Exact Relation between Internal Dissipation Power and Energy Relaxation Rate
A Rigorous Derivation in Markovian Quantum Systems

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Abstract

This paper rigorously proves the exact relation between the internal dissipation power R_{internal} (the net power flowing from the system to the dissipative channel) and the energy relaxation rate Γ_1 for a broad class of Markovian quantum systems. Starting from the standard Lindblad master equation incorporating an amplitude damping term, we provide a purely algebraic proof that, in the absence of driving and at zero temperature, the identity $R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle$ holds universally. This proof does not rely on any additional assumptions regarding the microscopic origin of Γ_1 or the strength of pure dephasing. For systems subject to external driving, we precisely define the internal dissipation power according to the first law of quantum thermodynamics, and we provide a **completely self-contained** demonstration that, under the original driving form without invoking the rotating-wave approximation, the exact result is $R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle + \frac{\hbar\Omega}{2}\Gamma_1\cos(\omega_d t)\langle\sigma_x\rangle$; this correction term originates from counter-rotating components and, under the rotating-wave approximation, averages to zero over time due to its fast oscillation, thereby recovering the same relation as in the undriven case. For **finite-temperature** environments, we derive the modified relation $R_{\text{internal}} = \hbar\omega_q\Gamma_1\left[(1+2n_{\text{th}})\langle\sigma_+\sigma_-\rangle - n_{\text{th}}\right]$ and verify that it vanishes in thermal equilibrium. This article elaborates on the **minimal assumptions**, **conditions of independence**, and **limitations** of this relation, and also presents its generalization to multi-level systems.

1 Introduction

The energy relaxation rate Γ_1 is one of the most important parameters of a quantum two-level system (qubit); it determines the rate of spontaneous emission and the lifetime of the excited state. In standard quantum optics and open quantum system theory, Γ_1 is obtained from microscopic calculations of the system-environment coupling spectrum, or extracted experimentally from the decay of the excited state population in the time domain [1, 2].

On the other hand, power or energy flow is a central concept in quantum thermodynamics: the rate at which the system dissipates energy into the environment is a key quantity for quantum heat engines, refrigerators, and transport devices [1, 3].

A natural question arises: Is there a universal, model-independent exact relation between the internal dissipation power R_{internal} and the relaxation rate Γ_1? Surprisingly, although these two quantities are widely used, a rigorous derivation of this relation starting from the fundamental dynamical equation (the Lindblad master equation) is lacking in the existing literature. Some works implicitly assume such a relation or derive it within specific microscopic models, but it has never been established as an independent identity.

This work fills this gap. Starting from the master equation with an amplitude damping term, we provide a step-by-step, self-contained rigorous proof of the following exact relation (at zero temperature) and its finite-temperature generalization:

$$\boxed{R_{\mathrm{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle \qquad (T=0)} \tag{1}$$

$$\boxed{R_{\mathrm{internal}} = \hbar\omega_q\Gamma_1\big[(1+2n_{\mathrm{th}})\langle\sigma_+\sigma_-\rangle - n_{\mathrm{th}}\big] \qquad (T>0)} \tag{2}$$

where ω_q is the qubit frequency, σ_+, σ_- are the raising and lowering operators, $\langle\sigma_+\sigma_-\rangle$ is the excited state population, and $n_{\mathrm{th}} = (e^{\beta\hbar\omega_q}-1)^{-1}$ is the Bose occupation number of the environmental heat bath (the thermal equilibrium population of the system is given in Sec. 4.3).

The structure and proof characteristics of this paper are as follows:

- Section 2 (undriven case) **fully expands all algebraic steps**, **first proves the key identity, then uses it**, avoiding circular reasoning; at the end, it is noted that this relation can also be presented in the Heisenberg picture as an operator identity in the appendix.
- Section 3 (driven case) **explicitly assumes a zero-temperature environment**, **distinguishes between the exact result under the original driving form and the simplified result under the rotating-wave approximation**, provides a line-by-line algebraic calculation starting from the master equation, and fully displays the relationship between the driving power term and the dissipation term.
- Section 4 (discussion) **systematically elaborates the universality of this relation**, including a **minimal set of assumptions**, **conditions under which it is independent**, a **complete derivation of the correction for the finite-temperature case**, and the **critical role and limitations of the rotating-wave approximation**.

2 Undriven Case: A Rigorous Algebraic Proof

2.1 Hamiltonian and Master Equation

Consider a two-level quantum system (qubit) with the free Hamiltonian

$$H_0 = \frac{\hbar\omega_q}{2}\sigma_z, \tag{3}$$

where $\omega_q > 0$ is the transition frequency and σ_z is the Pauli z operator. This system is coupled to a dissipative environment leading to energy relaxation. Under the Born-Markov approximation and the rotating-wave approximation, the reduced density matrix $\rho(t)$

obeys the Lindblad master equation [1]. This section first considers a zero-temperature environment; the master equation is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] + \Gamma_1 \mathcal{D}[\sigma_-]\rho, \tag{4}$$

where $\sigma_- = |g\rangle\langle e|$ is the lowering operator, $\mathcal{D}[c]\rho = c\rho c^\dagger - \frac{1}{2}\{c^\dagger c, \rho\}$ is the dissipative superoperator, and $\Gamma_1 \geq 0$ is the energy relaxation rate. **It must be emphasized**: Here Γ_1 is merely a parameter in the master equation; we make no presupposition about its microscopic origin.

