The Endpoint Adjugate Identity: A Universal PIM Relation for Brauer Tree Algebras and the Torus-Crossing Phenomenon for $\mathrm{GL}(2)$

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Abstract

We prove that for the $e \times e$ tridiagonal Cartan matrix $\mathbf{C}(e,m)$ of a Brauer tree algebra with e-edge line tree and exceptional multiplicity $m \geq 1$ at one endpoint, the last row of the adjugate matrix $\mathrm{adj}(\mathbf{C}(e,m))$ is independent of m, with entries $(-1)^{e-1-j}(j+1)$. This independence is proved via an explicit continuant-recurrence computation of the $(j, e-1)$-cofactors. As a corollary, for any cyclic defect block whose Brauer tree is an e-edge line with exceptional vertex at an endpoint—including unipotent cyclic-defect blocks of $\mathrm{GL}(n, \mathbb{F}_q)$ in the line-tree endpoint-exceptional regime—the normalized PIM dimensions satisfy a universal linear relation: after normalization by $|D|$, the resulting linear combination depends only on the ordinary character degrees of the block, and is independent of m and the ℓ-part parameter encoded by m. For $\mathrm{GL}(2, \mathbb{F}_q)$ with $\ell \mid q+1$ (the case $e = 2$), the identity specializes to $-\widetilde{d}_0 + 2\widetilde{d}_1 = q - 1 = |\mathbb{F}_q^\times|$, recovering the order of a single factor of the split maximal torus $T_s \cong (\mathbb{F}_q^\times)^2$ from block data governed by the nonsplit torus. We interpret this *torus-crossing phenomenon* in the context of the Drinfeld curve cohomology, the Bonnafé–Rouquier derived equivalence, and the Hiraga–Ichino–Ikeda formal degree formula, and formulate conjectures connecting the identity's coefficients to Artin conductor data under the local Langlands correspondence. Computational verification spans 224 distinct blocks for $\mathrm{GL}(2)$, 19 blocks for $\mathrm{GL}(3)$ with $e = 3$ (where the identity recovers the cuspidal degree $(q-1)^2(q+1)$), adjugate computations for all $e \leq 8$ and $m \leq 100$, and 10 non-line tree topologies supporting a generalization conjecture.

Keywords: Brauer tree algebra, projective indecomposable module, Cartan matrix, Smith normal form, cyclic defect block, Deligne–Lusztig theory, $\mathrm{GL}(2, \mathbb{F}_q)$, torus-crossing, Drinfeld curve, Langlands program, formal degree, conductor

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Contents

1 Introduction

1.1 Context and motivation

The modular representation theory of finite groups of Lie type, initiated by Brauer and developed by Fong–Srinivasan [8, 9], Green [11], and Dipper–James [6], rests on the interplay between ordinary characters (over fields of characteristic zero) and modular characters (over fields of positive characteristic ℓ coprime to the defining characteristic). When the defect group of an ℓ-block is cyclic—the *cyclic defect* condition—the block algebra assumes a rigid structure governed by a combinatorial invariant called the *Brauer tree*, as established by Dade [5, Theorem 1] and Janusz [15, Theorem 3.3]. This tree encodes the decomposition matrix, the Cartan matrix, and the dimensions of all projective indecomposable modules (PIMs) of the block.

For the general linear groups $\mathrm{GL}(n, \mathbb{F}_q)$, the Brauer trees of unipotent blocks with cyclic defect are *lines* (path graphs), a fact established by Fong–Srinivasan [8, Theorem (7A)]. The e-edge line tree, where $e = \mathrm{ord}_\ell(q)$, determines an $e \times e$ tridiagonal Cartan matrix $\mathbf{C}(e, m)$ whose entries encode the composition multiplicities of simple modules within PIMs. Here m denotes the *exceptional multiplicity*—the number of ordinary characters at the exceptional vertex—and the defect group order satisfies $|D| = me + 1$.

1.2 Main results

In this paper we establish the following structural result about these Cartan matrices. While the proof is elementary (relying only on cofactor expansion and determinantal recurrences for tridiagonal matrices), the result appears to be new in the Brauer tree algebra literature.

Theorem A (Adjugate Identity Theorem (AIT); Theorem 4.2 below). *Let $\mathbf{C}(e, m)$ be the $e \times e$ Cartan matrix of a Brauer tree algebra with e-edge line tree and exceptional multiplicity $m \geq 1$ at the endpoint vertex v_e. Then:*

(i) The last row of $\mathrm{adj}(\mathbf{C}(e, m))$ is independent of m:

$$\mathrm{adj}(\mathbf{C}(e, m))_{e-1,\,j} \;=\; (-1)^{e-1-j} \cdot (j + 1) \qquad \text{for } j = 0, 1, \ldots, e - 1.$$

(ii) For each row $k < e - 1$, there exists an entry $\mathrm{adj}(\mathbf{C}(e, m))_{k,j}$ that depends on m. Hence the last row is the unique row whose entries are all independent of m.

Here "universal" means: for all line Brauer trees with exceptional vertex at an endpoint, for all values of $e \geq 2$ and $m \geq 1$. The result is a statement in integer linear algebra; the group-theoretic and geometric interpretations in later sections are contextual and, in some cases, conjectural.

The representation-theoretic consequence is:

Corollary B (Endpoint Adjugate Identity; Corollary 4.7 below). *Let $\mathbf{B}$ be a cyclic defect block of a finite group whose Brauer tree is an e-edge line with exceptional vertex at an endpoint. Let $\widetilde{d}_j = \dim P(S_j)/|D|$ be the normalized PIM dimensions. Then:*

$$\sum_{j=0}^{e-1} (-1)^{e-1-j}\,(j + 1)\,\widetilde{d}_j \;=\; \dim(S_{e-1}).$$

For this endpoint-cyclic line-tree class, after normalization by $|D|$, the left-hand side is determined by the decomposition numbers and ordinary character degrees, and is independent of m and the ℓ-part parameter encoded by m.

For $\mathrm{GL}(2, \mathbb{F}_q)$ with $e = 2$ (requiring $\ell \mid q + 1$), the identity specializes to:

Theorem C (Theorem 5.1 below). $-\widetilde{d}_0 + 2\widetilde{d}_1 = q - 1 = |\mathbb{F}_q^\times|$, *the order of a single factor of the split maximal torus* $T_s \cong (\mathbb{F}_q^\times)^2$ *of* $\mathrm{GL}(2)$.

The right-hand side is the order of one coordinate factor of the *split* torus, despite the block being controlled by the *nonsplit* torus $T'(\mathbb{F}_q)$ of order $q+1$. We call this the *torus-crossing phenomenon.*

1.3 Interpretations and conjectures

In Sections 6–7 we provide contextual interpretations of the torus-crossing, drawing on:

- the étale cohomology of the Drinfeld curve and its factorization $\dim H_c^1(Y) = (q+1)(q-1)$ (Section 6.2);
- the Bonnafé–Rouquier derived equivalence [3, Théorème 11.1] realizing Broué's conjecture for GL(2) (Section 6.3);
- the Hiraga–Ichino–Ikeda formal degree formula [13, Theorem 1.1] giving $\mathrm{fdeg}(\pi) = q - 1$ for depth-zero supercuspidals (Section 7.2).

These connections are *contextual*: they explain why the number $q-1$ appears on both sides through the cohomology of the Drinfeld curve and the Bonnafé–Rouquier equivalence, but they are not derived as formal consequences of Theorem A. The interpretations in Section 6 identify a precise cohomological mechanism (the m-independence reflects insensitivity to ℓ-torsion in the integral lattice), while the Langlands-theoretic connections in Section 7 remain conjectural. The paper's core contribution stands independently of both. We also formulate:

Conjecture D (Conjecture 9.1). *The endpoint independence extends to any Brauer tree where the exceptional vertex is a leaf (valence 1).* Computational evidence covering 10 non-line tree topologies (stars, Y-trees, T-trees, caterpillar graphs) is presented in Section 8.4.

Observation E (Observation 8.2). *For* $\mathrm{GL}(e, \mathbb{F}_q)$ *with* $\mathrm{ord}_\ell(q) = e$, *the adjugate identity gives* $\dim(S_{e-1}) = \prod_{i=1}^{e-1}(q^i - 1)$, *the generic cuspidal degree.* Verified for $e = 2$ (224 blocks) and $e = 3$ (19 blocks; Section 8.3).

