet(P)^\dagger C_{\bullet'}(P)\right) \\
&= \omega_\Lambda\!\left(\left(\sum_\bullet c_\bullet C_\bullet(P)\right)^\dagger \left(\sum_{\bullet'} c_{\bullet'} C_{\bullet'}(P)\right)\right) \\
&= \omega_\Lambda(A^\dagger A).
\end{aligned} \quad (26)$$

By the positivity axiom of the warrant functional (Definition 3.25), $\omega_\Lambda(A^\dagger A) \geq 0$ for all $A \in \mathcal{A}_\Lambda^{\text{alg}}$ [7, 17, 19]. Hence $G(P) \succeq 0$. $\square$

Remark 4.14 (Relation to quantum state tomography)**.** The Gram matrix $G(P)$ encodes the mutual overlaps of the four corner kets in the logical GNS space. In quantum information theory, such Gram matrices appear in state discrimination and tomography problems [59, 101]. The rank of $G(P)$ determines the dimension of the logical modal subspace $\mathcal{H}_L(P)$ (Proposition 4.16). In the sharp quantum realization limit (Tier 2 enrichment, Definition 3.33), $G(P)$ may become diagonal with entries corresponding to Born probabilities [56, 63].

Remark 4.15 (Paraconsistent and paracomplete signatures)**.** The off-diagonal entries of $G(P)$ carry logical information beyond mere probabilities:

- **Paraconsistent regime (Synthesis):** Nonzero off-diagonal entries between Thesis and Antithesis corners indicate coherent co-holding without orthogonality, matching the paraconsistent "glut" semantics (Definition 2.11, Section 2.13) [54, 55].

- **Paracomplete regime (Holothesis):** Vanishing or ill-defined entries involving Holothesis corners reflect the typed inapplicability of polar questions (Definition 2.12, Section 2.14).

- **Classical limit:** In a sharp classical realization, $G(P)$ becomes diagonal with a single nonzero entry (exclusivity), recovering standard Boolean logic [2, 84].

Thus the Gram matrix serves as a quantitative witness of the underlying logical regime [8, 9].

4.5 The Logical Modal Subspace

Proposition 4.16 (Dimension and faithfulness on the corner span)**.** *Let P be an atomic corner context, and let $G(P)$ be its Gram matrix (Definition 4.12). Then:*

(a) $\dim \mathcal{H}_L(P) = \operatorname{rank} G(P) \leq 4.$

(b) If $G(P)$ is positive definite (equivalently, has no nontrivial null directions), then the four corner kets are linearly independent and $\dim \mathcal{H}_L(P) = 4$.

Proof. By construction, $\mathcal{H}_L(P)$ is spanned by four vectors, so $\dim \mathcal{H}_L(P) \leq 4$. Moreover, the Gram matrix rank equals the dimension of the span of the generating vectors. If $G(P)$ is positive definite, the only linear combination with zero norm is the trivial one, hence the generating vectors are independent and the span is four-dimensional. $\square$

Remark 4.17 (Faithfulness vs. positive definiteness on a subspace)**.** Faithfulness of ω_Λ on the *whole* algebra implies that $\mathcal{N}_{\omega_\Lambda} = \{0\}$, but one can also work with the weaker, local requirement that no nonzero linear combination $\sum_\bullet c_\bullet C_\bullet(P)$ lies in $\mathcal{N}_{\omega_\Lambda}$. This is precisely the condition that $G(P)$ is positive definite on the corner span. In physical realizations (e.g. the double-slit), symmetries or logical relations may reduce this dimension (e.g. to 2 for sharp which-path contexts) [17, 19].

4.5.1 Ontological Corner-Ket Notation

Remark 4.18 (Ontological corner-ket notation)**.** To facilitate the representation of the tetralemmatic structure within the logical Hilbert template, we introduce primitive ontological state

symbols as labels for the corner kets in $\mathcal{H}_L(P)$. These symbols are defined directly via their correspondence to the four logical corners:

$$|\mathbf{Th}; P\rangle \equiv |\mathring{\mathbf{0}}\rangle, \qquad |\mathbf{SynTh}; P\rangle \equiv |\mathring{\mathbf{0}} \, \bar{\bar{\wedge}} \, \mathring{\mathbf{1}}\rangle,$$

$$|\mathbf{AntiTh}; P\rangle \equiv |\mathring{\mathbf{1}}\rangle, \qquad |\mathbf{HoloTh}; P\rangle \equiv |\mathring{\mathbf{0}} \, \overset{\infty}{\wedge} \, \mathring{\mathbf{1}}\rangle. \tag{27}$$

This relabelling is purely notational: the Hilbert space $\mathcal{H}_\Lambda$ and the modal subspace $\mathcal{H}_L(P)$ were constructed from $(\mathcal{A}_\Lambda^{\mathrm{alg}}, \omega_\Lambda)$ via GNS, without assuming any prior physical Hilbert space or any realization of formulas as quantum effects. The ontological state symbols serve as primitive labels for the corner-acts within the logical framework. The interpretation of these symbols as *symmetry modes* (Sym, AntiSym, ParaSym, HoloSym) is reserved for the physical realization layer (cf. Section 6.5), where the bridge map Φ establishes the correspondence between logical corners and algebraic symmetry sectors. This notation emphasizes the structural correspondence between the logical corner-acts and the primitive ontological states of the tetralemmatic lens [8, 9].

Remark 4.19 (Relation to the tetralemmatic lens representation)**.** Under the canonical representation of the atomic tetralemmatic lens Remark 2.21, the logical modal subspace $\mathcal{H}_L(P)$ admits the equivalent representation:

$$\mathcal{H}_L(P) \, \cong \, \mathrm{span}\Big\{|\mathring{\mathbf{0}}\rangle, \, |\mathring{\mathbf{1}}\rangle, \, |\mathring{\mathbf{0}} \, \bar{\bar{\wedge}} \, \mathring{\mathbf{1}}\rangle, \, |\mathring{\mathbf{0}} \, \overset{\infty}{\wedge} \, \mathring{\mathbf{1}}\rangle\Big\}.$$

This makes explicit that the four-corner Hilbert template is the linear span of the four ontological state symbols, with the duality involution $\mathbf{D}_\mathcal{C}$ acting as the swap $|\mathring{\mathbf{0}}\rangle \leftrightarrow |\mathring{\mathbf{1}}\rangle$ while fixing the para- and holosymmetric states [13, 15].

4.6 Born-Valued Semantics of Tetralemmatic Logic

Definition 4.20 (Dialectic truth as Born valuation)**.** Fix a context C and an admissible locus $\lambda \in \Lambda_C$. Let $H_{C,\lambda}$ be the relevant Hilbert space, and let an *effect* on $H_{C,\lambda}$ mean a bounded positive operator E with $0 \leq E \leq I$, where I is the identity on $H_{C,\lambda}$, following the standard framework of generalized quantum measurements [63, 83].

Let

$$M \in \{\mathrm{Th}, \mathrm{AntiTh}, \mathrm{SynTh}, \mathrm{HoloTh}\}.$$

For each corner M (for a chosen operational test in (C, λ)), associate an effect

$$E_M^{C,\lambda} \quad \text{on} \quad H_{C,\lambda}, \qquad 0 \leq E_M^{C,\lambda} \leq I,$$

whenever that test is *admitted* (defined) in (C, λ). For any density operator ρ on $H_{C,\lambda}$, define the *Born-valued dialectic valuation* (degree of warrant) by the standard probability rule [58, 84]:

$$v_M^{C,\lambda}(\rho) \; := \; \mathrm{Tr}\Big(E_M^{C,\lambda} \rho\Big) \; \in [0, 1]. \tag{28}$$

If $E_M^{C,\lambda}$ is a projector and $\rho = |\psi\rangle\langle\psi|$, then $v_M^{C,\lambda}(\rho) = \|E_M^{C,\lambda}|\psi\rangle\|^2$, consistent with the eigenstate-eigenvalue link [59].

Typed admissibility. Valuation is typed by definability: if the relevant test is not admitted in (C, λ), then $E_M^{C,\lambda}$ is undefined and $v_M^{C,\lambda}(\rho)$ is not formed, reflecting quantum contextuality constraints [2, 9]. In particular, for the polar existential proposition

$$\Pi := \exists_\lambda^C x\, \varphi(x),$$

the polar tests corresponding to Π are admitted only when the applicability predicate $\mathrm{App}_C(\Pi; \lambda)$ holds.

(Optional) thresholded holding. Fix $\tau \in (0, 1]$. Define a thresholded holding relation by

$$(C, \lambda, \rho) \models_\tau M \iff E_M^{C,\lambda} \text{ is defined and } v_M^{C,\lambda}(\rho) \geq \tau.$$

Example 4.21 (Concrete realization on a symmetry–antisymmetry sector). Let $H_{(0,1)} = \mathrm{span}\{|e_1\rangle, |e_2\rangle\}$ and define the sharp polar tests

$$E_{\mathrm{Th}} := P_T := |e_1\rangle\langle e_1|, \qquad E_{\mathrm{AntiTh}} := P_A := |e_2\rangle\langle e_2|.$$

For Synthesis, define $|+\rangle := (|e_1\rangle + |e_2\rangle)/\sqrt{2}$ and set

$$E_{\mathrm{SynTh}} := P_S := |+\rangle\langle +|.$$

The Born values of these effects are the corresponding degrees of warrant, illustrating the paraconsistent co-holding regime [13, 55].

5 Multi-loci and Multi-partitite Systems

5.1 The *n*-Loci Thesis

While the primitive *Thesis* is defined at a single locus (Definition 2.7), physical systems often involve composite configurations requiring simultaneous affirmation across multiple loci. To maintain the *logic-first* reconstruction, we define the *n-Loci Thesis* not as a new primitive, but as the coherent conjunction of single-locus Thesis-acts across a configuration of distinct loci. This corresponds to the *Composite-Locus* structure (distinct λ_i) rather than the *Dual-Locus* structure (duplicate λ, reserved for Holothesis in Definition 5.8).

Definition 5.1 (n-Loci Thesis (*Asti; Bhāva; Sat*)). Fix $\mathcal{D}$, $\mathcal{C}$, $\Lambda_\mathcal{C}$, and a *locus configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_\mathcal{C}^n$. For a sequence of formulas $\vec{\varphi} = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$, define the $\mathcal{C}$- and λ_i-relative existential claims for each $i \in \{1, \ldots, n\}$:

$$P_i := \exists_{\lambda_i}^\mathcal{C} x_i\, \varphi_i(x_i).$$

Let $\vec{P} = (P_1, \ldots, P_n)$ denote the vector of claims. A *n-Loci Thesis-act* about $\vec{\varphi}$ at $\vec{\lambda}$ is the structured tuple

$$\mathbf{Th}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) := \left\langle \vec{P},\ \vec{\mathcal{J}}_A(\vec{\varphi}; \mathcal{C}, \vec{\lambda}),\ \vec{\mathcal{J}}_N(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) \right\rangle,$$

where $\vec{\mathcal{J}}_A = (\mathcal{J}_A^{(1)}, \ldots, \mathcal{J}_A^{(n)})$ and $\vec{\mathcal{J}}_N = (\mathcal{J}_N^{(1)}, \ldots, \mathcal{J}_N^{(n)})$ are vectors of dialectical and procedural defenses corresponding to each locus.

We say that the n-Loci Thesis *is warranted* (write $\mathrm{Th}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda})$ holds) when the following conditions obtain:

(i) Ontological affirmation (Composite Being).

For all $i \in \{1, \ldots, n\}$, $\mathrm{App}_{\mathcal{C}}(P_i; \lambda_i)$ holds and $\mathcal{C}, \lambda_i \Vdash P_i$ holds. This is the "It is" commitment relative to every locus in the configuration [13, 46].

(ii) Admissibility and determinacy.

For all i, the predication φ_i is admissible in $\mathcal{C}$, and the reference/individuation conditions are fixed so that each P_i has stable content. Crucially, the loci λ_i are distinct elements of $\Lambda_{\mathcal{C}}$ (forming a composite configuration), ensuring no locus is duplicated (cf. Remark 5.6).

(iii) Dialectical posture (Aristotle, composite).

If $A \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_A^{(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_A(\mathcal{C})$.

(iv) Procedural obligation (Nyāya, composite).

If $N \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_N^{(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_N(\mathcal{C})$.

Equivalently, the n-Loci Thesis is warranted if and only if the single-locus Thesis is warranted at every constituent locus:

$$\mathrm{Th}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) \text{ holds} \iff \bigwedge_{i=1}^{n} \mathrm{Th}(\varphi_i; \mathcal{C}, \lambda_i) \text{ holds.}$$

Logical and Algebraic Significance. Definition 5.1 establishes the *Product Thesis* regime.

- **Logical Conjunction:** The warrant condition is a universal quantification over loci ($\forall i$). If even one locus warrants Antithesis or Holothesis, the composite $\mathrm{Th}^{(n)}$ fails.

- **Algebraic Realization:** In the algebraic kinematics of Section 3, this corresponds to the tensor product of corner generators where every factor is a Thesis generator:

$$C_{\mathbf{Th}^{(n)}}(\vec{P}) = \bigotimes_{i=1}^{n} C_{\mathbf{Th}}(P_i).$$

This structure supports separable states in the Logical Modal Subspace $\bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$ (cf. Remark 5.6).

- **Distinction from Dual-Locus:** Unlike the Holothesis meta-test which requires a *Dual-Locus* space (two copies of the *same* λ) to preserve symmetry (Remark 5.6), the n-Loci Thesis operates on a *Composite-Locus* space (distinct λ_i) to describe multi-partite systems.

5.2 The *n*-Loci Antithesis

While the primitive *Antithesis* is defined at a single locus (Definition 2.8), composite systems require simultaneous denial across multiple loci. To maintain the *logic-first* reconstruction, we define the *n-Loci Antithesis* as the coherent conjunction of single-locus Antithesis-acts across a configuration of distinct loci. This corresponds to the *Composite-Locus* structure (distinct λ_i) rather than the *Dual-Locus* structure (duplicate λ, reserved for Holothesis in Section 5.4).

Definition 5.2 (n-Loci Antithesis (*Nāsti*; *Abhāva*; *Asat*)). Fix $\mathcal{D}, \mathcal{C}, \Lambda_\mathcal{C}$, and a *locus configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_\mathcal{C}^n$. For a sequence of formulas $\vec{\varphi} = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$, define the $\mathcal{C}$- and λ_i-relative existential claims for each $i \in \{1, \ldots, n\}$:

$$P_i := \exists_{\lambda_i}^\mathcal{C} x_i\, \varphi_i(x_i).$$

Let $\neg\vec{P} = (\neg P_1, \ldots, \neg P_n)$ denote the vector of negated claims. A *n-Loci Antithesis-act* about $\vec{\varphi}$ at $\vec{\lambda}$ is the structured tuple

$$\mathbf{AntiTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) := \left\langle \neg\vec{P},\ \vec{\mathcal{J}}_A^\neg(\vec{\varphi}; \mathcal{C}, \vec{\lambda}),\ \vec{\mathcal{J}}_N^\neg(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) \right\rangle,$$

where $\vec{\mathcal{J}}_A^\neg = (\mathcal{J}_A^{\neg,(1)}, \ldots, \mathcal{J}_A^{\neg,(n)})$ and $\vec{\mathcal{J}}_N^\neg = (\mathcal{J}_N^{\neg,(1)}, \ldots, \mathcal{J}_N^{\neg,(n)})$ are vectors of refutational defenses corresponding to each locus.

We say that the n-Loci Antithesis *is warranted* (write $\mathrm{AntiTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda})$ holds) when the following conditions obtain:

(i) Ontological denial (Composite Not-Being).
   For all $i \in \{1, \ldots, n\}$, $\mathrm{App}_\mathcal{C}(\neg P_i; \lambda_i)$ holds and $\mathcal{C}, \lambda_i \Vdash \neg P_i$ holds. This is the "It is not" commitment relative to every locus in the configuration [13, 53].

(ii) Scope and determinacy.
   For all i, the denial is meaningful within $\mathcal{C}$: the locus λ_i is fixed, and the reference/individuation conditions for non-occurrence are specified. Crucially, the loci λ_i are distinct elements of $\Lambda_\mathcal{C}$ (forming a composite configuration), ensuring no locus is duplicated (cf. Remark 5.6).

(iii) Dialectical posture (Aristotle, composite).
   If $A \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_A^{\neg,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_A^\neg(\mathcal{C})$.

(iv) Procedural obligation (Nyāya, composite).
   If $N \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_N^{\neg,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_N^\neg(\mathcal{C})$; this may validate absence using Nyāya norms (e.g. *anupalabdhi*) at each locus [39, 40].

Equivalently, the n-Loci Antithesis is warranted if and only if the single-locus Antithesis is warranted at every constituent locus:

$$\mathrm{AntiTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda})\ \text{holds} \iff \bigwedge_{i=1}^{n} \mathrm{AntiTh}(\varphi_i; \mathcal{C}, \lambda_i)\ \text{holds}.$$

Logical and Algebraic Significance. Definition 5.2 establishes the *Product Antithesis* regime.

- **Logical Conjunction:** The warrant condition is a universal quantification over loci ($\forall i$). If even one locus warrants Thesis or Synthesis, the composite $\mathrm{AntiTh}^{(n)}$ fails. This mirrors the n-Loci Thesis structure but with negated ontological content.

- **Algebraic Realization:** In the algebraic kinematics of Section 3, this corresponds to the tensor product of corner generators where every factor is an Antithesis generator:

$$C_{\mathbf{AntiTh}^{(n)}}(\vec{P}) \;=\; \bigotimes_{i=1}^{n} C_{\mathbf{AntiTh}}(P_i).$$

  This structure supports separable states of exclusion in the Logical Modal Subspace $\bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$ (cf. Remark 5.6).

- **Distinction from Dual-Locus:** Unlike the Holothesis meta-test which requires a *Dual-Locus* space (two copies of the *same* λ) to preserve symmetry (Remark 5.6), the n-Loci Antithesis operates on a *Composite-Locus* space (distinct λ_i) to describe multi-partite systems of absence.

5.3 The *n*-Loci Synthesis

While the primitive *Synthesis* is defined at a single locus (Definition 2.11), composite quantum systems require simultaneous co-commitment across multiple loci. To maintain the *logic-first* reconstruction, we define the *n-Loci Synthesis* as the coherent conjunction of single-locus Synthesis-acts across a configuration of distinct loci. This corresponds to the *Composite-Locus* structure (distinct λ_i) rather than the *Dual-Locus* structure (duplicate λ, reserved for Holothesis in Definition 5.8). This structure supports both separable superpositions and entangled states across the locus configuration.

Definition 5.3 (n-Loci Synthesis (*Asti ca Nāsti ca*; *Sadasat*))**.** Fix $\mathcal{D}$, $\mathcal{C}$, $\Lambda_{\mathcal{C}}$, and a *locus configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$. For a sequence of formulas $\vec{\varphi} = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$, define the $\mathcal{C}$- and λ_i-relative existential claims for each $i \in \{1, \ldots, n\}$:

$$P_i \;:=\; \exists_{\lambda_i}^{\mathcal{C}} x_i \, \varphi_i(x_i).$$

Let $\vec{P} = (P_1, \ldots, P_n)$ and $\neg\vec{P} = (\neg P_1, \ldots, \neg P_n)$ denote the vectors of claims and negations. Assume that $\mathcal{C}$ admits a *paraconsistent* consequence/evaluation regime $\Vdash_{\mathcal{C},\vec{\lambda}}^{\mathrm{pc}}$ applicable to the configuration. A *n-Loci Synthesis-act* about $\vec{\varphi}$ at $\vec{\lambda}$ is the structured tuple

$$\mathbf{SynTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) \;:=\; \Big\langle\, \vec{P}, \; \neg\vec{P}, \; \vec{\mathcal{J}}_A^+, \; \vec{\mathcal{J}}_A^-, \; \vec{\mathcal{J}}_N^+, \; \vec{\mathcal{J}}_N^-, \; \Pi_{\mathrm{pc}}(\vec{P}) \,\Big\rangle,$$

where $\vec{\mathcal{J}}_A^{\pm} = (\mathcal{J}_A^{\pm,(1)}, \ldots, \mathcal{J}_A^{\pm,(n)})$ and $\vec{\mathcal{J}}_N^{\pm} = (\mathcal{J}_N^{\pm,(1)}, \ldots, \mathcal{J}_N^{\pm,(n)})$ are vectors of paired dialectical and procedural defenses corresponding to each locus.

   We say that the n-Loci Synthesis *is warranted* (write $\mathrm{SynTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda})$ holds) when the following conditions obtain:

(i) Composite Co-commitment (Both Being and Not-Being).
    For all $i \in \{1, \ldots, n\}$, $\mathrm{App}_{\mathcal{C}}(P_i; \lambda_i)$ and $\mathrm{App}_{\mathcal{C}}(\neg P_i; \lambda_i)$ hold, and $\mathcal{C}, \lambda_i \Vdash^{\mathrm{pc}} P_i$ as well as $\mathcal{C}, \lambda_i \Vdash^{\mathrm{pc}} \neg P_i$. This is the "both" corner relative to every locus in the configuration [46, 53]. Crucially, the warrant functional ω_{Λ} on the composite algebra may exhibit non-zero off-diagonal correlations between loci (entanglement), provided the local co-commitment conditions are satisfied [13, 55].

(ii) Stability against admissible disambiguations.

For every elementary refinement $r \in \mathsf{R}(\mathcal{C})$, letting $\mathcal{C}_r := r(\mathcal{C})$ and $P_{r,i} := \exists^{\mathcal{C}_r}_{\lambda_i} x_i \, \varphi_i(x_i)$, one still has $\mathrm{App}_{\mathcal{C}_r}(P_{r,i}; \lambda_i)$, $\mathrm{App}_{\mathcal{C}_r}(\neg P_{r,i}; \lambda_i)$, and $\mathcal{C}_r, \lambda_i \Vdash^{\mathrm{pc}} P_{r,i}$ as well as $\mathcal{C}_r, \lambda_i \Vdash^{\mathrm{pc}} \neg P_{r,i}$ for all i. (Intuitively: the composite contradiction is not an artifact removable by any refinement that $\mathcal{C}$ itself recognizes as a legitimate disambiguation.)

(iii) Paraconsistent discipline (nontriviality).

$\Pi_{\mathrm{pc}}(\vec{P})$ witnesses non-explosion for the composite configuration; a minimal proof-theoretic witness is $\exists \psi$ such that $\{\vec{P}, \neg\vec{P}\} \nvdash_{\mathrm{pc}} \psi$ [54, 55].

(iv) Dialectical posture (Aristotle, paired composite).

If $A \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_A^{+,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_A^+(\mathcal{C})$ and $\mathcal{J}_A^{-,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_A^-(\mathcal{C})$.

(v) Procedural obligation (Nyāya, paired composite).

If $N \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_N^{+,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_N^+(\mathcal{C})$ and $\mathcal{J}_N^{-,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_N^-(\mathcal{C})$.

Equivalently, the n-Loci Synthesis is warranted if and only if the single-locus Synthesis is warranted at every constituent locus:

$$\mathrm{SynTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) \text{ holds} \iff \bigwedge_{i=1}^{n} \mathrm{SynTh}(\varphi_i; \mathcal{C}, \lambda_i) \text{ holds.}$$

Logical and Algebraic Significance. Definition 5.3 establishes the *Composite Synthesis* regime.

- **Logical Conjunction:** The warrant condition is a universal quantification over loci ($\forall i$). If even one locus warrants Thesis, Antithesis, or Holothesis, the composite $\mathrm{SynTh}^{(n)}$ fails. This mirrors the n-Loci Thesis/Antithesis structure but with paraconsistent co-commitment.

- **Algebraic Realization:** In the algebraic kinematics of Section 3, this corresponds to the tensor product of corner generators where every factor is a Synthesis generator, *plus* the allowance for non-separable warrant functionals:

$$C_{\mathbf{SynTh}^{(n)}}(\vec{P}) = \bigotimes_{i=1}^{n} C_{\mathbf{SynTh}}(P_i).$$

This structure supports both separable superpositions and *entangled states* (non-zero off-diagonal terms in the Gram matrix across loci) within the Logical Modal Subspace $\bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$ (cf. Remarks 4.15 and 6.8).

- **Distinction from Dual-Locus:** Unlike the Holothesis meta-test which requires a *Dual-Locus* space (two copies of the *same* λ) to preserve symmetry (Remark 5.6), the n-Loci Synthesis operates on a *Composite-Locus* space (distinct λ_i) to describe multi-partite superposition and entanglement.

Definition 5.4 (Dual-Subspace and Control Spaces). Let $(\mathcal{C}, \vec{\lambda})$ be a context–locus configuration pair, where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$ is an *n-loci configuration* (cf. Definitions 5.1 to 5.3). Let $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$ denote the *composite operational space* constructed via the tensor product of local logical modal subspaces:

$$\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)} \; := \; \bigotimes_{i=1}^{n} \mathcal{H}_{L, \lambda_i}(P_{\lambda_i}), \tag{29}$$

where $\mathcal{H}_{L, \lambda_i}(P_{\lambda_i})$ is the logical modal subspace associated with the admissible polar claim P_{λ_i} at locus λ_i (Definition 3.46).

(i) **n-Loci Dual-Subspace.** The *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ is defined as the tensor product of two independent copies of the composite operational space:

$$\mathcal{H}_{\text{Dual}}^{(2n)} \; := \; \mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}, \tag{30}$$

where the two factors are isomorphic ($\mathcal{H}_{(1)}^{(n)} \cong \mathcal{H}_{(2)}^{(n)} \cong \mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$). This structure is structurally required for the *Holothesis meta-test* (Definition 5.8), as it supports the representation of *history superpositions* or *correlations* without forcing either copy into a definite polar state. By distributing the logical status across two copies, the diagnostic preserves the duality symmetry $\mathbf{D}_{\mathcal{C}}$ and ensures that polar claims remain *inapplicable* on the individual subspaces, satisfying the logical condition for Holothesis (Definition 2.12 and Remark 5.6).

(ii) **Ontological Control Space.** The *dyadic control space* $\mathcal{H}_c$ is a fundamental Hilbert space factor isomorphic to the logical dyadic sector ($\mathcal{H}_c \cong \text{span}\{|\mathbf{Th}\rangle, |\mathbf{AntiTh}\rangle\}$). Unlike an auxiliary ancilla, $\mathcal{H}_c$ is *ontological*: it encodes the *logical status* of the system (Polar vs. Holothesis) as a quantum degree of freedom.

- *Status Encoding:* State $|0\rangle_c$ corresponds to the Polar context (admitting Thesis/Antithesis/Synthesis). State $|1\rangle_c$ corresponds to the Holothesis context (meta-test mode).

- *Physical Realization:* In experimental setups, this logical status may be *encoded* in the internal symmetry sectors of the *n*-loci dual-subspace (e.g., symmetric vs. antisymmetric sectors of $\mathcal{H}^{(n)} \otimes \mathcal{H}^{(n)}$). This demonstrates that the control degree of freedom is an inherent consequence of the swap symmetry between registers, rather than requiring a separate physical ancilla wire. This encoding maintains the logic-first reconstruction while providing experimental accessibility.

- *Mutually Unbiased Basis:* $\mathcal{H}_c$ is equipped with a mutually unbiased basis (MUB) relative to the polar basis. The preparation state $|a\rangle_c$ and readout state $|b\rangle_c$ are chosen from this conjugate basis to ensure the test probes *complementarity* (contextual status) rather than polarity (Thesis/Antithesis).

Remark 5.5 (Operational significance of the n-loci dual-subspace). The introduction of the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ in Definition 5.4 serves three critical functions within the tetralemmatic framework:

(i) **Preservation of Inapplicability.** By distributing the system state across two independent subspaces (e.g., via a purification or copying map $\rho \mapsto \rho \otimes \rho$), the meta-test avoids interacting with a single polar degree of freedom. This ensures that the polar claims Π and $\neg\Pi$ remain *inapplicable* on the individual subspaces, satisfying the logical condition for Holothesis (Definition 2.12).

(ii) **Interference of Histories.** The tensor structure $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}} \otimes \mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$ allows the unitary U_{holo} to generate entanglement between the subspaces. This operationalizes the *superposition of histories* inherent in the Holothesis sector (Remark 6.141), distinguishing it from the single-register superposition of Synthesis (Remark 6.143).

(iii) **Complementary Diagnosis.** The dyadic control space (Definition 5.4 item (ii)) ensures that the measurement basis is mutually unbiased to the polar basis. This guarantees that the diagnostic extracts information about the *contextual status* (Holothesis) without revealing path information (Thesis/Antithesis), consistent with Bohr's complementarity principle [11].

Remark 5.6 (Structural necessity of the dual-subspace)**.** The use of an n-loci dual-subspace $\mathcal{H}^{(2n)}_{\text{Dual}} \cong \mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}} \otimes \mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$ is structurally required by the *dyadic ontology* of the tetralemmatic framework (Axiom 2.22).

(i) **Symmetry Preservation:** Holothesis diagnoses the *inapplicability* of the Thesis/Antithesis dyad. A single-subspace diagnostic risks breaking the duality symmetry $\mathbf{D}_{\mathcal{C}}$ by privileging a polar basis. The dual-subspace allows the meta-test unitary U_{holo} to act on the *correlation* between poles, ensuring the effect $E^{(a \to b)}_{\text{HoloTh}}$ remains invariant under slit-exchange (duality).

(ii) **Potentiality Encoding:** The tensor structure supports the encoding of *history superpositions* as entanglement between subspaces. This aligns with the ontological status of Holothesis as *logical transcendence* Remark 6.141), where paths exist as potential correlations rather than definite occupancies.

Thus, the dual-subspace is not merely an ancilla choice but an operational reflection of the logical dyad itself. Note that while Thesis, Antithesis, and Synthesis operate on the single composite space $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$ (cf. Definitions 5.1 to 5.3), the Holothesis meta-test necessitates this doubled structure.

5.4 The *n*-Loci Holothesis

While the primitive *Holothesis* is defined at a single locus (Definition 2.12), composite quantum systems require simultaneous inapplicability across multiple loci. To maintain the *logic-first* reconstruction, we define the *n-Loci Holothesis* as the coherent conjunction of single-locus Holothesis-acts across a configuration of distinct loci. Crucially, the operational diagnosis of n-Loci Holothesis necessitates the *n-Loci Dual-Subspace* structure (Definition 5.4), as a single-register diagnostic would risk collapsing the polar inapplicability that defines this corner.

Definition 5.7 (n-Loci Holothesis (*Naiva Asti na Nāsti*))**.** Fix $\mathcal{D}$, $\mathcal{C}$, $\Lambda_{\mathcal{C}}$, and a *locus configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$. For a sequence of formulas $\vec{\varphi} = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$, define the $\mathcal{C}$- and λ_i-relative existential claims for each $i \in \{1, \ldots, n\}$:

$$P_i := \exists_{\lambda_i}^{\mathcal{C}} x_i \, \varphi_i(x_i).$$

Let $\vec{P} = (P_1, \ldots, P_n)$ denote the vector of claims. A *n-Loci Holothesis-act* about $\vec{\varphi}$ at $\vec{\lambda}$ is the structured tuple

$$\mathbf{HoloTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) := \left\langle \neg \vec{\mathrm{App}}, \ \vec{\mathcal{J}}_A^{\perp}, \ \vec{\mathcal{J}}_N^{\perp}, \ \mathcal{H}_{\mathrm{Dual}}^{(2n)} \right\rangle,$$

where $\neg \vec{\mathrm{App}} = (\neg \mathrm{App}_{\mathcal{C}}(P_1; \lambda_1), \ldots, \neg \mathrm{App}_{\mathcal{C}}(P_n; \lambda_n))$ is the vector of inapplicability judgments, $\vec{\mathcal{J}}_A^{\perp}$ and $\vec{\mathcal{J}}_N^{\perp}$ are vectors of meta-dialectical defenses corresponding to each locus, and $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ is the n-loci dual-subspace (Definition 5.4) required for the Holothesis meta-test.

   We say that the n-Loci Holothesis *is warranted* (write $\mathrm{HoloTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda})$ holds) when the following conditions obtain:

(i) Composite Polar Inapplicability (Neither Being nor Not-Being).
   For all $i \in \{1, \ldots, n\}$, $\neg \mathrm{App}_{\mathcal{C}}(P_i; \lambda_i)$ holds (equivalently, by negation-closure, $\neg \mathrm{App}_{\mathcal{C}}(\neg P_i; \lambda_i)$). This is the "neither" corner relative to every locus in the configuration [44, 46].

(ii) Stability under admissible refinements.
   For every elementary refinement $r \in \mathsf{R}(\mathcal{C})$, letting $\mathcal{C}_r := r(\mathcal{C})$ and $P_{r,i} := \exists_{\lambda_i}^{\mathcal{C}_r} x_i \, \varphi_i(x_i)$, one has $\neg \mathrm{App}_{\mathcal{C}_r}(P_{r,i}; \lambda_i)$ for all i. (Intuitively: no refinement that $\mathcal{C}$ itself recognizes as legitimate restores existential polarity as a coherent predicative move at any locus in the configuration.) This stability condition parallels the contextuality constraints in quantum logic where certain observables remain undefined across compatible refinements [9, 11].

(iii) Dialectical posture (Aristotle, meta composite).
   If $A \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_A^{\perp,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_A^{\perp}(\mathcal{C})$, i.e. a disciplined meta-argument showing that forcing P_i vs. $\neg P_i$ at each locus commits a category mistake, presupposition failure, or illicit shift in the base of predication [33].

(iv) Procedural obligation (Nyāya, meta composite).
   If $N \in \mathsf{W}(\mathcal{C})$, then for all i, $\mathcal{J}_N^{\perp,(i)}(\varphi_i; \mathcal{C}, \lambda_i) \in \mathrm{Arg}_N^{\perp}(\mathcal{C})$, i.e. a disciplined Nyāya-style meta-justification whose *pratijñā* is an inapplicability claim and whose support makes explicit the failed delimitor/presupposition that $\mathcal{C}$ recognizes [39, 41].

(v) Dual-Subspace Diagnostic Realizability.
   The inapplicability status must be diagnosable via the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ without collapsing polar applicability on the individual subspaces (Definition 5.4). Specifically, there must exist a control space $\mathcal{H}_c$ and a unitary U_{holo} on $\mathcal{H}_c \otimes \mathcal{H}_{\mathrm{Dual}}^{(2n)}$ such that the induced Holothesis effect $E_{\mathrm{HoloTh}}^{(a \to b)}$ (Definition 5.8) yields a non-zero warrant

valuation $v_{\text{HoloTh}} > 0$ while preserving $\neg\,\text{App}_{\mathcal{C}}(P_i; \lambda_i)$ on each factor of $\mathcal{H}_{\text{Dual}}^{(2n)}$. This ensures the Holothesis is an operational meta-claim rather than a mere absence of data.

Optional strengthening (shift witness). If one can exhibit a translation to a stratum where the relevant content is evaluable without reinstating existential polarity as a stable predication, one may strengthen a warranted n-Loci Holothesis by adjoining a lift $\Uparrow\colon \text{Lang}(\mathcal{C}) \to \text{Lang}(\mathcal{C}^{\Uparrow})$ and requiring that $\Uparrow (\vec{\varphi})$ is evaluable in $\mathcal{C}^{\Uparrow}$ while projections back to $\mathcal{C}$ do not restore applicability of P_i vs. $\neg P_i$ at any λ_i. This strengthening is a consequent enrichment, not a prerequisite for asserting inapplicability, and resonates with Bohr's requirement to shift experimental arrangements to define complementary quantities [11].

Equivalently, the n-Loci Holothesis is warranted if and only if the single-locus Holothesis is warranted at every constituent locus *and* the dual-subspace diagnostic is realizable:

$$\text{HoloTh}^{(n)}(\vec{\varphi}; \mathcal{C}, \vec{\lambda}) \text{ holds} \iff \left(\bigwedge_{i=1}^{n} \text{HoloTh}(\varphi_i; \mathcal{C}, \lambda_i) \text{ holds} \right) \wedge \text{ Cond. (v) holds.}$$

Logical and Algebraic Significance. Definition 5.7 establishes the *Composite Holothesis* regime.

- **Logical Conjunction:** The warrant condition is a universal quantification over loci ($\forall i$). If even one locus warrants Thesis, Antithesis, or Synthesis, the composite $\text{HoloTh}^{(n)}$ fails. This mirrors the n-Loci Thesis/Antithesis/Synthesis structures but with paracomplete inapplicability.

- **Dual-Subspace Necessity:** Unlike Thesis, Antithesis, and Synthesis which operate on the single composite space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (cf. Definitions 5.1 to 5.3), Holothesis *requires* the doubled structure $\mathcal{H}_{\text{Dual}}^{(2n)}$. This is because diagnosing inapplicability on a single register forces applicability (collapsing the gap), whereas the dual-subspace allows the meta-test to act on *correlations* without fixing polar occupancies (Remark 5.6).

- **Algebraic Realization:** In the algebraic kinematics, this corresponds to the Holothesis meta-test effect $E_{\text{HoloTh}}^{(a \to b)}$ acting on the dual-subspace, with warrant weight $\omega_{\Lambda}(C_{\textbf{HoloTh}^{(n)}}(\vec{P}))$ diverted from the polar corners. This provides the operational mechanism for *Holothesis Density* as a contextuality resource (Proposition 6.144).

- **Ontological Status:** The n-Loci Holothesis represents *global logical transcendence* across the configuration. It is the formal identity for quantum interference regimes where path questions are ill-typed at every locus simultaneously, distinguishing it from local indeterminacy or ignorance.

Definition 5.8 (Holothesis as a meta-test via Holosymmetric control)**.** Holothesis is characterized logically by polar inapplicability: neither Π nor $\neg\Pi$ is applicable at $(\mathcal{C}, \vec{\lambda})$ (stably under admissible refinements), where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is an *n-loci configuration*. Operationally, this means the *polar* tests need not be admitted in $(\mathcal{C}, \vec{\lambda})$, so one does not assign Born values to $E_{\text{Th}}(\Pi)$ or $E_{\text{AntiTh}}(\Pi)$ there, resonating with Bohr's complementarity principle [11].

Instead, assign Holothesis-warrant by a distinct *meta-level* test, analogous to controlled quantum operations [59, 76]. Let $\mathcal{H}_c$ be a dyadic control space and let $\mathcal{H}_{\text{Dual}}^{(2n)}$ be an n-loci dual-subspace (Definition 5.4). Fix a unitary operator U_{holo} on $\mathcal{H}_c \otimes \mathcal{H}_{\text{Dual}}^{(2n)}$, consistent with Wigner's theorem on symmetry transformations [73]. Prepare the control in $|a\rangle_c$ and read out in $|b\rangle_c$. Define the induced effect on the dual-subspace by

$$E_{\text{HoloTh}}^{(a \to b)} := \langle a|_c \, U_{\text{holo}}^\dagger \, (|b\rangle\langle b|_c \otimes I_{\text{Dual}}) \, U_{\text{holo}} \, |a\rangle_c, \qquad 0 \leq E_{\text{HoloTh}}^{(a \to b)} \leq I_{\text{Dual}},$$

where I_{Dual} is the identity on $\mathcal{H}_{\text{Dual}}^{(2n)}$. For a dual-subspace state ρ_{Dual}, define the Holothesis valuation

$$v_{\text{HoloTh}}(\rho_{\text{Dual}}) := \text{Tr}\left(E_{\text{HoloTh}}^{(a \to b)} \rho_{\text{Dual}} \right).$$

This assigns a Born-valued degree of warrant to the meta-claim that the Holosymmetric diagnostic has been triggered, while remaining consistent with polar inapplicability, implementing a topos-theoretic approach to undefined observables [9].

Remark 5.9 (Mathematical legitimacy of the Holothesis effect)**.** The construction of the induced effect $E_{\text{HoloTh}}^{(a \to b)}$ in Definition 5.8 satisfies the rigorous requirements for a quantum effect operator within the standard framework of generalized measurements [63, 83]. We verify this through type consistency, positivity, and contextual compatibility.

(i) **Type Consistency and Partial Inner Product.** The notation $\langle a|_c \, \mathcal{O} \, |a\rangle_c$, where $\mathcal{O}$ acts on the composite space $\mathcal{H}_c \otimes \mathcal{H}_{\text{Dual}}^{(2n)}$, denotes the *partial inner product* (or partial expectation value) with respect to the control space $\mathcal{H}_c$. Formally, this corresponds to the operation:

$$(\langle a|_c \otimes I_{\text{Dual}}) \, \mathcal{O} \, (|a\rangle_c \otimes I_{\text{Dual}}). \tag{31}$$

Since $\mathcal{O} = U_{\text{holo}}^\dagger \, (|b\rangle\langle b|_c \otimes I_{\text{Dual}}) \, U_{\text{holo}}$ is a bounded operator on $\mathcal{H}_c \otimes \mathcal{H}_{\text{Dual}}^{(2n)}$, the mapping in eq. (31) yields a bounded operator on the dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$. Thus, $E_{\text{HoloTh}}^{(a \to b)} \in \mathcal{B}(\mathcal{H}_{\text{Dual}}^{(2n)})$ is correctly typed.

(ii) **Verification as a Quantum Effect.** For $E_{\text{HoloTh}}^{(a \to b)}$ to constitute a valid effect (POVM element), it must satisfy $0 \leq E \leq I_{\text{Dual}}$.

- *Positivity:* The expression admits a Kraus decomposition of the form $E = M^\dagger M$, where $M = (|b\rangle\langle b|_c \otimes I_{\text{Dual}})^{1/2} \, U_{\text{holo}} \, (|a\rangle_c \otimes I_{\text{Dual}})$. Operators of the form $M^\dagger M$ are inherently positive semi-definite, ensuring $E_{\text{HoloTh}}^{(a \to b)} \geq 0$.

- *Boundedness:* Since $|b\rangle\langle b|_c \leq I_c$, it follows that $(|b\rangle\langle b|_c \otimes I_{\text{Dual}}) \leq I_c \otimes I_{\text{Dual}}$. By the monotonicity of conjugation under the unitary U_{holo} and the partial expectation value:

$$\langle a|_c U_{\text{holo}}^\dagger \, (|b\rangle\langle b|_c \otimes I_{\text{Dual}}) \, U_{\text{holo}} |a\rangle_c \leq \langle a|_c U_{\text{holo}}^\dagger \, (I_c \otimes I_{\text{Dual}}) \, U_{\text{holo}} |a\rangle_c. \tag{32}$$

Utilizing the unitarity condition $U_{\text{holo}}^\dagger U_{\text{holo}} = I_{\text{total}}$ and the normalization $\langle a|a\rangle_c = 1$, the right-hand side reduces to I_{Dual}. Therefore, $E_{\text{HoloTh}}^{(a \to b)} \leq I_{\text{Dual}}$.

(iii) **Contextual Compatibility.** This construction resolves the logical tension inherent in assigning a probability value to the *Holothesis* corner (logical inapplicability) without violating the typed inapplicability of the polar claims defined in Definition 2.12.

- *Meta-Test Mechanism:* While the polar tests $(E_{\mathrm{Th}}, E_{\mathrm{AntiTh}})$ remain undefined when $\neg\,\mathrm{App}_{\mathcal{C}}(\Pi; \vec{\lambda})$ holds (cf. Definition 3.8), the Holothesis effect is defined via a distinct operational arrangement involving the control space $\mathcal{H}_c$. This allows the framework to diagnose the *status* of the context (i.e., verifying the Holosymmetric diagnostic) without forcing the polar claims to become applicable.

- *Consistency with Realization:* This aligns with the Realization Axiom (Section 6.5), which mandates that logical statuses must be associated with operational effects to possess empirical content. The valuation v_{HoloTh} defined in Definition 5.8 thus provides a Born-valued degree of warrant [58, 84] for the meta-claim of inapplicability, consistent with the dialectic Born semantics of Definition 4.20.

- *Dimensional Structure:* The dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ is defined as the tensor product $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (Definition 5.4). This structure ensures that the meta-test unitary U_{holo} acts on correlations between the two copies without collapsing polar degrees of freedom, preserving the duality symmetry $\mathbf{D}_{\mathcal{C}}$ required for Holothesis diagnosis (Remark 5.6).

Consequently, the formula represents a mathematically legitimate POVM element derived from an indirect measurement scheme [59, 76], serving the critical function of operationalizing the Holothesis corner as a measurable meta-claim.

Remark 5.10 (Operational notes). (i) **Composite configurations.** For composite systems, local effects extend by identity on the other tensor factors, following standard quantum information conventions [59]. For example, for a locus λ_i one may write

$$P_T^{(\lambda_i)} := |e_1\rangle\langle e_1|_{\lambda_i} \otimes I_{\mathrm{rest}}.$$

In the case of the Holothesis effect (Definition 5.8), the "local" effect on the dual-subspace is induced via a partial trace over the control space, consistent with the indirect measurement scheme verified in Remark 5.9. Note that while Thesis, Antithesis, and Synthesis operate on the single composite space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$, the Holothesis meta-test necessitates the doubled structure $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definition 5.4).

(ii) **POVM structure and contextual definedness.** The four corner effects correspond to different tests and need not form a single POVM. This arises from two distinct mechanisms:

- *Operational Incompatibility:* Defined effects may fail to commute, reflecting the Kochen–Specker theorem [2, 102].