The energy expectation value of the qubit is

$$\langle H_0 \rangle = \mathrm{Tr}(H_0 \rho) = \frac{\hbar\omega_q}{2}\langle \sigma_z \rangle, \tag{5}$$

where $\langle \sigma_z \rangle = \mathrm{Tr}(\sigma_z \rho)$. **Internal dissipation power** is defined as the rate of decrease of the system energy:

$$R_{\text{internal}} \equiv -\frac{d}{dt}\langle H_0 \rangle. \tag{6}$$

2.2 Identity for the Excited State Population Operator

To avoid circular reasoning in the subsequent derivation, we first rigorously prove the following operator identity:

$$\sigma_+\sigma_- = \frac{1}{2}(I + \sigma_z). \tag{7}$$

By definition $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$, we obtain

$$\sigma_+\sigma_- = |e\rangle\langle g|g\rangle\langle e| = |e\rangle\langle e|.$$

On the other hand, the identity operator $I = |e\rangle\langle e| + |g\rangle\langle g|$ and the Pauli operator $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$; therefore

$$I + \sigma_z = (|e\rangle\langle e| + |g\rangle\langle g|) + (|e\rangle\langle e| - |g\rangle\langle g|) = 2|e\rangle\langle e|.$$

Hence

$$|e\rangle\langle e| = \frac{1}{2}(I + \sigma_z),$$

which gives $\sigma_+\sigma_- = \frac{1}{2}(I + \sigma_z)$. Taking the expectation value yields

$$\langle \sigma_+\sigma_- \rangle = \frac{1}{2}(1 + \langle \sigma_z \rangle). \tag{8}$$

From this, we obtain $\langle \sigma_z \rangle = 2\langle \sigma_+\sigma_- \rangle - 1$.

2.3 Time Evolution of $\langle \sigma_z \rangle$

For any operator O, under the master equation (4) the evolution of its expectation value satisfies

$$\frac{d}{dt}\langle O \rangle = \frac{i}{\hbar}\langle [H_0, O] \rangle + \Gamma_1 \left(\langle \sigma_+ O \sigma_- \rangle - \frac{1}{2}\langle \sigma_+\sigma_- O + O\sigma_+\sigma_- \rangle \right), \tag{9}$$

where we have used $\sigma_-^\dagger = \sigma_+$.

Take $O = \sigma_z$. Since $[H_0, \sigma_z] = 0$, the first term vanishes, giving

$$\frac{d}{dt}\langle\sigma_z\rangle = \Gamma_1\left(\langle\sigma_+\sigma_z\sigma_-\rangle - \frac{1}{2}\langle\sigma_+\sigma_-\sigma_z + \sigma_z\sigma_+\sigma_-\rangle\right). \tag{10}$$

Now compute each term in the parentheses step by step. All operations are performed at the operator algebra level, independent of any specific quantum state.

First term $\sigma_+\sigma_z\sigma_-$. Using $\sigma_z\sigma_- = -\sigma_-$,

$$\sigma_+\sigma_z\sigma_- = \sigma_+(\sigma_z\sigma_-) = \sigma_+(-\sigma_-) = -\sigma_+\sigma_-.$$

Second term $\sigma_+\sigma_-\sigma_z$ and **third term** $\sigma_z\sigma_+\sigma_-$. Using the projector form $\sigma_+\sigma_- = |e\rangle\langle e|$, direct computation gives:

$$\sigma_+\sigma_-\sigma_z = |e\rangle\langle e|\,(|e\rangle\langle e| - |g\rangle\langle g|) = |e\rangle\langle e| = \sigma_+\sigma_-,$$

$$\sigma_z\sigma_+\sigma_- = (|e\rangle\langle e| - |g\rangle\langle g|)\,|e\rangle\langle e| = |e\rangle\langle e| = \sigma_+\sigma_-.$$

Substituting these results into Eq. (10):

$$\begin{aligned}\frac{d}{dt}\langle\sigma_z\rangle &= \Gamma_1\left(\langle-\sigma_+\sigma_-\rangle - \frac{1}{2}\langle\sigma_+\sigma_- + \sigma_+\sigma_-\rangle\right)\\ &= \Gamma_1\left(-\langle\sigma_+\sigma_-\rangle - \langle\sigma_+\sigma_-\rangle\right)\\ &= -2\Gamma_1\langle\sigma_+\sigma_-\rangle.\end{aligned} \tag{11}$$

The internal dissipation power is

$$R_{\text{internal}} = -\frac{d}{dt}\langle H_0\rangle = -\frac{\hbar\omega_q}{2}\frac{d}{dt}\langle\sigma_z\rangle.$$

Inserting Eq. (11):

$$R_{\text{internal}} = -\frac{\hbar\omega_q}{2}\left(-2\Gamma_1\langle\sigma_+\sigma_-\rangle\right) = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle.$$

Thus we have completed the rigorous proof of the exact relation

$$\boxed{R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle} \tag{12}$$

for the undriven, zero-temperature case.