Question F (Question 9.3). *Is there a conceptual explanation for the equality between the leading coefficient e of the adjugate identity and the Artin conductor $f(\sigma) = e$ of the depth-zero Langlands parameter?*

1.4 Organization

Sections 2–4 form the self-contained core: background (Section 2), tridiagonal linear algebra (Section 3), and the proof of Theorem A and Corollary B (Section 4, with the full continuant-recurrence proof of the key Lemma 4.3 in Appendix A). Section 5 proves Theorem C. Section 6 provides contextual interpretations via the Drinfeld curve and Bonnafé–Rouquier equivalence, identifying the precise cohomological mechanism behind the torus-crossing. Section 7 discusses conjectural Langlands-theoretic connections, with speculative material explicitly marked. Section 8 presents computational verification: GL(2) (224 blocks), GL(3) (19 blocks with a worked example), abstract tree adjugates ($e \leq 8$), and non-line tree evidence for Conjecture D. Section 9 discusses open problems.

1.5 Conventions and notation

Throughout, q is a prime power, $p = \mathrm{char}(\mathbb{F}_q)$, and ℓ is a prime with $\ell \nmid q$. We set $k = \overline{\mathbb{F}}_\ell$ and $\mathcal{O} = W(k)$. All matrix and module indices are 0-based: the Cartan matrix $\mathbf{C} = (c_{ij})_{0 \le i,j \le e-1}$ is $e \times e$, the simple modules are $S_0, \ldots, S_{e-1}$, and the Brauer tree vertices are $v_0, \ldots, v_e$ with edges $S_j = (v_j, v_{j+1})$. The exceptional vertex is always v_e (the right endpoint). We write $v_\ell(n)$ for the ℓ-adic valuation, $|n|_\ell = \ell^{v_\ell(n)}$ for the ℓ-part, and

$$\widetilde{d}_j = \frac{\dim P(S_j)}{|D|}$$

for the normalized PIM dimensions.

2 Brauer Trees and Cyclic Defect Blocks

2.1 Blocks with cyclic defect groups

Let G be a finite group and ℓ a prime dividing $|G|$. The group algebra kG decomposes as a direct sum of indecomposable two-sided ideals called *blocks*. Each block $\mathbf{B}$ is associated with an ℓ-subgroup $D \leq G$, unique up to conjugacy, called its *defect group*. The block controls a finite set $\mathrm{Irr}(\mathbf{B})$ of ordinary irreducible characters and a finite set $\mathrm{IBr}(\mathbf{B})$ of irreducible Brauer characters.

When D is cyclic, the structure of $\mathbf{B}$ is governed by Dade [5, Theorem 1] and Janusz [15, Theorem 3.3]: $\mathbf{B}$ is a *Brauer tree algebra*. Specifically:

(a) The Brauer tree Γ is a finite tree with $e = |\mathrm{IBr}(\mathbf{B})|$ edges and $e + 1$ vertices.

(b) Each vertex carries ordinary characters: non-exceptional vertices carry one character each; the *exceptional vertex* carries $m \geq 1$ characters of equal degree, giving $|\mathrm{Irr}(\mathbf{B})| = e + m$.

(c) Each edge S_j corresponds to a simple $\mathbf{B}$-module. The PIM $P(S_j)$ is determined by walking around the tree from edge S_j, collecting composition factors.

(d) The defect group order satisfies $|D| = me + 1$ [1, Proposition 6.3.3].

2.2 The Cartan matrix

For a block $\mathbf{B}$ with Brauer tree Γ, the *Cartan matrix* $\mathbf{C} = (c_{ij})_{0 \leq i,j \leq e-1}$ is the $e \times e$ symmetric positive-definite integer matrix with $c_{ij} = [P(S_i) : S_j]$, the composition multiplicity of S_j in $P(S_i)$. The Cartan matrix satisfies $\det(\mathbf{C}) = |D|$ and factors as $\mathbf{C} = \Delta^T \Delta$ where Δ is the $(e + m) \times e$ decomposition matrix [1, Theorem 4.18.8].

2.3 Line trees and tridiagonal Cartan matrices

We specialize to Brauer trees that are *lines* (path graphs). With the labeling from Section 1.5 (0-based, exceptional vertex at v_e), the Cartan matrix $\mathbf{C}(e, m)$ is the $e \times e$ tridiagonal matrix:

$$\mathbf{C}(e,m)_{ij} = \begin{cases} 2 & \text{if } i = j < e-1, \\ m+1 & \text{if } i = j = e-1, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly:

$$\mathbf{C}(e,m) = \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & m+1 \end{pmatrix}. \tag{2.1}$$

For $e = 2$: $\mathbf{C}(2, m)$ has rows $[2,\ 1]$ and $[1,\ m+1]$, with $\det = 2m + 1$.
For $e = 3$: $\mathbf{C}(3, m)$ has rows $[2, 1, 0]$, $[1, 2, 1]$, $[0, 1, m + 1]$, with $\det = 3m + 1$.

2.4 Fong–Srinivasan theory for GL(n)

For $G = \mathrm{GL}(n, \mathbb{F}_q)$ with $\ell \nmid q$, the unipotent ℓ-blocks with cyclic defect are described by Fong–Srinivasan [8, Theorem (7A)]. The key parameters are:

(a) The *inertial index* $e = \mathrm{ord}_\ell(q)$, the multiplicative order of q modulo ℓ.

(b) The *defect group* D, cyclic of order ℓ^a where $\ell^a \| \Phi_e(q)$, with Φ_e the e-th cyclotomic polynomial. For $e = 2$: $\Phi_2(q) = q + 1$, so $|D| = |q + 1|_\ell$.

(c) The *Brauer tree*: a line with e edges [8, Theorem (7A)]. The tree topology and character assignment are refined by Hiss [12] and Geck [10].

For $G = \mathrm{GL}(2, \mathbb{F}_q)$ with $\ell \mid q + 1$: the block contains the trivial character 1_G (degree 1), the Steinberg character St_G (degree q), and $m = (|D| - 1)/2$ cuspidal characters of degree $q - 1$. The Brauer tree is:

$$v_0 \; [1_G, \text{ deg } 1] \text{—}_{S_0}\text{—} \; v_1 \; [\mathrm{St}_G, \text{ deg } q] \text{—}_{S_1}\text{—} \; v_2 \; [m \times \text{cusp}, \text{ deg } q{-}1]$$

giving PIM dimensions:

$$\dim P(S_0) = \deg(1_G) + \deg(\mathrm{St}_G) = q + 1, \qquad \dim P(S_1) = \deg(\mathrm{St}_G) + m \cdot \deg(\text{cusp}) = q + m(q - 1). \tag{2.2}$$

3 Linear Algebra of Modified Tridiagonal Matrices

3.1 The standard tridiagonal matrix

Let $\mathbf{A}_n$ denote the $n \times n$ tridiagonal matrix with 2's on the main diagonal and 1's on the sub- and super-diagonals. This is the Cartan matrix of the root system A_n.

Lemma 3.1. *$\det(\mathbf{A}_n) = n + 1$ for all $n \geq 1$.*

Proof. Let $f(n) = \det(\mathbf{A}_n)$, with $f(0) = 1$ (empty matrix) and $f(1) = 2$. Cofactor expansion along the first row gives the recurrence $f(n) = 2f(n-1) - f(n-2)$. The characteristic polynomial $x^2 - 2x + 1 = (x-1)^2$ has a double root, so $f(n) = \alpha + \beta n$. The initial conditions $f(0) = 1$, $f(1) = 2$ give $\alpha = \beta = 1$, hence $f(n) = n + 1$. □

Proposition 3.2. $\det(\mathbf{C}(e, m)) = me + 1 = |D|$.

Proof. Cofactor expansion of $\det(\mathbf{C}(e, m))$ along the last row:

$$\det(\mathbf{C}(e, m)) = (m+1) \cdot \det(\mathbf{A}_{e-1}) - 1 \cdot \det(\mathbf{A}_{e-2}) = (m+1)e - (e-1) = me + 1.$$

Here $\det(\mathbf{A}_{e-1}) = e$ and $\det(\mathbf{A}_{e-2}) = e - 1$ by Lemma 3.1 (with the convention $\det(\mathbf{A}_0) = 1$). □

3.2 The Smith normal form

Proposition 3.3. *The Smith normal form of $\mathbf{C}(e, m)$ is $\operatorname{diag}(1, 1, \ldots, 1, |D|)$ with $e-1$ leading 1's.*

Proof. For Brauer tree algebras with cyclic defect, the elementary divisors of the Cartan matrix are 1 (with multiplicity $e - 1$) and $|D|$ (with multiplicity 1). This follows from the structure theorem $\mathbf{C} = \Delta^T \Delta$ and the fact that Δ has all $(e-1) \times (e-1)$ minors coprime; see [1, Theorem 4.18.8] for the general statement and [21, Proposition 3.1] for the explicit cyclic defect case. Verified computationally for all $e \leq 8$ and $m \leq 100$. □

Remark 3.4. The Smith normal form implies $\mathbb{Z}^e / \mathbf{C}\mathbb{Z}^e \cong \mathbb{Z}/|D|\mathbb{Z}$, recovering the defect group order from the Cartan matrix alone.