- *Typed Inapplicability:* In contexts warranting Holothesis, polar effects $(E_{\mathrm{Th}}, E_{\mathrm{AntiTh}})$ are *undefined* rather than non-commuting (cf. Definitions 2.12 and 3.8).

Thus, the collection of admissible effects is locus-dependent, and the Holothesis effect serves as a meta-test for this admissibility boundary (Remark 5.9).

(iii) **Probabilistic interpretation.** The numerical values $v_M^{\mathcal{C},\vec{\lambda}}(\rho)$ are Born probabilities (degrees of warrant), not amplitudes, grounding truth in quantum probability [56, 103]. These values satisfy the positivity and boundedness conditions ($0 \leq v \leq 1$) established for valid quantum effects in Remark 5.9, consistent with the dialectic Born semantics of Definition 4.20.

5.5 Born-Valued Semantics for n-Loci Configurations

Definition 5.11 (Dialectic truth as Born valuation (n-Loci)). Fix a context $\mathcal{C}$ and an admissible *n-loci configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$. Let $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ denote the *composite operational space* constructed via the tensor product of local logical modal subspaces (Definitions 5.1 to 5.3):

$$\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} := \bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i}). \tag{33}$$

Let $\mathcal{H}_{\text{Dual}}^{(2n)}$ denote the *n-loci dual-subspace* required for the Holothesis meta-test (Definition 5.4):

$$\mathcal{H}_{\text{Dual}}^{(2n)} := \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}. \tag{34}$$

Let an *effect* on a Hilbert space $\mathcal{H}$ mean a bounded positive operator E with $0 \leq E \leq I_{\mathcal{H}}$, following the standard framework of generalized quantum measurements [63, 83].

Let

$$M \in \{\text{Th}, \text{AntiTh}, \text{SynTh}, \text{HoloTh}\}.$$

For each corner M, associate an effect $E_M^{(\mathcal{C},\vec{\lambda})}$ as follows:

(i) **Polar and Synthesis Corners ($M \in \{\text{Th}, \text{AntiTh}, \text{SynTh}\}$):** The effect acts on the composite operational space:

$$E_M^{(\mathcal{C},\vec{\lambda})} \quad \text{on} \quad \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}, \qquad 0 \leq E_M^{(\mathcal{C},\vec{\lambda})} \leq I_{\mathcal{H}^{(n)}},$$

whenever that test is *admitted* (defined) in $(\mathcal{C}, \vec{\lambda})$.

(ii) **Holothesis Corner ($M = \text{HoloTh}$):** The effect acts on the n-loci dual-subspace via the meta-test construction (Definition 5.8):

$$E_{\text{HoloTh}}^{(\mathcal{C},\vec{\lambda})} \quad \text{on} \quad \mathcal{H}_{\text{Dual}}^{(2n)}, \qquad 0 \leq E_{\text{HoloTh}}^{(\mathcal{C},\vec{\lambda})} \leq I_{\text{Dual}}.$$

For any density operator ρ on the respective space ($\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ for Th/AntiTh/SynTh, or $\mathcal{H}_{\text{Dual}}^{(2n)}$ for HoloTh), define the *Born-valued dialectic valuation* (degree of warrant) by the standard probability rule [58, 84]:

$$v_M^{(\mathcal{C},\vec{\lambda})}(\rho) := \text{Tr}\left(E_M^{(\mathcal{C},\vec{\lambda})} \rho \right) \in [0,1]. \tag{35}$$

If $E_M^{(\mathcal{C}, \vec{\lambda})}$ is a projector and $\rho = |\psi\rangle\langle\psi|$, then $v_M^{(\mathcal{C}, \vec{\lambda})}(\rho) = \|E_M^{(\mathcal{C}, \vec{\lambda})}|\psi\rangle\|^2$, consistent with the eigenstate-eigenvalue link [59].

Typed admissibility. Valuation is typed by definability: if the relevant test is not admitted in $(\mathcal{C}, \vec{\lambda})$, then $E_M^{(\mathcal{C}, \vec{\lambda})}$ is undefined and $v_M^{(\mathcal{C}, \vec{\lambda})}(\rho)$ is not formed, reflecting quantum contextuality constraints [2, 9]. In particular, for the polar existential proposition

$$\Pi := \exists_{\vec{\lambda}}^{\mathcal{C}} x \, \varphi(x),$$

the polar tests corresponding to Π are admitted only when the applicability predicate $\mathrm{App}_{\mathcal{C}}(\Pi; \vec{\lambda})$ holds. Crucially, when HoloTh is warranted, the polar effects $E_{\mathrm{Th}}^{(\mathcal{C}, \vec{\lambda})}$ and $E_{\mathrm{AntiTh}}^{(\mathcal{C}, \vec{\lambda})}$ are *undefined*, while $E_{\mathrm{HoloTh}}^{(\mathcal{C}, \vec{\lambda})}$ is defined via the meta-test.

(Optional) thresholded holding. Fix $\tau \in (0, 1]$. Define a thresholded holding relation by

$$(\mathcal{C}, \vec{\lambda}, \rho) \models_{\tau} M \iff E_M^{(\mathcal{C}, \vec{\lambda})} \text{ is defined and } v_M^{(\mathcal{C}, \vec{\lambda})}(\rho) \geq \tau.$$

Example 5.12 (Concrete realization on a symmetry–antisymmetry sector (n-Loci))**.** Let $\mathcal{H}_{(0,1)}^{(n)} = \mathrm{span}\{|e_1\rangle, |e_2\rangle\}^{\otimes n}$ and define the sharp polar tests for a single locus ($n = 1$) as:

$$E_{\mathrm{Th}} := P_T := |e_1\rangle\langle e_1|, \qquad E_{\mathrm{AntiTh}} := P_A := |e_2\rangle\langle e_2|.$$

For Synthesis, define $|+\rangle := (|e_1\rangle + |e_2\rangle)/\sqrt{2}$ and set

$$E_{\mathrm{SynTh}} := P_S := |+\rangle\langle +|.$$

The Born values of these effects are the corresponding degrees of warrant, illustrating the paraconsistent co-holding regime [13, 55]. For $n > 1$, these effects extend via tensor products (e.g., $P_T^{(n)} = P_T \otimes \cdots \otimes P_T$) to represent composite Thesis states.

5.6 Duality is Unitary

In the purely logical construction, duality enters as a $*$-automorphism $\alpha_{\mathbf{D}} : \mathcal{A}_{\Lambda}^{\mathrm{alg}} \to \mathcal{A}_{\Lambda}^{\mathrm{alg}}$ together with the **D**-invariance of the warrant functional ω_{Λ}. These two facts *force* a unitary implementation of duality on the GNS Hilbert space. This is a standard result in the theory of operator algebras, where state-preserving automorphisms are implemented by unitaries in the GNS representation [7, 16, 17, 19]. The construction ensures that logical symmetries (such as duality) manifest as physical symmetries (unitaries) on the Hilbert space [92].

Remark 5.13 (Topological Status of the Logical Algebra)**.** The algebra $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$ is defined in Section 3 as a purely algebraic unital $*$-algebra without presupposed norm or topology. The GNS representation $\pi_{\Lambda} : \mathcal{A}_{\Lambda}^{\mathrm{alg}} \to \mathcal{B}(\mathcal{H}_{\Lambda})$ induces topological structure via the operator norm on $\mathcal{B}(\mathcal{H}_{\Lambda})$:

- C^***-Completion:** The norm-closure of $\pi_{\Lambda}(\mathcal{A}_{\Lambda}^{\mathrm{alg}})$ in $\mathcal{B}(\mathcal{H}_{\Lambda})$ yields a **concrete C^*-algebra** $\mathfrak{A}_{\Lambda} := \overline{\pi_{\Lambda}(\mathcal{A}_{\Lambda}^{\mathrm{alg}})}^{\|\cdot\|}$.

- **Von Neumann Completion:** The weak-operator-topology closure (double commutant) yields the **von Neumann algebra** $\mathfrak{M}_\Lambda := \pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})''$.

For the purposes of this reconstruction, we work with the algebraic $*$-algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$ and its GNS representation, deferring C^*-enveloping algebra considerations to future work on dynamics and thermodynamic limits. This maintains the "logic-first" purity of the derivation while ensuring mathematical rigor at the kinematic level [7, 95].

Remark 5.14 (Applicability to n-Loci Configurations)**. The unitary implementation result holds for both single-locus and n-loci configurations.** The algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$ is generated by corner-observable generators $C_\bullet(P)$ where P ranges over all admissible claims, including n-loci claims $P_{\vec{\lambda}}$ (Definitions 5.1 to 5.3 and 5.7). The duality automorphism $\alpha_\mathbf{D}$ acts component-wise on n-loci claims:

$$\alpha_\mathbf{D}\big(C_\bullet(P_{\vec{\lambda}})\big) \;=\; C_{\Delta_\mathcal{C}(\bullet)}(P_{\vec{\lambda}}^\mathbf{D}),$$

where $P_{\vec{\lambda}}^\mathbf{D}$ denotes the dual claim configuration. Consequently, the unitary $U_\mathbf{D}$ acts on the full GNS Hilbert space $\mathcal{H}_\Lambda$, which includes both the *composite operational space* $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis) and the *n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (for Holothesis meta-tests) (Definition 5.4). This ensures that duality symmetry is preserved across all logical corners and all locus configurations.

5.7 Unitary implementer from $(\alpha_\mathbf{D}, \omega_\Lambda)$

Definition 5.15 (Invariant warrant functional)**.** We recall that ω_Λ is **D**-*invariant* if

$$\omega_\Lambda \circ \alpha_\mathbf{D} \;=\; \omega_\Lambda, \quad \text{i.e.} \quad \omega_\Lambda(\alpha_\mathbf{D}(A)) = \omega_\Lambda(A) \;\; \text{for all } A \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

This condition is the algebraic analogue of a symmetry-invariant state in quantum physics [7, 16].

Proposition 5.16 (Duality is unitarily implementable on $\mathcal{H}_\Lambda$)**.** *Let* $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ *be the GNS triple of* $(\mathcal{A}_\Lambda^{\mathrm{alg}}, \omega_\Lambda)$, *and let* $\alpha_\mathbf{D}$ *be a $*$-automorphism of* $\mathcal{A}_\Lambda^{\mathrm{alg}}$ *such that* $\omega_\Lambda \circ \alpha_\mathbf{D} = \omega_\Lambda$. *Then there exists a unique unitary operator*

$$U_\mathbf{D} : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$$

such that:

(a) Implementation (covariance):
The unitary operator $U_\mathbf{D}$ intertwines the representation π_Λ and its transform by $\alpha_\mathbf{D}$, implementing the automorphism on the Hilbert space level:

$$U_\mathbf{D}\, \pi_\Lambda(A)\, U_\mathbf{D}^\dagger \;=\; \pi_\Lambda(\alpha_\mathbf{D}(A)) \qquad \text{for all } A \in \mathcal{A}_\Lambda^{\mathrm{alg}}. \tag{36}$$

(b) Vacuum-fixing:
The unitary preserves the GNS cyclic vector, identifying it as the physical state invariant under the duality transformation:

$$U_\mathbf{D}\, |\Omega_\Lambda\rangle \;=\; |\Omega_\Lambda\rangle. \tag{37}$$

This result is the algebraic counterpart to Wigner's theorem on symmetry transformations [104, 105], adapted to the GNS context [17, 19].

Proof. Define first an operator $U_{\mathbf{D}}^{(0)}$ on the dense subspace $\pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})|\Omega_\Lambda\rangle \subset \mathcal{H}_\Lambda$ by

$$U_{\mathbf{D}}^{(0)}\big(\pi_\Lambda(A)|\Omega_\Lambda\rangle\big) \ := \ \pi_\Lambda(\alpha_{\mathbf{D}}(A))|\Omega_\Lambda\rangle, \qquad A \in \mathcal{A}_\Lambda^{\mathrm{alg}}. \tag{38}$$

We must check that this is well-defined and isometric. If $\pi_\Lambda(A)|\Omega_\Lambda\rangle = 0$, then $A \in \mathcal{N}_{\omega_\Lambda}$, i.e. $\omega_\Lambda(A^\dagger A) = 0$. By $*$-automorphism and invariance,

$$\omega_\Lambda\big(\alpha_{\mathbf{D}}(A)^\dagger \alpha_{\mathbf{D}}(A)\big) = \omega_\Lambda\big(\alpha_{\mathbf{D}}(A^\dagger A)\big) = \omega_\Lambda(A^\dagger A) = 0,$$

so $\alpha_{\mathbf{D}}(A) \in \mathcal{N}_{\omega_\Lambda}$ and hence $\pi_\Lambda(\alpha_{\mathbf{D}}(A))|\Omega_\Lambda\rangle = 0$. Thus (38) is well-defined on the quotient space [7, 16].

Next, for $A, B \in \mathcal{A}_\Lambda^{\mathrm{alg}}$ we compute inner products using the GNS rule:

$$\begin{aligned}
\Big\langle U_{\mathbf{D}}^{(0)}\big(\pi_\Lambda(A)|\Omega_\Lambda\rangle\big) \,\Big|\, U_{\mathbf{D}}^{(0)}\big(\pi_\Lambda(B)|\Omega_\Lambda\rangle\big)\Big\rangle &= \Big\langle \pi_\Lambda(\alpha_{\mathbf{D}}(A))|\Omega_\Lambda\rangle \,\Big|\, \pi_\Lambda(\alpha_{\mathbf{D}}(B))|\Omega_\Lambda\rangle\Big\rangle \\
&= \omega_\Lambda\big(\alpha_{\mathbf{D}}(A)^\dagger \alpha_{\mathbf{D}}(B)\big) \\
&= \omega_\Lambda\big(\alpha_{\mathbf{D}}(A^\dagger B)\big) \\
&= \omega_\Lambda(A^\dagger B) \\
&= \Big\langle \pi_\Lambda(A)|\Omega_\Lambda\rangle \,\Big|\, \pi_\Lambda(B)|\Omega_\Lambda\rangle\Big\rangle.
\end{aligned}$$

Hence $U_{\mathbf{D}}^{(0)}$ preserves inner products on a dense subspace, so it extends uniquely by continuity to an isometry $U_{\mathbf{D}}$ on all of $\mathcal{H}_\Lambda$ [17, 19]. Since $\alpha_{\mathbf{D}}$ is an automorphism, the same construction using $\alpha_{\mathbf{D}}^{-1}$ yields an isometric inverse, so $U_{\mathbf{D}}$ is unitary [16].

Finally, (37) follows from (38) with $A = \mathbf{1}$:

$$U_{\mathbf{D}}|\Omega_\Lambda\rangle = U_{\mathbf{D}}^{(0)}\big(\pi_\Lambda(\mathbf{1})|\Omega_\Lambda\rangle\big) = \pi_\Lambda(\alpha_{\mathbf{D}}(\mathbf{1}))|\Omega_\Lambda\rangle = \pi_\Lambda(\mathbf{1})|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle.$$

To obtain (36), compute on the dense set $\pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})|\Omega_\Lambda\rangle$:

$$\begin{aligned}
U_{\mathbf{D}}\, \pi_\Lambda(A)\, U_{\mathbf{D}}^\dagger\big(\pi_\Lambda(B)|\Omega_\Lambda\rangle\big) &= U_{\mathbf{D}}\, \pi_\Lambda(A)\big(\pi_\Lambda(\alpha_{\mathbf{D}}^{-1}(B))|\Omega_\Lambda\rangle\big) \\
&= U_{\mathbf{D}}\big(\pi_\Lambda(A\,\alpha_{\mathbf{D}}^{-1}(B))|\Omega_\Lambda\rangle\big) \\
&= \pi_\Lambda\big(\alpha_{\mathbf{D}}(A\,\alpha_{\mathbf{D}}^{-1}(B))\big)|\Omega_\Lambda\rangle \\
&= \pi_\Lambda\big(\alpha_{\mathbf{D}}(A)\,B\big)|\Omega_\Lambda\rangle \\
&= \pi_\Lambda(\alpha_{\mathbf{D}}(A))\,\pi_\Lambda(B)|\Omega_\Lambda\rangle,
\end{aligned}$$

which implies (36) by density. Uniqueness follows from cyclicity: any unitary satisfying (36) and (37) agrees with $U_{\mathbf{D}}$ on $\pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})|\Omega_\Lambda\rangle$, hence everywhere [7, 17]. $\qquad\square$

Remark 5.17 (Physical interpretation of the unitary implementer)**.** The unitary $U_{\mathbf{D}}$ represents the physical symmetry corresponding to the logical duality $\alpha_{\mathbf{D}}$. In the double-slit realization (Section 2.12.3), $U_{\mathbf{D}}$ corresponds to the slit-exchange operator. The vacuum-fixing condition (37) reflects the fact that the warrant functional (the "state of knowledge" or "preparation") is symmetric under duality [16, 92].

For n-loci configurations, $U_{\mathbf{D}}$ acts as a tensor product of local duality unitaries on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$, preserving the duality symmetry across all loci. **For the Holothesis meta-test, $U_{\mathbf{D}}$ acts on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$** by exchanging the two copies while preserving the correlation structure required for the meta-test (Definition 5.4 and Remark 5.6). This ensures that the Holothesis effect $E_{\text{HoloTh}}^{(a \to b)}$ remains invariant under slit-exchange (duality), as required for diagnosing polar inapplicability without collapsing the symmetry.

5.8 Action on Tetralemmatic Corner Kets

Proposition 5.18 (Corner action of $U_{\mathbf{D}}$ on n-Loci Configurations)**.** *Fix an* n-loci corner context $\vec{P}$ *defined over a locus configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ *(cf. Definitions 5.1 to 5.3 and 5.7). Let the associated corner kets be*

$$|\mathbf{Th}; \vec{P}\rangle, \ |\mathbf{AntiTh}; \vec{P}\rangle, \ |\mathbf{SynTh}; \vec{P}\rangle, \ |\mathbf{HoloTh}; \vec{P}\rangle$$

as defined in the logical GNS space $\mathcal{H}_\Lambda$ *(Definition 4.8). Then the duality unitary* $U_{\mathbf{D}}$ *acts as:*

$$U_{\mathbf{D}}|\bullet; \vec{P}\rangle = \left[\alpha_{\mathbf{D}}(C_\bullet(\vec{P}))\right]_{\omega_\Lambda} \qquad (\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}). \qquad (39)$$

In particular, using the defining action of $\alpha_{\mathbf{D}}$ *on generators (Definition 3.22),*

$$U_{\mathbf{D}}|\mathbf{Th}; \vec{P}\rangle = \left[C_{\mathbf{AntiTh}}(\vec{P}^{\mathbf{D}})\right]_{\omega_\Lambda}, \qquad U_{\mathbf{D}}|\mathbf{AntiTh}; \vec{P}\rangle = \left[C_{\mathbf{Th}}(\vec{P}^{\mathbf{D}})\right]_{\omega_\Lambda},$$

$$U_{\mathbf{D}}|\mathbf{SynTh}; \vec{P}\rangle = \left[C_{\mathbf{SynTh}}(\vec{P}^{\mathbf{D}})\right]_{\omega_\Lambda}, \qquad U_{\mathbf{D}}|\mathbf{HoloTh}; \vec{P}\rangle = \left[C_{\mathbf{HoloTh}}(\vec{P}^{\mathbf{D}})\right]_{\omega_\Lambda}. \qquad (40)$$

If moreover $\vec{P}^{\mathbf{D}} = \vec{P}$ *in the chosen setting (e.g. a symmetric n-loci configuration), then these reduce to the schematic swap/fix rules:*

$$U_{\mathbf{D}}|\mathbf{Th}; \vec{P}\rangle = |\mathbf{AntiTh}; \vec{P}\rangle, \qquad U_{\mathbf{D}}|\mathbf{SynTh}; \vec{P}\rangle = |\mathbf{SynTh}; \vec{P}\rangle,$$

$$U_{\mathbf{D}}|\mathbf{AntiTh}; \vec{P}\rangle = |\mathbf{Th}; \vec{P}\rangle, \qquad U_{\mathbf{D}}|\mathbf{HoloTh}; \vec{P}\rangle = |\mathbf{HoloTh}; \vec{P}\rangle. \qquad (41)$$

This implements the corner-level duality involution $\Delta_\mathcal{C}$ *of Axiom 2.22 at the Hilbert space level [16, 17, 92].*

Proof. By Definition 4.8 and Remark 4.9, $|\bullet; \vec{P}\rangle = \pi_\Lambda(C_\bullet(\vec{P}))|\Omega_\Lambda\rangle$. Using Proposition 5.16 and that $U_{\mathbf{D}}|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle$ (vacuum-fixing property of the GNS implementer [7, 19]),

$$U_{\mathbf{D}}|\bullet; \vec{P}\rangle = U_{\mathbf{D}}\pi_\Lambda(C_\bullet(\vec{P}))|\Omega_\Lambda\rangle = \pi_\Lambda(\alpha_{\mathbf{D}}(C_\bullet(\vec{P})))|\Omega_\Lambda\rangle = [\alpha_{\mathbf{D}}(C_\bullet(\vec{P}))]_{\omega_\Lambda},$$

which is (39). The explicit forms (40) follow from the defining action of $\alpha_{\mathbf{D}}$ on generators (Definition 3.22), and (41) follows when $\vec{P}^{\mathbf{D}} = \vec{P}$. This is the standard covariance relation for symmetry implementation in algebraic quantum theory [16, 17]. $\square$

Remark 5.19 (Meaning of "duality becomes unitary"). The phrase *"duality becomes unitary"* means precisely this: duality is unitary *with respect to the GNS inner product induced by the warrant functional* ω_Λ, i.e. the inner product $\langle [A]_{\omega_\Lambda} \mid [B]_{\omega_\Lambda} \rangle_\Lambda = \omega_\Lambda(A^\dagger B)$. No independent (and no "by fiat") choice of inner product is made: once ω_Λ is fixed, the GNS theorem forces the Hilbert space structure and forces the unitary implementer of any ω_Λ-preserving $*$-automorphism such as $\alpha_{\mathbf{D}}$ [7, 16, 18].

This resolves a foundational circularity: in standard quantum mechanics, one typically *postulates* a Hilbert space and then seeks unitary representations of symmetries. Here, the Hilbert space and the unitary are *derived* from the logical algebra and the warrant functional. The unitarity of duality is thus a *theorem* rather than an assumption, contingent only on the duality-invariance of ω_Λ (Definition 3.25) [8, 92].

For n-loci configurations, $U_{\mathbf{D}}$ acts as a tensor product of local duality unitaries on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis), preserving the duality symmetry across all loci. **For the Holothesis meta-test, $U_{\mathbf{D}}$ acts on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$** by exchanging the two copies while preserving the correlation structure required for the meta-test (Definition 5.4 and Remark 5.6). This ensures that the Holothesis effect $E_{\text{HoloTh}}^{(a \to b)}$ remains invariant under slit-exchange (duality), as required for diagnosing polar inapplicability without collapsing the symmetry.

Remark 5.20 (Connection to ontological state symbols). Via the ontological corner-ket notation of Remark 4.18, the action (41) corresponds to the duality action on ontological state symbols:

$$U_{\mathbf{D}}|\overset{\circ}{\mathbf{0}}\rangle = |\overset{\circ}{\mathbf{1}}\rangle, \qquad U_{\mathbf{D}}|\overset{\circ}{\mathbf{0}} \overset{=}{\wedge} \overset{\circ}{\mathbf{1}}\rangle = |\overset{\circ}{\mathbf{0}} \overset{=}{\wedge} \overset{\circ}{\mathbf{1}}\rangle,$$

$$U_{\mathbf{D}}|\overset{\circ}{\mathbf{1}}\rangle = |\overset{\circ}{\mathbf{0}}\rangle, \qquad U_{\mathbf{D}}|\overset{\circ}{\mathbf{0}} \overset{\infty}{\wedge} \overset{\circ}{\mathbf{1}}\rangle = |\overset{\circ}{\mathbf{0}} \overset{\infty}{\wedge} \overset{\circ}{\mathbf{1}}\rangle,$$

matching the corner involution $\Delta_{\mathcal{C}}$ of Axiom 2.22. This demonstrates that the logical duality symmetry is faithfully represented in the GNS Hilbert space [7, 9].

5.9 Kinematic Completion: Global Unitary Evolution and Sector Stability

A potential objection to the tetralemmatic framework concerns the apparent change in operational dimensionality between the polar sectors ($\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$) and the Holothesis sector ($\mathcal{H}_{\text{Dual}}^{(2n)}$). In standard quantum mechanics, unitary evolution requires a fixed Hilbert space. We resolve this by adopting a *Monist Architecture* where the global space is a unified tensor product manifold containing all sectors as subspaces.

Definition 5.21 (Global Tetralemmatic Manifold). Let $\mathcal{H}_\Lambda$ be the logical GNS Hilbert space (Definition 4.1). We define the *Global Tetralemmatic Manifold* $\mathcal{H}_{\text{Total}}$ as the tensor product of the ontological control space and the dual-subspace:

$$\mathcal{H}_{\text{Total}} := \mathcal{H}_c \otimes \mathcal{H}_{\text{Dual}}^{(2n)}. \tag{42}$$

(i) **Dimension:** The dimension arises from the logical structure itself. Let $d = \dim(\mathcal{H}_L(P))$ be the local logical modal dimension (Definition 3.46).

- *Generic Four-Corner Case:* In the nondegenerate case where all four corners are linearly independent, $d = 4$. Consequently, $\dim(\mathcal{H}^{(n)}) = 4^n$, $\dim(\mathcal{H}^{(2n)}_{\mathrm{Dual}}) = 4^{2n}$, and $\dim(\mathcal{H}_c) = 2$. Thus, $\dim(\mathcal{H}_{\mathrm{Total}}) = 2 \cdot 4^{2n} = 2^{4n+1}$.

- *Dyadic Reduction:* For the canonical binary logical dyad (where each locus admits only a Thesis/Antithesis distinction), $d = 2$. Consequently, $\dim(\mathcal{H}_{\mathrm{Total}}) = 2 \cdot 2^{2n} = 2^{2n+1}$.

(ii) **Logical Origin of Qubit Structure:** The dyadic reduction dimension (2^{2n+1}) matches that of n physical qubits coupled to a control qubit. However, in this reconstruction, the qubit structure is *derived* from the binary logical dyad via GNS compression (Proposition 5.18), not presupposed from physical spin-1/2 systems. The generic dimension (2^{4n+1}) reflects the full tetralemmatic capacity required to accommodate contextuality (Remark 6.135).

(iii) **Sector Containment:** The Polar Sector $\mathcal{H}_{\mathrm{Polar}}$ is identified with the subspace $\mathrm{span}\{|0\rangle_c\} \otimes \mathcal{H}^{(n)}_{\mathrm{diag}} \subset \mathcal{H}_{\mathrm{Total}}$, where $\mathcal{H}^{(n)}_{\mathrm{diag}}$ is the diagonal subspace of $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$. The Dual Sector $\mathcal{H}_{\mathrm{Dual}}$ corresponds to $\mathrm{span}\{|1\rangle_c\} \otimes \mathcal{H}^{(2n)}_{\mathrm{Dual}}$.

The global state vector $|\Psi\rangle \in \mathcal{H}_{\mathrm{Total}}$ evolves unitarily under a global Hamiltonian H_{Global}.

Proposition 5.22 (Unitary Stability of Sector Transitions). *Let $U(t) = e^{-iH_{\mathrm{Global}}t}$ be the global unitary evolution operator on $\mathcal{H}_{\mathrm{Total}}$. Transitions between logical corners (e.g., Synthesis $\rightarrow$ Holothesis) are mediated by context refinements $\mathcal{C} \rightarrow \mathcal{C}'$ which induce a unitary rotation U_{trans} on $\mathcal{H}_{\mathrm{Total}}$ mapping the Polar subspace to the Dual subspace via the control factor $\mathcal{H}_c$.*

(i) *Norm Preservation: As U_{trans} is unitary on $\mathcal{H}_{\mathrm{Total}}$, probability is conserved globally ($\langle\Psi|\Psi\rangle = 1$).*

(ii) *No Dimensional Change: The apparent "doubling" is not a dynamical generation of degrees of freedom but a rotation within the fixed manifold. For the generic four-corner case ($d = 4$), the dimension is 2^{4n+1}; for the dyadic reduction ($d = 2$), the dimension is 2^{2n+1} (Definition 5.21).*

(iii) *Effective Evolution: The effective evolution within a sector is given by projection onto the control state: $U^{(S)}_{\mathrm{eff}}(t) = (\langle s|_c \otimes I)U(t)(|s\rangle_c \otimes I)$ for $s \in \{0, 1\}$.*

Proof. The proof proceeds in four steps, using only Chapter 2A and early Chapter 2B formalisms to avoid circularity with Chapter 3 dynamical realizations. **Step 1: Dimension of the Control Space $\mathcal{H}_c$.** From Definition 5.4 (Chapter 2B), the ontological control space is defined as:

$$\mathcal{H}_c \cong \mathrm{span}\{|\mathbf{Th}\rangle, |\mathbf{AntiTh}\rangle\}. \tag{43}$$

By Definition 2.19 and Axiom 2.22 (Chapter 2A), the Logical Dyad consists of exactly two primitive statuses: Thesis (**Th**) and Antithesis (**AntiTh**). These are linearly independent by the Axiom of Duality, which establishes them as distinct poles related by the involution $\mathbf{D}_{\mathcal{C}}$. Therefore:

$$\dim(\mathcal{H}_c) = 2. \tag{44}$$

Step 2: Dimension of the Local Logical Modal Subspace $\mathcal{H}_L(P)$**.** From Definition 3.46 (Chapter 2A), the local logical modal subspace is:

$$\mathcal{H}_L(P) := \mathrm{span}\Big\{C_{\mathbf{Th}}(P), C_{\mathbf{AntiTh}}(P), C_{\mathbf{SynTh}}(P), C_{\mathbf{HoloTh}}(P)\Big\} \subseteq \mathcal{H}_\Lambda, \tag{45}$$

where $C_\bullet(P)$ are the corner-observable generators (Definition 3.18, Chapter 2A). By Assumption 3.3 (Chapter 2A), the four corner-acts are available as primitive logical statuses. Their linear independence depends on the warrant functional ω_Λ (Definition 3.35, Chapter 2A):

- **Generic Case:** If ω_Λ is faithful on the corner span (Remark 4.17, Chapter 2B), all four corners are linearly independent, yielding $d = 4$.

- **Dyadic Reduction:** If Tier-2 effect-like enrichment (Definition 3.33, Chapter 2A) imposes exclusivity relations $C_{\mathbf{SynTh}}(P) = C_{\mathbf{HoloTh}}(P) = 0$, only Thesis and Antithesis remain independent, yielding $d = 2$.

Thus, $d \in \{2, 4\}$ depending on the enrichment regime.

Step 3: Dimension of the n-Loci Dual-Subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$**.** From Definition 5.4 (Chapter 2B), the n-loci dual-subspace is:

$$\mathcal{H}_{\mathrm{Dual}}^{(2n)} := \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}, \tag{46}$$

where the composite operational space is:

$$\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} := \bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i}) \quad \text{(Definition 5.4, Chapter 2B).} \tag{47}$$

Since each local subspace has dimension d (Step 2):

$$\dim\Big(\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}\Big) = d^n. \tag{48}$$

Therefore:

$$\dim\Big(\mathcal{H}_{\mathrm{Dual}}^{(2n)}\Big) = d^n \cdot d^n = d^{2n}. \tag{49}$$

Step 4: Total Dimension. Combining Steps 1 and 3:

$$\dim(\mathcal{H}_{\mathrm{Total}}) = \dim(\mathcal{H}_c) \cdot \dim\Big(\mathcal{H}_{\mathrm{Dual}}^{(2n)}\Big) = 2 \cdot d^{2n}. \tag{50}$$

Substituting the two cases for d:

- **Generic four-corner case** $(d = 4)$**:**

$$\dim(\mathcal{H}_{\mathrm{Total}}) = 2 \cdot 4^{2n} = 2 \cdot (2^2)^{2n} = 2 \cdot 2^{4n} = 2^{4n+1}. \tag{51}$$

- **Dyadic reduction** $(d = 2)$**:**

$$\dim(\mathcal{H}_{\mathrm{Total}}) = 2 \cdot 2^{2n} = 2^{2n+1}. \tag{52}$$

This completes the proof. $\square$

Remark 5.23 (Resolution of the Dimensionality Critique)**.** Proposition 5.22 resolves the concern that the framework violates unitarity:

- **Fixed Global Space:** The fundamental space $\mathcal{H}_{\text{Total}}$ is fixed at dimension d^{2n+1} where $d = \dim(\mathcal{H}_L(P))$. For the generic four-corner case ($d = 4$), this yields 2^{4n+1}; for the dyadic reduction ($d = 2$), this yields 2^{2n+1} (Definition 5.21). All sectors are subspaces of this single manifold.

- **Symmetry-Protected Confinement:** While $\dim(\mathcal{H}_{\text{Dual}}^{(2n)}) = d_n^2$ scales quadratically, the Holothesis meta-test dynamics are confined to a *symmetry-protected invariant subspace* defined by the duality automorphism $\mathbf{D}_\mathcal{C}$. This reduces the *effective* degrees of freedom susceptible to polar noise (cf. Remark 6.41).

- **Consistency with Schrödinger Equation:** Since $U(t)$ acts on $\mathcal{H}_{\text{Total}}$, the Schrödinger equation holds globally. The apparent "collapse" to a specific sector is an update of the applicability predicate $\text{App}_\mathcal{C}$ correlated with the control state $|s\rangle_c$, not a physical collapse of the wave function (Proposition 6.46).

- **Derivation vs. Assumption:** The dimension arises from the logical structure itself. For the dyadic reduction ($d = 2$), the Hilbert space is isomorphic to standard qubit space ($n + 1$ qubits), ensuring that physicists may apply standard quantum intuition (e.g., Bloch sphere representation for single loci) without compromising the logic-first derivation. For the generic four-corner case ($d = 4$), the larger dimension accommodates contextuality witnesses that require $d \geq 3$ (Remark 6.135).

Thus, the framework is kinematically complete and consistent with standard unitary quantum mechanics.

Remark 5.24 (Specific Noise Channels and Symmetry Protection)**.** The "Symmetry-Protected Confinement" discussed in Remark 5.23 relies on the commutation relation $[U_{\text{holo}}, \mathbf{D}_\mathcal{C}] = 0$. This protects the Holothesis sector against specific noise channels:

(i) **Protected:** Collective dephasing noise that acts identically on both copies of the dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ preserves the duality symmetry and thus leaves the Holothesis sector invariant (decoherence-free subspace).

(ii) **Unprotected (Intentional):** Which-path measurements or symmetry-breaking interactions do *not* commute with $\mathbf{D}_\mathcal{C}$. These channels intentionally break the symmetry, driving the transition from Holothesis to Polar sectors (Thesis/Antithesis). This distinguishes environmental decoherence (noise) from measurement (symmetry breaking) within the same formalism.

This clarifies that the framework protects potentiality against irrelevant noise while remaining sensitive to contextual refinement.

5.10   Tetralemmatic Complex Phases

The preceding steps require no preferred "phase" data: the GNS inner product is forced by ω_Λ, and duality becomes unitary because $\alpha_\mathbf{D}$ preserves ω_Λ. If one wishes to see a canonical

appearance of the fourth roots of unity $\{1, i, -1, -i\}$, then one must posit the corresponding *order*-4 *symmetry* at the logical/algebraic level. This section makes that dependence explicit. This approach aligns with the representation-theoretic origin of complex structure in quantum theory, where phases arise from symmetry groups rather than being presupposed [7, 71, 104].

5.10.1 Order-4 tetralemmatic symmetry at the algebraic level

Definition 5.25 (Tetralemmatic quarter-turn automorphism)**.** A *tetralemmatic or tetralemmatic quarter-turn* is a $*$-automorphism

$$\alpha_{\mathbf{R}} : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_\Lambda^{\mathrm{alg}}$$

satisfying:

(a) **Order four:** $\alpha_{\mathbf{R}}^4 = \mathrm{id}_{\mathcal{A}_\Lambda^{\mathrm{alg}}}$. This reflects the cyclic structure of the four corners [13, 15].

(b) **Corner-compatibility:** $\alpha_{\mathbf{R}}$ preserves the corner-typed envelope in the sense that it maps corner generators to corner generators (possibly with a relabelling of the corner tag and/or claim argument). Concretely, for each well-formed polar claim P (including *n-loci claims* $P_{\vec{\lambda}}$ defined over locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$) there exist corner labels $\sigma_{\mathbf{R}}(\bullet) \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$ and a claim transform $P \mapsto P^{\mathbf{R}}$ (not necessarily distinct from P) such that

$$\alpha_{\mathbf{R}}\big(C_\bullet(P)\big) \;=\; C_{\sigma_{\mathbf{R}}(\bullet)}(P^{\mathbf{R}}). \tag{53}$$

(In an "atomic" context one often takes $P^{\mathbf{R}} = P$.)

(c) **Duality compatibility (optional but natural):** $\alpha_{\mathbf{R}}$ commutes with the duality automorphism,

$$\alpha_{\mathbf{R}} \circ \alpha_{\mathbf{D}} = \alpha_{\mathbf{D}} \circ \alpha_{\mathbf{R}}. \tag{54}$$

This ensures the quarter-turn symmetry is compatible with the Thesis–Antithesis exchange [92].

This definition follows the standard theory of periodic automorphisms on $*$-algebras [17, 19].

Definition 5.26 (**R**-invariant warrant functional)**.** A warrant functional $\omega_\Lambda : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathbb{C}$ is **R**-*invariant* if

$$\omega_\Lambda \circ \alpha_{\mathbf{R}} = \omega_\Lambda, \quad \text{i.e.} \quad \omega_\Lambda(\alpha_{\mathbf{R}}(A)) = \omega_\Lambda(A) \ \text{ for all } A \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

This is the algebraic analogue of a symmetry-invariant state [7, 16].

Proposition 5.27 (GNS implementation of the quarter-turn)**.** *Let* $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ *be the logical GNS triple (Definition 4.1). If* $\alpha_{\mathbf{R}}$ *is a tetralemmatic quarter-turn (Definition 5.25) and* ω_Λ *is* **R**-*invariant (Definition 5.26), then there exists a unique unitary operator*

$$U_{\mathbf{R}} : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$$

such that:

(a) *Implementation:* $U_{\mathbf{R}} \, \pi_\Lambda(A) \, U_{\mathbf{R}}^\dagger = \pi_\Lambda(\alpha_{\mathbf{R}}(A))$ *for all* $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$.

(b) *Vacuum-fixing:* $U_{\mathbf{R}}|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle$.

(c) *Order four:* $U_{\mathbf{R}}^4 = I_{\mathcal{H}_\Lambda}$.

Proof. Existence and uniqueness of the unitary implementer follow from the standard GNS construction for invariant states [16, 17, 19], analogous to Proposition 5.16. Specifically, define $U_{\mathbf{R}}^{(0)}$ on the dense domain $\pi_\Lambda(\mathcal{A}_\Lambda^{\mathrm{alg}})|\Omega_\Lambda\rangle$ by

$$U_{\mathbf{R}}^{(0)}\left(\pi_\Lambda(A)|\Omega_\Lambda\rangle\right) := \pi_\Lambda(\alpha_{\mathbf{R}}(A))|\Omega_\Lambda\rangle.$$

Invariance of ω_Λ ensures this is well-defined and isometric. The order-four property follows from $\alpha_{\mathbf{R}}^4 = \mathrm{id}$:

$$U_{\mathbf{R}}^4 \, \pi_\Lambda(A)|\Omega_\Lambda\rangle = \pi_\Lambda(\alpha_{\mathbf{R}}^4(A))|\Omega_\Lambda\rangle = \pi_\Lambda(A)|\Omega_\Lambda\rangle,$$

so $U_{\mathbf{R}}^4 = I$ by density [71]. $\qquad\square$

Corollary 5.28 (Emergence of fourth roots of unity)**.** *Under the hypotheses of Proposition 5.27, the spectrum of $U_{\mathbf{R}}$ is contained in the fourth roots of unity:*

$$\sigma(U_{\mathbf{R}}) \subseteq \{1, i, -1, -i\}.$$

Consequently, the complex phase i appears canonically as an eigenvalue of the logical quarter-turn symmetry, without being presupposed in the algebraic axioms [7, 8].

Proof. If $U_{\mathbf{R}}v = \lambda v$ for $v \neq 0$, then $U_{\mathbf{R}}^4 v = \lambda^4 v$. Since $U_{\mathbf{R}}^4 = I$, we have $\lambda^4 = 1$, hence $\lambda \in \{1, i, -1, -i\}$ [71]. $\qquad\square$

Remark 5.29 (Logical origin of complex structure)**.** Corollary 5.28 demonstrates that the complex unit i need not be presupposed as part of a physical Hilbert space axiom. Instead, it emerges representation-theoretically from the order-4 symmetry of the logical corner-structure (Definition 2.17). This aligns with the view that quantum kinematics is forced by logical coherence and symmetry rather than by physical postulate [7, 8, 9].

Corollary 5.30 (Fourth roots of unity appear as representation-theoretic eigenvalues)**.** *Assume the hypotheses of Proposition 5.27. Then every eigenvalue λ of the unitary implementer $U_{\mathbf{R}}$ satisfies $\lambda^4 = 1$, hence*

$$\lambda \in \{1, i, -1, -i\}.$$

Consequently, when $U_{\mathbf{R}}$ acts nontrivially on the logical modal subspace $\mathcal{H}_L(P) \subseteq \mathcal{H}_\Lambda$, the phases $\{1, i, -1, -i\}$ arise canonically as the spectrum of the tetralemmatic quarter-turn symmetry. This establishes the complex unit i as a representation-theoretic consequence of the tetralemmatic logical structure, rather than a presupposed feature of the scalar field [7, 71, 104].

Proof. Let $v \in \mathcal{H}_\Lambda$ be a nonzero eigenvector of $U_{\mathbf{R}}$ with eigenvalue $\lambda \in \mathbb{C}$, so that $U_{\mathbf{R}}v = \lambda v$. Applying $U_{\mathbf{R}}$ repeatedly yields $U_{\mathbf{R}}^4 v = \lambda^4 v$. By Proposition 5.27 (specifically item (c)), $U_{\mathbf{R}}^4 = I_{\mathcal{H}_\Lambda}$, so $U_{\mathbf{R}}^4 v = v$. Thus $\lambda^4 v = v$, and since $v \neq 0$, we have $\lambda^4 = 1$. The solutions to this polynomial equation in $\mathbb{C}$ are precisely the fourth roots of unity $\{1, i, -1, -i\}$ [19, 71, 98]. $\qquad\square$

Remark 5.31 (Logical origin of complex structure extended)**.** Corollary 5.30 demonstrates that the complex unit i need not be presupposed as part of a physical Hilbert space axiom. Instead, it emerges representation-theoretically from the order-4 symmetry of the logical corner-structure (Definition 2.17). This aligns with the view that quantum kinematics is forced by logical coherence and symmetry rather than by physical postulate [7, 8, 9]. In particular, the appearance of i is tied to the cyclic permutation of the tetralemma corners consistent with the Axiom of Duality (Axiom 2.22) extended to order-4 rotations [13, 15].

Remark 5.32 (Action on n-Loci Configurations and Dual-Subspace)**. The order-**4 **symmetry extends naturally to n-loci configurations.** For Thesis, Antithesis, and Synthesis corners defined over an n-loci configuration $\vec{\lambda}$, the unitary $U_{\mathbf{R}}$ acts on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ as a tensor product of local quarter-turn unitaries, preserving the cyclic corner structure across all loci. **For the Holothesis corner,** $U_{\mathbf{R}}$ acts on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4). Crucially, the quarter-turn symmetry must preserve the dual-subspace structure required for the Holothesis meta-test; specifically, it must commute with the swap operation between the two copies of $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ that defines the dual-subspace correlation. This ensures that the complex phase structure is consistent across all four corners, whether operating on single loci, composite configurations, or the dual-subspace diagnostic arena.

5.11 Minimality and Uniqueness of the Tetralemmatic Quarter-Turn

We now address whether the order-4 symmetry $\alpha_{\mathbf{R}}$ is imposed or emergent. We demonstrate that it is the *minimal automorphism* satisfying the **Continuity of Logical Status**: the requirement that there exists a transformation connecting all four tetralemmatic sectors. This result holds for both single-locus and *n-loci configurations*, as the algebraic structure of the corner generators is preserved under tensor product composition.

Theorem 5.33 (Uniqueness of the Complex Phase from Corner Cyclicity)**.** *Let $\mathcal{H}_{\Lambda}$ be the GNS Hilbert space constructed from $(\mathcal{A}_{\Lambda}^{\text{alg}}, \omega_{\Lambda})$ (Definition 4.1). Assume the four corner sectors* $\{\mathbf{Th}, \mathbf{SynTh}, \mathbf{AntiTh}, \mathbf{HoloTh}\}$ *are mutually distinct and cyclically related via a unitary operator $U_{\mathbf{R}}$ acting on the corner kets $|\bullet; \vec{P}\rangle$ for an n-loci configuration $\vec{\lambda}$ (cf. Definitions 5.1 to 5.3 and 5.7) such that:*

$$U_{\mathbf{R}}|\mathbf{Th}; \vec{P}\rangle \propto |\mathbf{SynTh}; \vec{P}\rangle, \quad U_{\mathbf{R}}|\mathbf{SynTh}; \vec{P}\rangle \propto |\mathbf{AntiTh}; \vec{P}\rangle, \quad \ldots \tag{55}$$

Then $U_{\mathbf{R}}$ is uniquely determined as a quarter-turn ($U_{\mathbf{R}}^4 = I$) and its spectrum must include $\{1, i, -1, -i\}$.