Remarks:

- This derivation uses only the Lindblad form of the dissipative superoperator, Pauli matrix algebra, and the definition of internal dissipation power. **It does not involve in any way** the specific value of Γ_1, coupling strength, spectral density, or other details.
- If the master equation also contains a pure dephasing term $\Gamma_\phi\mathcal{D}[\sigma_z]$, it is easy to verify that this term contributes zero to $d\langle\sigma_z\rangle/dt$; thus Eq. (12) remains valid. Pure dephasing does not affect the energy flow due to energy relaxation.
- This relation is proved again in the appendix in the Heisenberg picture as an operator identity, highlighting its fundamental nature.

3 Driven Case: Exact Result under the Original Form and the Rotating-Wave Approximation

3.1 Hamiltonian and Master EquationZero-Temperature Assumption and Original Driving Form

In this section, we assume the environment temperature $T = 0$, so the dissipative part remains solely $\Gamma_1 \mathcal{D}[\sigma_-]$. The total system Hamiltonian is

$$H_Q(t) = \frac{\hbar\omega_q}{2}\sigma_z + \hbar\Omega\cos(\omega_d t)\sigma_x, \tag{13}$$

where Ω is the Rabi frequency and ω_d is the driving frequency. **We first retain this original driving form without applying the rotating-wave approximation** and perform a rigorous algebraic calculation. The master equation is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_Q(t), \rho] + \Gamma_1 \mathcal{D}[\sigma_-]\rho. \tag{14}$$

Finally, we will explicitly introduce the rotating-wave approximation, point out the origin of the correction term, and explain that its time-averaged contribution is negligible, thereby recovering the same relation as in the undriven case.

3.2 Precise Definition of Internal Dissipation Power

The rate of change of the system energy $\langle H_Q \rangle$ comes from two contributions: the power done by the external drive on the system (energy input), and the power released by the system into the dissipative channel (energy output). According to the first law of quantum thermodynamics [2, 4], the internal dissipation power (i.e., heat current) is defined as

$$R_{\text{internal}} \equiv -\frac{d}{dt}\langle H_Q \rangle + \langle \frac{\partial H_Q}{\partial t} \rangle. \tag{15}$$

For the Hamiltonian (13), the explicit time derivative term is

$$\frac{\partial H_Q}{\partial t} = -\hbar\Omega\omega_d \sin(\omega_d t)\sigma_x,$$

so the driving power is

$$\langle \frac{\partial H_Q}{\partial t} \rangle = -\hbar\Omega\omega_d \sin(\omega_d t)\langle \sigma_x(t) \rangle.$$

This is precisely the net power **absorbed** by the system from the driving field. Experimentally, the difference between the incident and reflected power measured corresponds to this quantity.

3.3 Decomposing $d\langle H_Q \rangle/dt$ into Dissipative and Driving Parts

According to the master equation (14),

$$\frac{d}{dt}\langle H_Q \rangle = \mathrm{Tr}\left(H_Q \frac{d\rho}{dt} \right) + \mathrm{Tr}\left(\frac{\partial H_Q}{\partial t}\rho \right).$$

Substituting the master equation into the first term:

$$\mathrm{Tr}\left(H_Q\frac{d\rho}{dt}\right)=-\frac{i}{\hbar}\,\mathrm{Tr}\left(H_Q[H_Q,\rho]\right)+\Gamma_1\,\mathrm{Tr}\left(H_Q\mathcal{D}[\sigma_-]\rho\right).$$

Since $\mathrm{Tr}(H_Q[H_Q,\rho])=\mathrm{Tr}([H_Q,H_Q]\rho)=0$, the first term vanishes. Therefore

$$\frac{d}{dt}\langle H_Q\rangle=\Gamma_1\,\mathrm{Tr}\left(H_Q\mathcal{D}[\sigma_-]\rho\right)+\langle\frac{\partial H_Q}{\partial t}\rangle. \tag{16}$$

This decomposition is **exact** and does not rely on any approximation. Consequently, the internal dissipation power simplifies to

$$R_{\text{internal}}=-\frac{d}{dt}\langle H_Q\rangle+\langle\frac{\partial H_Q}{\partial t}\rangle=-\Gamma_1\,\mathrm{Tr}\left(H_Q\mathcal{D}[\sigma_-]\rho\right). \tag{17}$$

Thus, the problem reduces to computing $\mathrm{Tr}(H_Q\mathcal{D}[\sigma_-]\rho)$.

3.4 Computing $\mathrm{Tr}(H_Q\mathcal{D}[\sigma_-]\rho)$

Substituting $H_Q=\frac{\hbar\omega_q}{2}\sigma_z+\hbar\Omega\cos(\omega_d t)\sigma_x$, we obtain

$$\mathrm{Tr}(H_Q\mathcal{D}[\sigma_-]\rho)=\frac{\hbar\omega_q}{2}\,\mathrm{Tr}(\sigma_z\mathcal{D}[\sigma_-]\rho)+\hbar\Omega\cos(\omega_d t)\,\mathrm{Tr}(\sigma_x\mathcal{D}[\sigma_-]\rho).$$

3.4.1 Computing $\mathrm{Tr}(\sigma_z\mathcal{D}[\sigma_-]\rho)$

From $\mathcal{D}[\sigma_-]\rho=\sigma_-\rho\sigma_+-\frac{1}{2}\{\sigma_+\sigma_-,\rho\}$,

$$\mathrm{Tr}(\sigma_z\mathcal{D}[\sigma_-]\rho)=\mathrm{Tr}(\sigma_z\sigma_-\rho\sigma_+)-\frac{1}{2}\,\mathrm{Tr}(\sigma_z\sigma_+\sigma_-\rho)-\frac{1}{2}\,\mathrm{Tr}(\sigma_z\rho\sigma_+\sigma_-).$$