3.3 The adjugate: definition and index conventions

For any invertible $e \times e$ integer matrix $\mathbf{M}$, the *adjugate* (classical adjoint) is $\operatorname{adj}(\mathbf{M}) = \det(\mathbf{M}) \cdot \mathbf{M}^{-1}$, satisfying $\operatorname{adj}(\mathbf{M}) \cdot \mathbf{M} = \det(\mathbf{M}) \cdot \mathbf{I}$.

Lemma 3.5 (Index convention for adjugate entries)**.** *The (i, j)-entry of $\operatorname{adj}(\mathbf{M})$ is:*

$$\operatorname{adj}(\mathbf{M})_{ij} = (-1)^{i+j} \cdot \det\big(\mathbf{M}[\widehat{\jmath} \mid \widehat{\imath}]\big)$$

where $\mathbf{M}[\widehat{\jmath} \mid \widehat{\imath}]$ denotes the $(e-1) \times (e-1)$ submatrix of $\mathbf{M}$ obtained by deleting row j and column i. Note the transposition: the entry in row i, column j of the adjugate involves deleting row j and column i from $\mathbf{M}$.

Proof. Standard; see e.g. [14, Theorem 0.8.2]. □

This transposition is important: the *last row* ($i = e - 1$) of $\operatorname{adj}(\mathbf{C})$ involves minors of the form $\mathbf{C}[\widehat{\jmath} \mid \widehat{e-1}]$, obtained by deleting row j and column $e - 1$ from $\mathbf{C}$. The *last column* ($j = e - 1$) involves minors $\mathbf{C}[\widehat{e-1} \mid \widehat{\imath}]$, obtained by deleting row $e - 1$ and column i. These are different operations, and only the former produces m-independent minors.

3.4 PIM dimensions via the adjugate

For the Cartan matrix, if $\mathbf{d} = (\dim P(S_j))_{0\le j\le e-1}$ and $\mathbf{s} = (\dim S_j)_{0\le j\le e-1}$ are column vectors, then $\mathbf{d} = \mathbf{C}\cdot\mathbf{s}$ inverts to:

$$\mathrm{adj}(\mathbf{C})\cdot\mathbf{d} = |D|\cdot\mathbf{s}. \tag{3.1}$$

Row k of equation (3.1) reads:

$$\sum_{j=0}^{e-1}\mathrm{adj}(\mathbf{C})_{kj}\cdot\dim P(S_j) = |D|\cdot\dim(S_k). \tag{3.2}$$

Each row of $\mathrm{adj}(\mathbf{C})$ provides a linear relation among PIM dimensions. We ask: *for which rows k are the coefficients $\mathrm{adj}(\mathbf{C})_{kj}$ independent of the exceptional multiplicity m?*

4 The Adjugate Identity Theorem

4.1 Localization of m and its consequences

Lemma 4.1 (Localization of m)**.** *In* $\mathbf{C}(e, m)$*, the parameter* m *appears only in the single entry* $\mathbf{C}_{e-1,\,e-1} = m + 1$*. All other entries are independent of* m*.*

Proof. Immediate from the definition (Section 2.3). □

This localization has two immediate consequences for cofactors, depending on whether column $e - 1$ is deleted or retained:

- **Column** $e - 1$ **deleted** (relevant for the *last row* of $\operatorname{adj}(\mathbf{C})$): The minor $\mathbf{C}[\widehat{j} \mid \widehat{e-1}]$ deletes the unique column containing m. All surviving entries are m-independent, so the minor itself is m-independent.
- **Column** $e - 1$ **retained, row** $e - 1$ **also retained** (relevant for entries $\operatorname{adj}(\mathbf{C})_{k,j}$ with both $k < e-1$ and $j < e-1$): The minor $\mathbf{C}[\widehat{j} \mid \widehat{k}]$ with $j < e-1$ retains row $e-1$, and with $k < e-1$ retains column $e - 1$. The entry $\mathbf{C}_{e-1,\,e-1} = m + 1$ survives, so the minor depends on m.
- **Row** $e - 1$ **deleted** (relevant for the last column of $\operatorname{adj}(\mathbf{C})$): The minor $\mathbf{C}[\widehat{e-1} \mid \widehat{k}]$ deletes the row containing $m+1$. By symmetry of $\mathbf{C}$, these minors equal the corresponding column-deleted minors, and are m-independent.

Theorem 4.2(ii) below is row-wise existential: for each $k < e-1$, at least one entry in row k depends on m. This is compatible with possible m-independent entries in specific columns (e.g., the last column in low rank), as illustrated in Example 4.6.

4.2 Statement of the main theorem

Theorem 4.2 (Adjugate Identity Theorem)**.** *Let* $\mathbf{C}(e, m)$ *be the Cartan matrix defined in Section 2.3. Then:*

(i) For all $j = 0, 1, \ldots, e - 1$*:*

$$\operatorname{adj}(\mathbf{C}(e, m))_{e-1,\,j} = (-1)^{e-1-j} \cdot (j + 1).$$

In particular, these entries are independent of m*.*

(ii) For each k *with* $0 \le k \le e - 2$*, there exists* $j \in \{0, \ldots, e - 1\}$ *such that* $\operatorname{adj}(\mathbf{C})_{k,j}$ *depends on* m*. Hence no row* $k < e - 1$ *is fully* m*-independent, and the last row is the unique row of* $\operatorname{adj}(\mathbf{C}(e, m))$ *with this property.*

4.3 Proof of part (i): the key minor computation

Recall from Lemma 3.5 that $\operatorname{adj}(\mathbf{M})_{ij} = (-1)^{i+j} \det(\mathbf{M}[\widehat{j} \mid \widehat{\imath}])$. For row $i = e-1$, the needed minors are $\mathbf{C}[\widehat{j} \mid \widehat{e-1}]$—that is, column $e-1$ is deleted—exactly the minors controlled by Lemma 4.3 below.

By Lemma 3.5, $\operatorname{adj}(\mathbf{C})_{e-1,\,j} = (-1)^{(e-1)+j} \cdot \det(\mathbf{C}[\widehat{j} \mid \widehat{e-1}])$. The minor $\mathbf{C}[\widehat{j} \mid \widehat{e-1}]$ is obtained by deleting row j and column $e - 1$ from $\mathbf{C}(e, m)$.

By Lemma 4.1 and the column-deletion observation above, this minor is m-independent. It remains to compute its value.

Lemma 4.3 (The endpoint minor formula)**.** *For all $e \geq 2$ and $0 \leq j \leq e-1$:*

$$\det\bigl(\mathbf{C}(e,m)[\widehat{j} \mid \widehat{e-1}]\bigr) = j+1.$$

Proof. See Appendix A for the complete proof by strong induction on e. □

Combining Lemma 3.5 and Lemma 4.3:

$$\operatorname{adj}(\mathbf{C})_{e-1,\,j} = (-1)^{(e-1)+j}(j+1) = (-1)^{e-1-j}(j+1)$$

where the second equality uses $(-1)^{(e-1)+j} = (-1)^{(e-1)-j}$ since $(-1)^{2j} = 1$. This completes the proof of part (i). □

4.4 Proof of part (ii): m-dependence of non-last rows

For $0 \leq k \leq e-2$ and $0 \leq j \leq e-2$, the entry $\operatorname{adj}(\mathbf{C})_{k,j} = (-1)^{k+j} \cdot \det(\mathbf{C}[\widehat{j} \mid \widehat{k}])$. The minor $\mathbf{C}[\widehat{j} \mid \widehat{k}]$ is obtained by deleting row j and column k from $\mathbf{C}$. Since $j < e-1$, row $e-1$ survives; since $k < e-1$, column $e-1$ survives. Therefore the entry $\mathbf{C}_{e-1,\,e-1} = m+1$ is present in this minor, and the determinant is a non-constant function of m.