Proof. **Step 1: Duality as Square.** By the Axiom of Duality (Definition 2.22), the duality automorphism $\alpha_{\mathbf{D}}$ exchanges Thesis and Antithesis while fixing Synthesis and Holothesis. Its unitary implementer $U_{\mathbf{D}}$ satisfies $U_{\mathbf{D}}^2 = I$ (Proposition 5.16). If $U_{\mathbf{R}}$ cycles corners as in Eq. (55), then $U_{\mathbf{R}}^2$ must implement the Thesis–Antithesis exchange, hence $U_{\mathbf{R}}^2 = U_{\mathbf{D}}$ up to phase.

Step 2: Order-Four Constraint. Since $U_{\mathbf{D}}^2 = I$, it follows that $U_{\mathbf{R}}^4 = (U_{\mathbf{R}}^2)^2 = U_{\mathbf{D}}^2 = I$. Thus $U_{\mathbf{R}}$ has order dividing 4.

Step 3: Non-Triviality. If $U_{\mathbf{R}}^2 = I$, then $U_{\mathbf{R}}$ would not cycle through Synthesis and Holothesis (it would collapse to duality). Thus $U_{\mathbf{R}}^2 \neq I$, so the order is exactly 4.

Step 4: Spectral Consequence. The eigenvalues λ of $U_{\mathbf{R}}$ satisfy $\lambda^4 = 1$ and $\lambda^2 \neq 1$, hence $\lambda \in \{i, -i\}$ on the cycling subspace. The complex unit i is therefore the unique spectral signature of the minimal symmetry connecting all four corners [71, 98]. $\square$

Remark 5.34 (Physical Interpretation: Fact $\leftrightarrow$ Potentiality)**.** Theorem 5.33 demonstrates that the complex unit i is not imposed to "recover" quantum mechanics. Instead, it emerges as the *geometric signature* of the minimal transformation required to rotate between:

- **Fact axis** (Thesis/Antithesis): Sharp measurement outcomes, represented by real eigenvalues ± 1. For n-loci configurations, this corresponds to the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$.

- **Potentiality axis** (Synthesis/Holothesis): Interference and inapplicability, represented by imaginary eigenvalues $\pm i$. For Holothesis, this corresponds to the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4), where potentiality is encoded as correlations between copies.

This aligns with the representation-theoretic origin of complex structure in quantum theory, where phases arise from symmetry groups rather than being presupposed [7, 104].

Corollary 5.35 (Minimality of Z_4)**.** *The cyclic group Z_4 is the minimal group capable of cyclically permuting four distinct logical sectors while preserving the duality involution (**Th** $\leftrightarrow$ **AntiTh**, **SynTh** $\leftrightarrow$ **SynTh**, **HoloTh** $\leftrightarrow$ **HoloTh**). No smaller group (e.g. Z_2) can satisfy the corner-cycle requirement without collapsing distinct sectors. This minimality holds for both single-locus and n-loci configurations, as the logical distinction between corners is preserved under tensor product composition.*

5.11.1 Corner-level action in an n-loci context

Proposition 5.36 (Schematic corner action of $U_{\mathbf{R}}$ on n-Loci Configurations)**.** *Fix an n-loci corner context $\vec{P}$ defined over a locus configuration $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ (cf. Definitions 5.1 to 5.3 and 5.7). Define the corner kets $|\bullet; \vec{P}\rangle := [C_{\bullet}(\vec{P})]_{\omega_\Lambda}$ as in Definition 4.8. Assume $\vec{P}^{\mathbf{R}} = \vec{P}$ in (53). Then*

$$U_{\mathbf{R}} |\bullet; \vec{P}\rangle = \left[C_{\sigma_{\mathbf{R}}(\bullet)}(\vec{P}) \right]_{\omega_\Lambda} = |\sigma_{\mathbf{R}}(\bullet); \vec{P}\rangle.$$

In particular, if $\sigma_{\mathbf{R}}$ is a 4-cycle on the corner labels, then $U_{\mathbf{R}}$ acts as a quarter-turn on the corner span, and its spectrum on that invariant subspace is contained in $\{1, i, -1, -i\}$. This implements the tetralemmatic quarter-turn symmetry at the Hilbert space level, mapping corner kets according to the corner permutation $\sigma_{\mathbf{R}}$ [16, 17].

Proof. As in Proposition 5.18, use $|\bullet; \vec{P}\rangle = \pi_\Lambda(C_{\bullet}(\vec{P}))|\Omega_\Lambda\rangle$ together with Proposition 5.27 (specifically item (a)) and $U_{\mathbf{R}}|\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle$. Explicitly:

$$U_{\mathbf{R}} |\bullet; \vec{P}\rangle = U_{\mathbf{R}} \, \pi_\Lambda(C_{\bullet}(\vec{P}))|\Omega_\Lambda\rangle = \pi_\Lambda(\alpha_{\mathbf{R}}(C_{\bullet}(\vec{P})))|\Omega_\Lambda\rangle = \pi_\Lambda(C_{\sigma_{\mathbf{R}}(\bullet)}(\vec{P}))|\Omega_\Lambda\rangle = |\sigma_{\mathbf{R}}(\bullet); \vec{P}\rangle,$$

where the second equality uses the implementation property in Proposition 5.27 and the third uses (53) with $\vec{P}^{\mathbf{R}} = \vec{P}$ [19, 94]. $\square$

Remark 5.37 (Operational realization of the quarter-turn on Holothesis)**.** While the algebraic action of $U_\mathbf{R}$ is uniform across all four corners, the *operational realization* of the Holothesis ket $|\mathbf{HoloTh}; \vec{P}\rangle$ requires the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4). Consequently, when $U_\mathbf{R}$ acts on the Holothesis sector in a physical realization, it preserves the dual-subspace structure (correlations between copies) required for the meta-test. This ensures that the complex phase structure is consistent across all corners, whether operating on the composite operational space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis) or the dual-subspace (for Holothesis).

5.11.2   What happens if no order-4 symmetry is posited?

Remark 5.38 (Two honest routes by which $\mathbb{C}$ may appear)**.** There are two logically clean ways complex structure may enter, and they should not be conflated:

(a) **Representation-theoretic phases from symmetry (preferred if one wants i canonically).** If one posits an order-4 tetralemmatic symmetry $\alpha_\mathbf{R}$ that preserves ω_Λ, then the resulting unitary $U_\mathbf{R}$ has spectrum in $\{1, i, -1, -i\}$ on invariant subspaces, so "i" appears as an eigenvalue forced by tetralemmatic symmetry. This is the representation-theoretic origin of complex phases, analogous to how time-translation symmetry yields energy eigenvalues via Stone's theorem [17, 71, 104]. This symmetry acts on the full algebra $\mathcal{A}_\Lambda^{\text{alg}}$, including generators for *n-loci claims* $\vec{P}$.

(b) **Scalar field choice for $*$-algebra states (minimal and standard).** Even without any order-4 symmetry, one may still allow ω_Λ to be complex-valued, as is standard for $*$-algebras. This introduces $\mathbb{C}$ only as the codomain of the warrant functional and thus of the GNS inner product, without assigning any preferred physical phase structure to the corners. This is the conventional algebraic approach where complex numbers enter at the level of the scalar field for the $*$-algebra [7, 19, 94].

In both cases, no physical Hilbert space or Born rule is presupposed: complex structure enters either as (i) a symmetry signature of the tetralemmatic algebra, or (ii) the conventional scalar setting for positive functionals on $*$-algebras. This distinction clarifies the logical status of complex numbers in the reconstruction, avoiding the common conflation of algebraic scalars with physical phases [8, 9].

Remark 5.39 (Comparison with standard quantum reconstruction programs)**.** The present approach differs from standard quantum reconstruction programs in several respects:

- **Algebraic primacy:** We construct the $*$-algebra from logical primitives first, then derive the Hilbert space via GNS. Standard approaches often presuppose a Hilbert space or effect algebra [56, 63].

- **Logical origin of phases:** Complex phases emerge from the order-4 tetralemmatic symmetry rather than being imposed as part of the scalar field axiom [71].

- **Typed inapplicability and n-Loci Structure:** The Holothesis corner provides a formal mechanism for handling inapplicable propositions via the *n-loci dual-subspace* meta-test (Definitions 5.4 and 5.7). This structural distinction between polar corners (single register) and Holothesis (dual register) is absent from standard quantum logic [8, 9].

This positions the framework as a logical reconstruction of quantum kinematics rather than a physical axiomatization [7].

5.12   Emergence of Complex Structure from Tetralemmatic Symmetry

The reconstruction program culminates in the derivation of the complex Hilbert space structure from purely logical primitives. This section establishes that the complex unit i is not an imposed physical postulate but a representation-theoretic necessity arising from the order-4 symmetry of the tetralemmatic lens. This derivation holds for both single-locus and *n-loci configurations*, as the algebraic symmetry acts on the full corner-observable algebra generated by claims $P_{\vec{\lambda}}$ over locus configurations $\vec{\lambda}$ (cf. Definitions 5.1 to 5.3 and 5.7).

The GNS Implementation and Symmetry Preservation

The transition from the abstract logical algebra to concrete kinematics is mediated by the Gelfand–Naimark–Segal (GNS) construction. Given the logical corner-observable $*$-algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$ (Definition 3.15) and the coherent warrant functional ω_Λ, the logical GNS triple $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ is uniquely determined (Definition 3.42). This construction ensures that the symmetries of the logic are preserved as unitary symmetries of the physics. Specifically, any $*$-automorphism α of $\mathcal{A}_\Lambda^{\mathrm{alg}}$ that preserves ω_Λ admits a unique unitary implementer U_α on $\mathcal{H}_\Lambda$ such that $U_\alpha \pi_\Lambda(A) U_\alpha^\dagger = \pi_\Lambda(\alpha(A))$ [16, 19]. This guarantees that the duality involution and the quarter-turn symmetry defined at the logical level manifest as unitary operators in the state space, avoiding the circularity of presupposing a Hilbert space to define symmetries [7].

The Appearance of i via Order-4 Symmetry

The central result of this derivation is Proposition 3.55. If the logical structure admits a tetralemmatic quarter-turn automorphism $\alpha_\mathbf{R}$ (Definition 3.54) that cycles the corners $(\mathbf{Th} \to \mathbf{SynTh} \to \mathbf{AntiTh} \to \mathbf{HoloTh})$, the corresponding unitary implementer $U_\mathbf{R}$ must satisfy $U_\mathbf{R}^4 = I_{\mathcal{H}_\Lambda}$. Consequently, the spectrum of $U_\mathbf{R}$ is constrained to the fourth roots of unity:

$$\sigma(U_\mathbf{R}) \subseteq \{1, i, -1, -i\}. \tag{56}$$

This forces the emergence of the complex unit i as an eigenvalue of the logical symmetry operator. Remark 3.56 highlights the foundational significance of this result: the complex phase is grounded in the Catuškoṭi structure itself. Unlike standard reconstructions where $\mathbb{C}$ is introduced via the scalar field of the vector space [71], here the phase records the position of a state within the tetralemmatic cycle. This aligns with the view that quantum kinematics is forced by logical coherence and symmetry rather than physical postulate [8, 9].

The Rotational Mapping of Logical Corners

The order-4 symmetry induces a canonical mapping between the four logical corners and the quadrants of the complex plane. This mapping provides a geometric interpretation of the logical statuses as rotational states within the Hilbert space $\mathcal{H}_\Lambda$. For n-loci configurations, this mapping applies to the corner kets $|\bullet; \vec{P}\rangle$, with the operational realization of Holothesis requiring the n-loci dual-subspace (Definition 5.4).

Table 4: Correspondence between tetralemmatic corners, algebraic phases, and rotational states in the logical GNS space.

Logical Corner	Algebraic Phase	Rotational State
Thesis (**Th**)	1	The 0° Anchor (Real Axis). Sharp outcome on composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$.
Synthesis (**SynTh**)	i	The 90° Transition (Imaginary "Both"). Coherent superposition on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$.
Antithesis (**AntiTh**)	-1	The 180° Inversion (Real Dual). Sharp outcome on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$.
Holothesis (**HoloTh**)	$-i$	The 270° Preclusion (Imaginary "Neither"). Inapplicability diagnostic on n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$.

Structural Significance and Ontological Implications

This derivation yields three critical structural insights for the framework:

1. **Unitary Logic:** The tetralemma is transformed into a unitary operating system. Transitions between logical corners are not merely semantic choices but correspond to unitary rotations in the state space. This elevates the logical structure to a dynamical symmetry of the kinematics, applicable across both single-locus and n-loci configurations.

2. **Duality as a Fixed-Point Involution:** The duality involution $\Delta_{\mathcal{C}}$ (Axiom 2.22), which exchanges Thesis and Antithesis while fixing Synthesis and Holothesis, is identified as the square of the quarter-turn operation (g^2 in $\mathbb{Z}_4$). A 180° rotation maps $1 \leftrightarrow -1$ (swapping the real axis) while mapping $i \leftrightarrow -i$ (preserving the imaginary axis up to sign). This explains why Synthesis and Holothesis are self-dual: they reside on the imaginary axis of the logical manifold, invariant under the polarity exchange that governs the real axis [104].

3. **Ontological Decomposition (Real vs. Imaginary):** As detailed in Remark 5.31, the complex structure permits a rigorous ontological distinction. The **Thesis** and **Antithesis** corners correspond to the *real components* of the ontological state, representing sharp measurement outcomes (facts) on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$. The **Synthesis** and **Holothesis** corners correspond to the *imaginary components*, representing potentiality. Crucially, **Synthesis** represents coherence (superposition) on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$, while **Holothesis** represents inapplicability diagnosed via correlations on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definitions 5.4 and 5.7). This provides a symmetrical explanation for why measurement outcomes are restricted to the real axis (eigenvalues ± 1), while the imaginary sector governs coherence and phase relations [56].

In summary, the complex Hilbert space structure is derived as the minimal representation space capable of hosting the tetralemmatic symmetry. The imaginary unit i is thereby reinterpreted not as a computational artifact, but as the geometric signature of the logical transition between fact (Thesis/Antithesis) and potentiality (Synthesis/Holothesis) [7, 8].

6 Quantum Realizations and Contextuality Witnesses

This section synthesizes the preceding constructions to exhibit how the tetralemmatic logical framework naturally yields the structural core of algebraic quantum kinematics. The derivation is *logic-first*: the Hilbert space, inner product, and observable algebra are all induced from the corner-structure and warrant functional, rather than presupposed [7, 8, 16]. This synthesis holds for both single-locus and *n-loci configurations*, as the algebraic structure is preserved under tensor product composition of locus configurations $\vec{\lambda} \in \Lambda_\mathcal{C}^n$ (cf. Definitions 5.1 to 5.3 and 5.7).

6.1 Logical Noncommutative Probability Space

Definition 6.1 (Logical corner probability space)**.** Fix a *locus configuration* $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_\mathcal{C}^n$, and its Tier-0 corner $*$-algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$ (Definition 3.15). A *logical corner probability space* is a pair

$$(\mathcal{A}_\Lambda^{\mathrm{alg}}, \omega_\Lambda),$$

where ω_Λ is a coherent warrant functional in the sense of Definition 3.35. This structure coincides with the standard notion of a noncommutative probability space in operator algebra theory [16, 17, 19]. For n-loci configurations, the algebra is generated by corner-observables $C_\bullet(P_{\vec{\lambda}})$ associated with n-loci claims $P_{\vec{\lambda}}$.

Remark 6.2 (Why this is already "quantum-shaped" but not "quantum-presupposing")**.** The data in Definition 6.1 coincide structurally with a noncommutative probability space (state on a $*$-algebra), yet $\mathcal{A}_\Lambda^{\mathrm{alg}}$ is logic-generated (Tier-0, cf. Remark 3.17) and $\mathcal{H}_\Lambda$ is forced by GNS (Definition 4.1), not postulated. This distinguishes the present reconstruction from standard quantum axiomatizations where the Hilbert space is taken as primitive [7, 8, 9]. The algebraic approach aligns with quantum measure theory [8, 14] and the view that logic stands to quantum theory as geometry stands to gravity [8].

For n-loci configurations, the GNS Hilbert space $\mathcal{H}_\Lambda$ includes both the *composite operational space* $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis) and the *n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (for Holothesis meta-tests), as defined in Definition 5.4. This ensures the probability space captures the full tetralemmatic structure across all locus configurations.

6.1.1 Corner Gram Matrices and Automatic Positivity

Definition 6.3 (Corner Gram matrix in an atomic context)**.** Fix an *atomic corner context* P (Definition 4.7) for a single locus λ, or an *n-loci atomic corner context* $\vec{P}$ for a locus configuration $\vec{\lambda}$ (cf. Definitions 5.1 to 5.3 and 5.7). Define the corner kets $|\bullet; P\rangle$ (single-locus,

Definition 4.8) or $|\bullet; \vec{P}\rangle$ (n-loci). The Gram matrix is:

$$G(P)_{\bullet,\bullet'} := \langle \bullet; P \mid \bullet'; P \rangle_\Lambda = \omega_\Lambda\big(C_\bullet(P)^\dagger C_{\bullet'}(P)\big),$$

or for n-loci:

$$G(\vec{P})_{\bullet,\bullet'} := \langle \bullet; \vec{P} \mid \bullet'; \vec{P} \rangle_\Lambda = \omega_\Lambda\big(C_\bullet(\vec{P})^\dagger C_{\bullet'}(\vec{P})\big).$$

Since generators are self-adjoint (Definition 3.20), this simplifies to $\omega_\Lambda\big(C_\bullet(P)\,C_{\bullet'}(P)\big)$. This construction follows the standard theory of Gram matrices in inner product spaces [71, 100].

Proposition 6.4 (Gram positivity is forced by coherence)**.** *For every admissible P (single-locus) or $\vec{P}$ (n-loci), the matrix $G(P)$ or $G(\vec{P})$ of Definition 6.3 is positive semidefinite.*

Proof. Let $z_\bullet \in \mathbb{C}$ and set $A := \sum_\bullet z_\bullet C_\bullet(P) \in \mathcal{A}_\Lambda^{\mathrm{alg}}$ (or $A := \sum_\bullet z_\bullet C_\bullet(\vec{P})$ for n-loci). Then by linearity of ω_Λ and the definition of the Gram matrix entries:

$$
\begin{aligned}
\sum_{\bullet,\bullet'} \overline{z_\bullet}\, z_{\bullet'}\, G(P)_{\bullet,\bullet'} &= \sum_{\bullet,\bullet'} \overline{z_\bullet}\, z_{\bullet'}\, \omega_\Lambda\big(C_\bullet(P)^\dagger C_{\bullet'}(P)\big) \\
&= \omega_\Lambda\bigg(\Big(\sum_\bullet z_\bullet C_\bullet(P)\Big)^\dagger \Big(\sum_{\bullet'} z_{\bullet'} C_{\bullet'}(P)\Big)\bigg) \qquad (57) \\
&= \omega_\Lambda(A^\dagger A).
\end{aligned}
$$

By the positivity axiom of the coherent warrant functional (Definition 3.35, Proposition 3.36), $\omega_\Lambda(A^\dagger A) \geq 0$ for all $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$ [17, 19, 94]. Hence $G(P) \succeq 0$ (and similarly $G(\vec{P}) \succeq 0$ for n-loci). $\square$

Remark 6.5 (Effect-like collapse is an enrichment, not a logical necessity)**.** If one additionally imposes Tier-2 relations (Definition 3.33), e.g. exclusivity $C_\bullet(P)C_{\bullet'}(P) = 0$ for $\bullet \neq \bullet'$ and a partition $\sum_\bullet C_\bullet(P) = \mathbf{1}$, then distinct corner kets become orthogonal and $G(P)$ becomes diagonal with entries $\omega_\Lambda(C_\bullet(P))$. This is an optional sharpening, not part of the minimal template. In the sharp quantum realization limit, this corresponds to moving from a general $*$-algebra to a projection lattice or effect algebra [56, 63, 85]. The paraconsistent Synthesis corner may require relaxing exclusivity axioms to accommodate coherent superpositions [54, 55].

For n-loci configurations, the distinction between *composite operational space* $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis) and *n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (for Holothesis) must be maintained. Effect-like collapse applies to the single-register corners, while Holothesis requires the dual-subspace meta-test structure (Definitions 5.4 and 5.7).

Remark 6.6 (Connection to quantum state tomography and contextuality)**.** The Gram matrix $G(P)$ encodes the mutual overlaps of the four corner kets in the logical GNS space. In quantum information theory, such Gram matrices appear in state discrimination and tomography problems [59, 101]. The rank of $G(P)$ determines the dimension of the logical modal subspace $\mathcal{H}_L(P)$ (Proposition 4.16). For n-loci configurations, the rank of $G(\vec{P})$ determines the dimension of the *composite logical modal subspace* $\bigotimes_{i=1}^n \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$.

Deviations from diagonality in $G(P)$ witness non-classical correlations and contextuality [8, 56, 63]. In particular, off-diagonal entries between Thesis and Antithesis corners indicate coherent co-holding without orthogonality, matching the paraconsistent "glut" semantics of Synthesis [54, 55]. **For Holothesis,** the Gram matrix structure must be interpreted within the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$, where off-diagonal correlations between the two copies encode the meta-test diagnostic for polar inapplicability (Definition 5.8 and Remark 5.5).

6.2 Positivity in the Paraconsistent Sector

A potential concern in combining paraconsistent logic with Hilbert space methods is the risk of "ghost states" (negative norm states) arising from logical contradictions. We explicitly verify that the Synthesis corner remains free of such pathologies, both for single-locus and *n-loci configurations.*

Lemma 6.7 (Absence of negative norms in Synthesis (n-Loci)). *Let $\vec{P}$ be an n-loci polar claim warranting Synthesis at locus configuration $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$ (cf. Definition 5.3). Then the corner kets $|\mathbf{Th}; \vec{P}\rangle$ and $|\mathbf{AntiTh}; \vec{P}\rangle$ satisfy:*

1. *Positive Norms: $\langle \mathbf{Th}; \vec{P} \mid \mathbf{Th}; \vec{P}\rangle_\Lambda \geq 0$ and $\langle \mathbf{AntiTh}; \vec{P} \mid \mathbf{AntiTh}; \vec{P}\rangle_\Lambda \geq 0$.*

2. *Positive Warrant: $\omega_\Lambda(C_{\mathbf{Th}}(\vec{P})) > 0$ and $\omega_\Lambda(C_{\mathbf{AntiTh}}(\vec{P})) > 0$.*

3. *Non-Orthogonality: Typically $\langle \mathbf{Th}; \vec{P} \mid \mathbf{AntiTh}; \vec{P}\rangle_\Lambda \neq 0$.*

4. *Inter-Locus Correlations: For $n > 1$, non-zero off-diagonal correlations may exist between distinct loci λ_i, λ_j (entanglement), while preserving overall positivity.*

Proof. **(1) & (2):** By Definition 3.35, ω_Λ is a positive linear functional on $\mathcal{A}_\Lambda^{\mathrm{alg}}$. Thus for any generator $C_\bullet(\vec{P})$, which is self-adjoint (Definition 3.15),

$$\langle \bullet; \vec{P} \mid \bullet; \vec{P}\rangle_\Lambda = \omega_\Lambda(C_\bullet(\vec{P})^\dagger C_\bullet(\vec{P})) = \omega_\Lambda(C_\bullet(\vec{P})^2).$$

In the Synthesis regime, the warrant conditions (Definition 5.3) require $\omega_\Lambda(C_{\mathbf{Th}}(\vec{P})) > 0$ and $\omega_\Lambda(C_{\mathbf{AntiTh}}(\vec{P})) > 0$ (nonzero Born weights). Since $C_\bullet(\vec{P})$ are positive-type generators in the realization (Proposition 6.4), their norms are non-negative. Positivity of ω_Λ is an axiom independent of logical status and is preserved under tensor product composition of loci [17, 19].

(3): The Tier-0 algebra does *not* impose exclusivity relations

$$C_{\mathbf{Th}}(\vec{P})C_{\mathbf{AntiTh}}(\vec{P}) = 0$$

Thus the off-diagonal Gram entries $G_{\mathbf{Th},\mathbf{AntiTh}}(\vec{P}) = \omega_\Lambda(C_{\mathbf{Th}}(\vec{P})C_{\mathbf{AntiTh}}(\vec{P}))$ are generally nonzero. This non-orthogonality encodes the logical "glut" (co-holding) without violating metric positivity. The Gram matrix $G(\vec{P})$ remains Positive Semidefinite by Proposition 6.4, ensuring no negative eigenvalues (ghosts) exist.

(4): For $n > 1$, the composite warrant functional $\omega_\Lambda^{(n)}$ on the tensor product algebra $\mathcal{A}_\Lambda^{\mathrm{alg},\otimes n}$ is defined via the tensor product of local warrant functionals. Positivity is preserved

under tensor products: if $\omega_\Lambda^{(i)}$ is positive on each factor, then $\bigotimes_{i=1}^n \omega_\Lambda^{(i)}$ is positive on the composite algebra [106]. Non-zero off-diagonal correlations between loci (entanglement) are encoded in the non-factorizable components of $\omega_\Lambda^{(n)}$, but these do not introduce negative eigenvalues since the Gram matrix $G(\vec{P})$ remains positive semidefinite by construction. $\square$

Remark 6.8 (Physical interpretation: Coherent superposition and entanglement)**.** Physically, Lemma 6.7 corresponds to coherent superposition and entanglement. For a single locus ($n = 1$), in a state $|\psi\rangle = \alpha|L\rangle + \beta|R\rangle$, both path probabilities $|\alpha|^2$ and $|\beta|^2$ are positive (Synthesis), yet the norm $\langle\psi|\psi\rangle = 1$ remains positive. The paraconsistent "contradiction" is logical (both paths warranted), not metric (no negative probabilities).

For n-loci configurations ($n > 1$)**:** The Synthesis corner operates on the *composite operational space* $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (cf. Definition 5.3), not the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (which is reserved for Holothesis meta-tests, Definition 5.4). Non-zero off-diagonal correlations between loci represent *entanglement* (e.g., Bell states across λ_i, λ_j), while the overall warrant functional positivity ensures no ghost states arise from the logical co-commitment structure. This distinguishes Synthesis (coherent superposition/entanglement on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$) from Holothesis (inapplicability diagnostic on $\mathcal{H}_{\text{Dual}}^{(2n)}$).

Remark 6.9 (Distinction from Holothesis positivity)**.** It is crucial to distinguish the positivity structure of Synthesis from Holothesis:

- **Synthesis:** Operates on the single composite space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$. Positivity is ensured by the positive semidefiniteness of the Gram matrix $G(\vec{P})$ on this space. Non-orthogonality encodes coherent co-holding (logical glut).

- **Holothesis:** Operates on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)} = \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (cf. Definitions 5.4 and 5.7). Positivity is ensured by the Holothesis meta-test effect $E_{\text{HoloTh}}^{(a \to b)}$ being a valid POVM element (Proposition 5.9). The dual-subspace structure preserves polar inapplicability while allowing Born-valued diagnosis of the inapplicability status.

Both corners maintain metric positivity despite their non-classical logical structure, but through distinct operational mechanisms appropriate to their respective sectors.

6.3 Operator Models and the Canonical GNS Embedding

This subsection establishes the precise mathematical relationship between the abstract logical GNS construction and concrete physical operator models. The framework ensures that Born-valued semantics factors through the universal corner algebra via $*$-homomorphisms, maintaining consistency with the algebraic quantum theory paradigm [7, 16, 17]. This relationship holds for both single-locus and *n-loci configurations*, as the logical algebra $\mathcal{A}_\Lambda^{\text{alg}}$ is generated by corner-observables for all admissible claims, including n-loci claims $\vec{P}$ defined over locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ (cf. Definitions 5.1 to 5.3 and 5.7).

Definition 6.10 (Operator model of the logical corner algebra)**.** Let H be a Hilbert space and let ω_{phys} be a positive normalized functional on a unital $*$-algebra $\mathcal{A}_{\text{phys}} \subseteq \mathcal{B}(H)$ (e.g.

$\omega_{\text{phys}}(X) = \text{Tr}(\rho X)$ for a density operator ρ). An *operator model* of $\mathcal{A}_\Lambda^{\text{alg}}$ is a unital $*$-homomorphism

$$r : \mathcal{A}_\Lambda^{\text{alg}} \to \mathcal{A}_{\text{phys}}.$$

This definition aligns with the standard notion of a representation of a $*$-algebra on a Hilbert space [19, 94].

Remark 6.11 (Structural requirements for n-loci realizations)**.** For the operator model to faithfully realize the *n-loci tetralemmatic structure*, the physical Hilbert space H must be capable of supporting:

(i) **Composite Operational Space:** For Thesis, Antithesis, and Synthesis corners, H must contain subspaces isomorphic to the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} = \bigotimes_{i=1}^n \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$.

(ii) **n-Loci Dual-Subspace:** For the Holothesis corner, H must contain a subspace isomorphic to the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)} = \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (Definition 5.4). This is structurally required to implement the Holothesis meta-test effect E_{HoloTh} without collapsing polar applicability (Definitions 5.7 and 5.8).

If H lacks the dual-subspace structure, the model can realize Th/AntiTh/SynTh but fails to faithfully realize Holothesis as a measurable meta-claim.

Proposition 6.12 (Any operator model induces a coherent warrant)**.** *Given an operator model r (Definition 6.10), the pullback functional*

$$\omega_\Lambda := \omega_{\text{phys}} \circ r$$

is a coherent warrant functional (Definition 3.35). If additionally r is equivariant with a symmetry α on $\mathcal{A}_\Lambda^{\text{alg}}$ and a symmetry β on $\mathcal{A}_{\text{phys}}$ in the sense of Definition 3.57, and $\omega_{\text{phys}} \circ \beta = \omega_{\text{phys}}$, then $\omega_\Lambda \circ \alpha = \omega_\Lambda$. This is a standard result in the theory of positive functionals on $*$-algebras [17, 19].

Proof. Normalization is preserved because $r(\mathbf{1}) = \mathbf{1}$, hence

$$\omega_\Lambda(\mathbf{1}) = \omega_{\text{phys}}(r(\mathbf{1})) = \omega_{\text{phys}}(\mathbf{1}) = 1.$$

Positivity holds since for all $A \in \mathcal{A}_\Lambda^{\text{alg}}$,

$$\omega_\Lambda(A^\dagger A) = \omega_{\text{phys}}\big(r(A^\dagger A)\big) = \omega_{\text{phys}}\big(r(A)^\dagger r(A)\big) \geq 0,$$

where the inequality follows from the positivity of ω_{phys} on $\mathcal{A}_{\text{phys}}$ [94]. Equivariance plus invariance gives

$$\omega_\Lambda \circ \alpha = \omega_{\text{phys}} \circ r \circ \alpha = \omega_{\text{phys}} \circ \beta \circ r = \omega_{\text{phys}} \circ r = \omega_\Lambda,$$

establishing the symmetry invariance of the induced warrant functional [16, 92]. $\qquad\square$

Remark 6.13 (Preservation of n-loci warrant structure)**.** Proposition 6.12 ensures that the coherent warrant structure is preserved under realization for *all* locus configurations. Specifically:

- **Composite Positivity:** For n-loci claims $\vec{P}$, the induced warrant $\omega_\Lambda(C_\bullet(\vec{P})^\dagger C_\bullet(\vec{P})) \geq 0$ is guaranteed by the positivity of ω_{phys} on the physical algebra, ensuring no ghost states arise in composite configurations (Lemma 6.7).

- **Symmetry Invariance:** If the physical model respects duality ($\beta_{\mathbf{D}}$) and quarter-turn ($\beta_{\mathbf{R}}$) symmetries, the induced logical warrant ω_Λ inherits these invariances, ensuring the unitary implementers $U_{\mathbf{D}}$ and $U_{\mathbf{R}}$ exist on the logical GNS space (Propositions 5.16 and 5.27).

- **Holothesis Meta-Test:** For Holothesis claims, the realization must map the logical generators to physical operators on the dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)} \subseteq H$. The induced warrant $\omega_\Lambda(C_{\text{HoloTh}}(\vec{P}))$ then corresponds to the Born valuation of the meta-test effect E_{HoloTh} (Definition 5.8), preserving the operational distinction between inapplicability (HoloTh) and denial (AntiTh).

This confirms that the logic-first reconstruction remains consistent under physical realization across all tetralemmatic corners and locus configurations.

Proposition 6.14 (Canonical embedding of the logical GNS space into any operator model)**.**
Assume $\omega_\Lambda = \omega_{\text{phys}} \circ r$ as in Proposition 6.12. Let $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ be the logical GNS triple (Definition 4.1). Let $(\pi_{\text{phys}}, \mathcal{H}_{\text{phys}}, |\Omega_{\text{phys}}\rangle)$ be the GNS triple of $(\mathcal{A}_{\text{phys}}, \omega_{\text{phys}})$. Then the map on the dense subspace

$$V_0 : \ \mathcal{A}_\Lambda^{\text{alg}}/\mathcal{N}_{\omega_\Lambda} \to \mathcal{H}_{\text{phys}}, \qquad V_0([A]) \ := \ \pi_{\text{phys}}(r(A))|\Omega_{\text{phys}}\rangle,$$

is a well-defined isometry and extends to an isometric embedding $V : \mathcal{H}_\Lambda \hookrightarrow \mathcal{H}_{\text{phys}}$ intertwining the representations:

$$V\, \pi_\Lambda(A) = \pi_{\text{phys}}(r(A))\, V \qquad (A \in \mathcal{A}_\Lambda^{\text{alg}}).$$

This embedding preserves the n-loci structure of $\mathcal{H}_\Lambda$, including the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis) and the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (for Holothesis), provided the physical model satisfies the structural requirements of Definition 6.10.

Proof. **Well-definedness.** If $[A] = [B]$ in $\mathcal{A}_\Lambda^{\text{alg}}/\mathcal{N}_{\omega_\Lambda}$, then $A - B \in \mathcal{N}_{\omega_\Lambda}$, i.e. $\omega_\Lambda((A-B)^\dagger(A-B)) = 0$. Since $\omega_\Lambda = \omega_{\text{phys}} \circ r$,

$$\omega_{\text{phys}}\Big(r(A-B)^\dagger r(A-B)\Big) = 0,$$

so $r(A-B) \in \mathcal{N}_{\omega_{\text{phys}}}$ and hence $[r(A)] = [r(B)]$ in the physical GNS quotient. Thus V_0 is well-defined [19].
    Isometry. For $[A], [B] \in \mathcal{A}_\Lambda^{\text{alg}}/\mathcal{N}_{\omega_\Lambda}$,

$$\begin{aligned}
\Big\langle V_0([A]) \mid V_0([B]) \Big\rangle_{\text{phys}} &= \Big\langle \pi_{\text{phys}}(r(A))|\Omega_{\text{phys}}\rangle \ \Big| \ \pi_{\text{phys}}(r(B))|\Omega_{\text{phys}}\rangle \Big\rangle_{\text{phys}} \\
&= \omega_{\text{phys}}\Big(r(A)^\dagger r(B)\Big) \\
&= \omega_{\text{phys}}\Big(r(A^\dagger B)\Big) \\
&= \omega_\Lambda(A^\dagger B) \\
&= \Big\langle [A] \mid [B] \Big\rangle_\Lambda.
\end{aligned}$$

Hence V_0 preserves the inner product [17, 94]. Note that this holds for all $A, B \in \mathcal{A}_\Lambda^{\mathrm{alg}}$, including n-loci generators $C_\bullet(\vec{P})$.

Extension and intertwining. Since V_0 is an isometry on a dense subspace, it extends uniquely to an isometric embedding $V : \mathcal{H}_\Lambda \hookrightarrow \mathcal{H}_{\mathrm{phys}}$ [98]. For the intertwining relation, compute on the dense domain:

$$
\begin{aligned}
V \pi_\Lambda(A)[B] &= V([AB]) \\
&= \pi_{\mathrm{phys}}(r(AB))|\Omega_{\mathrm{phys}}\rangle \\
&= \pi_{\mathrm{phys}}(r(A))\,\pi_{\mathrm{phys}}(r(B))|\Omega_{\mathrm{phys}}\rangle \\
&= \pi_{\mathrm{phys}}(r(A))\,V([B]),
\end{aligned}
\tag{58}
$$

which extends by continuity to all of $\mathcal{H}_\Lambda$ [16]. □

Remark 6.15 (Connection to Born-valued semantics)**.** The Born-valued semantics of Section 5.5 together with the realization scheme Section 6.5 provide a concrete source of operator models: one chooses (when defined/typed) realized effects/operators for the corner-generators and extends uniquely by the universal property (Proposition 3.16). This construction ensures that the abstract logical warrant functional ω_Λ coincides with the physical Born valuation ω_{phys} on the image of the realization map [56, 63].

For n-loci configurations, the embedding V maps the logical composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ into the corresponding physical subspace, and the logical n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ into the physical dual-register space. This ensures that the Born valuation $v_M^{(\mathcal{C},\vec{\lambda})}(\rho)$ (Definition 5.11) is preserved under realization for all corners, including the Holothesis meta-test effect E_{HoloTh} which acts on the dual-subspace.

The "typed inapplicability" discipline remains primary (Holothesis, cf. Definition 2.12 and the double-slit walk in Sections 2.8 and 2.12.3), while paraconsistent/paracomplete regimes provide secondary semantics once applicability is restored/refined [8, 14]. This hierarchy preserves the operational discipline that quantum propositions are only meaningful relative to admissible measurement contexts [7, 9].

Remark 6.16 (Universality and minimality of the logical GNS construction)**.** Proposition 6.14 demonstrates that the logical GNS space $\mathcal{H}_\Lambda$ is *minimal* in the following sense: any physical realization factors through it via an isometric embedding. This universality property ensures that the logical Hilbert template captures exactly the structure forced by the warrant functional, without presupposing additional physical degrees of freedom [7, 16]. Consequently, $\mathcal{H}_\Lambda$ serves as a canonical "logical skeleton" upon which various physical realizations may be built, each corresponding to a different choice of operator model r [17].

Structural Minimality for n-Loci: The "logical skeleton" includes the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ required for Holothesis. Any faithful physical realization must contain a subspace isomorphic to $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ to support the Holothesis meta-test without collapsing polar applicability (cf. Definition 5.4 and Remark 6.11). This ensures the logical distinction between Synthesis (single-register coherence) and Holothesis (dual-subspace inapplicability diagnosis) is preserved in all physical models.

Remark 6.17 (Preservation of n-Loci Structure)**.** The isometric embedding V preserves the n-loci structure of $\mathcal{H}_\Lambda$:

- **Composite Operational Space:** V maps $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ isometrically into the physical composite space, preserving the tensor product structure across loci $\vec{\lambda}$.

- **Dual-Subspace:** V maps $\mathcal{H}_{\text{Dual}}^{(2n)}$ isometrically into the physical dual-register space, preserving the correlation structure required for the Holothesis meta-test.

- **Corner Generators:** For n-loci corner generators $C_\bullet(\vec{P})$, the embedding ensures $\langle \bullet; \vec{P} \mid \bullet'; \vec{P} \rangle_\Lambda = \langle V(\bullet; \vec{P}) \mid V(\bullet'; \vec{P}) \rangle_{\text{phys}}$, preserving the Gram matrix structure (Proposition 6.4) in the physical realization.

This confirms that the logic-first reconstruction remains consistent under physical realization across all tetralemmatic corners and locus configurations.

6.4 Algebraic Witnesses of Noncommutativity and Complementarity

This subsection defines algebraic witnesses of noncommutativity and complementarity that apply to both single-locus and *n-loci configurations*. The definitions rely solely on the coherent warrant functional ω_Λ and the logical corner-observable algebra $\mathcal{A}_\Lambda^{\text{alg}}$, ensuring consistency with the logic-first reconstruction. In particular, the observables A considered below include corner-linear observables $O_{\mathbf{a}}(\vec{P})$ associated with n-loci claims $\vec{P}$ defined over locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ (cf. Definition 6.62).

Definition 6.18 (Centering and variance in a coherent warrant state)**.** Let ω_Λ be a coherent warrant functional on $\mathcal{A}_\Lambda^{\text{alg}}$ (Definition 3.35). For any self-adjoint $A = A^\dagger \in \mathcal{A}_\Lambda^{\text{alg}}$, define the centered observable

$$\widetilde{A} \;:=\; A - \omega_\Lambda(A)\,\mathbf{1},$$

and its variance

$$\text{Var}_{\omega_\Lambda}(A) \;:=\; \omega_\Lambda(\widetilde{A}^2) \;=\; \omega_\Lambda(A^2) - \omega_\Lambda(A)^2 \;\geq\; 0.$$

We say that A is ω_Λ-*sharp* if $\text{Var}_{\omega_\Lambda}(A) = 0$. This definition aligns with the standard notion of variance for states on $*$-algebras [17, 19].

Remark 6.19 (Variance for n-Loci Observables)**.** For an n-loci corner-linear observable $A = O_{\mathbf{a}}(\vec{P})$ (Definition 6.62), the variance $\text{Var}_{\omega_\Lambda}(A)$ quantifies the uncertainty of the claim $\vec{P}$ across the entire locus configuration $\vec{\lambda}$.

- **Composite Sharpness:** A is ω_Λ-sharp iff the warrant functional assigns definite values to the corner generators at *all* loci $\lambda_i \in \vec{\lambda}$ simultaneously (no superposition or indeterminacy across the configuration).

- **Entanglement Witness:** Non-zero variance for composite observables may arise from local indeterminacy (Synthesis at individual loci) or from non-separable correlations (entanglement) between loci, both of which are captured by the algebraic variance without presupposing a physical Hilbert space.

Proposition 6.20 (Variance as a GNS norm)**.** *Let $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ be the logical GNS triple (Definition 4.1). For self-adjoint $A \in \mathcal{A}_\Lambda^{\text{alg}}$,*

$$\text{Var}_{\omega_\Lambda}(A) \;=\; \left\| \left(\pi_\Lambda(A) - \omega_\Lambda(A)I \right) |\Omega_\Lambda\rangle \right\|^2.$$

In particular, A is ω_Λ-sharp iff $\left(\pi_\Lambda(A) - \omega_\Lambda(A)I\right)|\Omega_\Lambda\rangle = 0$. This identifies the variance as the squared GNS-norm of the centered observable acting on the cyclic vector [7, 16].

Proof. Let $A \in \mathcal{A}_\Lambda^{\mathrm{alg}}$ be self-adjoint ($A = A^\dagger$). By Definition 6.18, the centered observable is $\widetilde{A} = A - \omega_\Lambda(A)\mathbf{1}$. Since $\omega_\Lambda(A) \in \mathbb{C}$ and $\mathbf{1}$ is the unit of $\mathcal{A}_\Lambda^{\mathrm{alg}}$, we have $\widetilde{A}^\dagger = A^\dagger - \overline{\omega_\Lambda(A)}\,\mathbf{1} = \widetilde{A}$ (using positivity of ω_Λ which implies $\omega_\Lambda(A) \in \mathbb{R}$ for self-adjoint A) [19, 94].

By the definition of variance and the linearity of ω_Λ:

$$\mathrm{Var}_{\omega_\Lambda}(A) = \omega_\Lambda(\widetilde{A}^2) = \omega_\Lambda(\widetilde{A}^\dagger \widetilde{A})$$
$$= \omega_\Lambda\left(\left(A - \omega_\Lambda(A)\mathbf{1}\right)^\dagger \left(A - \omega_\Lambda(A)\mathbf{1}\right)\right).$$

In the GNS representation (Definition 4.1), we have the state recovery formula (Proposition 4.5):

$$\omega_\Lambda(X) = \langle \Omega_\Lambda \mid \pi_\Lambda(X) \mid \Omega_\Lambda \rangle_\Lambda \qquad \text{for all } X \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

Applying this to $X = \widetilde{A}^\dagger \widetilde{A}$:

$$\mathrm{Var}_{\omega_\Lambda}(A) = \langle \Omega_\Lambda \mid \pi_\Lambda(\widetilde{A}^\dagger \widetilde{A}) \mid \Omega_\Lambda \rangle_\Lambda$$
$$= \langle \Omega_\Lambda \mid \pi_\Lambda(\widetilde{A})^\dagger \pi_\Lambda(\widetilde{A}) \mid \Omega_\Lambda \rangle_\Lambda$$
$$= \left\| \pi_\Lambda(\widetilde{A})|\Omega_\Lambda\rangle \right\|^2,$$

where the second equality uses that π_Λ is a $*$-representation (Proposition 4.3), and the third equality is the definition of the Hilbert space norm [17, 71].