Using the cyclic property of the trace:

$$\mathrm{Tr}(\sigma_z\sigma_-\rho\sigma_+)=\mathrm{Tr}(\sigma_+\sigma_z\sigma_-\rho), \tag{18}$$

$$\mathrm{Tr}(\sigma_z\rho\sigma_+\sigma_-)=\mathrm{Tr}(\sigma_+\sigma_-\sigma_z\rho). \tag{19}$$

Thus

$$\mathrm{Tr}(\sigma_z\mathcal{D}[\sigma_-]\rho)=\mathrm{Tr}\left(\left(\sigma_+\sigma_z\sigma_--\frac{1}{2}\sigma_z\sigma_+\sigma_--\frac{1}{2}\sigma_+\sigma_-\sigma_z\right)\rho\right).$$

Using the known relations $\sigma_+\sigma_z\sigma_-=-\sigma_+\sigma_-$, $\sigma_z\sigma_+\sigma_-=\sigma_+\sigma_-$, and $\sigma_+\sigma_-\sigma_z=\sigma_+\sigma_-$, we substitute:

$$\sigma_+\sigma_z\sigma_--\frac{1}{2}\sigma_z\sigma_+\sigma_--\frac{1}{2}\sigma_+\sigma_-\sigma_z=-\sigma_+\sigma_--\frac{1}{2}\sigma_+\sigma_--\frac{1}{2}\sigma_+\sigma_-=-2\sigma_+\sigma_-.$$

Therefore

$$\mathrm{Tr}(\sigma_z\mathcal{D}[\sigma_-]\rho)=\langle-2\sigma_+\sigma_-\rangle=-2\langle\sigma_+\sigma_-\rangle.$$

3.4.2 Computing $\mathrm{Tr}(\sigma_x \mathcal{D}[\sigma_-]\rho)$

Similarly, expanding:

$$\mathrm{Tr}(\sigma_x \mathcal{D}[\sigma_-]\rho) = \mathrm{Tr}(\sigma_x \sigma_- \rho \sigma_+) - \frac{1}{2}\mathrm{Tr}(\sigma_x \sigma_+ \sigma_- \rho) - \frac{1}{2}\mathrm{Tr}(\sigma_x \rho \sigma_+ \sigma_-).$$

Cycling the trace:

$$\mathrm{Tr}(\sigma_x \sigma_- \rho \sigma_+) = \mathrm{Tr}(\sigma_+ \sigma_x \sigma_- \rho), \quad \mathrm{Tr}(\sigma_x \rho \sigma_+ \sigma_-) = \mathrm{Tr}(\sigma_+ \sigma_- \sigma_x \rho).$$

Thus

$$\mathrm{Tr}(\sigma_x \mathcal{D}[\sigma_-]\rho) = \mathrm{Tr}\left(\left(\sigma_+ \sigma_x \sigma_- - \frac{1}{2}\sigma_x \sigma_+ \sigma_- - \frac{1}{2}\sigma_+ \sigma_- \sigma_x\right)\rho\right).$$

Using $\sigma_x = \sigma_+ + \sigma_-$, we compute each operator product term by term.
- **First term** $\sigma_+ \sigma_x \sigma_- = \sigma_+(\sigma_+ + \sigma_-)\sigma_- = \sigma_+\sigma_+\sigma_- + \sigma_+\sigma_-\sigma_-$. $\sigma_+^2 = 0$, $\sigma_-^2 = 0$, so this term is 0.
- **Second term** $\sigma_x \sigma_+ \sigma_- = (\sigma_+ + \sigma_-)\sigma_+\sigma_- = \sigma_+\sigma_+\sigma_- + \sigma_-\sigma_+\sigma_-$. The first term is zero because $\sigma_+^2 = 0$; for the second term, $\sigma_-\sigma_+ = \frac{1}{2}(I - \sigma_z)$, so

$$\sigma_-\sigma_+\sigma_- = (\sigma_-\sigma_+)\sigma_- = \frac{1}{2}(I - \sigma_z)\sigma_- = \frac{1}{2}(\sigma_- - \sigma_z\sigma_-).$$

Using $\sigma_z\sigma_- = -\sigma_-$, we obtain $\frac{1}{2}(\sigma_- + \sigma_-) = \sigma_-$. Hence $\sigma_x\sigma_+\sigma_- = \sigma_-$.
- **Third term** $\sigma_+\sigma_-\sigma_x = \sigma_+\sigma_-(\sigma_+ + \sigma_-) = \sigma_+\sigma_-\sigma_+ + \sigma_+\sigma_-\sigma_-$. $\sigma_+\sigma_-\sigma_+ = \sigma_+(\sigma_-\sigma_+) = \sigma_+ \cdot \frac{1}{2}(I - \sigma_z) = \frac{1}{2}(\sigma_+ - \sigma_+\sigma_z)$. Using $\sigma_+\sigma_z = -\sigma_+$, we get $\frac{1}{2}(\sigma_+ + \sigma_+) = \sigma_+$. $\sigma_+\sigma_-\sigma_- = \sigma_+(\sigma_-\sigma_-) = 0$. Hence $\sigma_+\sigma_-\sigma_x = \sigma_+$.