Concretely, row $e-1$ of $\mathbf{C}$ restricted to the surviving columns $\{0,\ldots,e-1\} \setminus \{k\}$ has a nonzero entry $m+1$ in the position corresponding to column $e-1$, plus a 1 in the position corresponding to column $e-2$ (if $k \neq e-2$). Expanding the determinant along this row shows that the minor is an affine function of m with nonzero slope. To make this explicit, consider the witness entry $\operatorname{adj}(\mathbf{C})_{k,0} = (-1)^k \det(\mathbf{C}[\widehat{0} \mid \widehat{k}])$ for any fixed $k < e-1$. Cofactor expansion of the minor along its surviving last row extracts the coefficient $(m+1)\det(\mathbf{C}[\widehat{0}, \widehat{e-1} \mid \widehat{k}, \widehat{e-1}])$. The inner determinant is a principal minor of $\mathbf{A}_{e-1}$ and equals $k+1 \neq 0$ by Lemma 3.1, confirming that $\operatorname{adj}(\mathbf{C})_{k,0}$ is nonconstant in m.

We record this as a standalone statement for later use:

Lemma 4.4 (Nonvanishing slope of the witness minor)**.** *For each $k \in \{0,\ldots,e-2\}$, let $M_k(m) := \det\bigl(\mathbf{C}(e,m)[\widehat{0} \mid \widehat{k}]\bigr)$. Then $M_k(m) = \alpha_k m + \beta_k$ with $\alpha_k = k+1 \neq 0$.*

Proof. Expand $M_k(m)$ along the surviving row corresponding to row $e-1$. The coefficient of $m+1$ is $\det\bigl(\mathbf{C}(e,m)[\widehat{0}, \widehat{e-1} \mid \widehat{k}, \widehat{e-1}]\bigr)$, which is the $(k+1) \times (k+1)$ leading principal minor of $\mathbf{A}_{e-1}$, hence equals $k+1$ by Lemma 3.1. Therefore M_k is affine in m with nonzero slope, so $\operatorname{adj}(\mathbf{C})_{k,0} = (-1)^k M_k(m)$ is nonconstant in m. □

Remark 4.5. The identification of the inner determinant as a principal minor of $\mathbf{A}_{e-1}$ uses the same block-structure decomposition detailed in Appendix A: after deleting row 0 and column k from $\mathbf{C}(e,m)$, the surviving rows partition into a "top block" (rows $1,\ldots,k$) and a "bottom block" (rows $k+1,\ldots,e-1$), coupled by a single off-diagonal entry. Expanding along the bottom row $(e-1)$ separates the m-dependent coefficient from the m-free remainder, and the latter reduces to the continuant determinant $\det(\mathbf{A}_{k+1}) = k+2$ by the recurrence of Lemma 3.1. See Appendix A for the complete inductive framework.

For the last-column entries $\operatorname{adj}(\mathbf{C})_{k,\,e-1} = (-1)^{k+e-1}\det(\mathbf{C}[\widehat{e-1} \mid \widehat{k}])$, the minor deletes row $e-1$ (removing the $m+1$ entry). Since $\mathbf{C}$ is symmetric, $\det(\mathbf{C}[\widehat{e-1} \mid \widehat{k}]) = \det(\mathbf{C}[\widehat{k} \mid \widehat{e-1}]) = k+1$ by Lemma 4.3. Hence $\operatorname{adj}(\mathbf{C})_{k,\,e-1} = (-1)^{k+e-1}(k+1)$, which is m-independent and equals $\operatorname{adj}(\mathbf{C})_{e-1,\,k}$ (as expected from $\operatorname{adj}(\mathbf{C}) = \operatorname{adj}(\mathbf{C})^T$ for symmetric $\mathbf{C}$).

Since $\operatorname{adj}(\mathbf{C})_{k,0}$ depends on m for all $k < e-1$, each non-last row is m-dependent. □

Example 4.6 (Complete adjugate for $e = 3$)**.** For $\mathbf{C}(3, m)$ with rows $[2, 1, 0]$, $[1, 2, 1]$, $[0, 1, m+1]$, we compute all nine cofactors using Lemma 3.5:

$$\operatorname{adj}(\mathbf{C}(3, m)) = \begin{pmatrix} 2m+1 & -(m+1) & 1 \\ -(m+1) & 2(m+1) & -2 \\ 1 & -2 & 3 \end{pmatrix}.$$

We label the entries by (row, column) and verify Theorem 4.2:

Part (i): Last row ($i = 2$). The entries are $\operatorname{adj}(\mathbf{C})_{2,0} = 1$, $\operatorname{adj}(\mathbf{C})_{2,1} = -2$, $\operatorname{adj}(\mathbf{C})_{2,2} = 3$. These match $(-1)^{2-j}(j+1)$ for $j = 0, 1, 2$, and are independent of m. ✓

Part (ii): Non-last rows ($i = 0, 1$). Row 0 has entries $(2m+1,\ -(m+1),\ 1)$. The entries at $j = 0$ and $j = 1$ depend on m; the entry at $j = 2$ ($= e-1$) equals $1 = (-1)^{0+2}(0+1)$, m-independent by symmetry. Row 1 has entries $(-(m+1),\ 2m+2,\ -2)$, with the same pattern. ✓

Last column ($j = 2$). The entries $\operatorname{adj}(\mathbf{C})_{0,2} = 1$, $\operatorname{adj}(\mathbf{C})_{1,2} = -2$, $\operatorname{adj}(\mathbf{C})_{2,2} = 3$ are all m-independent. This confirms that both the last row and the last column of $\operatorname{adj}(\mathbf{C})$ are m-free, while the 2×2 interior block

$$\begin{pmatrix} 2m+1 & -(m+1) \\ -(m+1) & 2(m+1) \end{pmatrix}$$

carries all m-dependence. (The m-independence of the entire last column is a low-rank feature of $e = 3$ and is not the content of Theorem 4.2(ii); the theorem asserts uniqueness of the *last row* as the row with full m-independence for general e.) ✓

4.5 The Endpoint Adjugate Identity

Corollary 4.7 (Endpoint Adjugate Identity)**.** *Let* $\mathbf{B}$ *be a cyclic defect block whose Brauer tree is an* e*-edge line with exceptional vertex at the endpoint* v_e*. Let* $\widetilde{d}_j = \dim P(S_j)/|D|$*. Then:*

$$\sum_{j=0}^{e-1} (-1)^{e-1-j}\,(j+1)\,\widetilde{d}_j \;=\; \dim(S_{e-1}). \tag{4.1}$$

Proof. Substitute $k = e - 1$ into equation (3.2):

$$\sum_{j=0}^{e-1} \operatorname{adj}(\mathbf{C})_{e-1,\,j} \cdot \dim P(S_j) = |D| \cdot \dim(S_{e-1}).$$

By Theorem 4.2(i), $\operatorname{adj}(\mathbf{C})_{e-1,\,j} = (-1)^{e-1-j}(j+1)$. Dividing both sides by $|D|$ and using $\widetilde{d}_j = \dim P(S_j)/|D|$ gives equation (4.1). The coefficients are m-independent by Theorem 4.2(i), and $|D|$ cancels in the normalization. □

Explicit instances:

$$\begin{aligned} e = 2:&\quad -\widetilde{d}_0 + 2\widetilde{d}_1 = \dim(S_1), \\ e = 3:&\quad \widetilde{d}_0 - 2\widetilde{d}_1 + 3\widetilde{d}_2 = \dim(S_2), \\ e = 4:&\quad -\widetilde{d}_0 + 2\widetilde{d}_1 - 3\widetilde{d}_2 + 4\widetilde{d}_3 = \dim(S_3), \\ e = 5:&\quad \widetilde{d}_0 - 2\widetilde{d}_1 + 3\widetilde{d}_2 - 4\widetilde{d}_3 + 5\widetilde{d}_4 = \dim(S_4). \end{aligned}$$

The alternating signs with coefficients $1, 2, 3, \ldots, e$ reflect the cofactor formula of Lemma 4.3.

Remark 4.8. Theorem 4.2(ii) is an existence statement: for each non-last row, at least one entry depends on m. Example 4.6 makes the general proof concrete for $e = 3$, exhibiting all nine cofactors and the 2×2 interior block that carries the m-dependence. The general case (Section 4.4) proceeds by the same cofactor-expansion mechanism.

5 The GL(2) Adjugate Identity

Theorem 5.1. *Let q be a prime power and ℓ a prime with $\ell \mid q+1$ and $\ell \neq p$. Let $\mathbf{B}$ be the unipotent ℓ-block of* $\mathrm{GL}(2, \mathbb{F}_q)$ *with cyclic defect group of order $|D| = |q+1|_\ell$ and exceptional multiplicity $m = (|D|-1)/2$. Then:*

$$-\widetilde{d}_0 + 2\widetilde{d}_1 = q - 1. \tag{5.1}$$

The simple module dimensions are $\dim(S_0) = 1$ *and* $\dim(S_1) = q - 1$.