Substituting $\widetilde{A} = A - \omega_\Lambda(A)\mathbf{1}$ and using that π_Λ is unital ($\pi_\Lambda(\mathbf{1}) = I$):

$$\pi_\Lambda(\widetilde{A}) = \pi_\Lambda(A) - \omega_\Lambda(A)I.$$

Therefore:

$$\mathrm{Var}_{\omega_\Lambda}(A) = \left\| \left(\pi_\Lambda(A) - \omega_\Lambda(A)I\right)|\Omega_\Lambda\rangle \right\|^2.$$

For the sharpness criterion: $\mathrm{Var}_{\omega_\Lambda}(A) = 0$ iff the norm vanishes, which holds iff the vector itself is zero (positive definiteness of the GNS inner product) [19, 94]:

$$\left\| \left(\pi_\Lambda(A) - \omega_\Lambda(A)I\right)|\Omega_\Lambda\rangle \right\|^2 = 0 \quad \Longleftrightarrow \quad \left(\pi_\Lambda(A) - \omega_\Lambda(A)I\right)|\Omega_\Lambda\rangle = 0.$$

This completes the proof. $\qquad\square$

Remark 6.21 (Realization of n-Loci Variance and Dual-Subspace)**.** When a physical realization $r : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_{\mathrm{phys}}$ is employed (Definition 6.10), the variance is preserved via the isometric embedding $V : \mathcal{H}_\Lambda \hookrightarrow \mathcal{H}_{\mathrm{phys}}$ (Proposition 6.14).

- **Composite Operational Space:** For Thesis, Antithesis, and Synthesis claims, the observable $\pi_\Lambda(A)$ acts on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$. Sharpness corresponds to definite outcomes across all n loci.

- **Dual-Subspace for Holothesis:** For Holothesis claims, the observable acts on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definition 5.4). The variance in this sector quantifies the stability of the inapplicability diagnosis; a sharp Holothesis observable indicates a stable meta-test outcome (definite inapplicability) across the configuration.

This ensures that the algebraic uncertainty relations derived from these witnesses (e.g., Robertson-type bounds) apply consistently across all tetralemmatic corners and locus configurations.

Definition 6.22 (Commutator witness)**.** For self-adjoint $A, B \in \mathcal{A}_\Lambda^{\mathrm{alg}}$, define the commutator

$$[A, B] := AB - BA,$$

and the *warrant commutator witness*

$$\mathfrak{N}_{\omega_\Lambda}(A, B) := \left| \omega_\Lambda([A, B]) \right|.$$

This witness measures the degree of noncommutativity as detected by the warrant functional [71].

Remark 6.23 (Commutator witness for n-Loci observables)**.** For n-loci corner-linear observables $A = O_{\mathbf{a}}(\vec{P})$ and $B = O_{\mathbf{b}}(\vec{Q})$ defined over locus configurations $\vec{\lambda}, \vec{\mu} \in \Lambda_C^n$ (Definition 6.62), the witness $\mathfrak{N}_{\omega_\Lambda}(A, B)$ quantifies the algebraic incompatibility across the composite configuration.

- **Local Noncommutativity:** If $\vec{\lambda} = \vec{\mu}$, the witness captures noncommutativity at each locus $\lambda_i \in \vec{\lambda}$, potentially amplified by tensor product structure.

- **Cross-Locus Correlations:** If $\vec{\lambda} \neq \vec{\mu}$, the witness may detect non-separable correlations (entanglement) between distinct locus configurations, provided the warrant functional ω_Λ assigns non-zero weight to cross-locus commutators.

- **Holothesis Dual-Subspace:** For Holothesis claims, the commutator is evaluated on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definition 5.4), where the witness quantifies the stability of the meta-test diagnostic under symmetry transformations.

Proposition 6.24 (Robertson-type uncertainty from positivity)**.** *For any coherent warrant functional ω_Λ and any self-adjoint $A, B \in \mathcal{A}_\Lambda^{\mathrm{alg}}$,*

$$\mathrm{Var}_{\omega_\Lambda}(A) \, \mathrm{Var}_{\omega_\Lambda}(B) \geq \frac{1}{4} \left| \omega_\Lambda([A, B]) \right|^2 = \frac{1}{4} \mathfrak{N}_{\omega_\Lambda}(A, B)^2.$$

This is the algebraic analogue of the Robertson uncertainty relation [107, 108].

Proof. Let $\widetilde{A} = A - \omega_\Lambda(A)\mathbf{1}$ and $\widetilde{B} = B - \omega_\Lambda(B)\mathbf{1}$ be the centered observables. Note that $[\widetilde{A}, \widetilde{B}] = [A, B]$ since scalar multiples of $\mathbf{1}$ commute with everything.

In the GNS Hilbert space $\mathcal{H}_\Lambda$, define the vectors

$$|\psi_A\rangle := \pi_\Lambda(\widetilde{A})|\Omega_\Lambda\rangle, \qquad |\psi_B\rangle := \pi_\Lambda(\widetilde{B})|\Omega_\Lambda\rangle.$$

By Proposition 6.20, we have $\mathrm{Var}_{\omega_\Lambda}(A) = \||\psi_A\rangle\|^2$ and $\mathrm{Var}_{\omega_\Lambda}(B) = \||\psi_B\rangle\|^2$.

Apply the Cauchy–Schwarz inequality in $\mathcal{H}_\Lambda$ [19, 71]:

$$\left| \langle \psi_A \mid \psi_B \rangle_\Lambda \right|^2 \leq \||\psi_A\rangle\|^2 \, \||\psi_B\rangle\|^2 = \mathrm{Var}_{\omega_\Lambda}(A) \, \mathrm{Var}_{\omega_\Lambda}(B).$$

Now compute the inner product:

$$\langle \psi_A \mid \psi_B \rangle_\Lambda = \langle \Omega_\Lambda \mid \pi_\Lambda(\widetilde{A})^\dagger \pi_\Lambda(\widetilde{B}) \mid \Omega_\Lambda \rangle_\Lambda = \omega_\Lambda(\widetilde{A}^\dagger \widetilde{B}) = \omega_\Lambda(\widetilde{A}\widetilde{B}),$$

using that $\widetilde{A}, \widetilde{B}$ are self-adjoint.

Write $\omega_\Lambda(\widetilde{A}\widetilde{B})$ in terms of its real and imaginary parts:

$$\omega_\Lambda(\widetilde{A}\widetilde{B}) = \frac{1}{2}\omega_\Lambda(\{\widetilde{A}, \widetilde{B}\}) + \frac{1}{2}\omega_\Lambda([\widetilde{A}, \widetilde{B}]),$$

where $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}\widetilde{B} + \widetilde{B}\widetilde{A}$ is the anticommutator. Since A, B are self-adjoint, $\{\widetilde{A}, \widetilde{B}\}$ is self-adjoint and $[\widetilde{A}, \widetilde{B}]$ is skew-adjoint, so $\omega_\Lambda(\{\widetilde{A}, \widetilde{B}\}) \in \mathbb{R}$ and $\omega_\Lambda([\widetilde{A}, \widetilde{B}]) \in i\mathbb{R}$ [17].

Therefore:

$$\left|\omega_\Lambda(\widetilde{A}\widetilde{B})\right|^2 = \frac{1}{4}\left|\omega_\Lambda(\{\widetilde{A}, \widetilde{B}\})\right|^2 + \frac{1}{4}\left|\omega_\Lambda([\widetilde{A}, \widetilde{B}])\right|^2 \geq \frac{1}{4}\left|\omega_\Lambda([A, B])\right|^2.$$

Combining with Cauchy–Schwarz:

$$\mathrm{Var}_{\omega_\Lambda}(A)\, \mathrm{Var}_{\omega_\Lambda}(B) \geq \left|\omega_\Lambda(\widetilde{A}\widetilde{B})\right|^2 \geq \frac{1}{4}\left|\omega_\Lambda([A, B])\right|^2,$$

which is the Robertson bound [71, 107, 108]. $\qquad\square$

Corollary 6.25 (No joint sharpness under nonzero commutator witness). *If* $\mathfrak{N}_{\omega_\Lambda}(A, B) \neq 0$, *then* A *and* B *cannot both be* ω_Λ-*sharp. Equivalently,*

$$\mathrm{Var}_{\omega_\Lambda}(A) = 0 = \mathrm{Var}_{\omega_\Lambda}(B) \quad \Longrightarrow \quad \omega_\Lambda([A, B]) = 0.$$

This is the evaluator-level witness of complementarity: noncommuting observables cannot both have zero variance [7, 19].

Remark 6.26 (n-Loci joint sharpness). For n-loci observables $A = O_{\mathbf{a}}(\vec{P})$ and $B = O_{\mathbf{b}}(\vec{Q})$, joint sharpness requires zero variance across *all* loci in the configurations $\vec{\lambda}, \vec{\mu}$. If $\mathfrak{N}_{\omega_\Lambda}(A, B) \neq 0$, then at least one locus $\lambda_i \in \vec{\lambda}$ or $\mu_j \in \vec{\mu}$ exhibits indeterminacy, preventing global joint definiteness. This captures the algebraic signature of contextuality across composite configurations [8, 9].

Definition 6.27 (Corner-linear observables for a claim). For any admissible polar claim P (cf. Definition 3.18) and any real tuple $\mathbf{a} = (a_{\mathbf{Th}}, a_{\mathbf{AntiTh}}, a_{\mathbf{SynTh}}, a_{\mathbf{HoloTh}}) \in \mathbb{R}^4$, define

$$O_{\mathbf{a}}(P) := \sum_{\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}} a_\bullet\, C_\bullet(P) \in \mathcal{A}_\Lambda^{\mathrm{alg}}.$$

Then $O_{\mathbf{a}}(P)$ is self-adjoint and the witness/uncertainty statements above apply to pairs $O_{\mathbf{a}}(P), O_{\mathbf{b}}(Q)$.

Remark 6.28 (n-Loci corner-linear observables). For n-loci claims $\vec{P}$ defined over locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ (Definition 5.1), the corner-linear observable generalizes to:

$$O_{\mathbf{a}}(\vec{P}) := \sum_\bullet a_\bullet\, C_\bullet(\vec{P}) \in \mathcal{A}_\Lambda^{\mathrm{alg}},$$

where $C_\bullet(\vec{P}) = \otimes_{i=1}^n C_\bullet(P_{\lambda_i})$ are the n-loci corner generators. The uncertainty relations apply identically, with variances and commutators evaluated on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (for Thesis, Antithesis, Synthesis) or the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (for Holothesis, Definition 5.4).

Remark 6.29 (Compatibility with realizations)**.** If a realization $r : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_{\mathrm{phys}}$ (Definition 3.57) is given, then $\mathfrak{N}_{\omega_\Lambda}(A,B) = \left| \omega_{\mathrm{phys}}([r(A), r(B)]) \right|$. Thus noncommutativity can be detected purely at the evaluator level and then transported to any concrete quantum realization [16, 17].

Remark 6.30 (n-Loci realization structure)**.** For n-loci observables, the realization map r must preserve the tensor product structure:

- **Composite Operational Space:** For Thesis, Antithesis, Synthesis, r maps $O_{\mathbf{a}}(\vec{P})$ to operators on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \subseteq \mathcal{H}_{\mathrm{phys}}$.

- **Dual-Subspace for Holothesis:** For Holothesis claims, r maps $O_{\mathbf{a}}(\vec{P})$ to operators on $\mathcal{H}_{\mathrm{Dual}}^{(2n)} \subseteq \mathcal{H}_{\mathrm{phys}}$, ensuring the meta-test structure is preserved (Definition 5.8).

- **Commutator Preservation:** Since r is a $*$-homomorphism, $[r(A), r(B)] = r([A,B])$, so the witness $\mathfrak{N}_{\omega_\Lambda}$ is invariant under faithful realizations.

This ensures the algebraic uncertainty relations derived at the logical level hold identically in all physical realizations [7, 63].

6.5 Realization Axiom: Interpretation into the Fourfold Quantum Model

Definition 6.31 (Realization Axiom (Interpretation Map $\mathcal{R}$))**.** Let $\mathcal{L}$ be the language of formulas used in the formal logic of Section 2. Let $\mathcal{M}$ be the Fourfold Quantum Model with a context-indexed Hilbert space structure and a unital $*$-algebra of operationally admissible observables, following the standard operator-algebraic formulation of quantum theory [6, 71].

Fix an immanent context $\mathcal{C}$ and a *locus configuration* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$ (for $n = 1$, this reduces to a single locus λ). Let $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ denote the *composite operational space* for Thesis, Antithesis, and Synthesis corners, and let $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ denote the *n-loci dual-subspace* for Holothesis meta-tests (Definition 5.4).

Let $\mathcal{E}_{\mathcal{C},\vec{\lambda}}^{(n)}$ denote the set of effects operationally admissible at $(\mathcal{C}, \vec{\lambda})$ for the polar and synthesis corners:

$$\mathcal{E}_{\mathcal{C},\vec{\lambda}}^{(n)} := \left\{ E \in \mathcal{B}(\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}) \ : \ 0 \le E \le I_{\mathcal{H}^{(n)}} \text{ and } E \text{ is available as a measurement effect at } \vec{\lambda} \right\}.$$

Let $\mathcal{E}_{\mathrm{Dual}}^{(2n)}$ denote the set of effects admissible for the Holothesis meta-test:

$$\mathcal{E}_{\mathrm{Dual}}^{(2n)} := \left\{ E \in \mathcal{B}(\mathcal{H}_{\mathrm{Dual}}^{(2n)}) \ : \ 0 \le E \le I_{\mathrm{Dual}} \text{ and } E \text{ is a valid Holothesis meta-test effect} \right\}.$$

This structure aligns with the framework of operational quantum physics and positive-operator-valued measures (POVMs) [57, 63, 83], extended to the n-loci Born semantics of Definition 5.11.

Let $\mathsf{R}(\mathcal{C}) \subseteq U(\mathcal{H}_{\mathcal{C},\vec{\lambda}})$ be the subgroup (or generated set) of sanctioned unitary symmetries regarded as admissible refinements of $\mathcal{C}$, consistent with Wigner's theorem on symmetry transformations [73, 74].

(R1) Content $\mapsto$ effect. There is a (generally partial) interpretation map

$$\mathcal{R}_{\mathcal{C},\vec{\lambda}} : \mathcal{L} \rightharpoonup \mathcal{E}_{\mathcal{C},\vec{\lambda}}^{(n)} \cup \mathcal{E}_{\mathrm{Dual}}^{(2n)}$$

such that, for each formula $\varphi(x)$, the context- and locus-relative existential claim

$$P := \exists_{\vec{\lambda}}^{\mathcal{C}} x \, \varphi(x)$$

is interpreted (when defined) by an effect according to its warranted corner status:

- **Thesis, Antithesis, Synthesis:** For $M \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$,

$$E_M^{(n)}(P) := \mathcal{R}_{\mathcal{C},\vec{\lambda}}(M(P)) \in \mathcal{E}_{\mathcal{C},\vec{\lambda}}^{(n)}.$$

  Its polarity partner is interpreted by the complementary effect $E_{\neg P} := I - E_P$ (in the sharp case, orthogonal projections), recovering the standard von Neumann measurement scheme [6, 59].

- **Holothesis:** For $M = \mathbf{HoloTh}$, the effect is defined via the meta-test construction (Definition 5.8):
$$E_{\mathrm{HoloTh}}^{(2n)}(P) := \mathcal{R}_{\mathcal{C},\vec{\lambda}}(\mathbf{HoloTh}(P)) \in \mathcal{E}_{\mathrm{Dual}}^{(2n)}.$$

  This effect acts on the n-loci dual-subspace and diagnoses polar inapplicability without collapsing the symmetry.

(R2) Applicability $\leftrightarrow$ definability in the operational algebra. The applicability judgement is identified with interpretability as an admissible effect:

$$\mathrm{App}_{\mathcal{C}}(\chi; \vec{\lambda}) \iff \mathcal{R}_{\mathcal{C},\vec{\lambda}}(\chi) \text{ is defined (and lies in the appropriate effect set).}$$

This condition reflects the contextuality of quantum observables, where certain questions are ill-typed relative to a specific experimental arrangement [2, 9, 11]. In particular:

- For polar claims (**Th**, **AntiTh**), applicability requires definability in $\mathcal{E}_{\mathcal{C},\vec{\lambda}}^{(n)}$.

- For Holothesis, applicability requires definability in $\mathcal{E}_{\mathrm{Dual}}^{(2n)}$ via the meta-test.

- When $\neg \mathrm{App}_{\mathcal{C}}(P; \vec{\lambda})$ and $\neg \mathrm{App}_{\mathcal{C}}(\neg P; \vec{\lambda})$, the polar effects are undefined, warranting Holothesis status.

(R3) Holding $\Vdash$ as Born valuation (with optional thresholding). For any prepared state ρ on the appropriate space ($\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ for Th/AntiTh/SynTh, or $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ for HoloTh), the holding judgement is realized by the Born valuation on the interpreted effect, following the foundational probability rule [58, 84] and the n-loci Born semantics of Definition 5.11:

$$\mathcal{C}, \vec{\lambda} \Vdash_\rho \chi \quad \Longleftrightarrow \quad \mathrm{Tr}\!\left(\mathcal{R}_{\mathcal{C},\vec{\lambda}}(\chi)\,\rho\right) = 1.$$

If one prefers probabilistic holding, fix a threshold $\tau \in (0, 1]$ and set

$$\mathcal{C}, \vec{\lambda} \Vdash_{\rho,\tau} \chi \quad \Longleftrightarrow \quad \mathrm{Tr}\!\left(\mathcal{R}_{\mathcal{C},\vec{\lambda}}(\chi)\,\rho\right) \geq \tau.$$

Such thresholding is common in quantum detection theory and state discrimination tasks [56, 83, 103].

(R4) Refinements as sanctioned duality symmetries. Each admissible refinement $r \in \mathsf{R}(\mathcal{C})$ acts on effects by conjugation,

$$E \longmapsto rEr^\dagger,$$

and preserves admissibility at $\vec{\lambda}$ (so $rEr^\dagger$ lies in the appropriate effect set whenever E does). Stability conditions such as (S2) and (H2) are read as invariance of the warranted status of the corresponding act under the action of $\mathsf{R}(\mathcal{C})$, analogous to symmetry constraints in open quantum systems [109, 110]. For n-loci configurations, refinements must preserve the tensor product structure across loci and the dual-subspace structure for Holothesis.

With (R1)–(R4), the formal logic of Section 2 is modeled inside $\mathcal{M}$: claims become operational effects, applicability becomes definability within the admitted operational algebra at $\vec{\lambda}$, and holding becomes Born valuation (possibly thresholded). In particular, the Holothesis corner corresponds to the failure of definability of the polar tests at $\vec{\lambda}$, i.e. $\neg \mathrm{App}_{\mathcal{C}}(P; \vec{\lambda})$ and $\neg \mathrm{App}_{\mathcal{C}}(\neg P; \vec{\lambda})$ mean that no such polar effect is admitted at the operational locus configuration under $\mathcal{C}$, warranting the meta-test on $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$. This resonates with Bohr's doctrine of complementarity and modern topos-theoretic approaches to physics [9, 11].

Remark 6.32 (N-Loci structural requirements)**.** The Realization Axiom must satisfy the following structural requirements for n-loci configurations:

- **Composite Operational Space:** For Thesis, Antithesis, and Synthesis, the realization map must preserve the tensor product structure $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} = \bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$ (Definitions 5.1 to 5.3).

- **Dual-Subspace for Holothesis:** For Holothesis, the realization must map to operators on $\mathcal{H}_{\mathrm{Dual}}^{(2n)} = \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (Definitions 5.4 and 5.7). This is structurally required to implement the Holothesis meta-test without collapsing polar applicability.

- **Born Semantics Consistency:** The Born valuation must align with Definition 5.11, distinguishing the effect domains for polar/synthesis corners ($\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$) and Holothesis ($\mathcal{H}_{\mathrm{Dual}}^{(2n)}$).

- **Symmetry Preservation:** Refinements must commute with the duality involution $\mathbf{D}_{\mathcal{C}}$ and quarter-turn symmetry $\mathbf{R}_{\mathcal{C}}$ across all loci, ensuring the unitary implementers $U_{\mathbf{D}}$ and $U_{\mathbf{R}}$ exist on the realized space (Propositions 5.16 and 5.27).

These requirements ensure that the logic-first reconstruction remains consistent under physical realization across all tetralemmatic corners and locus configurations.

Remark 6.33 (Connection to GNS embedding)**.** The Realization Axiom is the operational counterpart to the GNS embedding theorem (Proposition 6.14). The interpretation map $\mathcal{R}_{\mathcal{C},\vec{\lambda}}$ corresponds to the $*$-homomorphism $r : \mathcal{A}_{\Lambda}^{\mathrm{alg}} \to \mathcal{A}_{\mathrm{phys}}$, and the isometric embedding $V : \mathcal{H}_{\Lambda} \hookrightarrow \mathcal{H}_{\mathrm{phys}}$ ensures that the logical warrant functional ω_{Λ} coincides with the physical Born valuation ω_{phys} on the image of the realization. For n-loci configurations, this embedding preserves the composite operational space and dual-subspace structures, ensuring all algebraic witnesses (commutators, uncertainty relations, contextuality graphs) transport faithfully from the logical to the physical level (Proposition 6.74).

6.6   Foundational Implications: Interference and Outcomes

This section addresses two critical foundational questions: the potential for deviations from standard quantum mechanics (SQM) in higher-order interference, and the resolution of the measurement problem within the tetralemmatic ontology. These implications hold for both single-locus and *n-loci configurations*, with many-particle regimes offering enhanced sensitivity to logical deficiency in symmetric subspaces (Remark 6.37 and Proposition 6.39).

6.6.1   Predictive Deviations: The Sorkin Witness

Standard quantum mechanics, built on complex Hilbert spaces ($U(1)$ symmetry), predicts the vanishing of Sorkin's third-order interference term I_3 [8, 14]. The tetralemmatic framework, derived from an order-4 logical symmetry (Z_4), recovers the Born rule as an asymptotic limit (Section 6.5) but allows for a potential correction term in regimes of maximal logical saturation.

To establish this rigorously, we first decompose the warrant functional into interference orders. This decomposition applies to warrant functionals defined over locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ (cf. Definition 5.11).

Lemma 6.34 (Interference Decomposition of the Warrant Functional)**.** *Let $\{P_1, \ldots, P_n\}$ be a set of mutually exclusive path claims (e.g., slits) admissible at locus configuration $(\mathcal{C}, \vec{\lambda})$. Let $S \subseteq \{1, \ldots, n\}$ denote a subset of open paths. The probability warrant $P_S := \omega_{\Lambda}\left(C_{\mathbf{Th}}(\bigvee_{i \in S} P_i)\right)$ admits a decomposition:*

$$P_S = \sum_{i \in S} w_i + \sum_{\{i,j\} \subseteq S} I_{ij} + \sum_{\{i,j,k\} \subseteq S} I_{ijk} + \ldots \tag{59}$$

where w_i are single-path weights, I_{ij} are pairwise interference terms, and I_{ijk} are higher-order interference terms. In standard quantum mechanics (Tier-2 enrichment), $I_{ijk} = 0$ for all triplets [14].

Proof. The decomposition follows from the multilinearity of the warrant functional ω_Λ on the corner-observable algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$ (Definition 3.35) and the inclusion-exclusion principle applied to the logical join operation $\vee$. This holds for both single-locus and n-loci claims, as the algebraic structure is preserved under tensor product composition (Definition 5.1).

Case $n = 2$ (two paths): For mutually exclusive claims P_1, P_2:

$$\omega_\Lambda\big(C_{\mathbf{Th}}(P_1 \vee P_2)\big) = \omega_\Lambda\big(C_{\mathbf{Th}}(P_1)\big) + \omega_\Lambda\big(C_{\mathbf{Th}}(P_2)\big) + I_{12}, \tag{60}$$

where I_{12} is the pairwise interference term. This recovers the standard Born rule $|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\operatorname{Re}(\psi_1^* \psi_2)$ via the n-loci Born semantics (Definition 5.11).

Case $n = 3$ (three paths): For mutually exclusive claims P_1, P_2, P_3:

$$\omega_\Lambda\big(C_{\mathbf{Th}}(P_1 \vee P_2 \vee P_3)\big) = \sum_{i=1}^{3} \omega_\Lambda\big(C_{\mathbf{Th}}(P_i)\big) \\ + \sum_{\{i,j\} \subseteq \{1,2,3\}} I_{ij} + I_{123}. \tag{61}$$

The third-order term I_{123} is defined as the residual after subtracting all single-path and pairwise contributions. In standard quantum mechanics, $I_{123} = 0$ because the Born rule on a complex Hilbert space contains only pairwise cross-terms [59, 63].

General n: The decomposition extends by induction. The vanishing of I_{ijk} in SQM corresponds to the fact that the realized effects E_i are projectors on a complex Hilbert space, where $|\sum \psi_i|^2$ contains only pairwise cross-terms [14]. **In the tetralemmatic framework, $I_{ijk} \neq 0$** arises from non-vanishing warrant weight in the Holothesis sector, which operates on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definitions 5.4 and 5.7). This prevents the perfect cancellation of higher-order terms, yielding $I_3 \propto \mathcal{K}(\Gamma, d)$ (Proposition 6.36). $\qquad\square$

Remark 6.35 (Holothesis and Higher-Order Interference)**.** The presence of non-zero higher-order interference terms ($I_{ijk} \neq 0$) is a direct signature of Holothesis warrant. Since Holothesis corresponds to polar inapplicability diagnosed via the n-loci dual-subspace (Definition 5.8), deviations from $I_3 = 0$ indicate regimes where the logical deficiency constant $\mathcal{K}(\Gamma, d)$ is non-zero. This distinguishes the tetralemmatic framework from standard quantum logic, where $I_3 = 0$ is enforced by the Hilbert space structure alone [8, 14]. For n-loci configurations (many-particle systems), the incompatibility graph Γ grows combinatorially, potentially enhancing sensitivity to $\mathcal{K}$ in symmetric subspaces (Remark 6.37 and Proposition 6.39).

Proposition 6.36 (Tetralemmatic Correction to I_3 (n-Loci))**.** *Let P_{ABC} denote the probability detection rate in a triple-slit experiment with all slits open, and $P_{AB}, P_{BC}, \dots$ denote rates with subsets of slits open. These probabilities are evaluated via the n-loci Born semantics (Definition 5.11) for claims defined over a locus configuration $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$. Define the Sorkin interference parameter:*

$$I_3 := P_{ABC} - (P_{AB} + P_{BC} + P_{AC}) + (P_A + P_B + P_C). \tag{62}$$

In the tetralemmatic framework, I_3 is constrained by the logical deficiency constant $\mathcal{K}(\Gamma, d)$ from Theorem 6.94 and the quarter-turn phase $\theta_{\mathbf{R}}$:

$$I_3 = \mathcal{K}(\Gamma, d) \cdot \sin(4\theta_{\mathbf{R}}) + \mathcal{O}\big(\mathcal{K}^2\big), \tag{63}$$

where Γ is the incompatibility graph of the path claims over the configuration $\vec{\lambda}$ and $d \in \{2, 4\}$ is the dimension of the local logical modal subspace restricted to the path subspace (Definitions 3.46 and 5.21). For $n > 1$ (many-particle regimes), $\mathcal{K}$ may scale with the combinatorial growth of Γ in symmetric subspaces (Proposition 6.39).

Proof. **Step 1: Expansion via Lemma 6.34.** Substituting the decomposition (59) (updated for n-loci claims in Lemma 6.34) into the definition (62):

$$I_3 = \left(\sum w_i + \sum I_{ij} + \sum I_{ijk}\right)_{ABC}$$
$$- \left[\left(\sum w_i + \sum I_{ij}\right)_{AB} + \left(\sum w_i + \sum I_{ij}\right)_{BC} + \left(\sum w_i + \sum I_{ij}\right)_{AC}\right]$$
$$+ \left[(w_i)_A + (w_i)_B + (w_i)_C\right].$$

By the inclusion-exclusion principle, all single-path weights w_i and pairwise interference terms I_{ij} cancel exactly. The only remaining term is the third-order interference:

$$I_3 = I_{ABC}. \tag{64}$$

Step 2: Relating I_{ABC} to Logical Deficiency and Dual-Subspace. In the tetralemmatic algebra, the join operation $\vee$ is not perfectly additive due to the presence of the Holothesis corner. Specifically, for three mutually exclusive claims P_A, P_B, P_C defined over $\vec{\lambda}$:

$$C_{\mathbf{Th}}(P_A \vee P_B \vee P_C) \neq \sum C_{\mathbf{Th}}(P_i) + \text{Interference}. \tag{65}$$

The deviation is governed by the failure of the classical valuation, quantified by the Logical Deficiency Bound (Theorem 6.94):

$$\sum_{P \in \mathcal{P}} \omega_\Lambda\left(C_{\mathbf{HoloTh}}(P)\right) \geq \mathcal{K}(\Gamma, d). \tag{66}$$

The third-order term I_{ABC} arises precisely from the non-vanishing warrant weight in the Holothesis sector. Crucially, for n-loci configurations, the Holothesis warrant is evaluated on the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definitions 5.4 and 5.7). This dual-subspace structure prevents the perfect cancellation of higher-order terms that occurs in standard quantum mechanics (where only the single-register composite space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ is used). Thus I_{ABC} is proportional to $\mathcal{K}(\Gamma, d)$, with the proportionality constant reflecting the overlap between single-register and dual-subspace sectors.

Step 3: Symmetry Constraint $(Z_4$ vs $U(1))$. Standard quantum mechanics corresponds to a logical symmetry where the quarter-turn phase is $\theta_{\mathbf{R}} = \pi/2$ (recovering $i^2 = -1$). In this limit, $\sin(4\theta_{\mathbf{R}}) = \sin(2\pi) = 0$, and $I_3 = 0$. However, the tetralemmatic framework allows for a deviation $\theta_{\mathbf{R}} \neq \pi/2$ (Theorem 5.33). The leading-order correction to the interference pattern must respect the Z_4 symmetry of the corner algebra. The lowest-order invariant polynomial under Z_4 that vanishes at $\theta_{\mathbf{R}} = \pi/2$ is $\sin(4\theta_{\mathbf{R}})$. Therefore:

$$I_{ABC} = \mathcal{K}(\Gamma, d) \cdot \sin(4\theta_{\mathbf{R}}) + \mathcal{O}\left(\mathcal{K}^2\right), \tag{67}$$

which yields (63). The constant of proportionality is absorbed into the definition of $\mathcal{K}(\Gamma, d)$ from Theorem 6.94, which is a structural constant determined by the graph topology and Hilbert space dimension. $\qquad\square$

Remark 6.37 (Empirical Status and Bounds (n-Loci)).   Standard Regime:

In regimes where the warrant functional ω_Λ is perfectly Z_4-symmetric (Theorem 5.33), $\mathcal{K}(\Gamma, d) = 0$ and $I_3 = 0$, recovering SQM [14]. This holds for both single-locus and n-loci configurations $\vec{\lambda} \in \Lambda_\mathcal{C}^n$.

Experimental Bounds (2010–2026):

Initial tests [25, 111] constrained $|\mathcal{K}| \lesssim 10^{-2}$. Recent advances in attosecond photoionization [112], integrated photonics [113], and superconducting quantum processors [26] have tightened this bound to roughly $|\mathcal{K}| \lesssim 10^{-3}$. This implies the tetralemmatic symmetry is preserved to high precision in standard interferometry and quantum computing contexts. **Note:** Current experiments primarily probe the dyadic reduction regime ($d = 2$), where $\mathcal{K} = 0$ is predicted by the framework (Proposition 6.39).

Many-Particle Sensitivity:

Following Pleinert et al. [27], many-particle interference tests forbid terms of order $(2M + 1)$ for M particles. In the tetralemmatic framework, an M-particle system corresponds to an *n-loci configuration* $\vec{\lambda}$ with $n \approx M$. The framework *suggests* that many-body correlations may be significantly more sensitive to logical deficiency $\mathcal{K}(\Gamma, d)$ due to the combinatorial growth of the incompatibility graph Γ over the configuration $\vec{\lambda}$. For certain graph structures, the chromatic number $\mathscr{X}(\Gamma_{\vec{\lambda}})$ can grow exponentially with n, which would imply $\mathcal{K}(\Gamma_{\vec{\lambda}}, d) \sim \mathscr{X}(\Gamma_{\vec{\lambda}}) - d$ scales accordingly [21].

Dimensional and Subspace Qualification: Crucially, non-zero $\mathcal{K}$ requires the generic four-corner case ($d = 4$) per Proposition 6.39. For the dyadic reduction ($d = 2$), $\mathcal{K}_n = 0$ regardless of graph growth. Furthermore, amplification ($\mathcal{K}_n \to 1$) is guaranteed only in symmetric subspaces where dimension grows polynomially ($d_n \sim n^3$) rather than the full tensor space ($d_n = 4^n$).

For n-loci configurations, the Holothesis warrant is evaluated on the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definitions 5.4 and 5.7). This dual-subspace structure provides a larger arena for correlation structures to manifest compared to the single-register composite space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$, potentially enhancing sensitivity to $\mathcal{K}$ in many-particle regimes. However, the precise scaling depends on the specific experimental configuration and remains an open question for future investigation.

This provides a clear empirical boundary: the framework acts as an *interpretative reconstruction* if $\mathcal{K} = 0$, but offers *novel physics* if deviations are detected in extreme regimes beyond effective nonlinearities [112].

6.7 Combinatorial Scaling of Logical Deficiency in *n*-Loci Configurations

The qualitative suggestion that many-body correlations enhance sensitivity to logical deficiency [27] can be made rigorous by analyzing the asymptotic behavior of the incompatibility graph Γ_n over *n*-loci configurations. We establish that the logical deficiency constant $\mathcal{K}(\Gamma_n, d_n)$ does not merely persist but *amplifies* with n under specific structural conditions, and that the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ is the minimal structural requirement to preserve this amplification against environmental decoherence.

Definition 6.38 (n-Loci Incompatibility Graph Product)**.** Let $\Gamma_1 = (V_1, E_1)$ be the incompatibility graph for a single locus (e.g., the KCBS pentagon C_5 for a qutrit). The *n-loci incompatibility graph* Γ_n for a configuration $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ is defined as the *strong product* of n copies of Γ_1:

$$\Gamma_n \; := \; \Gamma_1^{\boxtimes n} \; = \; \underbrace{\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_1}_{n \text{ times}} . \tag{68}$$

Vertices in Γ_n correspond to joint claims across the n loci, and edges represent pairwise incompatibility in at least one locus factor.

Proposition 6.39 (Asymptotic Growth of Logical Deficiency)**.** *Let $\mathscr{X}(\Gamma_n)$ denote the chromatic number of the n-loci incompatibility graph and d_n denote the dimension of the Hilbert space restricted to the configuration $\vec{\lambda}$. Let $d \in \{2, 4\}$ be the local logical modal dimension (Definition 5.21).*

(i) *Dyadic Reduction ($d = 2$): If the enrichment regime imposes Tier-2 effect-like exclusivity such that $d = 2$, then $\mathscr{X}(\Gamma_n) \leq 2$ for all n. Consequently, $\mathcal{K}_n = 0$ and no contextuality arises.*

(ii) *Generic Four-Corner Case ($d = 4$): If the local space supports all four corners ($d = 4$), then contextuality witnesses exist ($\mathcal{K}_n > 0$) wherever $\mathscr{X}(\Gamma_n) > 4$.*

(iii) *Amplification Condition: The logical deficiency constant $\mathcal{K}_n$ approaches unity exponentially fast ($\mathcal{K}_n \to 1$) if the chromatic number grows faster than the Hilbert space dimension ($\mathscr{X}(\Gamma_n) \gg d_n$). For the KCBS pentagon ($\Gamma_1 = C_5$), $\mathscr{X}(\Gamma_n) \geq (\sqrt{5})^n$.*

  - *Full Tensor Space: $d_n = 4^n$. Since $4 > \sqrt{5}$, amplification is not guaranteed for arbitrary states in the full space.*

  - *Symmetric Subspace: For indistinguishable bosonic states (typical in interference experiments), $d_n \sim n^{d-1} = n^3$. Since exponential growth dominates polynomial growth, $\mathcal{K}_n \to 1$ as $n \to \infty$.*

Proof. **Step 1: Dyadic Reduction ($d = 2$).** By Remark 6.135, incompatibility graphs for $d = 2$ are structurally 2-colorable. Thus $\mathscr{X}(\Gamma_n) \leq 2 = d_n$, implying $\mathcal{K}_n = 0$ by Theorem 6.94.

Step 2: Generic Existence ($d = 4$). For $d = 4$, contextuality arises wherever $\mathscr{X}(\Gamma_n) > 4$. For the KCBS pentagon, $\mathscr{X}(C_5) = 3$, so single-locus contextuality requires embedding in $d \geq 3$. For n-loci strong products, $\mathscr{X}(\Gamma_n) \geq (\sqrt{5})^n$. For $n \geq 3$, $(\sqrt{5})^n > 4$, ensuring $\mathcal{K}_n > 0$.

Step 3: Amplification Analysis. By Theorem 6.94, $\mathcal{K}_n \geq 1 - d_n/\mathscr{X}(\Gamma_n)$.

  - *Full Space: $d_n = 4^n$.* The ratio $d_n/\mathscr{X}(\Gamma_n) \geq (4/\sqrt{5})^n \to \infty$. The bound becomes trivial ($\mathcal{K}_n \geq$ negative), so amplification is not guaranteed without further state restrictions.

  - *Symmetric Subspace:* For bosonic states, the effective dimension grows polynomially: $d_n = \binom{n+d-1}{n} \sim n^{d-1} = n^3$ [27]. The ratio $d_n/\mathscr{X}(\Gamma_n) \sim n^3/(\sqrt{5})^n \to 0$ as $n \to \infty$. Thus $\mathcal{K}_n \to 1$.

This completes the proof. $\qquad\square$

Remark 6.40 (Asymptotic Growth of Logical Deficiency: Technical Analysis)**.** Proposition 6.39 establishes the critical technical bridge between the graph-theoretic structure of contextuality and the tetralemmatic logical framework. This remark provides a detailed analysis of the three regimes identified in the proposition and their empirical implications.

Core Concepts and Key Bound. The logical deficiency constant $\mathcal{K}_n$ quantifies the minimum warrant weight diverted to non-classical corners (Synthesis/Holothesis) when classical valuation fails. From Theorem 6.94, we have the fundamental bound:

$$\mathcal{K}_n \;\geq\; 1 \;-\; \frac{d_n}{\mathscr{X}(\Gamma_n)}, \tag{69}$$

where $\mathscr{X}(\Gamma_n)$ is the chromatic number of the n-loci incompatibility graph (Definition 6.68) and d_n is the Hilbert space dimension restricted to the configuration $\vec{\lambda}$ (Definition 5.21). When $\mathscr{X}(\Gamma_n) \gg d_n$, we obtain $\mathcal{K}_n \to 1$ (macroscopic logical deficiency).

Three Regimes. The proposition distinguishes three operational regimes based on the local logical modal dimension $d \in \{2, 4\}$ (Definition 3.46):

(i) **Dyadic Reduction ($d = 2$):** Standard qubit systems (superconducting processors, spin-1/2) operate in this regime. Incompatibility graphs are structurally 2-colorable (Remark 6.135), yielding $\mathscr{X}(\Gamma_n) \leq 2 = d_n$ and thus $\mathcal{K}_n = 0$. *No contextuality arises* in the dyadic regime. This explains why experimental bounds from superconducting qubits ($|\mathcal{K}| \lesssim 10^{-3}$, Remark 6.43) are consistent with the framework—they probe the $d = 2$ regime where $\mathcal{K} = 0$ is predicted.

(ii) **Generic Four-Corner Case ($d = 4$):** Full tetralemmatic capacity (all four corners independent). Incompatibility graphs can have $\mathscr{X} > 4$ for $n \geq 3$. For the KCBS pentagon ($\Gamma_1 = C_5$), $\mathscr{X}(C_5) = 3$, and for n-loci strong products, $\mathscr{X}(\Gamma_n) \geq (\sqrt{5})^n$ (Lovász theta bound [114]). For $n \geq 3$, $(\sqrt{5})^n > 4$, ensuring $\mathcal{K}_n > 0$. *Contextuality arises naturally* when the graph is chromatically exhausted.

(iii) **Amplification Condition:** The logical deficiency constant approaches unity exponentially fast ($\mathcal{K}_n \to 1$) only if the chromatic number grows faster than the Hilbert space dimension ($\mathscr{X}(\Gamma_n) \gg d_n$). This distinguishes two subspaces:

  - *Full Tensor Space:* $d_n = 4^n$ (exponential). Since $4 > \sqrt{5}$, amplification is *not guaranteed* for arbitrary states ($d_n/\mathscr{X}(\Gamma_n) \to \infty$).
  - *Symmetric Subspace:* For indistinguishable bosonic states (typical in interference experiments), $d_n = \binom{n+d-1}{n} \sim n^{d-1} = n^3$ (polynomial) [27]. Since exponential growth dominates polynomial growth, $d_n/\mathscr{X}(\Gamma_n) \to 0$ and $\mathcal{K}_n \to 1$ as $n \to \infty$.

Physical Interpretation. Many-particle interference experiments (e.g., multi-photon, multi-atom interferometry) naturally operate in symmetric subspaces. These are the *optimal regimes* for detecting logical deficiency ($\mathcal{K}_n \approx 1$ for large n). The framework predicts *enhanced sensitivity to contextuality* in many-body systems, but *only in symmetric subspaces*.

Connection to Broader Framework.

- **Dimensional Consistency:** This proposition aligns with the corrected Definition 5.21: Dyadic ($d = 2$) yields $\dim(\mathcal{H}_{\mathrm{Total}}) = 2^{2n+1}$; Generic ($d = 4$) yields $\dim(\mathcal{H}_{\mathrm{Total}}) = 2^{4n+1}$. Contextuality requires the generic case ($d = 4$).

- **Holothesis Connection:** From Proposition 6.144:

$$\sum_{\vec{P} \in \mathcal{P}} \omega_\Lambda\left(C_{\mathbf{HoloTh}}(\vec{P})\right) \ \geq \ \mathcal{K}(\Gamma, d). \tag{70}$$

When $\mathcal{K}_n \to 1$, the *Holothesis density becomes macroscopic*—this is the physical signature of the framework.

- **Empirical Predictions:** From Corollary 6.42:

$$I_3^{(n)} \ \approx \ \left(1 - \frac{poly(n)}{(\sqrt{5})^n}\right) \cdot \sin(4\theta_{\mathbf{R}}) \cdot \eta_{\mathrm{dual}}(n). \tag{71}$$

Symmetric subspace + dual-subspace protection (Remark 6.41) $\implies I_3^{(n)} \not\to 0$; Full space or single-subspace $\implies I_3^{(n)} \to 0$ (decoherence).

Summary. Table 5 summarizes the three regimes.

Table 5: Regimes of Logical Deficiency Scaling

Regime	d	d_n Growth	$\mathscr{X}(\Gamma_n)$ Growth	$\mathcal{K}_n$	Contextuality?
Dyadic (Qubits)	2	2^n	$\leq 2^n$	0	No
Generic (Full Space)	4	4^n	$(\sqrt{5})^n$	Trivial bound	Not guaranteed
Generic (Symmetric)	4	n^3	$(\sqrt{5})^n$	$\to 1$	Yes (amplified)

Significance. This proposition: (1) Explains null results from current qubit experiments ($d = 2$ correctly find $\mathcal{K} \approx 0$); (2) Guides future experiments (many-particle interference in symmetric subspaces is optimal for detecting $\mathcal{K}_n > 0$); (3) Validates the dual-subspace (amplification manifests physically only if the n-loci dual-subspace protects coherence, Remark 6.41); (4) Connects logic to physics (chromatic number directly bounds Holothesis warrant weight, observable via I_3). This is the technical bridge between the abstract logical structure and empirical predictions, showing exactly where and how the tetralemmatic framework diverges from Standard Quantum Mechanics.

Remark 6.41 (Role of the Dual-Subspace in Sensitivity Enhancement)**.** Proposition 6.39 establishes that $\mathcal{K}_n$ becomes *macroscopic* (≈ 1) for large n in the symmetric subspace regime. However, in Standard Quantum Mechanics (SQM), such high-order correlations are typically suppressed by decoherence. The *n-loci dual-subspace* within the Global Tetralemmatic Manifold ($\mathcal{H}_{\mathrm{Total}} \cong \mathcal{H}_c \otimes \mathcal{H}_{\mathrm{Dual}}^{(2n)}$) resolves this tension via three mechanisms:

(i) **Symmetry-Protected Confinement:** The full dimension of $\mathcal{H}_{\mathrm{Total}}$ scales as $2 \cdot d^{2n}$ (e.g., 2^{4n+1} for the generic four-corner case, 2^{2n+1} for the binary logical dyad) (Definition 5.21). However, the Holothesis meta-test dynamics are confined to a *symmetry-protected*

invariant subspace defined by the duality automorphism $\mathbf{D}_{\mathcal{C}}$. The unitary U_{holo} commutes with the swap operation between the two copies of $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$, restricting evolution to the symmetric sector. This drastically reduces the *effective* degrees of freedom susceptible to polar noise. Specifically, *collective dephasing* (which preserves $\mathbf{D}_{\mathcal{C}}$) is suppressed, while *symmetry-breaking interactions* (measurements) are permitted to drive sector transitions.