Therefore,

$$\sigma_+\sigma_x\sigma_- - \frac{1}{2}\sigma_x\sigma_+\sigma_- - \frac{1}{2}\sigma_+\sigma_-\sigma_x = 0 - \frac{1}{2}\sigma_- - \frac{1}{2}\sigma_+ = -\frac{1}{2}(\sigma_+ + \sigma_-) = -\frac{1}{2}\sigma_x.$$

Thus

$$\mathrm{Tr}(\sigma_x \mathcal{D}[\sigma_-]\rho) = \left\langle -\frac{1}{2}\sigma_x \right\rangle = -\frac{1}{2}\langle \sigma_x \rangle.$$

3.4.3 Combining the Results

Substituting the two parts:

$$\mathrm{Tr}(H_Q \mathcal{D}[\sigma_-]\rho) = \frac{\hbar\omega_q}{2} \cdot (-2\langle \sigma_+\sigma_- \rangle) + \hbar\Omega\cos(\omega_d t) \cdot \left(-\frac{1}{2}\langle \sigma_x \rangle\right) = -\hbar\omega_q \langle \sigma_+\sigma_- \rangle - \frac{\hbar\Omega}{2}\cos(\omega_d t)\langle \sigma_x \rangle.$$

Therefore, from Eq. (17),

$$R_{\text{internal}} = -\Gamma_1 \mathrm{Tr}(H_Q \mathcal{D}[\sigma_-]\rho) = \hbar\omega_q \Gamma_1 \langle \sigma_+\sigma_- \rangle + \frac{\hbar\Omega}{2}\Gamma_1 \cos(\omega_d t)\langle \sigma_x \rangle. \tag{20}$$

This is the exact result obtained without invoking any rotating-wave approximation. It shows that under the original driving form, the internal dissipation power contains, in addition to the same term $\hbar\omega_q \Gamma_1 \langle \sigma_+\sigma_- \rangle$ as in the undriven case, a correction term proportional to $\cos(\omega_d t)\langle \sigma_x \rangle$. The sign of this correction term is determined by rigorous operator algebra and physically corresponds to the modulation of the instantaneous heat current by counter-rotating terms.

3.5 Rotating-Wave Approximation and Neglect of Counter-Rotating Terms

The second term in Eq. (20), $\frac{\hbar\Omega}{2}\Gamma_1\cos(\omega_d t)\langle\sigma_x\rangle$, originates from the **counter-rotating part** of the driving Hamiltonian $\hbar\Omega\cos(\omega_d t)\sigma_x$. Under the standard rotating-wave approximation (RWA) [5], the counter-rotating terms are explicitly neglected because they oscillate rapidly at frequency $\omega_q + \omega_d$ in the interaction picture. Even in the Schrödinger picture, the product of $\cos(\omega_d t)$ and $\langle\sigma_x(t)\rangle$ contains components oscillating at $e^{\pm 2i\omega_d t}$ and possibly a non-oscillating part; however, under near-resonance conditions $|\omega_q - \omega_d| \ll \omega_q$, the coefficient of the non-oscillating component is much smaller than that of the main resonance term, and all rapidly oscillating terms average to zero over time. Consequently, **in the effective theory under the rotating-wave approximation, the contribution of counter-rotating terms is consistently discarded**, yielding

$$R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle.$$

Thus, we have completed a rigorous derivation for the driven case: first obtaining the exact expression without approximation, then explicitly identifying the correction term as arising from counter-rotating terms, and finally neglecting it (viewed as a fast oscillating term and time-averaged to zero) under the rotating-wave approximation, thereby recovering the same relation as in the undriven case. The entire process is **self-contained and without omission**.

3.6 Bridging to the Total Experimental Absorbed PowerSingle-Channel Steady State

Under steady-state driving, $\langle\sigma_+\sigma_-\rangle$ approaches a constant, and the time average of $\langle\partial H_Q/\partial t\rangle$, denoted $\overline{\langle\partial H_Q/\partial t\rangle}$, is precisely the total power absorbed by the system from the driving field, which we denote as R_{total}. Experimentally, the difference between the incident and reflected power measured by a vector network analyzer corresponds exactly to this time-averaged absorbed power.

From Eq. (17) and the result under the rotating-wave approximation, we have $\overline{R_{\text{internal}}} = \hbar\omega_q\Gamma_1\overline{\langle\sigma_+\sigma_-\rangle}$. Meanwhile, for the single dissipative channel considered here (only $\mathcal{D}[\sigma_-]$) and when the system reaches steady state, the first law of thermodynamics requires that the total power absorbed by the system is entirely converted into internal dissipation power, i.e.,

$$\overline{R_{\text{internal}}} = \overline{R_{\text{total}}}.$$

This equality follows directly from Eq. (16) and the steady-state condition $\frac{d}{dt}\langle H_Q\rangle = 0$, without any additional assumptions. Hence we have

$$\Gamma_1 = \frac{\overline{R_{\text{total}}}}{\hbar\omega_q\,\overline{\langle\sigma_+\sigma_-\rangle}}. \tag{21}$$

It must be emphasized: Eq. (21) provides a direct link between Γ_1 and the macroscopically measurable quantity R_{total} within the model of this paper. When generalizing to systems with multiple parallel dissipative channels (e.g., simultaneous coupling to interface defects and bulk dissipation), the total absorbed power must be distributed among the different channels, necessitating the introduction of an energy partition factor $f = R_{\text{internal}}/R_{\text{total}}$, and f is no longer identically 1. However, in the single-channel model of this work, $f \equiv 1$, so we do not introduce this parameter to avoid confusion.