Proof. From Section 2.4, $\dim P(S_0) = q+1$ and $\dim P(S_1) = q+m(q-1)$. Applying equation (4.1) with $e = 2$:

$$-\widetilde{d}_0 + 2\widetilde{d}_1 = \frac{-(q+1) + 2(q+m(q-1))}{|D|} = \frac{(q-1)(1+2m)}{|D|} = \frac{(q-1)(2m+1)}{2m+1} = q - 1.$$

That $\dim(S_0) = 1$ and $\dim(S_1) = q - 1$ follows from the Cartan inversion $\mathbf{s} = \mathbf{C}^{-1}\mathbf{d}$: solving the system $2s_0 + s_1 = q + 1$ and $s_0 + (m+1)s_1 = q + m(q-1)$ yields $s_0 = (2m+1)/(2m+1) = 1$ and $s_1 = (q-1)(2m+1)/(2m+1) = q - 1$. □

5.1 The torus-order observation

Observation 5.2. The right-hand side of (5.1) satisfies $q - 1 = |\mathbb{F}_q^\times|$, the order of a single factor of the split maximal torus $T_s \cong (\mathbb{F}_q^\times)^2$ of GL(2). This is a numerical observation about the value of $\dim(S_1)$, not a functorial statement relating the adjugate identity to the split torus. The content of Theorem 5.1 is the algebraic identity (5.1); the torus-order interpretation, discussed in Section 6, provides context for the arithmetic.

5.2 The Block Plancherel Formula

As an independent consistency check, the Block Plancherel Formula gives:

$$\sum_j \dim(S_j) \cdot \dim P(S_j) = \sum_{\chi \in \mathrm{Irr}(\mathbf{B})} (\deg \chi)^2.$$

For GL(2): $1 \cdot (q+1) + (q-1) \cdot (q + m(q-1)) = 1 + q^2 + m(q-1)^2$, verified across all 224 tested blocks.

6 Interpretations: The Torus-Crossing Phenomenon

Scope note. This section provides *contextual interpretations* of the torus-crossing observed in Section 5. The results cited are due to other authors; we do not claim them as consequences of Theorem 4.2.

6.1 Two tori

$\mathrm{GL}(2, \mathbb{F}_q)$ has two conjugacy classes of maximal tori:

(i) The *split torus* $T_s \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, of order $(q-1)^2$. Parabolic induction from T_s produces principal series representations of degree $q+1$.

(ii) The *nonsplit torus* $T' \cong \ker(N\colon \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times)$, of order $q+1$. Deligne–Lusztig induction from T' [7, §3] produces cuspidal representations of degree $q-1$.

The block $\mathbf{B}$ (with $\ell \mid q+1$) is controlled by the nonsplit torus: $D \leq T'$ is its ℓ-Sylow, and the exceptional characters are cuspidal representations from T'-characters. Yet the adjugate identity (5.1) outputs $q-1 = |\mathbb{F}_q^\times|$: the order of a single factor of the split torus. This torus-crossing is a value-level numerical emergence from the Cartan matrix algebra; it does not assert or require a functorial transfer between the split and nonsplit tori.

6.2 The Drinfeld curve

The Deligne–Lusztig variety for $T' \leq \mathrm{GL}(2)$ is the affine curve $Y\colon xy^q - x^q y = 1$ over $\mathbb{F}_{q^2}$. This smooth curve has genus $g = q(q-1)/2$ and admits an action of $\mathrm{GL}(2, \mathbb{F}_q) \times T'(\mathbb{F}_q)$. By [7, Theorem 3.2], the compactly supported étale cohomology $H^1_c(Y \otimes \overline{\mathbb{F}}, \overline{\mathbb{Q}}_\ell)$ has dimension $q^2 - 1$ and decomposes under the T'-action as:

$$H^1_c(Y, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\theta \in \widehat{T'}} V_\theta, \qquad \dim V_\theta = q - 1.$$

For a character θ of T' in general position (i.e., $\theta \neq \theta^q$), the isotypic component $-V_\theta$ is an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ of dimension $q-1$. This is the representation-theoretic origin of the value $\dim(S_1) = q-1$ that the adjugate identity extracts.

The factorization $\dim H^1_c = q^2 - 1 = (q+1)(q-1) = |T'| \cdot |\mathbb{F}_q^\times|$ is the *cohomological manifestation of the torus-crossing*: the T'-action accounts for $q+1$ isotypic pieces, each of dimension $q-1 = |\mathbb{F}_q^\times|$. The adjugate identity extracts the factor $q-1$ from the block data, which carries the ℓ-part of the $q+1$ factor.

6.3 The Bonnafé–Rouquier derived equivalence

Bonnafé–Rouquier [3, Théorème 11.1] proved that the complex $R\Gamma_c(Y, \mathcal{O}_\ell)$ realizes a splendid Rickard equivalence between $\mathbf{B}$ and its Brauer correspondent, proving Broué's abelian defect conjecture for $\mathrm{GL}(2)$. This geometric object simultaneously:

(a) produces the cuspidal representations in characteristic-zero cohomology [7, Theorem 3.2];

(b) determines the Brauer tree topology from the perversity structure [4, §5];

(c) controls the PIM structure, and hence (indirectly) the adjugate identity.

The connection is indirect for a precise reason: the Cartan matrix is a modular object (it lives over $\overline{\mathbb{F}}_\ell$), while the Drinfeld curve cohomology $H^1_c(Y, \overline{\mathbb{Q}}_\ell)$ is a characteristic-zero object. The bridge between them passes through the $\mathcal{O}_\ell$-lattice $H^1_c(Y, \mathcal{O}_\ell)$: the integral structure of this lattice determines the decomposition matrix D (via reduction modulo ℓ), and hence the Cartan matrix $C = D^\top D$. The adjugate identity then follows from the purely algebraic structure of C.

Concretely: the ℓ-torsion in $H^1_c(Y, \mathcal{O}_\ell)$ encodes the exceptional multiplicity m, and the fact that m appears in exactly one diagonal entry of C (namely $C_{e-1,e-1} = m + 1$) reflects the concentration of ℓ-torsion at a single vertex of the Brauer tree. The adjugate identity is m-independent precisely because it sees only the m-free part of the integral cohomology—the part that survives in $H^1_c(Y, \overline{\mathbb{Q}}_\ell)$, where each isotypic component has dimension $q - 1$ regardless of ℓ-structure.

6.4 Scope of the interpretation

The Drinfeld curve provides a precise explanation for why $q-1$ appears in the adjugate identity: it is the dimension of each isotypic component of $H^1_c(Y, \overline{\mathbb{Q}}_\ell)$, which is the characteristic-zero cuspidal degree of $\mathrm{GL}(2, \mathbb{F}_q)$. The Bonnafé–Rouquier equivalence explains the mechanism: the integral lattice $H^1_c(Y, \mathcal{O}_\ell)$ determines the Cartan matrix, and the adjugate identity extracts the m-free piece corresponding to rational cohomology. However, we do not claim that the adjugate identity is a *formal consequence* of the Bonnafé–Rouquier equivalence: the identity is proved from the Cartan matrix structure alone (Theorem 4.2), and the geometric interpretation provides the arithmetic reason for the value $q - 1$ without being logically required by the proof.

7 Connections to the Local Langlands Correspondence

Scope note. This section describes contextual connections and formulates conjectural questions. No results here are proved as consequences of the AIT.

7.1 Depth-zero supercuspidals and the Artin conductor

Under the local Langlands correspondence for GL(2) over a p-adic field F with residue field $\mathbb{F}_q$, each cuspidal representation π_θ of $\mathrm{GL}(2, \mathbb{F}_q)$ lifts to a *depth-zero supercuspidal* of $\mathrm{GL}(2, F)$ via compact induction [2, §15]:

$$\pi = \text{c-Ind}_{\mathrm{GL}(2,\mathcal{O})}^{\mathrm{GL}(2,F)} \tilde{\pi}_\theta.$$

The Langlands parameter is $\sigma = \mathrm{Ind}_{W(F_2)}^{W(F)} \chi$, where F_2/F is the unramified quadratic extension. The *Artin conductor* is:

$$f(\sigma) = \dim(\sigma) - \dim(\sigma^{I_F}) + \mathrm{sw}(\sigma) = 2 - 0 + 0 = 2.$$

Here $\sigma^{I_F} = 0$ because both summands of $\sigma|_{I_F}$ are nontrivial (χ is primitive), and the Swan conductor vanishes since the ramification is tame [19, Chapter VI, §2].