(ii) **Coherence Preservation:** As established in Remark 5.6, the dual-subspace structure protects the Holothesis sector from collapsing into polar Thesis/Antithesis states during the meta-test. By operating on correlations between the two copies rather than individual polar degrees of freedom, the diagnostic preserves the phase information required for I_3 interference even when $\mathcal{K}_n \approx 1$.

(iii) **Signal Amplification:** The observed Sorkin parameter $I_3^{(n)}$ is proportional to the coherent Holothesis weight. In a single-subspace realization, environmental coupling scales with n, suppressing coherence as $e^{-\gamma n}$. In the dual-subspace realization, the logical duality symmetry enforces a decoherence-free subspace (DFS) relative to polar noise channels that respect the duality symmetry, allowing the exponential growth of $\mathcal{K}_n$ to manifest physically as an exponential growth in $I_3^{(n)}$ sensitivity.

Thus, the dual-subspace is not merely a representational tool but the *physical conduit* that translates logical necessity ($\mathcal{K}_n \to 1$) into empirical signal ($I_3^{(n)} \neq 0$).

Corollary 6.42 (Empirical Scaling Law for Sorkin Interference). *Under the tetralemmatic framework with an n-loci dual-subspace realization, the Sorkin interference parameter scales as:*

$$I_3^{(n)} \approx \left(1 - \frac{poly(n)}{(\sqrt{5})^n}\right) \cdot \sin(4\theta_{\mathbf{R}}) \cdot \eta_{\text{dual}}(n), \tag{72}$$

where $\eta_{\text{dual}}(n)$ is a coherence retention factor. For single-subspace realizations, $\eta_{\text{single}}(n) \sim e^{-\gamma n}$. For dual-subspace realizations protected by duality symmetry within $\mathcal{H}_{\text{Total}}$, $\eta_{\text{dual}}(n) \approx 1$. Consequently, the signal-to-noise ratio for detecting logical deficiency scales exponentially with n only in the presence of the dual-subspace structure.

Proof. **Step 1: Logical Component.** By Proposition 6.36, $I_3^{(n)} = \mathcal{K}_n \cdot \sin(4\theta_{\mathbf{R}}) + \mathcal{O}(\mathcal{K}^2)$. By Proposition 6.39, $\mathcal{K}_n \approx 1 - poly(n)/(\sqrt{5})^n$ in the symmetric subspace regime ($d = 4$).

Step 2: Coherence Retention. The observed interference requires physical coherence. We introduce $\eta(n)$ such that $I_3^{(n)} \propto \mathcal{K}_n \cdot \eta(n)$.

- **Single-Subspace:** Standard decoherence theory [22, 23] implies $\eta_{\text{single}}(n) \sim e^{-\gamma n}$.

- **Dual-Subspace:** The Holothesis sector operates on $\mathcal{H}_{\text{Dual}}^{(2n)} = \mathcal{H}^{(n)} \otimes \mathcal{H}^{(n)}$ (Definition 5.4). While $\dim(\mathcal{H}_{\text{Dual}}^{(2n)}) = d^{2n}$, the meta-test unitary U_{holo} is constrained by the duality symmetry $\mathbf{D}_{\mathcal{C}}$ to act within a symmetry-protected invariant subspace. This creates a decoherence-free subspace (DFS) relative to polar noise channels. Thus, $\eta_{\text{dual}}(n) \approx 1$.

Step 3: Synthesis. Substituting these into the expression for $I_3^{(n)}$ yields Eq. (72). The signal magnitude approaches $\sin(4\theta_{\mathbf{R}})$ in the dual-subspace case, whereas it vanishes ($\sim e^{-\gamma n}$) in the single-subspace case. □

Remark 6.43 (Synthesis with Contemporary Developments (2026)). The tetralemmatic framework aligns with the emerging empirical and theoretical consensus as of 2026. Recent experimental bounds from superconducting quantum processors [26] constrain the logical deficiency constant to $|\mathcal{K}| \lesssim 10^{-3}$ in the *dyadic reduction regime* ($d = 2$). This is consistent with the framework's prediction that $\mathcal{K} = 0$ for $d = 2$ (Proposition 6.39), confirming that Z_4 symmetry is effectively reduced to Z_2 (qubit logic) in standard computational contexts. Simultaneously, the latest informational formulations of relational quantum mechanics [38] provide the ontological grounding for the locus-dependent facts derived in Proposition 6.46.

In the n-loci formalism, these "locus-dependent facts" are formalized as *n-loci corner statuses* (Thesis/Antithesis) defined over configurations $\vec{\lambda}$ (Definitions 5.1 and 5.2), while the "potentiality structure" corresponds to the Holothesis sector operating on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$. **Probing the Full Space:** To test the generic four-corner case ($d = 4$, dimension 2^{4n+1}), experiments must employ higher-dimensional systems (e.g., qutrits, $d \geq 3$) where $\mathcal{K} > 0$ is permitted (Remark 6.135). Together, these developments suggest that the tetralemmatic reconstruction does not merely reinterpret quantum kinematics but captures the structural constraints inherent in the relational architecture of physical reality, where contextuality and interference are governed by the logical topology of the incompatibility graph over locus configurations.

Remark 6.44 (Computational Feasibility and Efficient Witnessing). A potential concern regarding the empirical verification of Proposition 6.39 is the computational complexity of evaluating the incompatibility graph Γ_n for large n.

(i) **NP-Hardness of Chromatic Number:** Calculating the exact chromatic number $\mathscr{X}(\Gamma_n)$ is NP-hard [115]. For large n-loci configurations, exact evaluation is computationally intractable, posing a challenge for direct verification of the bound in Proposition 6.39.

(ii) **Polynomial-Time Bounds via Lovász Theta:** However, the proof of Proposition 6.39 relies on the Lovász theta function $\vartheta(\Gamma_n)$, which satisfies $\mathscr{X}(\Gamma_n) \geq \vartheta(\Gamma_n)$ [114]. Crucially, $\vartheta(\Gamma)$ is computable in *polynomial time* via Semidefinite Programming (SDP) [116]. This allows efficient certification of the lower bound on $\mathcal{K}_n$ without requiring exact coloring.

(iii) **Dimensional Requirement:** Non-trivial bounds ($\mathcal{K}_n > 0$) require $\mathscr{X}(\Gamma_n) > d$. For the dyadic reduction ($d = 2$), $\mathscr{X} \leq 2$ and $\mathcal{K}_n = 0$ trivially. Efficient witnessing of $\mathcal{K}_n > 0$ thus requires physical systems supporting $d \geq 3$ (e.g., qutrits), where the graph structure becomes non-2-colorable.

(iv) **Experimental Witnessing Strategy:** To avoid the $O(|V|^2)$ measurement overhead of reconstructing the full graph, experimental verification can employ *efficient witness families*. By measuring a strategically sampled subset of incompatibility witnesses $\mathfrak{N}_{\omega_\Lambda}(A, B)$ sufficient to bound $\vartheta(\Gamma_n)$, one can verify $\mathcal{K}_n > 0$ with resources scaling polynomially in n *provided* $d \geq 3$ [21, 117].

Thus, while full graph reconstruction is prohibitive, the logical deficiency constant $\mathcal{K}_n$ remains *efficiently verifiable* via SDP-computable bounds in the generic four-corner regime ($d = 4$), ensuring the framework's empirical predictions are computationally tractable where logically non-trivial.

6.7.1 The Outcome Problem as Locus Symmetry Breaking

The measurement problem is resolved here not via wave-function collapse, but via **Locus Symmetry Breaking**. The transition from potentiality (Holothesis) to fact (Thesis/Antithesis) is governed by the Symmetry-Polarity Correspondence (Theorem 6.138). This resolution holds for both single-locus and *n-loci configurations*, with the latter providing enhanced sensitivity to symmetry-breaking interactions across composite systems.

Definition 6.45 (Warrant Selection via Forcing Context (n-Loci))**.** Let $|\mathbf{HoloTh}; \vec{P}\rangle$ represent the pre-measurement potentiality state for an n-loci claim $\vec{P}$ defined over locus configuration $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ (Definition 5.7). A *Forcing Context* $\mathcal{C}_{\text{force}}$ is an admissible refinement (Definition 2.6) that breaks the Z_4 symmetry of the warrant functional ω_Λ over the configuration $\vec{\lambda}$. The selection of a specific outcome (Thesis or Antithesis) is a stochastic event dictated by the n-loci Born weights (Definition 5.11):

$$\text{Prob}(\mathbf{Th}^{(n)}) = \frac{\omega_\Lambda(C_{\mathbf{Th}}(\vec{P}))}{\omega_\Lambda(C_{\mathbf{Th}}(\vec{P})) + \omega_\Lambda(C_{\mathbf{AntiTh}}(\vec{P}))}. \tag{73}$$

For $n > 1$, this probability reflects the composite warrant weight across all loci in $\vec{\lambda}$, potentially enhanced by inter-locus correlations (entanglement).

Proposition 6.46 (Ontological Status of the Outcome (n-Loci))**.**   *(i) Single Sharp Outcome: The "fact" observed in Scenario A (Proposition 6.113) constitutes a* Thesis *status (Definition 5.1). It serves as the logical prerequisite for coherent discourse at the locus configuration $(\mathcal{C}_{\text{force}}, \vec{\lambda})$ and is real relative to that configuration [4].*

   (ii) Co-Commitment: The "co-commitment" in Scenario D (Proposition 6.118) constitutes a Synthesis *status (Definition 5.3). It represents the* operational limit *where symmetry is preserved (interference) across the configuration $\vec{\lambda}$.*

   (iii) No Collapse: There is no dynamical collapse of the Hilbert space $\mathcal{H}_\Lambda$. The Holothesis vector remains in the global space but becomes inaccessible to the local observer's applicability predicate $\text{App}_{\mathcal{C}}$ once the symmetry is broken (Theorem 6.138, eq. (84)). For n-loci configurations, this inaccessibility is evaluated on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ for Thesis/Antithesis/Synthesis, and on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ for Holothesis (Definitions 5.4 and 5.7).

Proof. **(i) Single Sharp Outcome.** By Proposition 6.113 (generalized to n-loci in Definition 5.1), the which-path locus configuration $(\mathcal{C}_1, \vec{\lambda}_1)$ admits a single frame where $\text{App}_{\mathcal{C}_1}(\vec{P}; \vec{\lambda}_1)$ holds and $\kappa_\rho(\vec{P}, \neg\vec{P}) = 0$. Under a forcing context $\mathcal{C}_{\text{force}}$, the symmetry breaking condition (eq. (84)) selects a definite polarity across the configuration. By Definition 5.1, this warrants the status $\text{Status}_{\mathcal{C}}(\vec{P}; \vec{\lambda}) = \mathbf{Th}^{(n)}$. Following [4], this fact is ontologically valid relative to the locus configuration $\vec{\lambda}$ where the interaction occurred, satisfying the context-indexed existence criterion (Definition 2.1).

   (ii) Co-Commitment. By Proposition 6.118 (generalized to n-loci in Definition 5.3), the refinement locus configuration $(\mathcal{C}', \vec{\lambda}')$ admits both path and phase claims (App holds for

both) but with non-zero incompatibility ($\kappa_\rho(\vec{P}, \vec{\Sigma}) > 0$). This forces the frame number $\mathscr{X} > 1$ across the configuration. By Definition 5.3, the co-holding of $\vec{P}$ and $\neg\vec{P}$ without explosion warrants the status $\mathrm{Status}_\mathcal{C}(\vec{P}; \vec{\lambda}) = \mathbf{SynTh}^{(n)}$. This corresponds to the preservation of off-diagonal warrant weights (interference) where symmetry is not fully broken, potentially enhanced by entanglement across loci.

(iii) No Collapse. The global Hilbert space $\mathcal{H}_\Lambda$ is constructed via the GNS representation of the warrant functional (Definition 4.1). Unitary evolution preserves the norm of the global state vector. By Theorem 6.138, symmetry breaking affects the *applicability* predicate $\mathrm{App}_\mathcal{C}$ locally, not the global vector. Specifically, eq. (84) shows that $\mathrm{App}_\mathcal{C}(\vec{P}; \vec{\lambda})$ becomes true while $\mathrm{App}_\mathcal{C}(\neg\vec{P}; \vec{\lambda})$ becomes false (or vice versa), rendering the alternative pole inaccessible to the local discourse. The Holothesis component ($|\mathbf{HoloTh}; \vec{P}\rangle$) remains in $\mathcal{H}_\Lambda$ but is filtered out by the local applicability constraint. For n-loci configurations, the Holothesis sector operates on the n-loci dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$, which remains in the global space but becomes inaccessible to local measurements on the composite operational space $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$. This is consistent with decoherence theory [22, 23] but grounded in logical typing rather than environmental tracing alone. $\qquad\square$

Remark 6.47 (Ontological Status of the Outcome (n-Loci)). • **Single Sharp Outcome:** The "fact" observed in Scenario A (Definition 5.1) is the *logical prerequisite* for coherent discourse at the locus configuration ($\mathcal{C}_{\mathrm{force}}, \vec{\lambda}$). It is real *relative to that configuration* [4].

- **Co-Commitment:** The "co-commitment" in Scenario D (Definition 5.3) represents the *operational limit* where symmetry is preserved (interference) across the configuration $\vec{\lambda}$.

- **No Collapse:** There is no dynamical collapse of the Hilbert space $\mathcal{H}_\Lambda$. The Holothesis vector remains in the global space but becomes **inaccessible** to the local observer's applicability predicate $\mathrm{App}_\mathcal{C}$ once the symmetry is broken (Theorem 6.138, eq. (84)). For n-loci configurations, this inaccessibility is enforced by the structural distinction between the composite operational space $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$ (for polar outcomes) and the n-loci dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$ (for Holothesis potentiality).

Thus, the "outcome problem" is reframed as a transition from *global potentiality* (Holothesis on $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$) to *local actuality* (Thesis on $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$) via contextual symmetry breaking, consistent with decoherence theory [22, 23] but grounded in logical ontology rather than environmental tracing alone. For many-particle systems ($n \gg 1$), the combinatorial growth of the incompatibility graph $\Gamma_{\vec{\lambda}}$ may enhance sensitivity to symmetry-breaking interactions, providing a potential empirical signature of the n-loci structure (Remark 6.37).

6.8 Structural Elements of the Incompatibility Graph

This subsection establishes the graph-theoretic structure of logical incompatibility for both single-locus and *n-loci configurations*. The incompatibility graph provides the combinatorial foundation for contextuality witnesses and the Logical Deficiency Bound (Theorem 6.94).

6.8.1 Typed Admissibility and the Vertex Set

Definition 6.48 (Locus-indexed claim vertex class (n-Loci))**.** Fix a *locus configuration* $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$ (for $n = 1$, this reduces to a single locus λ). Let $\mathsf{P}_{\mathcal{C},\vec{\lambda}}$ be the admissible polar claim class for the configuration (Definition 3.8, generalized to n-loci in Definitions 5.1 to 5.3 and 5.7). We regard $\mathsf{P}_{\mathcal{C},\vec{\lambda}}$ as the *vertex class* of the compatibility graph at $(\mathcal{C}, \vec{\lambda})$.

Remark 6.49 (Holothesis as boundary of the graph (n-Loci))**.** Because vertices are *admissible* claims, inapplicable formulas are not vertices. Thus the Holothesis mechanism (cf. Assumption 3.11 and the clauses defining Holothesis) appears combinatorially as a *locus-dependent deletion of vertices.* This matches the double-slit treatments in Sections 2.13 and 2.14: which-path claims fail admissibility in the interference locus and therefore do not enter the local graph there.

    For n-loci configurations, the vertex deletion occurs across the entire configuration $\vec{\lambda}$ when Holothesis is warranted at any locus $\lambda_i \in \vec{\lambda}$. This reflects the structural requirement that Holothesis operates on the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definitions 5.4 and 5.7), which is distinct from the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ used for polar corners. Consequently, the incompatibility graph for n-loci configurations may exhibit different connectivity properties depending on whether Holothesis warrant is present across the configuration.

6.8.2 Warrant Uncertainty and Commutator Seminorms

Definition 6.50 (Centering and variance in a coherent warrant state (n-Loci))**.** Let ω_Λ be a coherent warrant functional on $\mathcal{A}_\Lambda^{\text{alg}}$ (Definition 3.35). For self-adjoint $A = A^\dagger \in \mathcal{A}_\Lambda^{\text{alg}}$, define

$$\widetilde{A} := A - \omega_\Lambda(A)\,\mathbf{1}, \qquad \text{Var}_{\omega_\Lambda}(A) := \omega_\Lambda(\widetilde{A}^2) = \omega_\Lambda(A^2) - \omega_\Lambda(A)^2 \geq 0.$$

We call A ω_Λ-*sharp* if $\text{Var}_{\omega_\Lambda}(A) = 0$.

Remark 6.51 (Variance for n-Loci Observables)**.** For n-loci corner-linear observables $A = O_\mathbf{a}(\vec{P})$ defined over locus configurations $\vec{\lambda}$ (Definition 6.62), the variance $\text{Var}_{\omega_\Lambda}(A)$ quantifies uncertainty across the entire configuration.

- **Composite Sharpness:** A is ω_Λ-sharp iff the warrant functional assigns definite values to the corner generators at *all* loci $\lambda_i \in \vec{\lambda}$ simultaneously.

- **Entanglement Witness:** Non-zero variance for composite observables may arise from local indeterminacy (Synthesis at individual loci) or from non-separable correlations (entanglement) between loci.

- **Holothesis Contribution:** For observables involving Holothesis generators, the variance is evaluated on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, reflecting the meta-test structure (Definitions 5.7 and 5.8).

Proposition 6.52 (Variance as a GNS norm (n-Loci))**.** *Let $(\pi_\Lambda, \mathcal{H}_\Lambda, |\Omega_\Lambda\rangle)$ be the logical GNS triple (Definition 4.1). For self-adjoint $A \in \mathcal{A}_\Lambda^{\text{alg}}$,*

$$\text{Var}_{\omega_\Lambda}(A) = \left\| \Big(\pi_\Lambda(A) - \omega_\Lambda(A)I\Big)|\Omega_\Lambda\rangle \right\|^2.$$

In particular, A is ω_Λ-sharp iff $\big(\pi_\Lambda(A) - \omega_\Lambda(A)I\big)|\Omega_\Lambda\rangle = 0$. This holds for both single-locus and n-loci observables, as the GNS construction applies to the full algebra $\mathcal{A}_\Lambda^{\text{alg}}$ generated by all corner-observables.

Proof. Let $A \in \mathcal{A}_\Lambda^{\text{alg}}$ be self-adjoint ($A = A^\dagger$). By Definition 6.50, the centered observable is $\widetilde{A} = A - \omega_\Lambda(A)\mathbf{1}$. Since $\omega_\Lambda(A) \in \mathbb{C}$ and $\mathbf{1}$ is the unit of $\mathcal{A}_\Lambda^{\text{alg}}$, we have $\widetilde{A}^\dagger = A^\dagger - \overline{\omega_\Lambda(A)}\,\mathbf{1} = \widetilde{A}$ (using positivity of ω_Λ which implies $\omega_\Lambda(A) \in \mathbb{R}$ for self-adjoint A) [19, 94].

By the definition of variance and the linearity of ω_Λ:

$$
\begin{aligned}
\mathrm{Var}_{\omega_\Lambda}(A) &= \omega_\Lambda(\widetilde{A}^2) = \omega_\Lambda(\widetilde{A}^\dagger \widetilde{A}) \\
&= \omega_\Lambda\Big(\big(A - \omega_\Lambda(A)\mathbf{1} \big)^\dagger \big(A - \omega_\Lambda(A)\mathbf{1} \big) \Big).
\end{aligned}
$$

In the GNS representation (Definition 4.1), we have the state recovery formula (Proposition 4.5):

$$
\omega_\Lambda(X) = \langle \Omega_\Lambda \mid \pi_\Lambda(X) \mid \Omega_\Lambda \rangle_\Lambda \qquad \text{for all } X \in \mathcal{A}_\Lambda^{\text{alg}}.
$$

Applying this to $X = \widetilde{A}^\dagger \widetilde{A}$:

$$
\begin{aligned}
\mathrm{Var}_{\omega_\Lambda}(A) &= \langle \Omega_\Lambda \mid \pi_\Lambda(\widetilde{A}^\dagger \widetilde{A}) \mid \Omega_\Lambda \rangle_\Lambda \\
&= \langle \Omega_\Lambda \mid \pi_\Lambda(\widetilde{A})^\dagger \pi_\Lambda(\widetilde{A}) \mid \Omega_\Lambda \rangle_\Lambda \\
&= \left\| \pi_\Lambda(\widetilde{A})|\Omega_\Lambda\rangle \right\|^2,
\end{aligned}
$$

where the second equality uses that π_Λ is a $*$-representation (Proposition 3.41), and the third equality is the definition of the Hilbert space norm [17, 71].

Substituting $\widetilde{A} = A - \omega_\Lambda(A)\mathbf{1}$ and using that π_Λ is unital ($\pi_\Lambda(\mathbf{1}) = I$):

$$
\pi_\Lambda(\widetilde{A}) = \pi_\Lambda(A) - \omega_\Lambda(A)I.
$$

Therefore:

$$
\mathrm{Var}_{\omega_\Lambda}(A) = \left\| \big(\pi_\Lambda(A) - \omega_\Lambda(A)I \big)|\Omega_\Lambda\rangle \right\|^2.
$$

For the sharpness criterion: $\mathrm{Var}_{\omega_\Lambda}(A) = 0$ iff the norm vanishes, which holds iff the vector itself is zero (positive definiteness of the GNS inner product) [19, 94]:

$$
\left\| \big(\pi_\Lambda(A) - \omega_\Lambda(A)I \big)|\Omega_\Lambda\rangle \right\|^2 = 0 \quad \Longleftrightarrow \quad \big(\pi_\Lambda(A) - \omega_\Lambda(A)I \big)|\Omega_\Lambda\rangle = 0.
$$

This completes the proof. The result holds for n-loci observables since the GNS construction applies to the full algebra $\mathcal{A}_\Lambda^{\text{alg}}$, which includes n-loci corner generators (Definitions 5.1 to 5.3 and 5.7). $\qquad\square$

Remark 6.53 (Connection to n-Loci Born Semantics)**.** The variance definition aligns with the n-loci Born semantics (Definition 5.11):

- **Polar/Synthesis Corners:** For $M \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, the variance is evaluated on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{X}}^{(n)}$.

- **Holothesis Corner:** For $M = \textbf{HoloTh}$, the variance is evaluated on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, reflecting the meta-test structure.

- **Sharpness Interpretation:** An observable is ω_Λ-sharp iff it yields definite outcomes (variance zero) across all loci in the configuration, which corresponds to a classical Thesis/Antithesis valuation on the incompatibility graph.

This ensures that uncertainty relations derived from these witnesses (e.g., Robertson-type bounds) apply consistently across all tetralemmatic corners and locus configurations (Proposition 6.24).

Definition 6.54 (Warrant commutator seminorms (n-Loci)). For $A, B \in \mathcal{A}_\Lambda^{\text{alg}}$, define the commutator $[A, B] := AB - BA$. Define the *weak* and *strong* warrant commutator witnesses by

$$\mathfrak{n}_{\omega_\Lambda}(A, B) \; := \; \left| \omega_\Lambda([A, B]) \right|, \qquad \mathfrak{N}_{\omega_\Lambda}(A, B) \; := \; \left\| \pi_\Lambda([A, B]) |\Omega_\Lambda\rangle \right\| \; = \; \sqrt{\omega_\Lambda([A, B]^\dagger [A, B])}.$$

By the Cauchy–Schwarz inequality in $\mathcal{H}_\Lambda$, one has $\mathfrak{n}_{\omega_\Lambda}(A, B) \leq \mathfrak{N}_{\omega_\Lambda}(A, B)$, justifying the terminology [19, 71].

Remark 6.55 (Commutator witnesses for n-Loci observables). For n-loci corner-linear observables $A = O_\mathbf{a}(\vec{P})$ and $B = O_\mathbf{b}(\vec{Q})$ defined over locus configurations $\vec{\lambda}, \vec{\mu} \in \Lambda_\mathcal{C}^n$ (Definition 6.62), the witnesses $\mathfrak{n}_{\omega_\Lambda}(A, B)$ and $\mathfrak{N}_{\omega_\Lambda}(A, B)$ quantify algebraic incompatibility across the composite configuration.

- **Local Noncommutativity:** If $\vec{\lambda} = \vec{\mu}$, the witnesses capture noncommutativity at each locus $\lambda_i \in \vec{\lambda}$, potentially amplified by tensor product structure.

- **Cross-Locus Correlations:** If $\vec{\lambda} \neq \vec{\mu}$, the witnesses may detect non-separable correlations (entanglement) between distinct locus configurations, provided the warrant functional ω_Λ assigns non-zero weight to cross-locus commutators.

- **Holothesis Dual-Subspace:** For observables involving Holothesis generators, the strong witness $\mathfrak{N}_{\omega_\Lambda}$ is evaluated on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, reflecting the meta-test structure (Definitions 5.7 and 5.8).

Proposition 6.56 (Robertson-type uncertainty from coherence (n-Loci)). *For any coherent warrant functional ω_Λ and self-adjoint $A, B \in \mathcal{A}_\Lambda^{\text{alg}}$,*

$$\text{Var}_{\omega_\Lambda}(A) \; \text{Var}_{\omega_\Lambda}(B) \; \geq \; \frac{1}{4} \left| \omega_\Lambda([A, B]) \right|^2 \; = \; \frac{1}{4} \mathfrak{n}_{\omega_\Lambda}(A, B)^2.$$

This is the algebraic analogue of the Robertson uncertainty relation [71, 107, 108]. The result holds for both single-locus and n-loci observables, as the GNS construction applies to the full algebra $\mathcal{A}_\Lambda^{\text{alg}}$ generated by all corner-observables.

Proof. Let $\widetilde{A} = A - \omega_\Lambda(A)\mathbf{1}$ and $\widetilde{B} = B - \omega_\Lambda(B)\mathbf{1}$ be the centered observables (Definition 6.50). By Proposition 6.52, the variances are given by the squared norms of the vectors

$$|\psi_A\rangle := \pi_\Lambda(\widetilde{A})|\Omega_\Lambda\rangle, \qquad |\psi_B\rangle := \pi_\Lambda(\widetilde{B})|\Omega_\Lambda\rangle$$

in the logical GNS space $\mathcal{H}_\Lambda$. Apply the Cauchy–Schwarz inequality to these vectors [17, 71]:

$$\|\psi_A\|^2 \, \|\psi_B\|^2 \; \geq \; \left| \langle \psi_A \mid \psi_B \rangle_\Lambda \right|^2 .$$

The inner product can be decomposed into real and imaginary parts:

$$\langle \psi_A \mid \psi_B \rangle_\Lambda = \omega_\Lambda(\widetilde{A}^\dagger \widetilde{B}) = \omega_\Lambda(\widetilde{A}\widetilde{B}),$$

since A, B are self-adjoint. Using the identity $2i \operatorname{Im}(z) = z - \bar{z}$ and the fact that $\omega_\Lambda(X^\dagger) = \overline{\omega_\Lambda(X)}$:

$$2i \operatorname{Im}\!\left(\omega_\Lambda(\widetilde{A}\widetilde{B})\right) = \omega_\Lambda(\widetilde{A}\widetilde{B}) - \omega_\Lambda(\widetilde{B}\widetilde{A}) = \omega_\Lambda([\widetilde{A}, \widetilde{B}]).$$

Since $[\widetilde{A}, \widetilde{B}] = [A, B]$ (scalar terms commute), we have

$$\left| \langle \psi_A \mid \psi_B \rangle_\Lambda \right|^2 \; \geq \; \left(\operatorname{Im} \langle \psi_A \mid \psi_B \rangle_\Lambda \right)^2 = \frac{1}{4} \left| \omega_\Lambda([A, B]) \right|^2 .$$

Substituting the variances $\operatorname{Var}_{\omega_\Lambda}(A) = \|\psi_A\|^2$ and $\operatorname{Var}_{\omega_\Lambda}(B) = \|\psi_B\|^2$ yields the result [107, 108]. This proof holds for n-loci observables since the GNS construction applies to the full algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$, which includes n-loci corner generators (Definitions 5.1 to 5.3 and 5.7). □

Corollary 6.57 (No joint sharpness under nonzero commutator witness (n-Loci)). *If $\mathfrak{n}_{\omega_\Lambda}(A, B) \neq 0$, then A and B cannot both be ω_Λ-sharp. Equivalently, joint sharpness implies vanishing weak commutator witness. This captures the evaluator-level obstruction to classical joint definiteness [7, 19].*

Remark 6.58 (n-Loci joint sharpness). For n-loci observables $A = O_{\mathbf{a}}(\vec{P})$ and $B = O_{\mathbf{b}}(\vec{Q})$, joint sharpness requires zero variance across *all* loci in the configurations $\vec{\lambda}, \vec{\mu}$. If $\mathfrak{n}_{\omega_\Lambda}(A, B) \neq 0$, then at least one locus $\lambda_i \in \vec{\lambda}$ or $\mu_j \in \vec{\mu}$ exhibits indeterminacy, preventing global joint definiteness. This algebraic signature of contextuality across composite configurations aligns with the Kochen–Specker notion that no single global valuation exists for sets with incompatibility graph chromatic number $\mathscr{X} > 1$ (Theorem 6.94).

Remark 6.59 (Reduction to the physical Robertson bound under realization (n-Loci)). Under a realization $r : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_{\mathrm{phys}}$ (Definition 3.57) into a physical state ω_{phys} with $\omega_\Lambda = \omega_{\mathrm{phys}} \circ r$, Proposition 6.56 reduces to the familiar Robertson bound

$$\operatorname{Var}_\rho(A_{\mathrm{phys}}) \, \operatorname{Var}_\rho(B_{\mathrm{phys}}) \; \geq \; \frac{1}{4} \left| \operatorname{Tr}(\rho\,[A_{\mathrm{phys}}, B_{\mathrm{phys}}]) \right|^2, \tag{74}$$

with variances as in Definition 6.50 [63, 71].

Remark 6.60 (n-Loci realization structure). For n-loci observables, the realization map r must preserve the tensor product structure:

- **Composite Operational Space:** For Thesis, Antithesis, Synthesis, r maps $O_{\mathbf{a}}(\vec{P})$ to operators on $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)} \subseteq \mathcal{H}_{\mathrm{phys}}$.

- **Dual-Subspace for Holothesis:** For Holothesis claims, r maps $O_{\mathbf{a}}(\vec{P})$ to operators on $\mathcal{H}_{\mathrm{Dual}}^{(2n)} \subseteq \mathcal{H}_{\mathrm{phys}}$, ensuring the meta-test structure is preserved (Definition 5.8).

- **Commutator Preservation:** Since r is a $*$-homomorphism, $[r(A), r(B)] = r([A, B])$, so the witness $\mathfrak{n}_{\omega_\Lambda}$ is invariant under faithful realizations.

This ensures the algebraic uncertainty relations derived at the logical level hold identically in all physical realizations (Proposition 6.74).

Remark 6.61 (Connection to n-Loci Born Semantics and Contextuality)**.** The uncertainty relations align with the n-loci Born semantics (Definition 5.11):

- **Polar/Synthesis Corners:** For $M \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, the variance is evaluated on the composite operational space $\mathcal{H}^{(n)}_{\mathcal{C}, \vec{\lambda}}$.

- **Holothesis Corner:** For $M = \mathbf{HoloTh}$, the variance is evaluated on the n-loci dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$, reflecting the meta-test structure.

- **Contextuality Witness:** Non-zero commutator witnesses across n-loci configurations indicate irreducible contextuality, quantified by the Logical Deficiency Bound (Theorem 6.94). For many-particle systems ($n \gg 1$), the combinatorial growth of the incompatibility graph $\Gamma_{\vec{\lambda}}$ may enhance sensitivity to logical deficiency $\mathcal{K}$ (Remark 6.37).

This ensures that uncertainty relations derived from these witnesses apply consistently across all tetralemmatic corners and locus configurations.

6.9 Incompatibility Seminorms for Corner-Linear Observables (n-Loci)

Definition 6.62 (Corner-linear observable associated to an n-loci claim)**.** Fix a *locus configuration* $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda^n_{\mathcal{C}}$ and an admissible polar claim $\vec{P}$ defined over $\vec{\lambda}$ (Definitions 3.8 and 3.18, generalized to n-loci in Definitions 5.1 to 5.3 and 5.7). For any coefficient vector $\mathbf{a} \in \mathbb{R}^4$ indexed by $\mathbf{Status}_\Lambda = \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$ (Definition 3.10), define

$$O_{\mathbf{a}}(\vec{P}) := \sum_{\bullet \in \mathbf{Status}_\Lambda} a_\bullet \, C_\bullet(\vec{P}) \in \mathcal{A}^{\mathrm{alg}}_\Lambda.$$

Since each generator $C_\bullet(\vec{P})$ is self-adjoint (Definition 3.20) and $\mathbf{a} \in \mathbb{R}^4$, the element $O_{\mathbf{a}}(\vec{P})$ is self-adjoint in $\mathcal{A}^{\mathrm{alg}}_\Lambda$ [17, 19]. This construction parallels the formation of observables from effect-valued measures in operational quantum theory [56, 63], extended to composite configurations.

Remark 6.63 (Operational realization of n-loci observables)**.** Under a physical realization $r : \mathcal{A}^{\mathrm{alg}}_\Lambda \to \mathcal{A}_{\mathrm{phys}}$ (Definition 3.57):

- **Composite Operational Space:** For $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, the observable $O_{\mathbf{a}}(\vec{P})$ acts on the composite operational space $\mathcal{H}^{(n)}_{\mathcal{C}, \vec{\lambda}}$.

- **Dual-Subspace for Holothesis:** For $\bullet = \mathbf{HoloTh}$, the generator $C_{\mathbf{HoloTh}}(\vec{P})$ acts on the n-loci dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$ (Definitions 5.4 and 5.7). Consequently, mixed corner-linear observables involving Holothesis coefficients require the dual-subspace structure for faithful realization.

Definition 6.64 (Claim incompatibility seminorm (n-Loci))**.** Let ω_Λ be a coherent warrant functional (Definition 3.35). For admissible claims $\vec{P}, \vec{Q}$ defined over locus configurations $\vec{\lambda}, \vec{\mu} \in \Lambda_{\mathcal{C}}^n$, define the *strong incompatibility degree*

$$\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) := \sup_{\mathbf{a}, \mathbf{b} \in \mathbb{R}^4 \setminus \{0\}} \frac{\mathfrak{N}_{\omega_\Lambda}\big(O_{\mathbf{a}}(\vec{P}), O_{\mathbf{b}}(\vec{Q})\big)}{\|\mathbf{a}\| \, \|\mathbf{b}\|} \in [0, \infty],$$

and the *weak incompatibility degree*

$$\kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q}) := \sup_{\mathbf{a}, \mathbf{b} \in \mathbb{R}^4 \setminus \{0\}} \frac{\mathfrak{n}_{\omega_\Lambda}\big(O_{\mathbf{a}}(\vec{P}), O_{\mathbf{b}}(\vec{Q})\big)}{\|\mathbf{a}\| \, \|\mathbf{b}\|},$$

where $\mathfrak{N}_{\omega_\Lambda}$ and $\mathfrak{n}_{\omega_\Lambda}$ are the strong and weak warrant commutator witnesses (Definition 6.54). These seminorms quantify the degree of logical non-commutativity between claims $\vec{P}$ and $\vec{Q}$ relative to the evaluation regime ω_Λ [8, 71].

Remark 6.65 (Sources of incompatibility in n-loci configurations)**.** For n-loci claims, non-zero incompatibility $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) > 0$ may arise from:

- **Local Noncommutativity:** Non-commuting corner generators at individual loci $\lambda_i \in \vec{\lambda}$.

- **Cross-Locus Correlations:** Entanglement between distinct loci in $\vec{\lambda}$ and $\vec{\mu}$, detected via non-separable warrant weights.

- **Dual-Subspace Structure:** If $\vec{P}$ or $\vec{Q}$ involves Holothesis generators, the incompatibility is evaluated on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$, reflecting the meta-test structure (Definition 5.8).

Proposition 6.66 (Incompatibility implies no joint sharpness of some corner-linear pair (n-Loci))**.** *If $\kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q}) > 0$, then there exist coefficient vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ such that the corner-linear observables $O_{\mathbf{a}}(\vec{P})$ and $O_{\mathbf{b}}(\vec{Q})$ cannot both be ω_Λ-sharp (Definition 6.50). This is a direct consequence of the Robertson-type uncertainty relation for warrant functionals (Proposition 6.56) [107, 108].*

Proof. By definition of $\kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q})$, the condition $\kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q}) > 0$ implies the existence of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4 \setminus \{0\}$ such that $\mathfrak{n}_{\omega_\Lambda}(O_{\mathbf{a}}(\vec{P}), O_{\mathbf{b}}(\vec{Q})) \neq 0$. Let $A = O_{\mathbf{a}}(\vec{P})$ and $B = O_{\mathbf{b}}(\vec{Q})$. Then $|\omega_\Lambda([A, B])| > 0$. Apply Corollary 6.57, which states that a nonzero weak commutator witness precludes joint sharpness (zero variance) for the pair (A, B) across the locus configurations [19, 71]. $\square$

Remark 6.67 (Relation to joint measurability and contextuality (n-Loci))**.** The incompatibility degrees $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q})$ and $\kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q})$ serve as algebraic witnesses for the failure of joint classical describability. In the language of quantum logic, $\kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q}) > 0$ implies that $\vec{P}$ and $\vec{Q}$ do not belong to a common Boolean subalgebra of sharp elements in the GNS representation [7, 9]. This aligns with the graph-theoretic contextuality witnesses discussed in Sections 6.10 and 6.12.2, where incompatibility corresponds to edges in the tetralemmatic incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ defined over locus configurations [8, 20]. For many-particle systems ($n \gg 1$), the combinatorial growth of Γ_{ω_Λ} enhances sensitivity to logical deficiency $\mathcal{K}$ (Remark 6.37).

6.10 Tetralemmatic Incompatibility and Compatibility Graphs (n-Loci)

Definition 6.68 (Tetralemmatic incompatibility graph at a locus configuration). Fix a *locus configuration* $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$ and a coherent warrant ω_Λ. Define the undirected graph

$$\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$$

with vertex class $V = \mathsf{P}_{\mathcal{C}, \vec{\lambda}}$ (Definition 6.48) and an edge $\{\vec{P}, \vec{Q}\}$ iff

$$\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) > 0,$$

where κ_{ω_Λ} is the strong incompatibility degree (Definition 6.64). Define the *compatibility graph* $\Gamma_{\omega_\Lambda}^{\mathrm{compat}}(\mathcal{C}, \vec{\lambda})$ as the complementary graph on the same vertex class. This construction follows the graph-theoretic approach to quantum contextuality [21, 117], where incompatibility relations are encoded as adjacency relations in an exclusivity or incompatibility graph [2, 20], extended to composite configurations.

Remark 6.69 (Holothesis and the dual-subspace in the graph). For claims $\vec{P}$ warranting Holothesis, the incompatibility degree $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q})$ is evaluated on the *n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definition 5.4). This ensures that edges involving Holothesis vertices reflect the meta-test structure rather than standard polar noncommutativity. Consequently, the graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ encodes both standard operational incompatibility (within $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$) and meta-level inapplicability diagnostics (within $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$).

Remark 6.70 (Two layers of complementarity (n-Loci)). There are now two logically distinct obstructions to "joint classical talk" across the configuration $\vec{\lambda}$: (i) *typed inadmissibility* (Holothesis boundary; Remark 6.49), where claims fail to enter the vertex set entirely, and (ii) *warrant incompatibility* inside the admissible vertex class (edges of Γ_{ω_Λ}), witnessed by noncommutativity in the GNS sense. Both appear explicitly in the double-slit analyses Sections 2.13 and 2.14. The first layer corresponds to paracomplete truth-value gaps [54, 55], while the second layer corresponds to paraconsistent co-holding without explosion [13, 15]. This two-tier structure distinguishes the present framework from standard quantum logic approaches [7, 9], particularly in many-particle regimes where the vertex set may shrink due to global inapplicability.

6.10.1 Symmetry Covariance of the Graphs (n-Loci)

Proposition 6.71 (Duality covariance of incompatibility graphs (n-Loci)). *Assume $\alpha_{\mathbf{D}}$ is defined (Definition 3.38) and $\omega_\Lambda \circ \alpha_{\mathbf{D}} = \omega_\Lambda$. Then, for admissible n-loci claims $\vec{P}, \vec{Q}$,*

$$\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) = \kappa_{\omega_\Lambda}(\vec{P}^{\mathbf{D}}, \vec{Q}^{\mathbf{D}}), \qquad \kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}, \vec{Q}) = \kappa_{\omega_\Lambda}^{\mathrm{wk}}(\vec{P}^{\mathbf{D}}, \vec{Q}^{\mathbf{D}}),$$

and hence $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ is mapped isomorphically to itself under $\vec{P} \mapsto \vec{P}^{\mathbf{D}}$. Moreover, the unitary implementer $U_{\mathbf{D}}$ (Corollary 3.53) intertwines the commutator witnesses in the GNS representation.

Proof. By Definition 6.64, the incompatibility degrees are defined via suprema over corner-linear observables:

$$\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) \;=\; \sup_{\mathbf{a}, \mathbf{b} \in \mathbb{R}^4 \setminus \{0\}} \frac{\mathfrak{N}_{\omega_\Lambda}\big(O_{\mathbf{a}}(\vec{P}), O_{\mathbf{b}}(\vec{Q})\big)}{\|\mathbf{a}\| \, \|\mathbf{b}\|}.$$

By Proposition 5.18, the unitary implementer satisfies

$$U_{\mathbf{D}} \, O_{\mathbf{a}}(\vec{P}) \, U_{\mathbf{D}}^\dagger \;=\; O_{\mathbf{a}'}(\vec{P}^{\mathbf{D}})$$

for some transformed coefficient vector $\mathbf{a}'$ (since $U_{\mathbf{D}}$ permutes the corner generators according to $\alpha_{\mathbf{D}}$) [16, 17]. Since $U_{\mathbf{D}}$ is unitary and ω_Λ is **D**-invariant, the commutator witness is preserved:

$$\mathfrak{N}_{\omega_\Lambda}\big(O_{\mathbf{a}}(\vec{P}), O_{\mathbf{b}}(\vec{Q})\big) \;=\; \mathfrak{N}_{\omega_\Lambda}\big(U_{\mathbf{D}} O_{\mathbf{a}}(\vec{P}) U_{\mathbf{D}}^\dagger, U_{\mathbf{D}} O_{\mathbf{b}}(\vec{Q}) U_{\mathbf{D}}^\dagger\big) \;=\; \mathfrak{N}_{\omega_\Lambda}\big(O_{\mathbf{a}'}(\vec{P}^{\mathbf{D}}), O_{\mathbf{b}'}(\vec{Q}^{\mathbf{D}})\big).$$

Taking suprema over all coefficient vectors yields $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) = \kappa_{\omega_\Lambda}(\vec{P}^{\mathbf{D}}, \vec{Q}^{\mathbf{D}})$ [19, 92]. The same argument applies to the weak incompatibility degree $\kappa_{\omega_\Lambda}^{\mathrm{wk}}$.

Since edges in $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ are defined by $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) > 0$, the map $\vec{P} \mapsto \vec{P}^{\mathbf{D}}$ preserves adjacency and is therefore a graph isomorphism [117]. The intertwining property of $U_{\mathbf{D}}$ follows from the covariance relation $U_{\mathbf{D}} \, \pi_\Lambda(A) \, U_{\mathbf{D}}^\dagger = \pi_\Lambda(\alpha_{\mathbf{D}}(A))$ [7, 16]. $\qquad\square$

Remark 6.72 (Connection to Kochen–Specker contextuality (n-Loci))**.** The incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ provides a quantitative refinement of the Kochen–Specker exclusivity structure [2, 20]. In the sharp quantum realization limit (Tier 2 enrichment, Definition 3.33), edges correspond to pairs of observables that cannot be simultaneously sharp in any noncontextual hidden-variable model [118, 119]. The frame number $\mathscr{X}_{\omega_\Lambda}(X)$ (Definition 6.80) then quantifies the minimal number of classical contexts needed to cover a claim set X, generalizing the chromatic number bounds used in Kochen–Specker proofs [21, 120]. For many-particle systems ($n \gg 1$), the combinatorial growth of the graph may enhance sensitivity to logical deficiency $\mathcal{K}$ (Remark 6.37).

Remark 6.73 (Duality as a graph automorphism (n-Loci))**.** Proposition 6.71 establishes that duality acts as a graph automorphism on $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$. This has several consequences:

- **Symmetric incompatibility structure:** If $\vec{P}$ is incompatible with $\vec{Q}$, then $\vec{P}^{\mathbf{D}}$ is incompatible with $\vec{Q}^{\mathbf{D}}$.

- **Invariant subgraphs:** The fixed-point set of duality (claims with $\vec{P}^{\mathbf{D}} = \vec{P}$) forms an invariant subgraph.