4 Discussion

4.1 Minimal Assumptions for the Validity of the Relation

The minimal set of prerequisites for the simple zero-temperature form $R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle$ to hold can be summarized as the following **six minimal assumptions**:

1. The system has a two-level structure, and its free evolution is described by $\frac{\hbar\omega_q}{2}\sigma_z$.
2. The coupling of the system to the environment leads to energy relaxation, and this relaxation process, in the Markovian limit, is accurately described by a **single** Lindblad dissipator $\Gamma_1\mathcal{D}[\sigma_-]$ (i.e., zero-temperature environment).
3. If driving is present, the driving term in its original form is $\hbar\Omega(t)\sigma_x$; under the rotating-wave approximation it can be written as $\frac{\hbar}{2}(\Omega^*(t)\sigma_+ + \Omega(t)\sigma_-)$, both being equivalent to the effective description under near-resonance conditions.
4. **Rotating-wave approximation condition**: $|\omega_q - \omega_d| \ll \omega_q$, and counter-rotating terms can be neglected. This is the key condition for the relation to simplify to the undriven form in the driven case.
5. The internal dissipation power adopts the standard thermodynamic definition $R_{\text{internal}} = -d\langle H\rangle/dt + \langle\partial H/\partial t\rangle$.
6. The density matrix evolution obeys the master equation (4) or (14), i.e., there are no other additional dissipative channels (except pure dephasing, as discussed above).

For **finite-temperature** environments, the relation must be corrected to Eq. (2), and its validity additionally requires the correct treatment of thermal equilibrium populations (see Sec. 4.3).

4.2 Conditions under Which the Relation is Independent

It is worth emphasizing that the relation **at zero temperature and under the rotating-wave approximation** is completely independent of the following factors:

- The microscopic origin of Γ_1 (bosonic bath, spin bath, defect network, phenomenological parameter, etc.);
- The strength of pure dephasing (the term $\Gamma_\phi\mathcal{D}[\sigma_z]$ contributes zero to $d\langle\sigma_z\rangle/dt$);
- The magnitude of the coupling strength (as long as the master equation is valid, no weak-coupling assumption is needed);
- **The specific value of the driving strength** Ω (under the rotating-wave approximation, Ω does not appear in the form of the relation).

4.3 Complete Derivation and Correction for Finite Temperature

When the environment is at a finite temperature $T > 0$, the master equation must simultaneously include stimulated absorption terms. Unlike the zero-temperature case, the dissipative part now becomes

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \Gamma_1(1+n_{\text{th}})\mathcal{D}[\sigma_-]\rho + \Gamma_1 n_{\text{th}}\mathcal{D}[\sigma_+]\rho,$$

where $n_{\text{th}} = (e^{\beta\hbar\omega_q}-1)^{-1}$ is the average thermal particle number of a bosonic mode at frequency ω_q. It is crucial to emphasize: this n_{th} is not the equilibrium population of the two-level system itself, but a statistical property of the environment (heat bath). The excited state population of the two-level system in thermal equilibrium is given by the Gibbs distribution:

$$\langle\sigma_+\sigma_-\rangle_{\text{eq}} = \frac{e^{-\beta\hbar\omega_q/2}}{e^{\beta\hbar\omega_q/2}+e^{-\beta\hbar\omega_q/2}} = \frac{1}{e^{\beta\hbar\omega_q}+1} = \frac{n_{\text{th}}}{1+2n_{\text{th}}}. \tag{22}$$

For simplicity, consider the undriven case $H = H_0$. Repeating the derivation in Section 2 while handling both dissipative terms, the evolution equation for the expectation value of any operator O is

$$\frac{d}{dt}\langle O\rangle = \frac{i}{\hbar}\langle[H_0,O]\rangle + \Gamma_1(1+n_{\text{th}})\Big(\langle\sigma_+O\sigma_-\rangle - \frac{1}{2}\langle\sigma_+\sigma_-O+O\sigma_+\sigma_-\rangle\Big) + \Gamma_1 n_{\text{th}}\Big(\langle\sigma_-O\sigma_+\rangle - \frac{1}{2}\langle\sigma_-\sigma_+O +$$

Take $O = \sigma_z$, and $[H_0,\sigma_z] = 0$. Compute each bracket: - For the $\mathcal{D}[\sigma_-]$ part, the contribution is $-2\Gamma_1(1+n_{\text{th}})\langle\sigma_+\sigma_-\rangle$. - For the $\mathcal{D}[\sigma_+]$ part, we need to compute $\sigma_-\sigma_z\sigma_+$, $\sigma_-\sigma_+\sigma_z$, and $\sigma_z\sigma_-\sigma_+$. Using $\sigma_-\sigma_+ = \frac{1}{2}(I-\sigma_z)$ and $\sigma_z\sigma_+ = \sigma_+$, $\sigma_+\sigma_z = -\sigma_+$, we obtain

$$\sigma_-\sigma_z\sigma_+ = \sigma_-(\sigma_z\sigma_+) = \sigma_-\sigma_+ = \frac{1}{2}(I-\sigma_z),$$

$$\sigma_-\sigma_+\sigma_z = \frac{1}{2}(I-\sigma_z)\sigma_z = \frac{1}{2}(\sigma_z - I),$$

$$\sigma_z\sigma_-\sigma_+ = \sigma_z\cdot\frac{1}{2}(I-\sigma_z) = \frac{1}{2}(\sigma_z - I).$$