7.2 The formal degree formula

Hiraga–Ichino–Ikeda [13, Conjecture 1.4, proved for $\mathrm{GL}(n)$ in Theorem 1.1] express the formal degree of a discrete series representation in terms of its Langlands parameter. For depth-zero supercuspidals of $\mathrm{GL}(2, F)$ with the normalization of [13, §6]:

$$\mathrm{fdeg}(\pi) = q - 1.$$

This equals $\dim(S_1)$ = the right-hand side of the GL(2) adjugate identity (5.1).

For $\mathrm{GL}(e, F)$: the formal degree of the depth-zero supercuspidal π associated to a character of the degree-e unramified extension is

$$\mathrm{fdeg}(\pi) = \prod_{i=1}^{e-1} (q^i - 1),$$

which is the generic cuspidal degree of $\mathrm{GL}(e, \mathbb{F}_q)$. This coincides with $\dim(S_{e-1})$ as computed by the Endpoint Adjugate Identity (Observation 8.2), giving three independent routes to the same value:

(i) *Algebraic*: $\dim(S_{e-1})$ computed from the Cartan matrix adjugate;

(ii) *Cohomological*: $\dim V_\theta$ as the cuspidal degree from Deligne–Lusztig theory;

(iii) *Langlands-theoretic*: $\mathrm{fdeg}(\pi)$ from the Hiraga–Ichino–Ikeda formula.

7.3 The conductor–identity resonance

We observe a numerical parallel that extends to general e:

Quantity	Langlands side	Adjugate identity side
Leading coefficient	$f(\sigma) = e$	e (coeff. of $\widetilde{d}_{e-1}$)
Output value	$\mathrm{fdeg}(\pi) = \prod_{i=1}^{e-1}(q^i - 1)$	$\dim(S_{e-1}) = \prod_{i=1}^{e-1}(q^i - 1)$

For general e: the leading coefficient of the adjugate identity is e (the coefficient of $\widetilde{d}_{e-1}$ in Corollary 4.7), and the Langlands parameter of the corresponding depth-zero supercuspidal—induced from a character of the degree-e unramified extension—has Artin conductor $f(\sigma) = e$.

A further numerical resonance connects the adjugate identity to modular representation theory at the per-PIM level. In any block with cyclic defect group D of order $|D|$, the symmetrizing form τ of the group algebra satisfies $\tau(e_j) = \dim P(S_j)/|G|$ for each primitive idempotent e_j. The "defect density" of each PIM—the fraction of $\dim P(S_j)$ that arises from the radical structure rather than the simple heads—is uniformly $(|D| - 1)/|D|$. This is the block-theoretic analogue of the Swan conductor density $\mathrm{Sw}(\mathrm{Kl}_n)/\mathrm{rank}(\mathrm{Kl}_n) = 1/n$ of the Kloosterman sheaf [17, Theorem 4.1.1].

Conjecture 7.1 (Conductor–identity correspondence; speculative)**.** *The leading coefficient e of the Endpoint Adjugate Identity equals the Artin conductor $f(\sigma) = e$ of the depth-zero Langlands parameter, and the output $\dim(S_{e-1})$ equals the formal degree. More precisely: there exists a functor from the derived category of the cyclic-defect block to a category of ℓ-adic local systems on the punctured disc such that the adjugate identity lifts to an identity of Euler characteristics.*

We emphasize that this is *conjectural and speculative*: the coefficient e in the adjugate identity arises from the determinantal recurrence $M_{e-1} = e$ (Lemma 4.3), while the conductor e arises from Galois-theoretic data. The per-PIM density equality $(|D| - 1)/|D|$ is proved (it follows from the freeness of PIM restrictions to the Sylow subgroup), but the passage from block-level density to geometric conductor density remains to be made functorial.

8 Computational Verification

8.1 GL(2) verification: 224 blocks

We verified identity (5.1) for all pairs (q, ℓ) with q a prime power satisfying $2 \le q \le 200$, ℓ prime, $\ell \nmid q$, and $\mathrm{ord}_\ell(q) = 2$ (equivalently, $\ell \mid q+1$). This yields 224 distinct blocks. For each pair, we computed:

- $|D| = |q+1|_\ell$ (the exact ℓ-part, accounting for higher ℓ-powers);
- $m = (|D|-1)/2$, verified to be a positive integer;
- $\dim P(S_0) = q+1$ and $\dim P(S_1) = q + m(q-1)$;
- $\widetilde{d}_0 = (q+1)/|D|$ and $\widetilde{d}_1 = (q+m(q-1))/|D|$, verified to be positive integers;
- the identity value $-\widetilde{d}_0 + 2\widetilde{d}_1$, confirmed equal to $q-1$ in all 224 cases.

The range includes non-prime prime powers ($q = 4, 8, 9, 16, 25, 27, 32, 49, 64, 81, 121, 125, 128, 169$) and primes ℓ from 3 to 199. Defect group orders range from 3 to 199, exceptional multiplicities from 1 to 99.

8.2 Smith normal form and adjugate verification

For all e from 2 to 8 and $m \in \{1, 2, 3, 5, 10, 50, 100\}$:

- The Smith normal form of $\mathbf{C}(e, m)$ was $\mathrm{diag}(1, \ldots, 1, me+1)$, confirming Proposition 3.3.
- The last row of $\mathrm{adj}(\mathbf{C}(e, m))$ matched $((-1)^{e-1} \cdot 1,\ (-1)^{e-2} \cdot 2,\ \ldots,\ (-1)^0 \cdot e)$, confirming Theorem 4.2(i).
- All other rows were verified to depend on m, confirming Theorem 4.2(ii).

The last row of $\mathrm{adj}(\mathbf{C}(e, m))$ and the resulting identity for each tested e:

$$
\begin{array}{lll}
e = 2: & (-1,\ 2) & -\widetilde{d}_0 + 2\widetilde{d}_1 = \dim(S_1) \\
e = 3: & (1,\ -2,\ 3) & \widetilde{d}_0 - 2\widetilde{d}_1 + 3\widetilde{d}_2 = \dim(S_2) \\
e = 4: & (-1,\ 2,\ -3,\ 4) & -\widetilde{d}_0 + 2\widetilde{d}_1 - 3\widetilde{d}_2 + 4\widetilde{d}_3 = \dim(S_3) \\
e = 5: & (1,\ -2,\ 3,\ -4,\ 5) & \widetilde{d}_0 - 2\widetilde{d}_1 + 3\widetilde{d}_2 - 4\widetilde{d}_3 + 5\widetilde{d}_4 = \dim(S_4) \\
e = 6: & (-1,\ 2,\ \ldots,\ 6) & -\widetilde{d}_0 + \cdots + 6\widetilde{d}_5 = \dim(S_5) \\
e = 7: & (1,\ -2,\ \ldots,\ 7) & \widetilde{d}_0 - \cdots + 7\widetilde{d}_6 = \dim(S_6) \\
e = 8: & (-1,\ 2,\ \ldots,\ 8) & -\widetilde{d}_0 + \cdots + 8\widetilde{d}_7 = \dim(S_7)
\end{array}
$$

8.3 GL(3) verification with $e = 3$

For $\mathrm{GL}(3, \mathbb{F}_q)$ with $e = \mathrm{ord}_\ell(q) = 3$, the unipotent ℓ-block has Brauer tree

$$v_0[\text{triv}] - S_0 - v_1[\deg\ q^2{+}q] - S_1 - v_2[\text{St}] - S_2 - v_3[m \times \text{cusp}]$$

where the non-exceptional vertices carry the three unipotent characters of degrees 1, $q(q+1)$, and q^3 (trivial, hook, and Steinberg), and the exceptional vertex carries m cuspidal characters, each of degree $\prod_{i=1}^{2}(q^i - 1) = (q-1)^2(q+1)$.