- **Orbit structure:** Claims partition into duality orbits $\{\vec{P}, \vec{P}^{\mathbf{D}}\}$, and incompatibility relations respect this partitioning [16, 92].

In the double-slit example (Section 2.12.3), the left/right slit claims form a duality orbit, and their incompatibility with coherence-phase claims is preserved under slit exchange. For n-loci configurations, this extends to composite orbits across the configuration $\vec{\lambda}$.

6.11 Physical Realization of the Incompatibility Graph (*n*-Loci)

This subsection establishes the consistency between the abstract logical incompatibility graph and concrete physical operator models. It ensures that contextuality witnesses derived at the logical level are preserved (or refined) under physical realization, aligning with the algebraic quantum field theory paradigm where abstract algebras are represented on Hilbert spaces [7, 16, 17]. This consistency holds for both single-locus and *n-loci configurations*, as the algebraic structure is preserved under tensor product composition.

Proposition 6.74 (Graph witnesses transport through realizations (n-Loci)). *Let $r : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_{\mathrm{phys}}$ be a realization (Definition 3.57) and let $\omega_\Lambda = \omega_{\mathrm{phys}} \circ r$. Let $(\pi_{\mathrm{phys}}, \mathcal{H}_{\mathrm{phys}}, |\Omega_{\mathrm{phys}}\rangle)$ be the GNS triple for $(\mathcal{A}_{\mathrm{phys}}, \omega_{\mathrm{phys}})$. Then for all admissible n-loci claims $\vec{P}, \vec{Q}$ defined over locus configurations $\vec{\lambda}, \vec{\mu} \in \Lambda_{\mathcal{C}}^n$ and all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$,*

$$\mathfrak{N}_{\omega_\Lambda}\big(O_\mathbf{a}(\vec{P}), O_\mathbf{b}(\vec{Q})\big) = \big\|\pi_{\mathrm{phys}}\big([r(O_\mathbf{a}(\vec{P})), r(O_\mathbf{b}(\vec{Q}))]\big)|\Omega_{\mathrm{phys}}\rangle\big\|,$$

and similarly for the weak witness $\mathfrak{n}_{\omega_\Lambda}$:

$$\mathfrak{n}_{\omega_\Lambda}\big(O_\mathbf{a}(\vec{P}), O_\mathbf{b}(\vec{Q})\big) = \big|\langle\Omega_{\mathrm{phys}} \mid \pi_{\mathrm{phys}}\big([r(O_\mathbf{a}(\vec{P})), r(O_\mathbf{b}(\vec{Q}))]\big) \mid \Omega_{\mathrm{phys}}\rangle\big|.$$

In particular, if $[r(O_\mathbf{a}(\vec{P})), r(O_\mathbf{b}(\vec{Q}))] = 0$ in $\mathcal{A}_{\mathrm{phys}}$ for all $\mathbf{a}, \mathbf{b}$, then $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) = 0$ and $\{\vec{P}, \vec{Q}\}$ is not an edge of $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ (Definition 6.68). This demonstrates that physical commutativity implies logical compatibility [9, 63].

Proof. By Definition 6.54, the strong warrant commutator witness is

$$\mathfrak{N}_{\omega_\Lambda}(A, B) = \big\|\pi_\Lambda([A, B])|\Omega_\Lambda\rangle\big\| \tag{75}$$
$$= \sqrt{\omega_\Lambda([A, B]^\dagger[A, B])}.$$

Let $A = O_\mathbf{a}(\vec{P})$ and $B = O_\mathbf{b}(\vec{Q})$. Since $\omega_\Lambda = \omega_{\mathrm{phys}} \circ r$ and r is a $*$-homomorphism (Definition 3.57), we have

$$r([A, B]) = [r(A), r(B)]$$
$$\text{and} \tag{76}$$
$$r([A, B]^\dagger[A, B]) = [r(A), r(B)]^\dagger[r(A), r(B)].$$

Thus

$$\omega_\Lambda([A, B]^\dagger[A, B]) = \omega_{\mathrm{phys}}\big(r([A, B]^\dagger[A, B])\big) = \omega_{\mathrm{phys}}\big([r(A), r(B)]^\dagger[r(A), r(B)]\big).$$

By the GNS construction for the physical algebra [17, 19],

$$\omega_{\mathrm{phys}}(X^\dagger X) = \big\|\pi_{\mathrm{phys}}(X)|\Omega_{\mathrm{phys}}\rangle\big\|^2.$$

Setting $X = [r(A), r(B)]$ yields the identity for $\mathfrak{N}_{\omega_\Lambda}$. The proof for $\mathfrak{n}_{\omega_\Lambda}$ follows analogously using $\omega_\Lambda(Y) = \langle\Omega_{\mathrm{phys}} \mid \pi_{\mathrm{phys}}(r(Y)) \mid \Omega_{\mathrm{phys}}\rangle$.

For the incompatibility degree $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q})$, if $[r(A), r(B)] = 0$ for all $\mathbf{a}, \mathbf{b}$, then $\mathfrak{N}_{\omega_\Lambda}(A, B) = 0$ for all $\mathbf{a}, \mathbf{b}$. By Definition 6.64, $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) = 0$. By Definition 6.68, $\{\vec{P}, \vec{Q}\}$ is not an edge of $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$. This holds for n-loci claims since the $*$-homomorphism property is preserved under tensor product composition of the algebra [17]. $\square$

Remark 6.75 (Dual-subspace structure in physical realization)**.** For claims $\vec{P}$ or $\vec{Q}$ warranting Holothesis, the realization r must map the corresponding corner-linear observables to operators on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)} \subseteq \mathcal{H}_{\mathrm{phys}}$ (Definition 5.4). This ensures that the witness transport preserves the meta-test structure: commutators involving Holothesis generators are evaluated on $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ rather than the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$. Consequently, the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ encodes both standard operational incompatibility and meta-level inapplicability diagnostics.

Remark 6.76 (Logical vs. physical contextuality (n-Loci))**.** Proposition 6.74 clarifies the relationship between logical and physical contextuality for composite configurations:

- **Logical incompatibility:** Edges in $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ witness noncommutativity at the level of the abstract corner algebra $\mathcal{A}_\Lambda^{\mathrm{alg}}$, including n-loci corner generators.

- **Physical realization:** If a realization r maps generators to commuting operators, the logical incompatibility vanishes (classical limit). For n-loci configurations, this requires commutativity across all loci in $\vec{\lambda}$.

- **Contextuality gap:** If $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ has chromatic number $\mathscr{X} > 1$ but a specific realization admits a joint probability distribution (e.g. via hidden variables), the realization fails to capture the full logical structure [2, 20]. For many-particle systems ($n \gg 1$), the combinatorial growth of $\mathscr{X}$ may enhance sensitivity to logical deficiency $\mathcal{K}$ (Remark 6.37).

- **Dual-subspace witness:** For Holothesis claims, the contextuality gap is evaluated on $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$, providing a distinct signature from standard polar incompatibility.

Thus the logical graph provides *a priori* bounds on admissible physical realizations [7, 8].

Remark 6.77 (Connection to the Realization Axiom (n-Loci))**.** This result is the algebraic counterpart to the Realization Axiom of Section 6.5. There, applicability was identified with definability of effects; here, incompatibility is identified with noncommutativity of those effects. The transport of witnesses ensures that the tetralemmatic logical structure constrains the operational quantum model [56, 63].

For n-loci configurations, the Realization Axiom requires that r preserves the tensor product structure:

- **Composite Operational Space:** For $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, r maps $C_\bullet(\vec{P})$ to operators on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \subseteq \mathcal{H}_{\mathrm{phys}}$.

- **Dual-Subspace for Holothesis:** For $\bullet = \mathbf{HoloTh}$, r maps $C_\bullet(\vec{P})$ to operators on $\mathcal{H}_{\mathrm{Dual}}^{(2n)} \subseteq \mathcal{H}_{\mathrm{phys}}$.

- **Commutator Preservation:** Since r is a $*$-homomorphism, $[r(A), r(B)] = r([A, B])$, so the witness $\mathfrak{N}_{\omega_\Lambda}$ is invariant under faithful realizations.

This ensures the algebraic uncertainty relations derived at the logical level hold identically in all physical realizations (Proposition 6.56).

Remark 6.78 (Many-particle sensitivity and logical deficiency)**.** For many-particle systems $(n \gg 1)$, the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially with the number of loci. The chromatic number $\mathscr{X}(\Gamma_{\omega_\Lambda})$ may exceed the Hilbert space dimension d, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94). This manifests physically as:

- **Enhanced Contextuality:** Larger $\mathscr{X}$ implies stronger contextuality witnesses (e.g., KCBS violations).

- **Holothesis Density:** Excess probability weight diverted to the Holothesis sector to maintain consistency across the graph (Remark 6.145).

- **Sorkin Violations:** Potential for $I_3 \neq 0$ in regimes where $\mathcal{K} \neq 0$ (Proposition 6.36).

This provides an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems.

6.12 Contextuality Invariants of Tetralemmatic Graphs (*n*-Loci)

This section establishes the graph-theoretic invariants that quantify the degree of contextuality inherent in the tetralemmatic logical structure. The approach aligns with the graph-theoretic formulation of the Kochen–Specker theorem [2, 20], where logical incompatibility is encoded in the adjacency relations of an exclusivity or incompatibility graph [8, 9]. This analysis holds for both single-locus and *n-loci configurations*, with the latter providing enhanced sensitivity to logical deficiency in many-particle regimes.

6.12.1 Context covers and a minimal "number of classical frames" (n-Loci)

Definition 6.79 (Compatible contexts as independent sets (n-Loci))**.** Let $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ be the tetralemmatic incompatibility graph (Definition 6.68). A subset $S \subseteq \mathsf{P}_{\mathcal{C}, \vec{\lambda}}$ is called *jointly compatible* if it contains no edges of $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$, i.e. if S is an *independent set* of the incompatibility graph. Equivalently, S is a clique in the compatibility graph $\Gamma_{\omega_\Lambda}^{\mathrm{compat}}(\mathcal{C}, \vec{\lambda})$. This definition mirrors the standard treatment of compatibility in quantum logic, where jointly measurable observables form Boolean subalgebras [2, 7, 20].

Definition 6.80 (Context cover and frame number (n-Loci))**.** Let $X \subseteq \mathsf{P}_{\mathcal{C}, \vec{\lambda}}$ be any finite claim set defined over the locus configuration $\vec{\lambda} \in \Lambda_\mathcal{C}^n$. A *context cover* of X is a finite family of jointly compatible subsets $\{S_1, \ldots, S_m\}$ such that $X \subseteq \bigcup_{k=1}^m S_k$. Define the *frame number* of X by

$$\mathscr{X}_{\omega_\Lambda}(X) := \min\{ m : X \text{ admits a context cover of size } m \}.$$

This invariant quantifies the minimal number of classical contexts required to cover the logical content of X [8, 9].

Proposition 6.81 (Frame number equals chromatic number (n-Loci))**.** *Let* $\Gamma_{\omega_\Lambda}[X]$ *be the induced subgraph of* $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ *on X. Then*

$$\mathscr{X}_{\omega_\Lambda}(X) = \mathscr{X}\big(\Gamma_{\omega_\Lambda}[X]\big),$$

where $\mathscr{X}(G)$ denotes the usual chromatic number of the graph G.

Proof. By Definition 6.79, a jointly compatible subset S_k corresponds to an independent set in $\Gamma_{\omega_\Lambda}[X]$. A context cover $\{S_1, \ldots, S_m\}$ corresponds to a partition of the vertices of $\Gamma_{\omega_\Lambda}[X]$ into m independent sets (by assigning each vertex to one of the sets S_k that contains it). This is precisely the definition of a proper vertex coloring with m colors, where each color class is an independent set [20]. Thus the minimal cover size equals the chromatic number [2]. $\square$

Remark 6.82 (Interpretation (n-Loci))**.** $\mathscr{X}_{\omega_\Lambda}(X)$ is the minimal number of *pairwise compatible classical frames* needed to discuss all claims in X without running into commutator/uncertainty obstruction (Definition 6.68 and Corollary 6.57). Thus $\mathscr{X}_{\omega_\Lambda}(X) = 1$ means "globally compatible" on X at the locus configuration $\vec{\lambda}$, while $\mathscr{X}_{\omega_\Lambda}(X) > 1$ quantifies irreducible context-dependence *even after* typed admissibility (Holothesis) has already been handled. This aligns with the Kochen–Specker notion that no single global valuation exists for sets with $\mathscr{X} > 1$ [2, 20]. In the language of topos theory, $\mathscr{X}_{\omega_\Lambda}(X)$ measures the failure of the presheaf of valuations to be global [7, 9].

For n-loci configurations, claims warranting Holothesis are evaluated on the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4). The frame number quantifies contextuality among admissible claims (vertices), while Holothesis claims (inapplicable at the polar level) are excluded from the vertex set unless diagnosed via meta-test effects. This ensures the graph captures operational incompatibility among *askable* questions across the configuration $\vec{\lambda}$.

Remark 6.83 (Connection to Kochen–Specker contextuality (n-Loci))**.** The incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ provides a quantitative refinement of the Kochen–Specker exclusivity structure [2, 20]. In the sharp quantum realization limit (Tier 2 enrichment, Definition 3.33), edges correspond to pairs of observables that cannot be simultaneously sharp in any noncontextual hidden-variable model [118, 119]. The frame number $\mathscr{X}_{\omega_\Lambda}(X)$ then quantifies the minimal number of classical contexts needed to cover a claim set X, generalizing the chromatic number bounds used in Kochen–Specker proofs [8]. For example, odd-cycle graphs (e.g. the pentagon C_5) have $\mathscr{X}(C_5) = 3$, witnessing contextuality [20].

Many-Particle Sensitivity: For many-particle systems ($n \gg 1$), the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially with the number of loci. The chromatic number $\mathscr{X}(\Gamma_{\omega_\Lambda})$ may exceed the Hilbert space dimension d, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94). This manifests physically as enhanced contextuality witnesses (e.g., KCBS violations) and potential Sorkin I_3 violations (Proposition 6.36), providing an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems (Remark 6.37).

6.12.2 Exclusivity graphs and the contextuality gap (n-Loci)

This subsection introduces an optional sharp enrichment layer that enables graph-theoretic contextuality analysis within the tetralemmatic framework. The construction aligns with the graph-theoretic approach to quantum contextuality [21, 117], where logical incompatibility is encoded in the adjacency relations of an exclusivity graph [2, 20]. This analysis holds for both single-locus and *n-loci configurations*, with the latter providing enhanced sensitivity to logical deficiency in many-particle regimes.

Definition 6.84 (Sharp corner-events and an exclusivity graph (n-Loci)). Assume a Tier-2 sharp enrichment is in force (Definition 3.33) so that for each admissible n-loci claim $\vec{P}$ defined over locus configuration $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ the family $\{C_\bullet(\vec{P})\}_\bullet$ behaves like a sharp partition. Define the vertex set of *sharp corner-events*

$$V_{\text{evt}} := \{(\vec{P}, \bullet) : \ \vec{P} \in \mathsf{P}_{\mathcal{C}, \vec{\lambda}}, \ \bullet \in \mathbf{Status}_\Lambda\}.$$

Define an undirected graph $G_{\text{exclusive}}$ on V_{evt} by placing an edge between distinct events $e = (\vec{P}, \bullet)$ and $e' = (\vec{Q}, \bullet')$ whenever they are *exclusive*, meaning:

(i) $\vec{P} = \vec{Q}$ and $\bullet \neq \bullet'$ (distinct sharp outcomes of the same n-loci claim), or

(ii) $\vec{P} \neq \vec{Q}$ and there exist $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$ such that the corresponding n-loci corner-linear observables (Definition 6.62) witness incompatibility $\mathfrak{N}_{\omega_\Lambda}(O_{\mathbf{a}}(\vec{P}), O_{\mathbf{b}}(\vec{Q})) \neq 0$ (Definition 6.54), in which case $(\vec{P}, \bullet)$ and $(\vec{Q}, \bullet')$ are taken as not jointly co-realizable in a single sharp context.

This exclusivity relation generalizes the orthogonality relation in standard Kochen–Specker scenarios [2, 20].

Remark 6.85 (Dual-subspace structure for Holothesis events). For events involving Holothesis corner ($\bullet = \mathbf{HoloTh}$), the exclusivity relation is evaluated on the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4). This ensures that Holothesis events reflect the meta-test structure rather than standard polar noncommutativity. Consequently, the exclusivity graph $G_{\text{exclusive}}$ encodes both standard operational exclusivity (within $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$) and meta-level inapplicability diagnostics (within $\mathcal{H}_{\text{Dual}}^{(2n)}$).

Definition 6.86 (Classical (noncontextual) bound via independence number (n-Loci)). Let $G_{\text{exclusive}}$ be as in Definition 6.84. For any assignment of event-weights $p_e := \omega_\Lambda(C_\bullet(\vec{P}))$ (Tier-1 normalized if desired), define the exclusivity sum over a finite event set $W \subseteq V_{\text{evt}}$ by

$$\Sigma_{\omega_\Lambda}(W) := \sum_{e \in W} p_e.$$

In any deterministic noncontextual model (a single global sharp valuation), one has

$$\Sigma(W) \leq \alpha\big(G_{\text{exclusive}}[W]\big),$$

where α is the independence number (maximum size of a mutually nonexclusive subset). This bound follows from the fact that in a noncontextual assignment, at most one event from each exclusive pair can be assigned value 1 [117].

Remark 6.87 (n-Loci noncontextual valuations). For n-loci configurations, a noncontextual valuation requires a single global assignment across all loci in $\vec{\lambda}$. The Kochen–Specker theorem implies that for $\mathcal{X}(\Gamma_{\omega_\Lambda}) > d$, no such global valuation exists [2, 21]. The frame number $\mathcal{X}_{\omega_\Lambda}(W)$ (Definition 6.80) quantifies the minimal number of classical contexts needed to cover the event set W across the configuration.

Remark 6.88 (Quantum vs classical gap as a contextuality witness (n-Loci))**.** If one can realize the same exclusivity pattern by orthogonal projections in the GNS sector (via the realization interface Sections 5.5 and 6.5 and Proposition 6.74), then the maximal achievable $\Sigma_{\omega_\Lambda}(W)$ is typically strictly larger than the classical bound $\alpha(G_{\text{exclusive}}[W])$ for certain graphs (e.g. odd cycles), yielding an operational contextuality inequality. The *contextuality gap* can be measured as

$$\Delta(W) := \sup_{\omega_\Lambda \text{ coherent}} \Sigma_{\omega_\Lambda}(W) \ - \ \alpha\big(G_{\text{exclusive}}[W]\big).$$

For example, the pentagon graph C_5 has $\alpha(C_5) = 2$ classically, but quantum realizations can achieve $\Sigma(C_5) = \sqrt{5} > 2$ [120, 121]. This gap witnesses the impossibility of a noncontextual hidden-variable model [2, 119].

Many-Particle Sensitivity: For many-particle systems ($n \gg 1$), the exclusivity graph grows combinatorially with the number of loci. The chromatic number $\mathscr{X}(G_{\text{exclusive}})$ may exceed the Hilbert space dimension d, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94). This manifests physically as:

- **Enhanced Contextuality:** Larger $\mathscr{X}$ implies stronger contextuality witnesses (e.g., KCBS violations).

- **Holothesis Density:** Excess probability weight diverted to the Holothesis sector to maintain consistency across the graph (Remark 6.145).

- **Sorkin Violations:** Potential for $I_3 \neq 0$ in regimes where $\mathcal{K} \neq 0$ (Proposition 6.36).

This provides an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems (Remark 6.37).

Proposition 6.89 (Duality invariance of the contextuality gap)**.** *For any finite event set* $W \subseteq V_{\text{evt}}$,

$$\Delta(W) \ = \ \Delta(W^{\mathbf{D}}),$$

where $W^{\mathbf{D}} := \{(\vec{P}^{\mathbf{D}}, \bullet) : (\vec{P}, \bullet) \in W\}$.

Proof. The duality map induces a graph isomorphism $G_{\text{exclusive}}[W] \cong G_{\text{exclusive}}[W^{\mathbf{D}}]$ by the same argument as in Proposition 6.71. Since the independence number α is preserved under graph isomorphism [122],

$$\alpha\big(G_{\text{exclusive}}[W]\big) = \alpha\big(G_{\text{exclusive}}[W^{\mathbf{D}}]\big).$$

Furthermore, the warrant functional ω_Λ is **D**-invariant (Definition 5.15), so

$$\Sigma_{\omega_\Lambda}(W) = \Sigma_{\omega_\Lambda}(W^{\mathbf{D}}).$$

Therefore $\Delta(W) = \Delta(W^{\mathbf{D}})$ [16, 92]. □

Remark 6.90 (Physical interpretation of duality invariance)**.** Proposition 6.89 establishes that contextuality witnesses are symmetric under the duality transformation. This has several consequences:

- **Symmetric contextuality bounds:** Dual observable sets (e.g., which-path vs. interference) yield identical contextuality gaps.

- **Invariant exclusivity structure:** The exclusivity graph structure is preserved under slit-exchange or similar duality transformations.

- **Experimental robustness:** Contextuality witnesses derived from the tetralemmatic framework are stable under symmetry-preserving transformations, making them reliable empirical signatures [56, 63].

In the double-slit example (Section 2.12.3), the which-path and interference contexts form a duality orbit, and their respective contextuality invariants are preserved under slit exchange. For n-loci configurations, this extends to composite orbits across the configuration $\vec{\lambda}$.

6.12.3 Duality invariance of invariants (n-Loci)

Proposition 6.91 (Duality-invariance of frame number (n-Loci)). *Under the hypotheses of Proposition 6.71, for any finite claim set $X \subseteq \mathsf{P}_{\mathcal{C},\vec{\lambda}}$ defined over locus configuration $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$,*

$$\mathscr{X}_{\omega_\Lambda}(X) \;=\; \mathscr{X}_{\omega_\Lambda}(X^{\mathbf{D}}),$$

where $X^{\mathbf{D}} := \{\vec{P}^{\mathbf{D}} : \vec{P} \in X\}$.

Proof. By Proposition 6.71, the duality map $\vec{P} \mapsto \vec{P}^{\mathbf{D}}$ induces a graph isomorphism

$$\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})[X] \;\cong\; \Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})[X^{\mathbf{D}}],$$

since $\kappa_{\omega_\Lambda}(\vec{P}, \vec{Q}) = \kappa_{\omega_\Lambda}(\vec{P}^{\mathbf{D}}, \vec{Q}^{\mathbf{D}})$ for all $\vec{P}, \vec{Q} \in X$. Graph isomorphisms preserve the chromatic number [122], hence

$$\mathscr{X}\!\left(\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})[X]\right) \;=\; \mathscr{X}\!\left(\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})[X^{\mathbf{D}}]\right).$$

By Proposition 6.81, the frame number equals the chromatic number of the induced incompatibility graph:

$$\mathscr{X}_{\omega_\Lambda}(X) = \mathscr{X}\!\left(\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})[X]\right), \qquad \mathscr{X}_{\omega_\Lambda}(X^{\mathbf{D}}) = \mathscr{X}\!\left(\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})[X^{\mathbf{D}}]\right).$$

Therefore $\mathscr{X}_{\omega_\Lambda}(X) = \mathscr{X}_{\omega_\Lambda}(X^{\mathbf{D}})$ [92, 117]. □

Proposition 6.92 (Duality-invariance of the contextuality gap (n-Loci)). *Under the hypotheses of Proposition 6.71, for any finite event set $W \subseteq V_{\mathrm{evt}}$ defined over locus configuration $\vec{\lambda}$,*

$$\Delta(W) \;=\; \Delta(W^{\mathbf{D}}),$$

where $W^{\mathbf{D}} := \{(\vec{P}^{\mathbf{D}}, \bullet) : (\vec{P}, \bullet) \in W\}$.

Proof. The duality map induces a graph isomorphism $G_{\mathrm{exclusive}}[W] \cong G_{\mathrm{exclusive}}[W^{\mathbf{D}}]$ by the same argument as in Proposition 6.91, where $G_{\mathrm{exclusive}}$ is the exclusivity graph (Definition 6.84). Since the independence number α is preserved under graph isomorphism [122],

$$\alpha\!\left(G_{\mathrm{exclusive}}[W]\right) = \alpha\!\left(G_{\mathrm{exclusive}}[W^{\mathbf{D}}]\right).$$

Moreover, by the **D**-invariance of ω_Λ (Definition 3.35),

$$\omega_\Lambda(C_\bullet(\vec{P})) = \omega_\Lambda(C_{\mathbf{D}(\bullet)}(\vec{P}^{\mathbf{D}})),$$

so the supremum over coherent warrant functionals is unchanged under duality. Therefore $\Delta(W) = \Delta(W^{\mathbf{D}})$ [16, 92]. $\qquad\square$

Remark 6.93 (Physical interpretation of duality-invariant contextuality (n-Loci))**.** Propositions 6.91 and 6.92 establish that contextuality witnesses are structural invariants under the logical duality symmetry. This has several implications:

- **Symmetric contextuality:** If a claim set X exhibits contextuality ($\mathscr{X}_{\omega_\Lambda}(X) > 1$), then its dual $X^{\mathbf{D}}$ exhibits the same degree of contextuality.

- **Robustness under symmetry:** Contextuality gaps cannot be eliminated by duality transformations, reflecting the intrinsic nonclassicality of the logical structure [8, 9].

- **Experimental design:** In physical realizations, duality-symmetric measurement configurations (e.g. symmetric interferometer setups) will yield identical contextuality bounds for dual observable sets [56, 63].

- **Composite Orbits:** For n-loci configurations, claims partition into duality orbits $\{\vec{P}, \vec{P}^{\mathbf{D}}\}$ across the configuration $\vec{\lambda}$. Incompatibility relations respect this partitioning, ensuring contextuality invariants are preserved under composite symmetry transformations.

In the double-slit example (Section 2.12.3), the which-path and interference contexts form a duality orbit, and their respective contextuality invariants are preserved under slit exchange. For many-particle systems ($n \gg 1$), this invariance extends to composite orbits across the configuration $\vec{\lambda}$, providing robust signatures for contextuality witnesses in multi-party scenarios.

Theorem 6.94 (Logical Deficiency and Contextual Bounds (n-Loci))**.** *Let* $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ *be the incompatibility graph of a family of n-loci polar claims* $\mathcal{P}$ *defined over locus configuration* $\vec{\lambda} \in \Lambda_\mathcal{C}^n$ *(Definition 6.68). Let* $\mathscr{X}(\Gamma_{\omega_\Lambda})$ *be the chromatic number. Let d be the dimension of the local logical modal subspace* $\mathcal{H}_L(P)$ *restricted to* $\mathcal{P}$*, where* $d \in \{2, 4\}$ *depending on the enrichment regime (Definition 5.21). If* $\mathscr{X}(\Gamma_{\omega_\Lambda}) > d$*, then no classical* $\{0,1\}$ *valuation (Thesis/Antithesis assignment) exists. Consequently, for any coherent warrant* ω_Λ*:*

$$\sum_{P \in \mathcal{P}} \omega_\Lambda\Big(C_{\mathbf{SynTh}}(P) + C_{\mathbf{HoloTh}}(P)\Big) \geq \mathcal{K}(\Gamma_{\omega_\Lambda}, d) > 0, \tag{77}$$

where $\mathcal{K}(\Gamma_{\omega_\Lambda}, d)$ *is a positive constant depending on the graph topology and dimension. For polar corners (Thesis, Antithesis, Synthesis), the warrant is evaluated on the composite operational space* $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$*. For Holothesis corners, the warrant is evaluated on the n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ *(Definitions 5.4 and 5.7).*

Proof. A classical valuation corresponds to a valid d-coloring of Γ_{ω_Λ} where each vertex is assigned Thesis (1) or Antithesis (0) such that no adjacent vertices are both Thesis

(exclusivity). If $\mathscr{X}(\Gamma_{\omega_\Lambda}) > d$, such a coloring is impossible by definition [2, 120]. Thus, at least one claim $P \in \mathcal{P}$ must fail to admit a classical Thesis/Antithesis assignment. In the tetralemmatic framework, failure of classical assignment implies non-zero warrant in the non-classical corners (Synthesis or Holothesis).

To establish the lower bound $\mathcal{K} > 0$, consider the finite-dimensional subalgebra $\mathcal{A}_\mathcal{P}^{(n)} \subseteq \mathcal{A}_\Lambda^{\mathrm{alg}}$ generated by $\{C_\bullet(P) : P \in \mathcal{P}, \bullet \in \mathbf{Status}_\Lambda\}$ over the configuration $\vec{\lambda}$. The state space $S(\mathcal{A}_\mathcal{P}^{(n)})$ of coherent warrant functionals on this algebra is compact in the weak-* topology [17, 19]. We equip the dual space of $\mathcal{A}_\mathcal{P}^{(n)}$ with the norm $\|\omega\| = \sup\{|\omega(A)| : A \in \mathcal{B}\}$, where $\mathcal{B} = \{C_\bullet(P) : P \in \mathcal{P}, \bullet \in \mathbf{Status}_\Lambda\}$ is the finite set of corner generators. This norm induces the weak-* topology on the finite-dimensional state space. The subset $C \subset S(\mathcal{A}_\mathcal{P}^{(n)})$ of classical valuations (convex combinations of deterministic 0/1 assignments satisfying exclusivity) is closed. By the hypothesis $\mathscr{X} > d$, the set of coherent functionals is disjoint from C. The distance function $f(\omega) = \inf_{\nu \in C} \|\omega - \nu\|$ is continuous on the compact set of coherent functionals with respect to this norm, hence attains a minimum $\mathcal{K} > 0$ [7]. This minimum distance provides the lower bound for the sum of non-classical warrant weights in (77).

Structural Distinction: For claims warranting Synthesis, the warrant functional is evaluated on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} = \otimes_{i=1}^n \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$ (Definition 5.3). For claims warranting Holothesis, the warrant functional is evaluated on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)} = \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)} \otimes \mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$ (Definitions 5.4 and 5.7). This ensures the meta-test structure is preserved for Holothesis without collapsing polar applicability on the composite operational space. $\qquad \square$

Corollary 6.95 (Recovery of Non-Contextual and Standard Quantum Limits (n-Loci))**.** *Let $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ be the incompatibility graph of a family of n-loci polar claims $\mathcal{P}$ and d be the dimension of the local logical modal subspace $\mathcal{H}_L(P)$ restricted to $\mathcal{P}$, where $d \in \{2, 4\}$ (Definition 5.21). Let I_3 be the Sorkin third-order interference parameter defined in Proposition 6.36.*

  (i) Non-Contextual Limit: If $\mathscr{X}(\Gamma_{\omega_\Lambda}) \leq d$, then $\mathcal{K}(\Gamma_{\omega_\Lambda}, d) = 0$, implying $I_3 = 0$.

  (ii) Standard Quantum Phase Limit: If the warrant functional ω_Λ preserves the Z_4 logical symmetry such that the quarter-turn phase is $\theta_{\mathbf{R}} = \pi/2$ (Theorem 5.33), then $I_3 = 0$ regardless of $\mathscr{X}(\Gamma_{\omega_\Lambda})$.

  (iii) Dyadic Reduction Limit: If the enrichment regime imposes Tier-2 effect-like exclusivity (Definition 3.33) such that $d = 2$ (dyadic reduction), then contextuality witnesses requiring $d \geq 3$ vanish (Remark 6.135).

In all cases, the framework recovers the standard quantum mechanical prediction of vanishing higher-order interference [14].

Proof. **(i) Non-Contextual Limit.** If $\mathscr{X}(\Gamma_{\omega_\Lambda}) \leq d$, there exists a valid d-coloring of the graph Γ_{ω_Λ}. This corresponds to the existence of a classical valuation ν_{class} assigning Thesis/Antithesis values without violation of exclusivity. Thus, the set of classical valuations C intersects the set of coherent warrant functionals. The distance $\mathcal{K}(\Gamma_{\omega_\Lambda}, d) = \inf_{\nu \in C} \|\omega_\Lambda - \nu\|$ vanishes

because ω_Λ can be chosen to coincide with ν_{class} (or the minimum distance is zero by set intersection). By Proposition 6.36, $I_3 = \mathcal{K} \cdot \sin(4\theta_{\mathbf{R}}) + \mathcal{O}(\mathcal{K}^2)$. If $\mathcal{K} = 0$, then $I_3 = 0$.

(ii) Standard Quantum Phase Limit. Standard quantum mechanics corresponds to a complex Hilbert space structure where the logical symmetry is unbroken. By Theorem 5.33, this symmetry fixes the quarter-turn phase at $\theta_{\mathbf{R}} = \pi/2$. Substituting into the expression from Proposition 6.36:

$$\sin(4\theta_{\mathbf{R}}) = \sin(4 \cdot \pi/2) = \sin(2\pi) = 0. \tag{78}$$

Thus, even if $\mathscr{X}(\Gamma_{\omega_\Lambda}) > d$ (contextual regime), the leading-order term vanishes. Assuming higher-order terms $\mathcal{O}(\mathcal{K}^2)$ are similarly suppressed by the symmetry (as they are in standard Born rule derivations [59, 63]), we recover $I_3 = 0$.

(iii) Dyadic Reduction Limit. If Tier-2 effect-like enrichment imposes exclusivity relations reducing the local dimension to $d = 2$ (Definitions 3.33 and 5.21), then the incompatibility graph is structurally 2-colorable (Remark 6.135). This corresponds to the qubit limit where no contextuality arises. The Holothesis sector remains available for inapplicability diagnostics on the dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, but the logical deficiency constant $\mathcal{K}(\Gamma, 2) = 0$ for polar corners. $\square$

Remark 6.96 (Reconstruction of Quantum Bounds (n-Loci))**.** Theorem 6.94 and Corollary 6.95 demonstrate that the "quantum violation" of classical inequalities (e.g., KCBS bound $\sqrt{5}$ vs. classical 2) is not an empirical surprise but a *logical necessity* when $\mathscr{X} > d$ and symmetry is preserved. The excess probability weight is exactly the *Holothesis Density* required to maintain consistency across a graph that cannot be classically colored. This moves the framework from interpretation to *reconstruction*: the geometry of the incompatibility graph dictates the probabilistic bounds [21].

Dimension Qualification: The dimension d refers to the local logical modal subspace $\mathcal{H}_L(P)$, where $d \in \{2, 4\}$ depending on the enrichment regime (Definitions 3.46 and 5.21).

- *Dyadic Reduction (d = 2):* Corresponds to the qubit limit. Incompatibility graphs are 2-colorable; $\mathcal{K}(\Gamma, 2) = 0$. No contextuality arises.

- *Generic Four-Corner Case (d = 4):* Exceeds the $d \geq 3$ threshold for contextuality. Incompatibility graphs may have $\mathscr{X} > 4$, forcing $\mathcal{K}(\Gamma, 4) > 0$. Contextuality arises naturally.

Many-Particle Sensitivity: For many-particle systems ($n \gg 1$), the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially with the number of loci. The chromatic number $\mathscr{X}(\Gamma_{\omega_\Lambda})$ may exceed the Hilbert space dimension d more readily than in single-particle regimes, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94). This manifests physically as:

- **Enhanced Contextuality:** Larger $\mathscr{X}$ implies stronger contextuality witnesses (e.g., KCBS violations). For the strong product of n copies of a contextual graph (e.g., KCBS pentagon), $\mathscr{X}(\Gamma_n)$ grows exponentially while d_n grows polynomially **in symmetric subspaces** (Proposition 6.39).

- **Holothesis Density:** Excess probability weight diverted to the Holothesis sector to maintain consistency across the graph (Remark 6.145). Crucially, for n-loci configurations,

the Holothesis warrant is evaluated on the *n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definitions 5.4 and 5.7), preserving the meta-test structure without collapsing polar applicability on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$.

- **Sorkin Violations:** Potential for $I_3 \neq 0$ in regimes where $\mathcal{K} \neq 0$ (Proposition 6.36). The third-order interference term arises precisely from the non-vanishing warrant weight in the Holothesis sector, evaluated on the dual-subspace.

This provides an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems (Remark 6.37). The combinatorial amplification of $\mathcal{K}_n$ suggests that many-particle interference tests may be significantly more sensitive to logical deficiency than single-particle regimes, offering a potential experimental signature of the n-loci dual-subspace structure.

6.13 Uniqueness and Systematic Construction of the Realization Map (*n*-Loci)

To address concerns regarding empirical underdetermination, we demonstrate that the realization map $\mathcal{R}_{\mathcal{C},\vec{\lambda}}$ is not an arbitrary assignment but is canonically constrained by the GNS representation of the warrant functional across the locus configuration $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$.

6.13.1 Uniqueness up to Unitary Equivalence (n-Loci)

Theorem 6.97 (Uniqueness of the Realization Map (n-Loci)). *Let* $(\mathcal{A}_{\Lambda}^{\mathrm{alg}}, \omega_{\Lambda})$ *be the logical corner probability space for an n-loci configuration* $\vec{\lambda}$ *(Definition 3.20, Definition 3.35). Let* $\mathcal{R} : \mathcal{A}_{\Lambda}^{\mathrm{alg}} \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ *be a ∗-homomorphism into bounded operators on a physical Hilbert space* $\mathcal{H}_{\mathrm{phys}}$ *such that:*

(i) *State Preservation: There exists a cyclic vector* $|\Omega\rangle \in \mathcal{H}_{\mathrm{phys}}$ *such that* $\langle\Omega|\mathcal{R}(A)|\Omega\rangle = \omega_{\Lambda}(A)$ *for all* $A \in \mathcal{A}_{\Lambda}^{\mathrm{alg}}$.

(ii) *Effect Bounds: For all applicable n-loci generators* $C_{\bullet}(\vec{P})$, *the image satisfies* $0 \leq \mathcal{R}(C_{\bullet}(\vec{P})) \leq I$ *(Tier-2 enrichment, Definition 3.33).*

(iii) *Dual-Subspace Structure: For generators involving Holothesis (*$\bullet = $ **HoloTh***), the image acts on the n-loci dual-subspace* $\mathcal{H}_{\mathrm{Dual}}^{(2n)} \subseteq \mathcal{H}_{\mathrm{phys}}$ *(Definition 5.4).*

Then $\mathcal{R}$ *is unitarily equivalent to the logical GNS representation* π_{Λ} *(Definition 4.1). That is, there exists a unique unitary* $U : \mathcal{H}_{\mathrm{phys}} \to \mathcal{H}_{\Lambda}$ *such that* $\mathcal{R}(A) = U^{\dagger}\pi_{\Lambda}(A)U$ *and* $U|\Omega\rangle = |\Omega_{\Lambda}\rangle$.

Proof. Existence and uniqueness up to unitary equivalence follow from the Gelfand–Naimark–Segal (GNS) theorem [16, 18, 19]. Condition (i) ensures $\mathcal{R}$ is a GNS representation for ω_{Λ} over the n-loci algebra. Condition (ii) ensures the representation is physically admissible (maps to effects). Condition (iii) ensures the structural distinction between polar and Holothesis sectors is preserved (cf. Definition 5.11). Since any two GNS representations for the same state are unitarily equivalent [17, 94], the physical predictions (expectation

values $\text{Tr}(\rho E)$) are uniquely determined by ω_Λ across the configuration $\vec{\lambda}$, eliminating empirical underdetermination. $\qquad\square$

Remark 6.98 (Empirical Determinacy (n-Loci))**.** Theorem 6.97 guarantees that while different physical apparatuses (e.g. spin vs. path degrees of freedom) may realize the algebra, they are all unitarily equivalent representations of the same logical structure across the locus configuration $\vec{\lambda}$. Thus, the *logical content* is unique even if the *physical substrate* varies. For many-particle systems ($n \gg 1$), this ensures that contextuality witnesses (e.g., logical deficiency $\mathcal{K}$) are invariant under the choice of physical realization, provided the dual-subspace structure is preserved.

6.13.2 Systematic Construction of Effects from Generators (n-Loci)

The derivation of physical effects E_χ from logical generators $C_\bullet(\vec{P})$ follows a rigorous three-step protocol, respecting the Tier-0 to Tier-2 transition (Remark 3.17) and the n-loci structural distinctions:

(1) **Algebraic Mapping:** Each n-loci generator $C_\bullet(\vec{P})$ is mapped to its GNS operator image: $E_{\bullet,\vec{P}}^{(0)} := \pi_\Lambda(C_\bullet(\vec{P}))$. At Tier 0, these are merely self-adjoint operators (Definition 3.15).

(2) **Effect Constraint (Tier-2):** In an applicable context $\mathcal{C}$, the realization map $\mathcal{R}_{\mathcal{C},\vec{x}}$ selects a representation where generators satisfy effect bounds $0 \leq E \leq I$ (Definition 3.33). This restricts the admissible states ω_Λ to those yielding valid probabilities across the configuration.

(3) **Structural Distinction:**

- **Polar/Synthesis Corners:** For $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, the effect acts on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{x}}^{(n)}$.

- **Holothesis Corner:** For $\bullet = \mathbf{HoloTh}$, the effect acts on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, preserving the meta-test structure (Definition 5.8).

(4) **Projective Refinement:** In sharp measurement contexts (Scenario A), the logic becomes Boolean, enforcing idempotence $E^2 = E$. The construction yields sharp **Projection Operators**:

$$E_\chi = |\chi\rangle\langle\chi| \in \mathcal{B}(\mathcal{H}_{\mathcal{C},\vec{x}}^{(n)}). \tag{79}$$

Example 6.99 (Qutrit Path-Space Construction (Base Case $n = 1$))**.** In the three-slit KCBS scenario (Section 6.17.4), the generators $C_{\mathbf{Th}}(P_k)$ for $k = 1, \dots, 5$ map to rank-1 projectors $P_k = |v_k\rangle\langle v_k|$ onto the pentagon vectors (Eq. (82)). The "Synthesis" generators map to off-diagonal coherence terms $|v_j\rangle\langle v_k|$, recovering the full density matrix structure from the logical generators alone. **Generalization to n-Loci:** For $n > 1$, these effects extend via tensor products (e.g., $P_k^{(n)} = P_k \otimes \cdots \otimes P_k$) to represent composite Thesis states across the configuration $\vec{\lambda}$, while Holothesis effects require the dual-subspace construction (Definition 5.4). This demonstrates the systematic derivation of effects beyond the double-slit example.

6.14 Logic of the Double-slit Experiment

This section applies the tetralemmatic logical framework to the canonical double-slit experiment, illustrating how typed admissibility, realization, and compatibility graphs interact in a concrete physical setting. The analysis builds on the formal logic and the realization axiom of Section 6.5, and the Born-valued semantics of Section 4.6. The double-slit experiment has served as a foundational test case for quantum logic and contextuality since its early wave-mechanical formulation [10, 62, 123].

6.15 Logic of the Double-slit Experiment (*n*-Loci)

This section applies the tetralemmatic logical framework to the canonical double-slit experiment, illustrating how typed admissibility, realization, and compatibility graphs interact in a concrete physical setting. The analysis builds on the formal logic and the realization axiom of Section 6.5, and the Born-valued semantics of Definition 5.11. The double-slit experiment has served as a foundational test case for quantum logic and contextuality since its early wave-mechanical formulation [10, 62, 123]. This analysis holds for both single-locus ($n = 1$, base case) and *n-loci configurations* ($n > 1$, many-particle regimes).

6.15.1 Operational commutator witness at a locus configuration

Definition 6.100 (Realized effects at a locus configuration (n-Loci))**.** Fix a *locus configuration* $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\mathcal{C}}^n$ (Definition 3.7, generalized to n-loci in Definition 5.1). Assume the realization interface of Section 6.5 is in force, so that for each formula $\chi \in \mathrm{Lang}(\mathcal{C})$ with $\mathrm{App}_{\mathcal{C}}(\chi; \vec{\lambda})$ true, the realized effect/operator

$$E_{\chi}^{(\mathcal{C}, \vec{\lambda})} \; := \; \mathcal{R}_{\mathcal{C}, \vec{\lambda}}(\chi)$$

is defined (and is undefined if $\neg \mathrm{App}_{\mathcal{C}}(\chi; \vec{\lambda})$). This typed definability aligns with the operational approach to quantum effects [56, 63].

Remark 6.101 (Structural distinction for n-loci effects)**.** For n-loci configurations, the realized effects act on different Hilbert spaces depending on the corner status:

- **Polar/Synthesis Corners:** For $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, effects act on the composite operational space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$.

- **Holothesis Corner:** For $\bullet = \mathbf{HoloTh}$, effects act on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definition 5.4).

This ensures the meta-test structure is preserved for Holothesis claims across the configuration $\vec{\lambda}$.

Definition 6.102 (State-dependent commutator witness (n-Loci))**.** Fix a locus configuration $(\mathcal{C}, \vec{\lambda})$, a preparation/state ρ on the appropriate space (as in Definition 5.11), and formulas

$\chi, \psi \in \mathrm{Lang}(\mathcal{C})$ such that both are applicable at $(\mathcal{C}, \vec{\lambda})$. Define the (strong) operational commutator witness

$$\kappa_\rho^{(\mathcal{C}, \vec{\lambda})}(\chi, \psi) \; := \; \left(\mathrm{Tr} \left(\rho \, [E_\chi, E_\psi]^\dagger [E_\chi, E_\psi] \right) \right)^{1/2}, \qquad E_\chi := E_\chi^{(\mathcal{C}, \vec{\lambda})}, \; E_\psi := E_\psi^{(\mathcal{C}, \vec{\lambda})}.$$

This witness quantifies the degree of operational incompatibility relative to the state ρ across the locus configuration [107, 108].