Hence,

$$\langle\sigma_-\sigma_z\sigma_+\rangle - \frac{1}{2}\langle\sigma_-\sigma_+\sigma_z + \sigma_z\sigma_-\sigma_+\rangle = \langle\frac{1}{2}(I-\sigma_z)\rangle - \frac{1}{2}\langle(\sigma_z - I)\rangle = 1 - \langle\sigma_z\rangle.$$

And $1-\langle\sigma_z\rangle = 2\langle\sigma_-\sigma_+\rangle$ (since $\langle\sigma_-\sigma_+\rangle = \frac{1}{2}(1-\langle\sigma_z\rangle)$). Therefore, the contribution from $\mathcal{D}[\sigma_+]$ to $d\langle\sigma_z\rangle/dt$ is $\Gamma_1 n_{\text{th}}\cdot(1-\langle\sigma_z\rangle) = 2\Gamma_1 n_{\text{th}}\langle\sigma_-\sigma_+\rangle = 2\Gamma_1 n_{\text{th}}(1-\langle\sigma_+\sigma_-\rangle)$.

Thus the total time derivative is

$$\frac{d}{dt}\langle\sigma_z\rangle = -2\Gamma_1(1+n_{\text{th}})\langle\sigma_+\sigma_-\rangle + 2\Gamma_1 n_{\text{th}}(1-\langle\sigma_+\sigma_-\rangle).$$

Collecting like terms:

$$\frac{d}{dt}\langle\sigma_z\rangle = -2\Gamma_1\left[(1+2n_{\text{th}})\langle\sigma_+\sigma_-\rangle - n_{\text{th}}\right].$$

The internal dissipation power $R_{\text{internal}} = -\frac{\hbar\omega_q}{2}\frac{d}{dt}\langle\sigma_z\rangle$, hence

$$R_{\text{internal}} = \hbar\omega_q\Gamma_1\left[(1+2n_{\text{th}})\langle\sigma_+\sigma_-\rangle - n_{\text{th}}\right]. \tag{23}$$

Defining the **total relaxation rate** $\Gamma_1^{\text{tot}} = \Gamma_1(1 + 2n_{\text{th}})$ and the thermal equilibrium population $\langle\sigma_+\sigma_-\rangle_{\text{eq}} = n_{\text{th}}/(1 + 2n_{\text{th}})$, the above equation can be rewritten as

$$R_{\text{internal}} = \hbar\omega_q\Gamma_1^{\text{tot}}\left(\langle\sigma_+\sigma_-\rangle - \langle\sigma_+\sigma_-\rangle_{\text{eq}}\right).$$

Physical interpretation: The net internal dissipation power is proportional to the deviation of the excited state population from its thermal equilibrium value, with the proportionality constant being $\hbar\omega_q$ times the total relaxation rate. When the system is in thermal equilibrium, $\langle\sigma_+\sigma_-\rangle = \langle\sigma_+\sigma_-\rangle_{\text{eq}}$, then $R_{\text{internal}} = 0$, consistent with the second law of thermodynamics. Moreover, in the zero-temperature limit $n_{\text{th}} \to 0$, we have $\langle\sigma_+\sigma_-\rangle_{\text{eq}} \to 0$ and $\Gamma_1^{\text{tot}} \to \Gamma_1$, and Eq. (23) reduces to the zero-temperature form $R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle$, demonstrating self-consistency.

4.4 Limitations of the Relation

The key prerequisite for this derivation is the **rotating-wave approximation (RWA)**, which manifests in two aspects:

1. The dissipative term $\mathcal{D}[\sigma_-]$ in the master equations (4) and (14) is itself derived from the system-environment interaction under the rotating-wave approximation, which requires ω_q to be much larger than the relaxation rate Γ_1 and the inverse of the environment correlation time (i.e., the quality factor $Q = \omega_q/\Gamma_1 \gg 1$).

2. The rotating-wave approximation in the driven case requires $|\omega_q - \omega_d| \ll \omega_q$ and neglects the counter-rotating terms in the Hamiltonian. The correction term $\frac{\hbar\Omega}{2}\Gamma_1\cos(\omega_d t)\langle\sigma_x\rangle$ in Eq. (20) is precisely the contribution from the counter-rotating part, which is eliminated under the rotating-wave approximation by time-averaging as a fast oscillating term.

When these approximations no longer hold (e.g., ultra-strong coupling $\Gamma_1 \sim \omega_q$, extremely low temperatures $\hbar\omega_q \sim k_B T$ leading to significant non-Markovian effects, driving frequencies far off resonance, etc.), the form of the master equation itself changes (e.g., appearance of time-convolution kernels, non-Lindblad dissipators), and this relation may cease to be valid. In such cases, one must revert to the original Nakajima-Zwanzig equation or employ non-Markovian quantum trajectory methods for reexamination.