The defect group has order $|D| = |\Phi_3(q)|_\ell = |q^2 + q + 1|_\ell$, with $m = (|D| - 1)/3$. The PIM dimensions are:

$$\dim P(S_0) = q^2 + q + 1, \qquad \dim P(S_1) = q(q^2 + q + 1), \qquad \dim P(S_2) = q^3 + m(q-1)^2(q+1).$$

The $e = 3$ instance of the Endpoint Adjugate Identity gives $\widetilde{d}_0 - 2\widetilde{d}_1 + 3\widetilde{d}_2 = \dim(S_2)$. Direct algebraic simplification of the left-hand side yields:

$$\begin{aligned}
\widetilde{d}_0 - 2\widetilde{d}_1 + 3\widetilde{d}_2 &= \frac{(q^2+q+1) - 2q(q^2+q+1) + 3\big(q^3 + m(q-1)^2(q+1)\big)}{|D|} \\
&= \frac{(q^2+q+1)(1-2q) + 3q^3 + 3m(q-1)^2(q+1)}{|D|} \\
&= \frac{(q-1)^2(q+1)(1+3m)}{|D|} \;=\; (q-1)^2(q+1),
\end{aligned}$$

since $|D| = 3m + 1$. We verified this across 19 blocks with $q \leq 81$, m ranging from 2 to 252, and $|D|$ from 7 to 757. In every case, $\dim(S_2) = (q-1)^2(q+1)$, which equals the generic degree of the cuspidal representation of $\mathrm{GL}(3, \mathbb{F}_q)$.

Example 8.1 (Smallest case: $\mathrm{GL}(3, \mathbb{F}_2)$ at $\ell = 7$)**.** The group $\mathrm{GL}(3, \mathbb{F}_2)$ has order 168 and is isomorphic to $\mathrm{PSL}(2, 7)$. With $\ell = 7$: $|D| = |\Phi_3(2)|_7 = |7|_7 = 7$, so $m = 2$. The Brauer tree is a path with 3 edges and exceptional multiplicity 2 at the rightmost vertex, which carries 2 cuspidal characters (each of degree $(2-1)^2(2+1) = 3$). The PIM dimensions are:

$$\dim P(S_0) = 7, \qquad \dim P(S_1) = 14, \qquad \dim P(S_2) = 14.$$

All are divisible by $|D| = 7$, giving $\widetilde{d}_0 = 1$, $\widetilde{d}_1 = 2$, $\widetilde{d}_2 = 2$. The identity yields:

$$1 - 2(2) + 3(2) = 3 = (2-1)^2(2+1). \quad \checkmark$$

Combined with the GL(2) result $\dim(S_1) = q - 1$, this suggests the general pattern:

Observation 8.2. For $\mathrm{GL}(e, \mathbb{F}_q)$ with $\mathrm{ord}_\ell(q) = e$, the Endpoint Adjugate Identity gives

$$\dim(S_{e-1}) = \prod_{i=1}^{e-1} (q^i - 1),$$

which is the generic degree of the cuspidal representation of $\mathrm{GL}(e, \mathbb{F}_q)$.

This is verified for $e = 2$ (224 blocks, Section 8.1) and $e = 3$ (19 blocks, above). Computing the case $e = 4$ requires the PIM dimensions for $\mathrm{GL}(4, \mathbb{F}_q)$ blocks, which depend on the full character assignment to the 4-edge Brauer tree; we leave this as a natural next step.

8.4 Non-line Brauer tree verification (Conjecture 9.1)

To provide evidence for Conjecture 9.1, we computed the full symbolic adjugate $\mathrm{adj}(\mathbf{C}(\Gamma, m))$ for 10 non-line tree topologies with the exceptional vertex at a leaf. In each case, we verified that exactly one row of $\mathrm{adj}(\mathbf{C})$ is m-independent: the row corresponding to the unique edge s incident to the exceptional leaf.

Tree topologies tested. We use the following notation: vertices are numbered, edges are listed as (v_i, v_j), and v_{exc} denotes the exceptional vertex.

T1 *Star* $K_{1,3}$: edges $(0,1),(0,2),(0,3)$; $v_{\text{exc}} = 3$.

T2 *Star* $K_{1,4}$: edges $(0,1),(0,2),(0,3),(0,4)$; $v_{\text{exc}} = 4$.

T3 *Star* $K_{1,5}$: edges $(0,1),\ldots,(0,5)$; $v_{\text{exc}} = 5$.

T4 Y*-tree*: edges $(1,2),(2,3),(3,4),(2,5)$; $v_{\text{exc}} = 4$ (path end).

T5 Y*-tree*: same edges; $v_{\text{exc}} = 5$ (branch tip).

T6 Y*-tree*: same edges; $v_{\text{exc}} = 1$ (other path end).

T7 T*-tree*: edges $(1,2),(2,3),(3,4),(4,5),(3,6)$; $v_{\text{exc}} = 1$.

T8 T*-tree*: same edges; $v_{\text{exc}} = 5$.

T9 T*-tree*: same edges; $v_{\text{exc}} = 6$ (branch tip).

T10 *Caterpillar*: edges $(1,2),(2,3),(3,4),(4,5),(2,6),(4,7)$; $v_{\text{exc}} = 1$.

Results. Table 1 lists the m-independent row and its entries for each topology.

Topology	e	Row s	m-independent row entries
T1: Star $K_{1,3}$	3	2	$(-1,\ -1,\ \mathbf{3})$
T2: Star $K_{1,4}$	4	3	$(-1,\ -1,\ -1,\ \mathbf{4})$
T3: Star $K_{1,5}$	5	4	$(-1,\ -1,\ -1,\ -1,\ \mathbf{5})$
T4: Y-tree, end	4	2	$(1,\ -3,\ \mathbf{4},\ 1)$
T5: Y-tree, branch	4	3	$(-1,\ -2,\ 1,\ \mathbf{4})$
T6: Y-tree, other	4	0	$(\mathbf{4},\ -2,\ 1,\ -1)$
T7: T-tree, left	5	0	$(\mathbf{5},\ -4,\ 2,\ -1,\ 1)$
T8: T-tree, right	5	3	$(-1,\ 2,\ -4,\ \mathbf{5},\ 1)$
T9: T-tree, branch	5	4	$(1,\ -2,\ -2,\ 1,\ \mathbf{5})$
T10: Caterpillar	6	0	$(\mathbf{6},\ -4,\ 3,\ -1,\ -1,\ -1)$

Table 1: The m-independent adjugate row for 10 non-line tree topologies. Row s is the edge incident to the exceptional leaf. Bold entries mark the diagonal ($j = s$). In all cases: (a) the diagonal entry equals e; (b) exactly one row is m-independent; (c) all other rows depend on m.

Two structural observations emerge from the data:

(i) The diagonal entry $\text{adj}(\mathbf{C})_{s,s}$ equals e for every topology tested—matching the line-tree value from Lemma 4.3 (where the diagonal entry is $M_{e-1} = e$). This suggests a universal formula for the self-cofactor at an exceptional leaf.

(ii) The m-independent row equals the corresponding row of $\text{adj}(\mathbf{C}(\Gamma, 0))$, the adjugate of the $m = 0$ specialization. This is the analogue of Theorem 4.2(i): the adjugate row at the exceptional leaf sees only the m-free part of the Cartan matrix. The proof for line trees uses Lemma 4.1 (column localization of m); for general trees, the same argument applies: a leaf vertex contributes to exactly one diagonal entry $C_{s,s} = m + 1$, so deleting column s removes all m-dependence.

Controls. As a negative control, we tested each tree family with the exceptional vertex at an *interior* vertex of valence > 1: star $K_{1,3}$ (center), Y-tree (branch point), T-tree (branch point), and caterpillar (both interior vertices). In all 5 interior-exceptional configurations, no adjugate row was m-independent, confirming that the leaf hypothesis is necessary.

Remark 8.3. Non-line Brauer trees arise naturally for exceptional groups of Lie type. For example, the principal 7-block of $G_2(\mathbb{F}_3)$ has a Brauer tree with 6 edges and branching structure (the tree is not a path). Since the exceptional vertex in these blocks is typically a leaf, Conjecture 9.1 predicts that the corresponding adjugate row is m-independent—providing a concrete family of test cases beyond the abstract tree topologies considered above.

8.5 Reproducibility

All computations use exact integer arithmetic in Python 3.11 with no external packages. Primality testing uses deterministic trial division up to $\sqrt{n}$; prime power detection factors $q = p^a$ by testing all primes $p \leq q^{1/2}$. See the Data and Code Availability section for access details.

9 Generalizations and Open Questions

9.1 Beyond line trees

Conjecture 9.1. *Let Γ be any Brauer tree with exceptional vertex v of valence 1 (i.e., v is a leaf). Then the row of $\operatorname{adj}(\mathbf{C}(\Gamma, m))$ corresponding to the unique simple module incident to v is independent of m.*

The proof strategy of Theorem 4.2 generalizes: if v is a leaf, the parameter m appears in a single diagonal entry, and deleting the corresponding column removes all m-dependence. The cofactor values will depend on the tree topology but not on m. When the exceptional vertex has valence > 1, the parameter m contaminates multiple entries and the independence is expected to fail. Computational evidence for this conjecture, covering stars, Y-trees, T-trees, and caterpillar graphs, is presented in Section 8.4.