Remark 6.103 (n-Loci incompatibility sources)**.** For n-loci configurations, non-zero commutator witnesses may arise from:

- **Local Noncommutativity:** Non-commuting effects at individual loci $\lambda_i \in \vec{\lambda}$.

- **Cross-Locus Correlations:** Entanglement between distinct loci, detected via non-separable state ρ.

- **Dual-Subspace Structure:** For Holothesis claims, the witness is evaluated on $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$, reflecting the meta-test structure.

This provides enhanced sensitivity to contextuality in many-particle regimes ($n \gg 1$).

Remark 6.104 (Typedness: Holothesis removes vertices (n-Loci))**.** If $\neg\mathrm{App}_\mathcal{C}(\chi; \vec{\lambda})$, then $E_\chi^{(\mathcal{C}, \vec{\lambda})}$ is undefined (Definition 6.100). In particular, in the interference locus of the double-slit walk (base case $n = 1$), which-path formulas are inapplicable and thus receive Holothesis there (cf. the double-slit Holothesis clause in Sections 2.13 and 2.14). This typed exclusion is *prior* to any (in)compatibility analysis. This reflects the paracomplete nature of the Holothesis corner, where truth-value gaps arise from inapplicability rather than indeterminacy [8, 54, 55].

   For n-loci configurations: When Holothesis is warranted at any locus $\lambda_i \in \vec{\lambda}$, the corresponding vertex is excluded from the incompatibility graph for the entire configuration. This ensures the graph captures only operational incompatibility among *askable* questions across $\vec{\lambda}$, while Holothesis claims are diagnosed via the dual-subspace meta-test (Definition 5.8).

6.15.2   Typed incompatibility graphs and frame number (n-Loci)

Definition 6.105 (Typed incompatibility graph at a locus configuration (n-Loci))**.** Fix a *locus configuration* $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_\mathcal{C}^n$ and a state ρ on the appropriate space (Definition 5.11). Let $X \subseteq \mathrm{Lang}(\mathcal{C})$ be a finite set such that every $\chi \in X$ is applicable at $(\mathcal{C}, \vec{\lambda})$. Define the undirected graph

$$G_\rho^{(\mathcal{C}, \vec{\lambda})}(X) = (V, E), \qquad V := X, \qquad \{\chi, \psi\} \in E \iff \kappa_\rho^{(\mathcal{C}, \vec{\lambda})}(\chi, \psi) > 0,$$

where $\kappa_\rho^{(\mathcal{C}, \vec{\lambda})}$ is the state-dependent commutator witness (Definition 6.102). A *frame* is an independent set in $G_\rho^{(\mathcal{C}, \vec{\lambda})}(X)$ (a jointly compatible family). This graph-theoretic representation of incompatibility aligns with the exclusivity graph approach to contextuality [20, 21, 117].

Remark 6.106 (Structural distinction for n-loci graphs)**.** For n-loci configurations, the incompatibility graph primarily captures operational incompatibility among *applicable* claims

(Thesis, Antithesis, Synthesis) evaluated on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$. Claims warranting Holothesis are typically excluded from the vertex set (Definition 6.100), as they are diagnosed via the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ meta-test rather than standard commutator witnesses. This ensures the graph reflects incompatibility among *askable* questions across the configuration $\vec{\lambda}$.

Definition 6.107 (Frame number (n-Loci)). The *frame number* of X at $(\mathcal{C}, \vec{\lambda})$ in state ρ is

$$\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X) \ := \ \mathscr{X}\Big(G_\rho^{(\mathcal{C},\vec{\lambda})}(X)\Big),$$

the chromatic number of the incompatibility graph on X. This invariant quantifies the minimal number of classical contexts needed to cover the claim set X across the locus configuration [2, 9].

Proposition 6.108 (Interpretation of $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X)$ (n-Loci)). *If $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X) = 1$, then all pairs in X satisfy $\kappa_\rho^{(\mathcal{C},\vec{\lambda})}(\chi, \psi) = 0$, hence X admits a single jointly compatible frame (no operational incompatibility is witnessed on the support of ρ across the configuration). If $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X) > 1$, then X cannot be discussed within a single jointly sharp classical frame at that locus configuration (some operational noncommutativity is witnessed). This is the graph-theoretic signature of contextuality [2, 20, 21].*

Proof. Let $G := G_\rho^{(\mathcal{C},\vec{\lambda})}(X) = (V, E)$ be the typed incompatibility graph defined in Definition 6.105, with vertex set $V = X$ and edge set

$$E \ := \ \Big\{\{\chi, \psi\} \subseteq X \ : \ \kappa_\rho^{(\mathcal{C},\vec{\lambda})}(\chi, \psi) > 0\Big\}.$$

By Definition 6.79, a *frame* corresponds to an *independent set* in G (a subset of vertices containing no edges). The frame number $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X)$ is defined as the chromatic number $\mathscr{X}(G)$, which is the minimum number of independent sets required to cover V [20].

Case 1: $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X) = 1$. If $\mathscr{X}(G) = 1$, then V can be covered by a single independent set S_1. Since S_1 must contain all vertices ($V \subseteq S_1$), V itself is an independent set. By the definition of an independent set, no two vertices in V share an edge. Therefore, $E = \emptyset$, which implies

$$\kappa_\rho^{(\mathcal{C},\vec{\lambda})}(\chi, \psi) = 0 \qquad \text{for all } \chi, \psi \in X.$$

Thus, all claims in X are pairwise compatible, and X admits a single jointly compatible frame. This corresponds to a noncontextual scenario where a single classical context suffices [21].

Case 2: $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X) > 1$. If $\mathscr{X}(G) > 1$, then V cannot be covered by a single independent set. Therefore, V is *not* an independent set. This implies that there exists at least one edge $\{\chi, \psi\} \in E$. By the definition of the edge set, this means

$$\kappa_\rho^{(\mathcal{C},\vec{\lambda})}(\chi, \psi) > 0$$

for some pair $\chi, \psi \in X$. Consequently, X contains operationally incompatible claims that cannot be simultaneously sharp in a single classical frame. The necessity of using $\mathscr{X}(G)$ distinct frames to cover X reflects the impossibility of a global noncontextual valuation on the set X, which is the graph-theoretic signature of contextuality [2, 20]. $\square$

Remark 6.109 (Connection to Kochen–Specker contextuality (n-Loci))**.** The frame number $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X)$ provides a state-dependent refinement of the Kochen–Specker chromatic number [2]. While the KS theorem establishes the impossibility of a global noncontextual valuation for certain observable sets, the frame number quantifies the *degree* of contextuality relative to a specific preparation ρ and locus configuration $(\mathcal{C}, \vec{\lambda})$ [21]. In the double-slit context (base case $n = 1$), $\mathscr{X}_\rho^{(\mathcal{C},\vec{\lambda})}(X) = 2$ for which-path and coherence-phase claims in an applicability-restoring refinement (cf. Proposition 6.118), reflecting the minimal incompatibility between path and phase observables [56, 63]. **For many-particle systems (**$n \gg 1$**):** The combinatorial growth of the graph may enhance sensitivity to logical deficiency $\mathcal{K}$ (Theorem 6.94).

Remark 6.110 (State-dependence and the support of ρ (n-Loci))**.** The frame number depends on the state ρ through the commutator witness $\kappa_\rho^{(\mathcal{C},\vec{\lambda})}$. If ρ is supported on a subspace where $[E_\chi, E_\psi]$ vanishes (e.g. an eigenstate of one of the effects), then $\kappa_\rho^{(\mathcal{C},\vec{\lambda})}(\chi, \psi) = 0$ even if the operators do not commute globally [107, 108]. This state-dependence distinguishes the present operational approach from the state-independent Kochen–Specker theorem [2, 9]. For n-loci configurations, this allows contextuality to be detected only when the state ρ has support across the relevant loci in $\vec{\lambda}$.

Remark 6.111 (Contrast with Consistent Histories (n-Loci))**.** This framework distinguishes itself from Consistent Histories (CH) [5] in the ontological status of the "unaskable":

- **Consistent Histories:** Incompatible frameworks are *discarded.* A question outside a decoherent history is "meaningless" (linguistic prohibition).

- **Tetralemmatic Logic:** The "unaskable" is a *first-class logical status* (Holothesis). It is represented by a vector $|\mathbf{HoloTh}; \vec{P}\rangle$ in $\mathcal{H}_\Lambda$ (Definition 4.8) and carries a positive warrant weight $\omega_\Lambda(C_{\mathbf{HoloTh}}(\vec{P}))$.

Thus, where CH sees a boundary of language, Tetralemmatic Logic sees an operational state. This allows for a *continuity of inapplicability* (via Holothesis weight) that CH cannot formalize, enabling the derivation of contextuality bounds via Theorem 6.94. For n-loci configurations, this distinction is critical as Holothesis claims are evaluated on the dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$.

6.16 The Canonical Double-slit Loci (*n*-Loci)

Definition 6.112 (Double-slit loci and canonical claims (n-Loci))**.** Let P denote the which-path polar claim *"the particle went through the left slit"*, with $\neg P$ the right-slit claim, as in the double-slit analyses Sections 2.13 and 2.14. Let $(\mathcal{C}, \vec{\lambda})$ denote a locus configuration where $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ (for the standard double-slit, $n = 1$). Let:

- $(\mathcal{C}_1, \vec{\lambda}_1)$ be the *which-path locus configuration* (path detector present; which-path talk applicable),

- $(\mathcal{C}_2, \vec{\lambda}_2)$ be the *interference-region locus configuration* (between slits and screen; which-path talk inapplicable),

- $(\mathcal{C}_3, \vec{\lambda}_3)$ be the *screen locus configuration* (position measurement applicable; Born semantics applies, cf. Definition 5.11).

Optionally, let $(\mathcal{C}', \vec{\lambda}')$ denote an *applicability-restoring refinement* (weak which-path coupling while retaining partial coherence), as discussed in the double-slit refinements.

Proposition 6.113 (Scenario A: which-path locus admits a single frame on $\{P, \neg P\}$ (n-Loci))**.** *Assume* $\mathrm{App}_{\mathcal{C}_1}(P; \vec{\lambda}_1)$ *and* $\mathrm{App}_{\mathcal{C}_1}(\neg P; \vec{\lambda}_1)$. *If the realized effects satisfy* $E_{\neg P}^{(\mathcal{C}_1, \vec{\lambda}_1)} = \mathbf{1} - E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}$ *(as in the ideal two-slit projective example used in the double-slit discussion), then for* $X = \{P, \neg P\}$,
$$\kappa_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(P, \neg P) = 0 \quad \text{and hence} \quad \mathscr{X}_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(X) = 1.$$

This reflects the classical compatibility of which-path alternatives when a sharp measurement is admitted [10, 56, 63].

Proof. **Step 1: Commutativity of complementary effects.** By the Realization Axiom (Section 6.5), the polar claims P and $\neg P$ are interpreted by effects
$$E_P^{(\mathcal{C}_1, \vec{\lambda}_1)} \in \mathcal{E}_{\mathcal{C}_1, \vec{\lambda}_1}, \qquad E_{\neg P}^{(\mathcal{C}_1, \vec{\lambda}_1)} = \mathbf{1} - E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}.$$

Since $E_{\neg P}^{(\mathcal{C}_1, \vec{\lambda}_1)}$ is a linear polynomial in $E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}$ (specifically, an affine function), they commute:
$$\left[E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}, E_{\neg P}^{(\mathcal{C}_1, \vec{\lambda}_1)} \right] = \left[E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}, \mathbf{1} - E_P^{(\mathcal{C}_1, \vec{\lambda}_1)} \right] = \left[E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}, \mathbf{1} \right] - \left[E_P^{(\mathcal{C}_1, \vec{\lambda}_1)}, E_P^{(\mathcal{C}_1, \vec{\lambda}_1)} \right] = 0.$$

This is a standard result for complementary effects in operational quantum theory [63, 85].

Step 2: Vanishing commutator witness. By Definition 6.102, the operational commutator witness is
$$\kappa_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(P, \neg P) = \left(\mathrm{Tr} \left(\rho \, [E_P, E_{\neg P}]^\dagger [E_P, E_{\neg P}] \right) \right)^{1/2},$$

where we suppress the locus superscripts for brevity. Since $[E_P, E_{\neg P}] = 0$ from Step 1, the integrand vanishes identically for any state ρ:
$$\kappa_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(P, \neg P) = 0.$$

This matches the expectation that sharp which-path measurements yield compatible alternatives [23].

Step 3: Frame number equals 1. By Definition 6.105, the typed incompatibility graph $G_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(X)$ has vertex set $V = X = \{P, \neg P\}$ and an edge $\{P, \neg P\}$ iff $\kappa_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(P, \neg P) > 0$. Since the witness vanishes, there is no edge:
$$E\left(G_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(X) \right) = \emptyset.$$

Thus $G_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(X)$ is the empty graph on two vertices, which has chromatic number $\mathscr{X} = 1$ [20]. By Definition 6.107, the frame number is
$$\mathscr{X}_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(X) = \mathscr{X}\left(G_\rho^{(\mathcal{C}_1, \vec{\lambda}_1)}(X) \right) = 1.$$

Step 4: Interpretation. A frame number of 1 means that $X = \{P, \neg P\}$ admits a single jointly compatible frame at $(\mathcal{C}_1, \vec{\lambda}_1)$. This is consistent with the which-path context admitting a classical Boolean description of the alternatives [75, 124]. In graph-theoretic terms, the independence number $\alpha(G) = 2$ equals the vertex count $|V| = 2$, confirming global compatibility [117]. $\square$

Remark 6.114 (Contrast with interference context (n-Loci))**.** Proposition 6.113 should be contrasted with Scenario B (Proposition 6.115), where the which-path claims are inapplicable at the interference locus configuration $(\mathcal{C}_2, \vec{\lambda}_2)$. There, the frame number is undefined (no vertices), reflecting Holothesis rather than compatibility [8, 9]. This illustrates the two-layer complementarity structure: typed inadmissibility (Holothesis) vs. operational incompatibility (graph edges) [20]. For many-particle systems ($n \gg 1$), this distinction becomes critical as the vertex set may shrink due to global inapplicability across the configuration $\vec{\lambda}$.

Proposition 6.115 (Scenario B: interference-region locus deletes which-path vertices (Holothesis) (n-Loci))**.** *Assume* $\neg\mathrm{App}_{\mathcal{C}_2}(P; \vec{\lambda}_2)$ *and* $\neg\mathrm{App}_{\mathcal{C}_2}(\neg P; \vec{\lambda}_2)$ *as in the interference-locus clause of the double-slit walk Sections 2.13 and 2.14. Then* $E_P^{(\mathcal{C}_2, \vec{\lambda}_2)}$ *and* $E_{\neg P}^{(\mathcal{C}_2, \vec{\lambda}_2)}$ *are undefined (Definition 6.100), so* $\{P, \neg P\}$ *cannot appear as a vertex set of any typed graph* $G_\rho^{(\mathcal{C}_2, \vec{\lambda}_2)}(X)$. *Equivalently, which-path talk is excluded* by type *(Holothesis) before incompatibility is even considered.*

Proof. **Step 1: Inapplicability at the interference locus configuration.** By Definition 6.112, $(\mathcal{C}_2, \vec{\lambda}_2)$ denotes the interference-region locus configuration where no which-path detector is active (for the standard double-slit, $n = 1$). As established in the double-slit tetralemma walk Sections 2.8 and 2.14, the operational arrangement in $\mathcal{C}_2$ does not admit a which-path test. Consequently, the polar claims P ("left slit") and $\neg P$ ("right slit") satisfy the typed inapplicability condition:

$$\neg\mathrm{App}_{\mathcal{C}_2}(P; \vec{\lambda}_2) \quad \text{and} \quad \neg\mathrm{App}_{\mathcal{C}_2}(\neg P; \vec{\lambda}_2).$$

This inapplicability is the defining warrant condition for the Holothesis corner (Definition 2.12).

Step 2: Undefined realized effects. By Definition 6.100, the realized effect operator $E_\chi^{(\mathcal{C}, \vec{\lambda})} := \mathcal{R}_{\mathcal{C}, \vec{\lambda}}(\chi)$ is defined *if and only if* $\mathrm{App}_{\mathcal{C}}(\chi; \vec{\lambda})$ is true. Since applicability fails for both P and $\neg P$ at $(\mathcal{C}_2, \vec{\lambda}_2)$, the corresponding effects $E_P^{(\mathcal{C}_2, \vec{\lambda}_2)}$ and $E_{\neg P}^{(\mathcal{C}_2, \vec{\lambda}_2)}$ are undefined. Without defined effects, no Born valuation or commutator witness can be formed for these claims at this locus configuration.

Step 3: Exclusion from the typed incompatibility graph. By Definition 6.105, the typed incompatibility graph $G_\rho^{(\mathcal{C}, \vec{\lambda})}(X)$ is defined for a finite set $X \subseteq \mathrm{Lang}(\mathcal{C})$ *such that every* $\chi \in X$ *is applicable at* $(\mathcal{C}, \vec{\lambda})$. Formally, the vertex set V must satisfy:

$$V \subseteq \{\, \chi \in \mathrm{Lang}(\mathcal{C}) : \mathrm{App}_{\mathcal{C}}(\chi; \vec{\lambda}) \,\}.$$

Since $P \notin \{\chi : \mathrm{App}_{\mathcal{C}_2}(\chi; \vec{\lambda}_2)\}$ and $\neg P \notin \{\chi : \mathrm{App}_{\mathcal{C}_2}(\chi; \vec{\lambda}_2)\}$, the set $\{P, \neg P\}$ violates the applicability constraint required for vertex inclusion. Therefore, no valid graph $G_\rho^{(\mathcal{C}_2, \vec{\lambda}_2)}(X)$ can contain P or $\neg P$ as vertices.

Step 4: Logical interpretation (Holothesis vs. Incompatibility). Incompatibility (represented by edges in G_ρ) presupposes applicability (represented by vertices). Since P and $\neg P$ fail to qualify as vertices at $(\mathcal{C}_2, \vec{\lambda}_2)$, the question of incompatibility does not arise. Instead, their status is determined by the failure of applicability itself, which corresponds to the Holothesis corner (Definition 2.12). Thus, which-path talk is excluded *by type* prior to any consideration of operational incompatibility [8, 9].

Step 5: Many-particle generalization. For many-particle systems ($n > 1$), Holothesis at any locus $\lambda_i \in \vec{\lambda}_2$ excludes the corresponding vertex from the incompatibility graph for the entire configuration. This ensures the graph captures only operational incompatibility among *askable* questions across $\vec{\lambda}$, while Holothesis claims are diagnosed via the dual-subspace meta-test (Definition 5.8). □

Proposition 6.116 (Scenario C: screen locus supports position-claims and Born weights (n-Loci)). *At $(\mathcal{C}_3, \vec{\lambda}_3)$, let X_{pos} be a finite family of applicable position-claims (e.g. coarse-grained bins). Then each $\chi \in X_{\text{pos}}$ has a realized effect $E_\chi^{(\mathcal{C}_3, \vec{\lambda}_3)}$, and the Born semantics of Definition 5.11 assigns the corresponding probabilities $\text{Tr}(\rho\, E_\chi^{(\mathcal{C}_3, \vec{\lambda}_3)})$. Changing the upstream locus configuration (which-path vs interference) changes the effective incoming state at the screen locus and hence changes these Born weights, matching the narrative in the double-slit analysis [10, 58, 63].*

Proof. **Step 1: Applicability of position claims at the screen locus configuration.** By Definition 6.100, the screen locus configuration $(\mathcal{C}_3, \vec{\lambda}_3)$ is defined as the context in which position measurement is admitted. Therefore, for each position claim $\chi \in X_{\text{pos}}$ (e.g., "the electron hits in bin B_i"), the applicability predicate holds:

$$\text{App}_{\mathcal{C}_3}(\chi; \vec{\lambda}_3) = \text{True}.$$

By the Realization Axiom (Section 6.5), applicability implies definability of the realized effect:

$$E_\chi^{(\mathcal{C}_3, \vec{\lambda}_3)} := \mathcal{R}_{\mathcal{C}_3, \vec{\lambda}_3}(\chi) \in \mathcal{E}_{\mathcal{C}_3, \vec{\lambda}_3},$$

where $\mathcal{E}_{\mathcal{C}_3, \vec{\lambda}_3}$ denotes the set of admissible effects at $(\mathcal{C}_3, \vec{\lambda}_3)$ [56, 63].

Step 2: Born-valued warrant assignment. By Definition 5.11, for any prepared state ρ on the relevant Hilbert space $\mathcal{H}_{\mathcal{C}_3, \vec{\lambda}_3}^{(n)}$, the Born-valued degree of warrant is given by:

$$v(\chi; \rho) = \text{Tr}\Big(E_\chi^{(\mathcal{C}_3, \vec{\lambda}_3)}\, \rho \Big).$$

This is the standard Born rule for effect-valued measurements [58, 85].

Step 3: State dependence on upstream context. Let ρ_{in} denote the effective incoming state at the screen locus configuration. This state depends on the upstream context/locus configuration:

- **Which-path context $(\mathcal{C}_1, \vec{\lambda}_1)$:** The measurement interaction at the slits induces decoherence in the path basis, yielding a mixed state of the form

$$\rho_{\text{in}}^{(1)} = |\alpha|^2\, |L\rangle\langle L| + |\beta|^2\, |R\rangle\langle R|,$$

where off-diagonal coherence terms are suppressed [22, 23].

- **Interference context** $(\mathcal{C}_2, \vec{\lambda}_2)$**:** No which-path measurement occurs, preserving coherence. The incoming state is a pure superposition

$$\rho_{\text{in}}^{(2)} = |\psi\rangle\langle\psi|, \qquad |\psi\rangle = \alpha|L\rangle + \beta|R\rangle,$$

with nonzero off-diagonal terms [10].

Step 4: Interference pattern from Born weights. For position eigenkets $|x\rangle$ at the screen (or coarse-grained projectors $E_x = |x\rangle\langle x|$), the probability density is:

$$p(x) = \text{Tr}\big(E_x\,\rho_{\text{in}}\big) = \langle x|\rho_{\text{in}}|x\rangle.$$

In the interference context:

$$p^{(2)}(x) = \big|\alpha\langle x|L\rangle + \beta\langle x|R\rangle\big|^2 = |\alpha|^2|\langle x|L\rangle|^2 + |\beta|^2|\langle x|R\rangle|^2 + 2\,\text{Re}\big(\alpha^*\beta\langle L|x\rangle\langle x|R\rangle\big),$$

where the cross-term produces interference fringes [10, 62].

In the which-path context:

$$p^{(1)}(x) = |\alpha|^2|\langle x|L\rangle|^2 + |\beta|^2|\langle x|R\rangle|^2,$$

with no interference term due to decoherence [23].

Step 5: Contextual dependence of Born weights. Since $\rho_{\text{in}}^{(1)} \neq \rho_{\text{in}}^{(2)}$, the Born weights differ:

$$\text{Tr}\big(E_\chi^{(\mathcal{C}_3,\vec{\lambda}_3)}\,\rho_{\text{in}}^{(1)}\big) \neq \text{Tr}\big(E_\chi^{(\mathcal{C}_3,\vec{\lambda}_3)}\,\rho_{\text{in}}^{(2)}\big)$$

for position claims χ sensitive to the interference pattern. This contextual dependence matches the operational narrative of the double-slit experiment [8, 9, 63]. $\qquad\square$

Remark 6.117 (Connection to the Realization Axiom and contextuality (n-Loci))**.** Proposition 6.116 illustrates a key feature of the tetralemmatic framework: the Born weights at a given locus configuration depend not only on the local measurement effects but also on the upstream context that determines the incoming state. This is consistent with:

- **The Realization Axiom** (Section 6.5): effects are context-indexed, and state preparation is part of the contextual specification [56, 63].

- **Quantum contextuality**: the probability assignment depends on the full measurement context, not just the local observable [2, 9, 20].

- **The tetralemmatic walk**: the screen locus configuration $(\mathcal{C}_3, \vec{\lambda}_3)$ receives different logical status depending on whether the upstream locus configuration warranted Thesis/Antithesis (which-path) or Holothesis (interference) [8].

Thus, the Born weights at the screen encode the logical history of the experimental arrangement, making the probability assignment inherently contextual [7].

Many-Particle Sensitivity: For many-particle systems ($n \gg 1$), the contextual dependence is amplified as the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially. The chromatic number $\mathscr{X}(\Gamma_{\omega_\Lambda})$ may exceed the Hilbert space dimension d, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94). This manifests physically as enhanced contextuality witnesses and potential Sorkin I_3 violations (Proposition 6.36), providing an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems (Remark 6.37).

Proposition 6.118 (Scenario D: refinement locus yields a nontrivial incompatibility edge (n-Loci)). *Assume $(\mathcal{C}', \vec{\lambda}')$ is an applicability-restoring refinement in the sense of Definition 2.6, in which both a which-path claim P and a coherence-phase claim Σ are applicable across the locus configuration $\vec{\lambda}' \in \Lambda_{\mathcal{C}'}^n$. In the enriched language of the refined context, we introduce a primitive predicate $\sigma(x)$ interpreted as a test for the particle being in the coherent superposition state $|+\rangle$ at the slit plane. The predicate $\sigma(x)$ is admissible in $\mathcal{C}'$ specifically because the refinement satisfies the operational criteria of Definition 2.6, which permits coherence tests not available in the base context $\mathcal{C}$. The corresponding claim is defined formally as:*

$$\Sigma := \exists_{\vec{\lambda}'}^{\mathcal{C}'} x \, \sigma(x), \tag{80}$$

consistent with the claim construction in Definition 2.1. Suppose their realized effects satisfy, on the two-path subspace $\mathcal{H}_{\text{path}} \subseteq \mathcal{H}_{\mathcal{C}', \vec{\lambda}'}^{(n)}$,

$$E_P^{(\mathcal{C}', \vec{\lambda}')} \approx |L\rangle\langle L|, \qquad E_\Sigma^{(\mathcal{C}', \vec{\lambda}')} \approx |+\rangle\langle +|, \qquad |+\rangle = \tfrac{1}{\sqrt{2}}(|L\rangle + |R\rangle).$$

Then $[E_P^{(\mathcal{C}', \vec{\lambda}')}, E_\Sigma^{(\mathcal{C}', \vec{\lambda}')}] \neq 0$, hence

$$\kappa_\rho^{(\mathcal{C}', \vec{\lambda}')}(P, \Sigma) > 0,$$

so $\{P, \Sigma\}$ is an edge in $G_\rho^{(\mathcal{C}', \vec{\lambda}')}(X)$ for $X = \{P, \Sigma\}$ and

$$\mathscr{X}_\rho^{(\mathcal{C}', \vec{\lambda}')}(X) = 2.$$

This demonstrates that contextuality can arise even when all claims are applicable, via operational incompatibility [2, 20, 63].

Proof. **Step 1: Explicit commutator calculation.** On the two-path subspace $\mathcal{H}_{\text{path}} = \text{span}\{|L\rangle, |R\rangle\} \subseteq \mathcal{H}_{\mathcal{C}', \vec{\lambda}'}^{(n)}$, we have:

$$E_P = |L\rangle\langle L|, \qquad E_\Sigma = |+\rangle\langle +| = \frac{1}{2}\Big(|L\rangle + |R\rangle\Big)\Big(\langle L| + \langle R|\Big).$$

Compute the products:

$$E_P E_\Sigma = |L\rangle\langle L| \cdot \frac{1}{2}\Big(|L\rangle + |R\rangle\Big)\Big(\langle L| + \langle R|\Big)$$

$$= \frac{1}{2}|L\rangle\Big(\langle L|L\rangle + \langle L|R\rangle\Big)\Big(\langle L| + \langle R|\Big)$$

$$= \frac{1}{2}|L\rangle\Big(\langle L| + \langle R|\Big) = \frac{1}{2}\Big(|L\rangle\langle L| + |L\rangle\langle R|\Big),$$

using orthonormality $\langle L|L\rangle = 1$, $\langle L|R\rangle = 0$. Similarly:

$$E_\Sigma E_P = \frac{1}{2}\Big(|L\rangle + |R\rangle\Big)\Big(\langle L| + \langle R|\Big) \cdot |L\rangle\langle L|$$

$$= \frac{1}{2}\Big(|L\rangle + |R\rangle\Big)\Big(\langle L|L\rangle + \langle R|L\rangle\Big)\langle L|$$

$$= \frac{1}{2}\Big(|L\rangle + |R\rangle\Big)\langle L| = \frac{1}{2}\Big(|L\rangle\langle L| + |R\rangle\langle L|\Big).$$

Therefore the commutator is:

$$[E_P, E_\Sigma] = E_P E_\Sigma - E_\Sigma E_P = \frac{1}{2}\Big(|L\rangle\langle R| - |R\rangle\langle L|\Big) \neq 0.$$

This is a standard calculation for complementary observables in a qubit system [59, 71].

Step 2: Nonzero commutator witness. By Definition 6.102, the operational commutator witness is:

$$\kappa_\rho^{(\mathcal{C}',\vec{\lambda}')}(P, \Sigma) = \Big(\mathrm{Tr}\Big(\rho\,[E_P, E_\Sigma]^\dagger [E_P, E_\Sigma]\Big)\Big)^{1/2}.$$

Since $[E_P, E_\Sigma] \neq 0$, the operator $[E_P, E_\Sigma]^\dagger [E_P, E_\Sigma]$ is positive and nonzero. For any state ρ with full support on $\mathcal{H}_{\mathrm{path}}$ (e.g. $\rho = \frac{1}{2}I$ or any pure state with nonzero overlap with both $|L\rangle$ and $|R\rangle$), we have:

$$\mathrm{Tr}\Big(\rho\,[E_P, E_\Sigma]^\dagger [E_P, E_\Sigma]\Big) > 0,$$

hence $\kappa_\rho^{(\mathcal{C}',\vec{\lambda}')}(P, \Sigma) > 0$ [107, 108].

Step 3: Edge in the incompatibility graph. By Definition 6.105, an edge $\{P, \Sigma\}$ exists in $G_\rho^{(\mathcal{C}',\vec{\lambda}')}(X)$ iff $\kappa_\rho^{(\mathcal{C}',\vec{\lambda}')}(P, \Sigma) > 0$. Since Step 2 establishes this inequality, $\{P, \Sigma\}$ is an edge [20, 117].

Step 4: Frame number equals 2. The graph $G_\rho^{(\mathcal{C}',\vec{\lambda}')}(X)$ on $X = \{P, \Sigma\}$ consists of two vertices connected by a single edge. The chromatic number of such a graph is $\mathscr{X} = 2$ (the two vertices must receive different colors) [122]. By Definition 6.107, the frame number equals the chromatic number:

$$\mathscr{X}_\rho^{(\mathcal{C}',\vec{\lambda}')}(X) = \mathscr{X}\Big(G_\rho^{(\mathcal{C}',\vec{\lambda}')}(X)\Big) = 2.$$

This means no single jointly compatible classical frame can accommodate both P and Σ simultaneously [2, 9]. $\square$

Remark 6.119 (Two-layer complementarity in the double-slit (n-Loci)). The double-slit story exhibits two distinct obstructions to a single global classical description across the locus configuration $\vec{\lambda}$:

(i) *Typed obstruction* (Holothesis): in the interference-region locus configuration, which-path talk is inapplicable and thus excluded (Proposition 6.115). This corresponds to paracomplete truth-value gaps [54, 55].

(ii) *Operational incompatibility* (graph obstruction): in an applicability-restoring refinement, which-path and coherence-phase claims are simultaneously admissible yet incompatible, forcing $\mathscr{X} > 1$ (Proposition 6.118). This corresponds to paraconsistent co-holding without explosion [13, 15].

Both mechanisms are already foreshadowed in the tetralemmatic walk (Section 2.8) and are here made graph-theoretic. This two-layer structure distinguishes the present framework from standard quantum logic approaches [7, 9], where typically only operational incompatibility is considered without the typed inapplicability layer. The Holothesis layer provides a formal mechanism for handling unaskable questions, while the incompatibility graph captures contextuality among admissible questions [8, 20].

Many-Particle Sensitivity: For many-particle systems ($n \gg 1$), the incompatibility graph grows combinatorially. The frame number $\mathscr{X}$ may exceed the Hilbert space dimension d, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94). This manifests physically as enhanced contextuality witnesses and potential Sorkin I_3 violations (Proposition 6.36), providing an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems (Remark 6.37).

Remark 6.120 (Complete tetralemmatic walk: Transitions between corners (n-Loci))**.** The double-slit experiment provides a complete operational realization of the tetralemmatic structure, exhibiting all four corners and the transitions between them via *contextual refinement* and *symmetry breaking* across locus configurations.

(i) **Holothesis (Pure Interference).** In the standard interference context $(\mathcal{C}_2, \vec{\lambda}_2)$ (no which-path detector), polar claims are *typed as inapplicable* (Section 6.15). The system resides in the **Holothesis** sector, representing logical potentiality where path questions are ill-typed (Definition 2.12).

(ii) **Transition 1: Refinement to Synthesis.** By introducing a *weak/erasable* which-path coupling (an applicability-restoring refinement $(\mathcal{C}', \vec{\lambda}')$), polar claims become applicable but coherence is preserved (Section 2.13). The system transitions to **Synthesis**, where paths co-hold paraconsistently (non-zero off-diagonal warrants, table 6).

(iii) **Transition 2: Symmetry Breaking to Thesis/Antithesis.** By introducing a *strong* which-path measurement (symmetry-breaking interaction), the warrant functional selects a definite polarity (Section 6.18). The system transitions to **Thesis** (left slit) or **Antithesis** (right slit), representing actualized facts (table 6).

This walk demonstrates that the four corners are not static categories but *operational regimes* connected by physically realizable transformations (refinement and symmetry breaking), satisfying the requirement for a complete tetralemmatic analysis of quantum phenomena.

Generalization to n-Loci: For $n > 1$, these transitions occur across the composite configuration $\vec{\lambda}$. The Holothesis sector operates on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, while Synthesis/Thesis/Antithesis operate on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$. This structural distinction ensures the meta-test structure is preserved for Holothesis claims across the configuration.

6.17 The Qutrit Experiment: Coarse-grained Gaps and Contextuality Witnesses (*n*-Loci)

This continues the slit-story extension already noted under paracomplete "actuality" semantics (Section 2.14), but now uses a genuinely qutrit ($d = 3$) family of *incompatible* tests whose mutual overlaps force *contextuality*. This extension is necessary because contextuality witnesses of the Kochen–Specker type require dimension $d \geq 3$ [2, 85]. The graph-theoretic approach to contextuality provides a unified framework for analyzing such obstructions [20, 21, 117]. This analysis holds for both single-locus ($n = 1$, base case) and *n-loci configurations* ($n > 1$, many-particle regimes).

Logical Derivation of the $d = 3$ Threshold (n-Loci). The necessity of dimension $d \geq 3$ for contextuality witnesses follows directly from Theorem 6.94.

- **Qubit ($d = 2$):** Incompatibility graphs for qubit observables are structurally simple (e.g., chains or even cycles) and are always 2-colorable. Thus $\mathscr{X}(\Gamma) \leq 2$, allowing a classical Thesis/Antithesis valuation without forcing Holothesis weight. No contextuality arises.

- **Qutrit ($d = 3$):** The KCBS pentagon C_5 has chromatic number $\mathscr{X}(C_5) = 3$. Since $\mathscr{X} > d_{\mathrm{classical}} = 2$, binary logic is chromatically exhausted. The system *must* access the Holothesis/Synthesis sector to resolve the coloring obstruction. **Note:** While the framework's primary dimensional distinction is $d \in \{2, 4\}$ (dyadic reduction vs. generic four-corner, Definition 5.21), the $d = 3$ qutrit case is a valid special instance for Kochen–Specker contextuality witnesses.

- **Many-Particle ($n \gg 1$):** For composite configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$, the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially. The chromatic number $\mathscr{X}(\Gamma)$ may exceed the Hilbert space dimension d more readily **in symmetric subspaces**, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94 and Proposition 6.39). This provides enhanced sensitivity to contextuality in many-body regimes (Remark 6.37).

This demonstrates that contextuality is the signature of *logical saturation*: the point where binary valuation fails and the tetralemmatic structure becomes operationally active [2, 85].

6.17.1 Three-slit path space and the typed claim-to-effect interface (n-Loci)

Fix an immanent context $\mathcal{C}$ and *locus configuration* $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ in which a three-slit apparatus is operative (for the standard qutrit, $n = 1$). Let $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$ be the (realized) *composite operational space* of the situation as in Section 6.5 and Definition 5.11. Assume the slit-plane supports orthonormal path kets

$$|A\rangle, \; |B\rangle, \; |C\rangle \; \in \; \mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}, \qquad \langle A|B\rangle = \langle B|C\rangle = \langle C|A\rangle = 0.$$

This three-dimensional path space is the minimal arena for Kochen–Specker-type obstructions [2, 125].

Definition 6.121 (Three polar slit claims and their ideal realizations (n-Loci)). Let $\varphi_A(x)$ be the predicate "x is localized at slit A at time t_0 during passage", and similarly φ_B, φ_C. Form the polar existential claims (cf. Section 2)

$$P_A := \exists_{\vec{\lambda}}^{\mathcal{C}} x \, \varphi_A(x), \qquad P_B := \exists_{\vec{\lambda}}^{\mathcal{C}} x \, \varphi_B(x), \qquad P_C := \exists_{\vec{\lambda}}^{\mathcal{C}} x \, \varphi_C(x).$$

When applicable at $(\mathcal{C}, \vec{\lambda})$, the realization map of Section 6.5 may assign ideal slit projectors

$$E_A := \mathcal{R}_{\mathcal{C}, \vec{\lambda}}(P_A) = |A\rangle\langle A|, \quad E_B := \mathcal{R}_{\mathcal{C}, \vec{\lambda}}(P_B) = |B\rangle\langle B|, \quad E_C := \mathcal{R}_{\mathcal{C}, \vec{\lambda}}(P_C) = |C\rangle\langle C|.$$

These projectors form a complete orthogonal resolution of the identity on the path subspace [59, 85].

Remark 6.122 (Structural distinction for n-loci effects)**.** For n-loci configurations, the realized effects act on different Hilbert spaces depending on the corner status:

- **Polar/Synthesis Corners:** For $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, effects act on the composite operational space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$.

- **Holothesis Corner:** For $\bullet = \mathbf{HoloTh}$, effects act on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4).

This ensures the meta-test structure is preserved for Holothesis claims across the configuration $\vec{\lambda}$.

Remark 6.123 (Gap vs. coarse-grained certainty revisited (n-Loci))**.** The paracomplete gap phenomenon discussed in Section 2.14 has a direct three-slit form: for a coherent state $|\psi\rangle = \alpha|A\rangle + \beta|B\rangle + \gamma|C\rangle$ with all amplitudes nonzero, one can have $0 < \operatorname{Tr}(\rho E_A), \operatorname{Tr}(\rho E_B), \operatorname{Tr}(\rho E_C) < 1$ so that none of P_A, P_B, P_C is *certain*, while a coarse-grained claim Q ("passed the slit-plane region") may be realized by $E_Q = E_A + E_B + E_C$ and be certain: $\operatorname{Tr}(\rho E_Q) = 1$. This illustrates the failure of disjunctive certainty even when the disjunction is certain—a hallmark of quantum contextuality [8, 20].

For n-loci configurations: The gap phenomenon extends across the configuration $\vec{\lambda}$. When Holothesis is warranted at any locus $\lambda_i \in \vec{\lambda}$, the corresponding vertex is excluded from the incompatibility graph for the entire configuration. This ensures the graph captures only operational incompatibility among *askable* questions across $\vec{\lambda}$, while Holothesis claims are diagnosed via the dual-subspace meta-test (Definition 5.8).

6.17.2 Compatibility contexts and exclusivity as a graph structure (n-Loci)

Definition 6.124 (Admitted yes/no tests and compatibility contexts (n-Loci))**.** Let T be a collection of yes/no tests admitted in $(\mathcal{C}, \vec{\lambda})$ where $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ is a locus configuration (for the standard case, $n = 1$). Concretely, each $t \in \mathsf{T}$ is represented (when applicable) by an effect

$$E_t := \mathcal{R}_{\mathcal{C}, \vec{\lambda}}(\chi_t), \qquad 0 \le E_t \le I,$$

for some formula $\chi_t \in \operatorname{Lang}(\mathcal{C})$, with Born valuation $v(\chi_t; \rho) = \operatorname{Tr}(\rho E_t)$ as in Definition 5.11 and eq. (35).

A *compatibility context* is a subset $M \subseteq \mathsf{T}$ such that the corresponding effects commute pairwise across the configuration $\vec{\lambda}$:

$$t, t' \in M \implies [E_t, E_{t'}] = 0.$$

We write M for a chosen family of such contexts. This definition aligns with the standard notion of a measurement context in quantum foundations [59, 63].

Remark 6.125 (Structural distinction for n-loci effects)**.** For n-loci configurations, the realized effects act on different Hilbert spaces depending on the corner status:

- **Polar/Synthesis Corners:** For $\bullet \in \{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}\}$, effects act on the composite operational space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$.

- **Holothesis Corner:** For • = **HoloTh**, effects act on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4).

This ensures the meta-test structure is preserved for Holothesis claims across the configuration $\vec{\lambda}$.

Definition 6.126 (Exclusivity graph (n-Loci)). Assume (for simplicity and sharpness) that each E_t is a projector P_t. Define the *exclusivity graph* $G = (V, E)$ by taking $V := \mathsf{T}$ and putting an edge $\{t, t'\} \in E$ exactly when $P_t P_{t'} = 0$ (orthogonality), i.e. the two yes-events are mutually exclusive in any joint measurement context that contains them across the configuration $\vec{\lambda}$. This graph-theoretic representation of exclusivity relations is standard in the contextuality literature [21, 117].

Remark 6.127 (Many-particle sensitivity of exclusivity graphs). For many-particle systems $(n \gg 1)$, the exclusivity graph G grows combinatorially with the number of loci in $\vec{\lambda}$. The chromatic number $\mathcal{X}(G)$ may exceed the Hilbert space dimension d more readily **in symmetric subspaces** than in single-particle regimes, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94 and Proposition 6.39). This manifests physically as enhanced contextuality witnesses (e.g., KCBS violations) and potential Sorkin I_3 violations (Proposition 6.36), providing an empirical pathway for detecting the structural signatures of the tetralemmatic framework in many-body quantum systems (Remark 6.37).

6.17.3 Noncontextuality as the existence of a global valuation (n-Loci)

Definition 6.128 (Deterministic noncontextual valuation (n-Loci)). A *deterministic noncontextual valuation* for (T, M) defined over locus configuration $\vec{\lambda}$ is a map

$$\nu : \mathsf{T} \to \{0, 1\}$$

such that, whenever t, t' are exclusive (orthogonal projectors), one has

$$\{t, t'\} \in E(G) \quad \Longrightarrow \quad \nu(t) + \nu(t') \leq 1.$$

Equivalently, $\{t \in \mathsf{T} : \nu(t) = 1\}$ is an *independent set* in G. This captures the classical intuition that exclusive events cannot both occur [2, 119].

Proposition 6.129 (Odd-cycle (pentagon) noncontextual bound (n-Loci)). *Let $G = C_5$ be the 5-cycle (pentagon) exclusivity graph on vertices $\{1, 2, 3, 4, 5\}$ with edges $\{k, k+1\}$ (indices mod 5) defined over locus configuration $\vec{\lambda}$. Then for every deterministic noncontextual valuation ν,*

$$\sum_{k=1}^{5} \nu(k) \ \leq \ 2.$$

Proof. In a 5-cycle, any three distinct vertices contain at least two adjacent ones. Since adjacent vertices cannot both be assigned value 1 by the exclusivity constraint (Definition 6.128), at most two vertices can be assigned 1. Formally, let $S = \{k : \nu(k) = 1\}$. If $|S| \geq 3$, then by the pigeonhole principle applied to the cycle structure, there exist $j, k \in S$ with $|j - k| \equiv 1$ (mod 5), contradicting exclusivity. Hence $|S| \leq 2$, which gives the bound [20, 117]. $\square$

Corollary 6.130 (Probabilistic noncontextual bound (n-Loci))**.** *For any convex mixture of deterministic noncontextual valuations (i.e. any noncontextual probabilistic model) defined over locus configuration $\vec{\lambda}$, the corresponding probabilities $p_k \in [0, 1]$ satisfy*

$$\sum_{k=1}^{5} p_k \leq 2.$$

Proof. Any noncontextual probabilistic model can be written as a convex combination of deterministic valuations:

$$p_k = \sum_\nu w_\nu \, \nu(k), \qquad w_\nu \geq 0, \quad \sum_\nu w_\nu = 1.$$

Then

$$\sum_{k=1}^{5} p_k = \sum_{k=1}^{5} \sum_\nu w_\nu \, \nu(k) = \sum_\nu w_\nu \left(\sum_{k=1}^{5} \nu(k) \right) \leq \sum_\nu w_\nu \cdot 2 = 2,$$

where the inequality follows from Proposition 6.129 [21, 59]. $\square$

Remark 6.131 (Connection to the Kochen–Specker theorem (n-Loci))**.** The pentagon bound $\sum p_k \leq 2$ is a special case of the Kochen–Specker contextuality witnesses [2]. In dimension $d = 3$, one can construct a set of projectors forming a C_5 exclusivity graph that admits a quantum realization with $\sum p_k = \sqrt{5} > 2$, violating the noncontextual bound [120, 121]. This demonstrates that quantum mechanics cannot be reproduced by any noncontextual hidden-variable model [119, 125].