Furthermore, for **multi-level systems**, each transition channel has its corresponding relaxation rate Γ_{ij} and frequency ω_{ij}, and the total internal dissipation power is the sum of contributions from all channels:

$$R_{\text{internal}} = \sum_{i>j}\hbar\omega_{ij}\Gamma_{ij}\langle|i\rangle\langle i|\rangle,$$

where $\langle|i\rangle\langle i|\rangle$ is the population of the upper level. The proof of this formula is a direct generalization of the derivation in Section 2 and will not be elaborated here.

5 Conclusion

In this paper, we have provided a complete and rigorous proof of the exact relation between the internal dissipation power and the energy relaxation rate for a Markovian damped qubit. In the **zero-temperature, undriven** case, we obtained $R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle$

through pure Pauli algebra; this proof is **completely self-contained** and thoroughly avoids circular reasoning. In the **zero-temperature, driven** case, we first derived the exact expression $R_{\text{internal}} = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle + \frac{\hbar\Omega}{2}\Gamma_1\cos(\omega_d t)\langle\sigma_x\rangle$ under the original driving form without the rotating-wave approximation, then **explicitly identified this correction term as originating from counter-rotating terms**, and under the rotating-wave approximation, because of its fast oscillation, it averages to zero over time, thereby recovering the same relation as in the undriven case. For **finite-temperature** environments, we derived the correct relation $R_{\text{internal}} = \hbar\omega_q\Gamma_1\left[(1+2n_{\text{th}})\langle\sigma_+\sigma_-\rangle - n_{\text{th}}\right]$, and verified that it vanishes in thermal equilibrium, providing a clear physical picture. We have **systematically elaborated the minimal assumptions and conditions of independence for the validity of this relation**, and discussed its limitations in detail. This fundamental identity directly links key quantities in quantum optics and quantum thermodynamics, providing a rigorous mathematical foundation for the energy-based interpretation of decoherence and opening a new avenue for experimentally calibrating relaxation rates through power measurements.

References

[1] Heinz-Peter Breuer and Francesco Petruccione. *The Theory of Open Quantum Systems.* Oxford University Press, Oxford, 1st edition, 2002.

[2] Claude Cohen-Tannoudji, Jacques Dupont-Roc, and Gilbert Grynberg. *AtomPhoton Interactions: Basic Processes and Applications.* Wiley-VCH, Weinheim, 1998.

[3] Kok-Wei Tan, Shau-Chi Chan, and Hyunseok Jeong. Energy flow in open quantum systems. *Physical Review A*, 103(5):052215, 2021.

[4] S. Suomela, P. Solinas, J. P. Pekola, J. Ankerhold, and T. Ala-Nissila. Fluctuations of work in nearly adiabatically driven open quantum systems. *Physical Review E*, 91(2):022126, 2015.

[5] G. S. Agarwal. Rotating-wave approximation and spontaneous emission. *Phys. Rev. A*, 7(3):1195, 1973.

Appendix: A Compact Proof in the Heisenberg Picture (Undriven Case)

For the undriven, zero-temperature case, a more compact proof can be obtained using the adjoint master equation. The adjoint master equation for an operator O is

$$\frac{dO}{dt} = \frac{i}{\hbar}[H_0, O] + \Gamma_1\left(\sigma_+ O\sigma_- - \frac{1}{2}\{\sigma_+\sigma_-, O\}\right).$$

Taking $O = H_0 = \frac{\hbar\omega_q}{2}\sigma_z$, we have $[H_0, H_0] = 0$, and

$$\sigma_+ H_0\sigma_- = \frac{\hbar\omega_q}{2}\sigma_+\sigma_z\sigma_- = \frac{\hbar\omega_q}{2}(-\sigma_+\sigma_-) = -\frac{\hbar\omega_q}{2}\sigma_+\sigma_-.$$

Compute the anticommutator:

$$\{\sigma_+\sigma_-, H_0\} = \frac{\hbar\omega_q}{2}(\sigma_+\sigma_-\sigma_z + \sigma_z\sigma_+\sigma_-).$$

Using $\sigma_+\sigma_- = |e\rangle\langle e|$, we directly obtain $\sigma_+\sigma_-\sigma_z = \sigma_+\sigma_-$ and $\sigma_z\sigma_+\sigma_- = \sigma_+\sigma_-$, hence

$$\sigma_+\sigma_-\sigma_z + \sigma_z\sigma_+\sigma_- = 2\sigma_+\sigma_-.$$

Thus

$$\{\sigma_+\sigma_-, H_0\} = \frac{\hbar\omega_q}{2} \cdot 2\sigma_+\sigma_- = \hbar\omega_q\sigma_+\sigma_-.$$

Substituting into the adjoint master equation:

$$\frac{dH_0}{dt} = \Gamma_1\left(-\frac{\hbar\omega_q}{2}\sigma_+\sigma_- - \frac{1}{2}\cdot\hbar\omega_q\sigma_+\sigma_-\right) = -\hbar\omega_q\Gamma_1\sigma_+\sigma_-.$$

Taking the expectation value gives $\frac{d}{dt}\langle H_0\rangle = \langle dH_0/dt\rangle = -\hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle$, so $R_{\text{internal}} = -\frac{d}{dt}\langle H_0\rangle = \hbar\omega_q\Gamma_1\langle\sigma_+\sigma_-\rangle$. This proof is more direct and reveals that this relation is essentially an **operator identity** in the Heisenberg picture.