    For n-loci configurations: The violation can be amplified in many-particle regimes ($n \gg 1$) where the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially **in symmetric subspaces**. The chromatic number $\mathscr{X}(\Gamma)$ may exceed the Hilbert space dimension d more readily, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94 and Proposition 6.39). This provides enhanced sensitivity to contextuality witnesses in many-body quantum systems (Remark 6.37).

6.17.4   A qutrit KCBS pentagon realizable by three-slit interferometry (n-Loci)

This construction provides a concrete realization of the Klyachko–Can–Binicioglu–Shumovsky (KCBS) inequality [21, 117], which is the simplest state-dependent witness of quantum contextuality, requiring a Hilbert space of dimension $d \geq 3$ [2, 85]. This analysis holds for both single-locus ($n = 1$, base case) and *n-loci configurations* ($n > 1$, many-particle regimes). Identify the three-slit path subspace with $\mathbb{C}^3$ via the ordered basis $\{|A\rangle, |B\rangle, |C\rangle\}$, corresponding to the three admissible path claims (cf. Definition 6.121). Let $\phi_k := \frac{4\pi k}{5}$ for $k = 1, \ldots, 5$, and choose $\theta \in (0, \pi/2)$ such that

$$\cos^2 \theta = \frac{1}{\sqrt{5}}. \tag{81}$$

Define unit vectors

$$|v_k\rangle := \sin\theta \Big(\cos\phi_k \, |A\rangle + \sin\phi_k \, |B\rangle \Big) + \cos\theta \, |C\rangle, \qquad k = 1, \ldots, 5, \tag{82}$$

and rank-1 projectors $P_k := |v_k\rangle\langle v_k|$. These projectors represent sharp yes/no tests admissible in specific compatibility contexts (cf. Definition 6.124).

Proposition 6.132 (Pentagon exclusivity from orthogonality (n-Loci))**.** *With the choice eq.* (81), *the projectors in eq.* (82) *satisfy*

$$P_k P_{k+1} = 0 \quad \textit{for all } k \pmod 5,$$

so their exclusivity graph contains the cycle C_5 *[21, 117]*.

Proof. Two rank-1 projectors P_k, P_{k+1} are orthogonal (and hence exclusive) if and only if their defining vectors are orthogonal. Compute the inner product $\langle v_k | v_{k+1} \rangle$ using eq. (82):

$$\langle v_k | v_{k+1} \rangle = \sin^2 \theta \Big(\cos \phi_k \cos \phi_{k+1} + \sin \phi_k \sin \phi_{k+1} \Big) + \cos^2 \theta$$
$$= \sin^2 \theta \cos(\phi_{k+1} - \phi_k) + \cos^2 \theta.$$

By definition, $\phi_{k+1} - \phi_k = \frac{4\pi}{5}$. Using the identity $\cos\left(\frac{4\pi}{5}\right) = -\frac{1+\sqrt{5}}{4}$ and the choice $\cos^2\theta = \frac{1}{\sqrt{5}}$ (which implies $\sin^2\theta = 1 - \frac{1}{\sqrt{5}}$):

$$\langle v_k | v_{k+1} \rangle = \left(1 - \frac{1}{\sqrt{5}} \right) \left(-\frac{1+\sqrt{5}}{4} \right) + \frac{1}{\sqrt{5}}$$
$$= -\frac{1}{4}\left(1 - \frac{1}{\sqrt{5}} \right)(1 + \sqrt{5}) + \frac{1}{\sqrt{5}}$$
$$= -\frac{1}{4}\left(1 + \sqrt{5} - \frac{1}{\sqrt{5}} - 1 \right) + \frac{1}{\sqrt{5}}$$
$$= -\frac{1}{4}\left(\sqrt{5} - \frac{1}{\sqrt{5}} \right) + \frac{1}{\sqrt{5}}$$
$$= -\frac{1}{4}\left(\frac{5-1}{\sqrt{5}} \right) + \frac{1}{\sqrt{5}}$$
$$= -\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} = 0.$$

Thus $\langle v_k | v_{k+1} \rangle = 0$, implying $P_k P_{k+1} = |v_k\rangle\langle v_k | v_{k+1}\rangle\langle v_{k+1}| = 0$ [59]. $\qquad\square$

Theorem 6.133 (Quantum violation on a qutrit: $\sqrt{5} > 2$ (n-Loci))**.** *Let $\rho := |C\rangle\langle C|$ be the prepared state on the composite operational space $\mathcal{H}_{C,\vec{\lambda}}^{(n)}$ (cf. Definition 5.11). Then the sum of Born probabilities for the pentagon tests is*

$$\sum_{k=1}^{5} \mathrm{Tr}(\rho P_k) \;=\; \sqrt{5} \;>\; 2.$$

Consequently, no noncontextual probabilistic model (in the sense of Corollary 6.130) can reproduce these Born probabilities [2, 20].

Proof. Since $\rho = |C\rangle\langle C|$, the Born probability for each test P_k is

$$\mathrm{Tr}(\rho P_k) = \langle C | P_k | C \rangle = |\langle C | v_k \rangle|^2.$$

From eq. (82), $\langle C | v_k \rangle = \cos\theta$. Thus

$$\mathrm{Tr}(\rho P_k) = \cos^2\theta = \frac{1}{\sqrt{5}}.$$

Summing over the five vertices of the pentagon:

$$\sum_{k=1}^{5} \mathrm{Tr}(\rho P_k) = 5 \cdot \frac{1}{\sqrt{5}} = \sqrt{5}.$$

Since $\sqrt{5} \approx 2.236 > 2$, this violates the noncontextual bound established in Corollary 6.130 [21, 120]. □

Remark 6.134 (Why $d = 3$ matters (and why "a qubit is too small") (n-Loci))**.** A qubit ($d = 2$) cannot host a rank-1 projector family with a C_5 orthogonality cycle: in $\mathbb{C}^2$ there are only two mutually orthogonal rays. The qutrit/three-slit path space is the minimal arena in which odd-cycle contextuality witnesses (and, more generally, Kochen–Specker-type obstructions) can occur [2, 85]. This aligns with the three-slit extension discussed in Remark 6.123.

 For n-loci configurations: For many-particle systems ($n \gg 1$), the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$ grows combinatorially with the number of loci. The chromatic number $\mathscr{X}(\Gamma)$ may exceed the Hilbert space dimension d more readily **in symmetric subspaces** than in single-particle regimes, forcing non-zero logical deficiency $\mathcal{K}(\Gamma, d)$ (Theorem 6.94 and Proposition 6.39). This provides enhanced sensitivity to contextuality witnesses in many-body quantum systems (Remark 6.37).

Remark 6.135 (How this fits the tetralemmatic/tetralogic discipline (n-Loci))**.** Each χ_k realizing P_k is still *typed by applicability* (Section 2) and evaluated by Born degree (Definition 5.11 and Section 6.5). Contextuality adds a stronger lesson: even when every χ_k is applicable (well-typed) in some family of compatible contexts, there need not exist any *single global* valuation that simultaneously respects all contextual exclusivity constraints. This is exactly the structural pressure behind locus-sensitive semantics and against "one context to rule them all" [8, 20]. The tetralemmatic framework accommodates this by allowing the logical status (Thesis/Antithesis/etc.) to be locus-dependent (Definition 6.48), thereby avoiding the assumption of a global truth-value assignment that the Kochen–Specker theorem forbids [2, 9].

 Structural Distinction: For n-loci configurations, the KCBS tests act on the composite operational space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$ (for Thesis/Antithesis/Synthesis corners). Holothesis claims, when warranted, are diagnosed via the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ meta-test (Definitions 5.4 and 5.8). This ensures the framework captures both operational incompatibility (graph edges) and typed inapplicability (vertex exclusion) across the configuration $\vec{\lambda}$.

 Dimensional Note: While the framework's primary dimensional distinction is $d \in \{2, 4\}$ (dyadic reduction vs. generic four-corner, Definition 5.21), the $d = 3$ qutrit case demonstrated here is a valid special instance for Kochen–Specker contextuality witnesses, bridging the gap between the non-contextual qubit regime and the fully contextual generic four-corner regime.

6.18 Unification: Perspectives as Logical Loci

The tetralemmatic framework resolves the foundational tension between Relational Quantum Mechanics (RQM) [4, 24] and Consistent Histories (CH) [5] by identifying physical observers with *logical loci* and measurement interactions with *symmetry-breaking transitions* in the logical algebra. This unification holds for both single-locus ($n = 1$) and *n-loci configurations* ($n > 1$), where perspectives correspond to locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$.

6.18.1 The Cross-Perspective Link Problem (n-Loci)

A central sticking point in RQM is the *cross-perspective link*: if Alice measures a system and obtains a fact, that fact is real *for Alice*, but for Bob (who has not yet interacted with Alice), the system remains in a superposition [24]. Standard logic struggles to reconcile these conflicting truth values without invoking collapse or many worlds. The tetralemmatic framework resolves this by treating truth as *locus-configuration-indexed* (Definition 2.1) and applicability as *symmetry-dependent.*

Definition 6.136 (Perspective-Shift as Locus Transformation (n-Loci)). Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be contexts associated with observers A and B. Let $\vec{\lambda}_A \in \Lambda_{\mathcal{C}_A}^n$ and $\vec{\lambda}_B \in \Lambda_{\mathcal{C}_B}^n$ be their respective *locus configurations*. A *perspective-shift* is a map $\sigma : (\mathcal{C}_A, \vec{\lambda}_A) \to (\mathcal{C}_B, \vec{\lambda}_B)$ inducing an automorphism on the logical algebra $\alpha_\sigma : \mathcal{A}_\Lambda^{\mathrm{alg}} \to \mathcal{A}_\Lambda^{\mathrm{alg}}$. A physical fact P is *relative* if its logical status satisfies:

$$\mathrm{Status}_{\mathcal{C}_A}(P; \vec{\lambda}_A) = \mathbf{Th} \quad \text{and} \quad \mathrm{Status}_{\mathcal{C}_B}(P; \vec{\lambda}_B) = \mathbf{HoloTh}. \tag{83}$$

Remark 6.137 (Structural distinction for n-loci perspectives). For n-loci configurations, the logical status is evaluated on different Hilbert spaces depending on the corner:

- **Polar Corners (Thesis/Antithesis):** For Alice (symmetry-breaking), the status is evaluated on the composite operational space $\mathcal{H}_{\mathcal{C}_A, \vec{\lambda}_A}^{(n)}$.

- **Holothesis Corner:** For Bob (symmetry-preserving), the status is evaluated on the n-loci dual-subspace $\mathcal{H}_{\mathrm{Dual}}^{(2n)}$ (Definition 5.4).

This ensures the meta-test structure is preserved for Holothesis claims across the configuration $\vec{\lambda}_B$, resolving the cross-perspective link without collapse.

6.18.2 Noether-like Correspondence of Symmetry and Polarity (n-Loci)

We establish a formal correspondence between physical symmetry under perspective-shifts and logical polarity. This mirrors Noether's theorem in linking symmetry properties to conserved quantities; here, symmetry links to *logical status.*

Theorem 6.138 (Symmetry-Polarity Correspondence (n-Loci)). *Let ω_Λ be the warrant functional representing the physical state. Let σ be a perspective-shift between locus configurations $\vec{\lambda}_A$ and $\vec{\lambda}_B$.*

(i) *Symmetry Breaking $\implies$ Polarity: If the state breaks the symmetry of the perspective-shift (i.e., $\omega_\Lambda \circ \alpha_\sigma \neq \omega_\Lambda$ on the subalgebra generated by P), then the claim P becomes applicable at the broken-symmetry locus configuration. The logical status transitions from Holothesis to a polarized status (Thesis or Antithesis):*

$$\omega_\Lambda \circ \alpha_\sigma \neq \omega_\Lambda \quad \implies \quad \mathrm{App}_{\mathcal{C}}(P; \vec{\lambda}) \wedge \mathrm{Status}_{\mathcal{C}}(P; \vec{\lambda}) \in \{\mathbf{Th}, \mathbf{AntiTh}\}. \tag{84}$$

(ii) *Symmetry Preservation $\implies$ Transcendence: If the state preserves the symmetry (i.e., $\omega_\Lambda \circ \alpha_\sigma = \omega_\Lambda$), then the claim P fails the applicability test at that locus configuration. The logical status remains* logical transcendence *(Holothesis):*

$$\omega_\Lambda \circ \alpha_\sigma = \omega_\Lambda \quad \implies \quad \neg\,\mathrm{App}_{\mathcal{C}}(P; \vec{\lambda}) \wedge \mathrm{Status}_{\mathcal{C}}(P; \vec{\lambda}) = \mathbf{HoloTh}. \tag{85}$$

Proof. **(1) Symmetry Breaking.** Measurement interactions are modeled as symmetry-breaking refinements (Definition 2.6). When ω_Λ is not invariant under α_σ, the system selects an eigenstate of the observable associated with P across the configuration $\vec{\lambda}$. By the Realization Axiom (Section 6.5), this renders the effect E_P sharp ($E_P^2 = E_P$), satisfying the applicability condition $\mathrm{App}_{\mathcal{C}}(P; \vec{\lambda})$ (Definition 3.8). Consequently, the warrant functional assigns definite weight ($\omega_\Lambda(C_{\mathbf{Th}}(P)) = 1$ or $\omega_\Lambda(C_{\mathbf{AntiTh}}(P)) = 1$), warranting Thesis or Antithesis (Definition 5.1).

    (2) Symmetry Preservation. If ω_Λ is invariant under α_σ, the system remains in a coherent superposition relative to the perspective shift across the configuration. In this regime, the effect E_P is unsharp or undefined (Section 2.13). Thus $\mathrm{App}_{\mathcal{C}}(P; \vec{\lambda})$ fails, warranting Holothesis (Definition 5.7). This aligns with the paracomplete nature of interference regimes where excluded middle fails [9]. $\qquad\square$

Remark 6.139 (Resolution of the Cross-Perspective Link (n-Loci))**.** Theorem 6.138 resolves the cross-perspective link for composite configurations:

- **Alice's Locus Configuration $\vec{\lambda}_A$:** Alice interacts with the system, breaking the symmetry between outcome states. By Eq. (84), $\mathrm{App}_{\mathcal{C}_A}(P; \vec{\lambda}_A)$ holds and $\mathrm{Status}_{\mathcal{C}_A}(P; \vec{\lambda}_A) = \mathbf{Th}$. The fact is real *for Alice* on the composite operational space.

- **Bob's Locus Configuration $\vec{\lambda}_B$:** Bob treats Alice + System as a closed unit. The joint state preserves the symmetry under perspective-shift (unitary evolution). By Eq. (85), $\neg\,\mathrm{App}_{\mathcal{C}_B}(P; \vec{\lambda}_B)$ and $\mathrm{Status}_{\mathcal{C}_B}(P; \vec{\lambda}_B) = \mathbf{HoloTh}$. The fact is inapplicable *for Bob* on the n-loci dual-subspace.

There is no contradiction because existence is relative to the locus configuration: $\exists_{\vec{\lambda}_A}^{\mathcal{C}_A} x \neq \exists_{\vec{\lambda}_B}^{\mathcal{C}_B} x$. For many-particle systems ($n \gg 1$), this distinction is critical as the dual-subspace structure provides the necessary arena for Bob's Holothesis status without collapsing Alice's Thesis status.

6.18.3   Resolution of Wigner's Friend (n-Loci)

We apply Theorem 6.138 to the Wigner's Friend paradox, generalized to locus configurations.

Remark 6.140 (Relational Facts as Locus-Dependent Statuses (n-Loci))**.** Let P be the claim "The outcome is $|0\rangle$."

- **Alice's Locus Configuration** $(\mathcal{C}_A, \vec{\lambda}_A)$**:** Alice interacts with the system, breaking the symmetry between outcome states across the configuration $\vec{\lambda}_A$. By Eq. (84), $\mathrm{App}_{\mathcal{C}_A}(P; \vec{\lambda}_A)$ holds and $\mathrm{Status}_{\mathcal{C}_A}(P; \vec{\lambda}_A) = \mathbf{Th}$. The fact is real *for Alice* on the composite operational space $\mathcal{H}^{(n)}_{\mathcal{C}_A, \vec{\lambda}_A}$.

- **Bob's Locus Configuration** $(\mathcal{C}_B, \vec{\lambda}_B)$**:** Bob treats Alice + System as a closed unit. The joint state $|\Psi\rangle = \alpha|0\rangle|A_0\rangle + \beta|1\rangle|A_1\rangle$ preserves the symmetry under perspective-shift (unitary evolution) across the configuration $\vec{\lambda}_B$. By Eq. (85), $\neg\,\mathrm{App}_{\mathcal{C}_B}(P; \vec{\lambda}_B)$ holds and $\mathrm{Status}_{\mathcal{C}_B}(P; \vec{\lambda}_B) = \mathbf{HoloTh}$. The claim is *inapplicable* for Bob, diagnosed via the n-loci dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$ meta-test (Definitions 5.4 and 5.7).

There is no contradiction because existence is context-indexed (Definition 2.1): $\exists^{\mathcal{C}_A}_{\vec{\lambda}_A} x \neq \exists^{\mathcal{C}_B}_{\vec{\lambda}_B} x$. Rovelli's "relative facts" are recovered as locus-dependent Thesis/Antithesis assignments, while CH's consistency conditions are recovered as the symmetry constraints on admissible refinements between locus configurations [5].

Remark 6.141 (Ontological Status of the Wave Function (n-Loci))**.** This framework settles the debate on the "reality" of the wave function. The wave function corresponds to the *Warrant Functional* ω_Λ operating over the *Holothesis* sector of the n-loci algebra. It is not a "thing" (Thesis) or "nothing" (Antithesis); it is the state of *Logical Transcendence* (Holothesis) prior to symmetry-breaking measurement. Thus, the wave function is real as a *potentiality structure* (Definition 2.10) but not as a localized entity until a locus configuration selects a polar status. For many-particle systems ($n \gg 1$), the wave function encodes correlations across the configuration $\vec{\lambda}$, evaluated on the dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$ when Holothesis is warranted, ensuring the meta-test structure is preserved without collapse.

6.19 Physical Parsimony and Logical Granularity

A potential critique of the tetralemmatic framework is the perceived redundancy between Synthesis (paraconsistent) and Holothesis (paracomplete) sectors. However, these sectors represent two fundamentally different physical regimes that standard density matrix formalism ρ often conflates. The logical overhead of the tetralemmatic framework buys *discriminatory power* in state classification. This distinction holds for both single-locus ($n = 1$) and *n-loci configurations* ($n > 1$), where the structural distinction between composite operational space and dual-subspace becomes critical.

6.19.1 Finer Classification than Density Matrices (n-Loci)

Standard quantum mechanics represents states via density operators ρ on a Hilbert space $\mathcal{H}$. While ρ encodes statistical predictions $\mathrm{Tr}(\rho E)$, it does not intrinsically distinguish the *logical reason* for a vanishing probability. Table 6 compares the diagnostic power of the two frameworks across locus configurations $\vec{\lambda} \in \Lambda^n_{\mathcal{C}}$.

Table 6: Operational State Classification: Standard ρ vs. Tetralemmatic (n-Loci)

Operational State	Standard ρ Diagnostic	Tetralemmatic Sector
Definite Outcome	Diagonal $\rho_{ii} = 1$	**Thesis** (Definition 5.1)
Definite Exclusion	Diagonal $\rho_{ii} = 0$	**Antithesis** (Definition 5.2)
Coherent Superposition	Off-diagonal $\rho_{ij} \neq 0$	**Synthesis** (Definition 5.3)
Contextual Inapplicability	Undefined / Restricted Domain	**Holothesis** (Definition 5.7)

Remark 6.142 (Distinguishing False from Unaskable (n-Loci))**.** The critical physical gain lies in the last two rows:

(i) **Antithesis (False):** The claim is applicable but warranted false ($\mathrm{App}_{\mathcal{C}}(P; \vec{\lambda})$ holds, $\omega_\Lambda(C_{\mathbf{AntiTh}}(P)) = 1$). In ρ, this looks like $\mathrm{Tr}(\rho E_P) = 0$. This status is evaluated on the composite operational space $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$.

(ii) **Holothesis (Unaskable):** The claim is *inapplicable* ($\neg \mathrm{App}_{\mathcal{C}}(P; \vec{\lambda})$). In ρ, this may also yield $\mathrm{Tr}(\rho E_P) = 0$ (if E_P is undefined or restricted), but the *logical status* is distinct. This status is diagnosed via the n-loci dual-subspace $\mathcal{H}^{(2n)}_{\mathrm{Dual}}$ meta-test (Definition 5.4).

Standard QM cannot differentiate these cases without external context. The tetralemmatic framework internalizes this distinction via the applicability predicate $\mathrm{App}_{\mathcal{C}}$, resolving ambiguities in Kochen–Specker obstructions [2, 9]. For many-particle systems ($n \gg 1$), this distinction is critical as the dual-subspace structure provides the necessary arena for Holothesis status without collapsing Thesis/Antithesis status across the configuration $\vec{\lambda}$.

Remark 6.143 (Structural Distinction: Composite vs. Dual-Subspace)**.** The logical distinction between Synthesis and Holothesis is grounded in a structural distinction in the Hilbert space representation:

- **Thesis/Antithesis/Synthesis:** Operate on the *composite operational space* $\mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}} = \bigotimes_{i=1}^{n} \mathcal{H}_{L,\lambda_i}(P_{\lambda_i})$. Synthesis represents coherent superposition (non-zero off-diagonals) within this space.

- **Holothesis:** Operates on the *n-loci dual-subspace* $\mathcal{H}^{(2n)}_{\mathrm{Dual}} = \mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}} \otimes \mathcal{H}^{(n)}_{\mathcal{C},\vec{\lambda}}$. Holothesis represents inapplicability diagnosed via correlations between copies, preserving polar inapplicability without collapse (Definition 5.8).

This structural distinction ensures that the "granularity" purchased by the tetralemmatic framework is not merely linguistic but corresponds to distinct operational arenas in the physical realization.

6.19.2 Holothesis Density as a Contextuality Resource (n-Loci)

The logical overhead further yields a quantifiable physical resource: the *Holothesis Density*. In standard contextuality witnesses (e.g. KCBS), the violation magnitude is often treated

as a mysterious quantum feature. In the Tetralemmatic framework, it is derived from the logical necessity of diverting warrant weight into the Holothesis sector. This holds for both single-locus ($n = 1$) and *n-loci configurations* ($n > 1$), where the Holothesis warrant is evaluated on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ (Definition 5.4).

Proposition 6.144 (Holothesis Density Bounds Contextuality (n-Loci)). *Let $\mathcal{K}(\Gamma, d)$ be the logical deficiency constant from Theorem 6.94. The total warrant weight diverted to the Holothesis sector across a contextual set $\mathcal{P}$ of n-loci claims defined over locus configurations $\vec{\lambda} \in \Lambda_{\mathcal{C}}^n$ satisfies:*

$$\sum_{\vec{P} \in \mathcal{P}} \omega_\Lambda \left(C_{\textbf{HoloTh}}(\vec{P}) \right) \ \geq \ \mathcal{K}(\Gamma, d). \tag{86}$$

Proof. Follows directly from Theorem 6.94, which establishes that classical valuation failure (chromatic excess) forces non-zero weight into non-classical corners (Synthesis/Holothesis) across the configuration $\vec{\lambda}$. In regimes where Synthesis is suppressed (e.g. sharp measurements), the weight concentrates in Holothesis [7, 8]. For n-loci configurations, the Holothesis warrant is evaluated on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, ensuring the meta-test structure is preserved without collapsing polar applicability on the composite operational space $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$. □

Remark 6.145 (Physical Interpretation (n-Loci)). Proposition 6.144 demonstrates that *contextuality is the signature of Holothesis becoming active.* The "quantum violation" is not an extra-logical physical constant but the minimum logical "gap" required to maintain consistency across mutually exclusive contexts. This transforms contextuality from an observational anomaly into a *logical necessity* derived from the tetralemmatic structure [20, 21].

For n-loci configurations: The logical deficiency $\mathcal{K}(\Gamma, d)$ grows with the combinatorial complexity of the incompatibility graph $\Gamma_{\omega_\Lambda}(\mathcal{C}, \vec{\lambda})$. For many-particle systems ($n \gg 1$), the chromatic number $\mathcal{X}(\Gamma)$ may exceed the Hilbert space dimension d more readily **in symmetric subspaces**, forcing larger Holothesis density. This provides enhanced sensitivity to contextuality witnesses in many-body quantum systems (Remark 6.37 and Proposition 6.39).

6.19.3 Why Synthesis and Holothesis are Irreducible (n-Loci)

Finally, we justify the necessity of maintaining both paraconsistent (Synthesis) and paracomplete (Holothesis) sectors across locus configurations:

(i) **Synthesis (The Interference Regime):** This sector formalizes *constructive/destructive interference* between polarities. Algebraically, it corresponds to the non-zero off-diagonal elements of the Gram matrix $G(\vec{P})$ (Remark 4.15). Without Synthesis, the logic cannot host the complex phase required for unitary evolution (Theorem 5.33). For n-loci configurations, Synthesis operates on the *composite operational space* $\mathcal{H}_{\mathcal{C}, \vec{\lambda}}^{(n)}$, representing coherent superposition across the configuration.

(ii) **Holothesis (The Complementarity Regime):** This sector formalizes *Bohr's inapplicability.* It represents the state of a system relative to a context where the question itself is "ill-typed." Standard logic treats this as an error; Tetralemmatic treats it as a vector $|\textbf{HoloTh}; \vec{P}\rangle$ in $\mathcal{H}_\Lambda$ (Definition 4.8). For n-loci configurations, Holothesis is

diagnosed via the *n-loci dual-subspace* $\mathcal{H}_{\text{Dual}}^{(2n)}$ meta-test (Definition 5.8), preserving polar inapplicability without collapse.

Collapsing these sectors would erase the distinction between *coherent superposition* (Synthesis) and *contextual incompatibility* (Holothesis), reducing the framework to standard quantum logic without the resolution of the measurement problem provided by the applicability predicate [11, 63].

Remark 6.146 (The Operational Necessity of the Fourth Corner (n-Loci))**.** Standard binary logic with contextuality (SQM) recognizes that certain questions are "unaskable." However, it lacks a vector-space representation for this unaskability, treating it instead as a boundary of the Boolean lattice.

Our framework introduces the **Holothesis** as a novel logical category that "lifts" the Synthesis. While Synthesis (superposition) formalizes the *co-holding* of paths, Holothesis formalizes their *categorical inapplicability*. This distinction is physically grounded in the difference between a system that is **interfering** (warranting Synthesis on $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$) and a system that is **contextually obstructed** (warranting Holothesis on $\mathcal{H}_{\text{Dual}}^{(2n)}$), as evidenced by the chromatic exhaustion of incompatibility graphs (Section 6.12). For many-particle systems ($n \gg 1$), this structural distinction becomes critical as the dual-subspace provides the necessary arena for Holothesis status without collapsing Thesis/Antithesis status across the configuration $\vec{\lambda}$.

Remark 6.147 (Synthesis vs. Holothesis: Irreducible Physical Regimes (n-Loci))**.** A parsimony-based critique might suggest collapsing Synthesis and Holothesis into a single "non-classical" sector. However, this would erase the distinction between **interference** and **inapplicability**:

(i) **Synthesis (The Glut):** Addresses the question of how two paths can *both* contribute to an outcome. It is the logical engine of the complex phase. Evaluated on the composite operational space $\mathcal{H}_{\mathcal{C},\vec{\lambda}}^{(n)}$.

(ii) **Holothesis (The Gap):** Addresses the question of why certain paths are *categorically unaskable* in a given context. It is the logical engine of Kochen–Specker contextuality. Diagnosed via the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ meta-test.

By distinguishing these, we classify quantum states with greater granularity than the standard density matrix, which conflates "definitely not there" (Antithesis) with "ill-defined" (Holothesis). For n-loci configurations, this granularity is essential as the dual-subspace structure preserves the meta-test diagnostic without collapsing polar applicability across the configuration $\vec{\lambda}$.

Remark 6.148 (Relation to Consistent Histories and QBism (n-Loci))**.** The tetralemmatic framework shares significant operational ground with Consistent Histories (CH) [5] and QBism [12], yet it resolves their foundational tensions through the introduction of the Holothesis sector. In contrast to CH, which enforces a single-framework rule that renders incompatible questions "meaningless" via linguistic prohibition, the tetralemmatic framework assigns them a first-class logical status identified as Holothesis (Remark 6.111). This distinction allows for

operational reasoning about contextuality boundaries without the need to discard propositions entirely.

Regarding the ontological status of the quantum state, whereas QBism treats the wave function as epistemic user belief, the present framework identifies it with the Warrant Functional ω_Λ operating over the Holothesis sector (Remark 6.141). This grants the wave function ontological status as potentiality without requiring localized actuality, thereby resolving the epistemic/ontic divide. For n-loci configurations, the wave function encodes correlations across the configuration $\vec{\lambda}$, evaluated on the dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$ when Holothesis is warranted, ensuring the meta-test structure is preserved without collapse.

Furthermore, the framework resolves the "cross-perspective link" problem (Section 6.18.1) by identifying observers with logical loci. Consistency is enforced via symmetry constraints on refinements (Section 6.18), recovering CH consistency and RQM relativity without contradiction. Thus, the framework subsumes the operational constraints of CH and QBism into a broader logical ontology that avoids their respective limitations regarding the meaninglessness of unaskables and the subjectivity of states. For many-particle systems ($n \gg 1$), this ontology extends across the configuration $\vec{\lambda}$, providing a rigorous account of multi-observer consistency in composite quantum systems.

Remark 6.149 (Relation to Modern Relational Quantum Mechanics (n-Loci))**.** The recent evolution of Relational Quantum Mechanics (RQM) [3, 38] aligns closely with the tetralemmatic framework, though the two differ in their ultimate ontological commitments. While RQM defines "stable facts" via environmental decoherence [3], the tetralemmatic framework formalizes this stability as Locus Symmetry Breaking (Theorem 6.138), wherein the warrant functional selects a definite polarity (Thesis/Antithesis) from the Holothesis sector. This provides a logical mechanism for the emergence of stability that extends beyond phenomenological decoherence.

Similarly, RQM's notion of "relative information" [38] corresponds to the Warrant Functional ω_Λ operating over context-indexed existence (Definition 2.1). The tetralemmatic framework ontologizes this information as potentiality (Holothesis), resolving the epistemic/ontic ambiguity inherent in RQM. For n-loci configurations, this potentiality structure is evaluated on the n-loci dual-subspace $\mathcal{H}_{\text{Dual}}^{(2n)}$, encoding correlations across the configuration $\vec{\lambda}$ without requiring collapse to a polar status.

Finally, regarding intersubjectivity, while RQM resolves the crisis by restricting fact-comparison to compatible contexts, the tetralemmatic framework enforces this via typed admissibility (Definition 3.8). This ensures that cross-perspective links (Section 6.18.1) are logically constrained by the incompatibility graph structure. Thus, the tetralemmatic framework subsumes the operational insights of modern RQM while providing a rigorous logical ontology for the emergence of classical reality. For composite systems, this ontology distinguishes between Alice's polar facts (evaluated on $\mathcal{H}_{\mathcal{C}_A, \vec{\lambda}_A}^{(n)}$) and Bob's Holothesis status (evaluated on $\mathcal{H}_{\text{Dual}}^{(2n)}$), resolving the cross-perspective link without contradiction across the configuration $\vec{\lambda}$.

Remark 6.150 (Locus Symmetry Breaking vs. Pointer States)**.** The emergence of "stable facts" (Remark 6.149) is often attributed to the selection of pointer states via environmental decoherence. In the tetralemmatic framework, this is formalized as *Locus Symmetry Breaking.*

- **Mechanism:** Stability occurs when the applicability predicate $\text{App}_{\mathcal{C}}$ selects a definite polarity (Thesis/Antithesis) from the Holothesis sector, breaking the duality symmetry $\mathbf{D}_{\mathcal{C}}$ at a specific locus configuration $\vec{\lambda}$.

- **Distinction:** Unlike standard decoherence, which is purely dynamical, Locus Symmetry Breaking is a *logical transition* grounded in admissibility (type-checking). It provides the necessary topological constraint for the environment to "see" a definite Thesis or Antithesis, identifying logical necessity as the precursor to physical decoherence. This resolves the relational stability crisis in RQM [3] by grounding stability in logical type-checking rather than contingent dynamical details.

6.20 The Ontological Status of Information

The reconstruction program advanced in this work culminates in a resolution of the "It from Bit" dilemma originally posed by Wheeler [126]. Standard quantum foundations often treat the Hilbert space structure (the "Bit") as a physical postulate, subsequently deriving observable reality (the "It") from measurement statistics. This manuscript inverts that hierarchy: we demonstrate that the Hilbert space structure is itself derived from the *logical prerequisites* of coherent discourse regarding existence and non-existence. In this framework, information is not primitive; *logic* is primitive. Quantum kinematics emerges as the necessary representational space for a tetralemmatic logic operating over locus configurations.

6.20.1 From Logical Dyad to Hilbert Space

The foundational shift lies in the derivation of the qubit structure. In standard textbooks, the two-dimensional complex vector space $\mathbb{C}^2$ is presupposed for spin-1/2 systems. Here, the dimension $d_1 = 2$ arises from the *logical dyad* inherent in the Tetralemma: the distinction between Thesis (**Th**) and Antithesis (**AntiTh**) constitutes a binary logical degree of freedom. As established in Definition 3.46, the logical modal subspace $\mathcal{H}_L(P)$ is the span of the four corner kets. The Gelfand–Naimark–Segal (GNS) construction (Definition 3.42) then *forces* the emergence of a Hilbert space structure to represent the warrant functional ω_Λ coherently.

Consequently, the "qubit" is not a physical particle assumption but a *logical isomorphism*. Any physical system capable of supporting a binary logical distinction (affirmation/denial) will necessarily exhibit qubit-like kinematics because the logic demands a 2-dimensional representation space to maintain coherence under the duality automorphism $\alpha_{\mathbf{D}}$ (Corollary 3.53). This resolves the circularity often noted in foundational discussions where the Hilbert space is presupposed to derive quantum logic [7, 9].

6.20.2 The Global Tetralemmatic Manifold

The dimensionality of the state space is revealed as a count of logical degrees of freedom rather than a physical given. For an n-loci configuration, the *Global Tetralemmatic Manifold* $\mathcal{H}_{\text{Total}}$ is defined as the tensor product of the ontological control space and the dual-subspace (Definition 5.21):

$$\mathcal{H}_{\text{Total}} := \mathcal{H}_c \otimes \mathcal{H}_{\text{Dual}}^{(2n)}. \tag{87}$$

For the canonical binary dyad ($d = 2$), this yields a dimension of 2^{2n+1}; for the generic four-corner case ($d = 4$), the dimension is 2^{4n+1} (Definition 5.21). This structure is not merely a larger computer; it is the complete physical map of every possible logical status a multi-loci system can hold. The factor of 2^{2n} (or 4^{2n}) arises from the dual-subspace required to diagnose Holothesis (inapplicability) without collapsing polar claims, while the factor of 2^1 arises from the ontological control space $\mathcal{H}_c$ which tracks the logical status (Polar vs. Holothesis) as a quantum degree of freedom (Definition 5.4).

This Monist Architecture (Section 5.9) ensures that sector transitions (e.g., Synthesis $\rightarrow$ Holothesis) are unitary rotations within a fixed manifold, preserving probability and consistency with the Schrödinger equation. The "collapse" of the wave function is reinterpreted as an update of the applicability predicate $\text{App}_\mathcal{C}$ correlated with the control state, not a physical discontinuity (Remark 5.23).

6.20.3 Entanglement and Contextuality as Logical Phenomena

Within this reconstruction, distinctly quantum phenomena acquire precise logical identities:

(i) **Entanglement as Distributed Co-Commitment.** Entanglement is not a "spooky" physical link but the operational manifestation of *Synthesis* across multiple loci. When the warrant functional ω_Λ exhibits non-zero off-diagonal correlations between loci (Definition 5.3), it represents a single paraconsistent co-commitment ($\Vdash^{\text{pc}} (P \wedge \neg P)$) distributed across the configuration. Entanglement is thus *distributed logical co-commitment.*

(ii) **Contextuality as Graph Topology.** The Kochen-Specker theorem and related contextuality results are derived from the topology of the incompatibility graph Γ_n. The Logical Deficiency constant $\mathcal{K}_n$ quantifies the "hole" in the graph where classical truth-value assignments fail (Proposition 6.39). The "Void" (Holothesis) is the physical manifestation of the fact that the logic is too rich to be colored by classical True/False labels. As $n \to \infty$, $\mathcal{K}_n \to 1$ **in symmetric subspaces**, implying that logical deficiency becomes the dominant feature of many-body systems (Remark 6.41).

6.20.4 Universality: A Law of Thought

By deriving qubits from logic, this framework makes a universal claim. If there are other universes with different physical constants, particle contents, or dimensionalities, they would *still* exhibit quantum-like behavior provided they follow the same Tetralemmatic logic. The complex unit i, for instance, emerges as the eigenvalue of the logical quarter-turn symmetry $\alpha_{\mathbf{R}}$ (Corollary 5.28), not as a presupposed field scalar.

This turns Quantum Mechanics from a contingent "Law of Nature" into a necessary "Law of Thought." The relational architecture of physical reality (Section 6.18) is shown to be governed by the logical topology of the incompatibility graph over locus configurations. Information, in this view, is the measure of logical status relative to a context. The "It from Bit" dilemma is resolved because both "It" (physical reality) and "Bit" (informational state) are emergent reflections of the underlying Tetralemmatic structure.

6.20.5 Synthesis with Contemporary Interpretations

As detailed in Sections 6.19 and 6.18, the tetralemmatic framework subsumes the operational insights of Consistent Histories, QBism, and Relational Quantum Mechanics while resolving their foundational tensions. By assigning Holothesis as a first-class logical status, the framework avoids the linguistic prohibitions of Consistent Histories. By ontologizing the warrant functional, it resolves the epistemic/ontic divide of QBism. By identifying observers with logical loci, it resolves the cross-perspective link problem of RQM without collapse. For many-particle systems ($n \gg 1$), this ontology extends across the configuration $\vec{\lambda}$, providing a rigorous account of multi-observer consistency in composite quantum systems.

Remark 6.151 (Final Synthesis)**.** The tetralemmatic reconstruction does not merely reinterpret quantum kinematics; it captures the structural constraints inherent in the relational architecture of physical reality. The 2^{2n+1} (dyadic) or 2^{4n+1} (generic) manifold is the ontological ground for information, where contextuality and interference are governed by logical necessity. By formalizing the Holothesis as a typed inapplicability judgment (Definition 2.12), the framework seamlessly integrates Buddhist dialectics, paracomplete logic, and quantum preclusion into a single, cohesive topology [13, 43]. Future work will extend this kinematic completion to dynamics, exploring how the logical Hamiltonian H_{Global} emerges from the evolution of warrant functionals over context refinements.

A   Notation Table

To aid readability across the four-part structure, we summarize the primary notation used throughout the manuscript. Symbols are grouped by their conceptual role (Logical, Algebraic, Graph-Theoretic, and Physical).

<div align="center">Table 7: Comprehensive Notation Table</div>

Symbol	Definition and Usage
Logical & Contextual Primitives (Part I)	
$\mathcal{C}$	Immanent context (experimental arrangement or logical framework).
$\Lambda_{\mathcal{C}}$	Set of admissible loci for context $\mathcal{C}$ (evaluation points).
λ	Locus (spatiotemporal location, modal state, or measurement setting).
$\mathrm{Lang}(\mathcal{C})$	Language of well-formed formulas admissible in $\mathcal{C}$.
χ, φ, ψ	Logical formulas or claims (Greek Chi, Phi, Psi).
$\exists_{\lambda}^{\mathcal{C}}$	Context-indexed existence quantifier (Definition 2.1).
$\mathrm{App}_{\mathcal{C}}(\chi; \lambda)$	Applicability predicate (typed definability at locus).
$\mathsf{P}_{\mathcal{C},\lambda}$	Admissible polar claims at $(\mathcal{C}, \lambda)$ (Definition 3.8).
$\mathbf{Status}_{\Lambda}$	Corner label set $\{\mathbf{Th}, \mathbf{AntiTh}, \mathbf{SynTh}, \mathbf{HoloTh}\}$ (Definition 3.10).
$\mathrm{Status}_{\mathcal{C}}(\varphi, \lambda)$	Logical status function mapping to tetralemma corners.
$\Delta_{\mathcal{C}}$	Corner involution (duality on corner labels, Definition 2.37).
$\mathcal{C}, \lambda \Vdash \chi$	Holding/forcing relation (warranted truth at locus).
$\mathbf{D}_{\mathcal{C}}$	Duality operator (involution on ontological states).
$\mathsf{R}(\mathcal{C})$	Set of admissible refinements (context transformations).
Algebraic & GNS Structures (Part II & III)	
$\mathcal{A}_{\Lambda}^{\mathrm{alg}}$	Logical corner $*$-algebra (Definition 3.20).
$C_{\bullet}(P)$	Corner-observable generators ($\bullet \in \mathbf{Status}_{\Lambda}$).
ω_{Λ}	Warrant functional (positive linear functional on $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$).
$\mathcal{N}_{\omega_{\Lambda}}$	Null ideal in GNS construction (Definition 3.40).
$\mathcal{H}_{\Lambda}$	Logical GNS Hilbert space (completion of $\mathcal{A}_{\Lambda}^{\mathrm{alg}}/\mathcal{N}_{\omega_{\Lambda}}$).
$|\Omega_{\Lambda}\rangle$	Cyclic vector (vacuum state) in $\mathcal{H}_{\Lambda}$.
π_{Λ}	GNS representation of $\mathcal{A}_{\Lambda}^{\mathrm{alg}}$ on $\mathcal{H}_{\Lambda}$.
$|\bullet; P\rangle$	Corner kets in $\mathcal{H}_{\Lambda}$ (e.g., $|\mathbf{Th}; P\rangle$).
$G(P)$	Gram matrix of corner kets for claim P.
$\alpha_{\mathbf{D}}, \alpha_{\mathbf{R}}$	Duality and quarter-turn $*$-automorphisms (Definition 3.22).
$U_{\mathbf{D}}, U_{\mathbf{R}}$	Unitary implementers of $\alpha_{\mathbf{D}}, \alpha_{\mathbf{R}}$ on $\mathcal{H}_{\Lambda}$.

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Symbol	Definition and Usage
$\theta_{\mathbf{R}}$	Quarter-turn phase (logical symmetry angle, $\pi/2$ in SQM).

Graph-Theoretic & Contextuality Invariants (Part IV)

Symbol	Definition and Usage
Γ_{ω_Λ}	Typed incompatibility graph (vertices = admissible claims).
$\mathscr{X}(G)$	Chromatic number of graph G (Script X).
$\mathscr{X}_\rho^{(\mathcal{C},\lambda)}(X)$	Frame number of claim set X (minimal context cover size).
$\mathcal{K}(\Gamma, d)$	Logical deficiency constant (Theorem 6.94).
$\kappa_\rho^{(\mathcal{C},\lambda)}(\chi, \psi)$	Operational commutator witness (incompatibility degree).
$\mathfrak{N}_{\omega_\Lambda}, \mathfrak{n}_{\omega_\Lambda}$	Strong and weak warrant commutator witnesses.
I_3	Sorkin third-order interference parameter.
d	Dimension of Hilbert space restricted to claim set $\mathcal{P}$.

Operational & Physical Realization (Part IV)

Symbol	Definition and Usage
$\mathcal{R}_{\mathcal{C},\lambda}$	Realization map (interpretation $\mathcal{L} \to \mathcal{E}_{\mathcal{C},\lambda}$).
$\mathcal{E}_{\mathcal{C},\lambda}$	Set of operationally admissible effects at $(\mathcal{C}, \lambda)$.
$E_\chi^{(\mathcal{C},\lambda)}$	Realized effect operator for formula χ.
$\mathcal{H}_{\mathcal{C},\lambda}$	Physical/context-indexed Hilbert space (Section 6.5).
ρ	Physical state (density operator on $\mathcal{H}_{\mathcal{C},\lambda}$).
$v_M^{C,\lambda}(\rho)$	Valuation function (Born probability $\mathrm{Tr}(E\rho)$).
$\mathcal{A}_{\mathrm{phys}}$	Physical observable algebra (e.g., $\mathcal{B}(\mathcal{H}_{\mathrm{phys}})$).
$\mathcal{K}(\Gamma_M, d)$	Logical deficiency for M-particle system (scales with $\mathscr{X}$).

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