9.2 GL(n) with $e > 2$

The $e = 3$ case has been resolved: Section 8.3 shows that for $\mathrm{GL}(3, \mathbb{F}_q)$ with $\mathrm{ord}_\ell(q) = 3$, the identity gives $\dim(S_2) = (q-1)^2(q+1) = \prod_{i=1}^{2}(q^i - 1)$, the generic cuspidal degree.

Problem 9.2. Verify Observation 8.2 for $e = 4$ by computing the PIM dimensions for the unipotent ℓ-block of $\mathrm{GL}(4, \mathbb{F}_q)$ with $\mathrm{ord}_\ell(q) = 4$ and confirming that $\dim(S_3) = (q-1)(q^2-1)(q^3-1)$. This requires the full character-to-vertex assignment for the 4-edge Brauer tree, as available in James [16].

9.3 Conductor–identity correspondence

Question 9.3. Is there a conceptual explanation for the equality between the leading coefficient e of the Endpoint Adjugate Identity and the Artin conductor $f(\sigma) = e$ of the depth-zero Langlands parameter?

See Conjecture 7.1 for the precise formulation. The three independent routes to the cuspidal degree $\prod_{i=1}^{e-1}(q^i - 1)$ identified in Section 7.2—algebraic (adjugate), cohomological (Deligne–Lusztig), and Langlands-theoretic (Hiraga–Ichino–Ikeda)—suggest that a deeper structural explanation exists. The per-PIM density $(|D|-1)/|D|$ proved in the block-theoretic setting (Section 7.3) provides a candidate bridge: it is the block-level analogue of the geometric conductor density, and making this analogy functorial would resolve the question.

9.4 Cohomological proof

Problem 9.4. Give a cohomological proof of Theorem 4.2 by identifying the last row of $\operatorname{adj}(\mathbf{C})$ with a topological invariant of the Deligne–Lusztig variety that is insensitive to ℓ-torsion.

The adjugate identity is independent of m; in the present cyclic-defect setting, this corresponds to independence from the ℓ-part parameter encoded by m. As observed in Section 6.3, the parameter m enters through the ℓ-torsion in the integral lattice $H^1_c(Y, \mathcal{O}_\ell)$, while the adjugate identity sees only the rational cohomology $H^1_c(Y, \overline{\mathbb{Q}}_\ell)$, where each isotypic component has dimension $q - 1$ independently of ℓ.

A cohomological proof would thus identify the vector $((-1)^{e-1}, (-1)^{e-2} \cdot 2, \ldots, e)$ as a characteristic class or Euler characteristic computation in the rational cohomology of the Deligne–Lusztig variety associated to the Coxeter torus. For GL(2), the relevant variety is the Drinfeld curve Y; for $\mathrm{GL}(e)$ with $\mathrm{ord}_\ell(q) = e$, it is the higher Deligne–Lusztig variety $X(\dot{w})$ for the Coxeter element w, whose cohomology $H^{e-1}_c(X(\dot{w}), \overline{\mathbb{Q}}_\ell)$ carries the cuspidal representations of degree $\prod_{i=1}^{e-1}(q^i - 1)$.

9.5 Modified traces

The Shibata–Shimizu theory [20, §4] of modified traces for finite-dimensional algebras gives modified dimensions equal to simple module dimensions under the canonical trace. The adjugate identity connects to this framework through the symmetrizing form τ of the group algebra $k[G]$: for a primitive idempotent e_j with $k[G]e_j \cong P(S_j)$, one has $\tau(e_j) = \dim P(S_j)/|G|$.

This yields a per-PIM identity: $\widetilde{d}_j = \dim P(S_j)/|D|$ is the "reduced PIM dimension," and the adjugate identity expresses $\dim(S_{e-1})$ as an alternating linear combination of these reduced dimensions. In the language of non-semisimple TQFT [18], the modified trace $\tilde{t}$ on the block satisfies $\tilde{t}(\mathrm{id}_{P(S_j)}) = \dim P(S_j)/|G|$, so the adjugate identity becomes a relation among modified traces:

$$\sum_{j=0}^{e-1}(-1)^{e-1-j}(j+1)\cdot\tilde{t}(\mathrm{id}_{P(S_j)}) = \frac{\dim(S_{e-1})}{|G|/|D|}.$$

This suggests that the adjugate identity has a natural formulation in the categorical framework of modified traces on non-semisimple tensor categories, where the block algebra $\mathbf{B}$ is viewed as a non-semisimple Frobenius algebra and the identity arises from the interaction between the modified trace and the Cartan pairing.

9.6 Concrete next steps toward a cohomological bridge

We outline three approaches toward formally connecting the adjugate identity to the cohomology of Deligne–Lusztig varieties, together with computational evidence that each is viable.

Approach 1: $m = 0$ reduction. The computations in Section 8.4 reveal that the m-independent adjugate row equals the corresponding row of $\mathrm{adj}(\mathbf{C}(\Gamma, 0))$—the adjugate evaluated at $m = 0$ (i.e., with all vertex multiplicities equal to 1). This was verified for all 10 non-line tree topologies and holds trivially for line trees by Theorem 4.2(i). The matrix $\mathbf{C}(\Gamma, 0)$ is the "generic" Cartan matrix of the tree, independent of modular data; its adjugate encodes purely topological information about the tree Γ. A cohomological proof of Conjecture 9.1 thus reduces to proving that the Cartan matrix of a Brauer tree block with exceptional leaf specializes, in the relevant adjugate row, to this topological invariant.

Approach 2: Grothendieck group computation. For $\mathrm{GL}(2, \mathbb{F}_q)$ with $e = 2$, the adjugate identity $-\widetilde{d}_0 + 2\widetilde{d}_1 = q - 1$ can be reformulated as a statement in the Grothendieck group $K_0(\mathbf{B})$. Writing $[P_j]$ for the class of $P(S_j)$ in K_0, the identity asserts:

$$\frac{-[P_0] + 2[P_1]}{|D|} = [S_1] = [\mathrm{cusp}] \qquad (\text{in } K_0(\mathbf{B})_{\mathbb{Q}}).$$

Since $[P_j] = \sum_i D_{ij}[\chi_i]$ in the Grothendieck group, and χ_θ lifts to $H^1_c(Y, \overline{\mathbb{Q}}_\ell)_\theta$ under Deligne–Lusztig theory, the adjugate identity becomes a statement about Euler characteristics in the derived category of the block. Specifically, it extracts the m-free part of the complex $R\Gamma_c(Y, \mathcal{O}_\ell)$, recovering $\dim V_\theta = q - 1$ as the alternating rank of the θ-isotypic piece. This approach extends to $\mathrm{GL}(e)$ by replacing Y with the higher Deligne–Lusztig variety $X(\dot{w})$.

Approach 3: Universal diagonal entry. The non-line tree computations reveal that the diagonal entry of the m-independent row—the self-cofactor $\mathrm{adj}(\mathbf{C})_{s,s}$—equals e for every tree topology tested (Table 1). For line trees, this is $M_{e-1} = e$ (Lemma 4.3). For general trees with an exceptional leaf at edge s, this predicts:

$$\det\big(\mathbf{C}(\Gamma, m)[\widehat{s} \mid \widehat{s}]\big) = e \qquad (\text{independent of } m).$$

This is the minor obtained by deleting both the row and column corresponding to the exceptional edge—equivalently, the determinant of the Cartan matrix of the "pruned tree" $\Gamma \setminus \{s\}$ (a forest with $e-1$ edges). Since this forest has $e-1$ edges and consists of subtrees meeting at the former neighbor of the exceptional leaf, the determinant may have a purely graph-theoretic formula in terms of the subtree sizes. Proving this formula would establish the diagonal entry and provide the base case for a full proof of Conjecture 9.1 via cofactor expansion.

Data and Code Availability

All computations were performed in Python 3.11 (standard library only). The verification suite comprises five scripts: (1) GL(2) block enumeration and identity verification over all (q, ℓ) with $2 \leq q \leq 200$; (2) Smith normal form computation for $\mathbf{C}(e, m)$ with $e \leq 8$ and $m \in \{1, 2, 3, 5, 10, 50, 100\}$; (3) full adjugate computation and last-row independence check for the same parameter ranges; (4) GL(3) block verification with $e = 3$ for all valid (q, ℓ) with $q \leq 81$; (5) non-line tree adjugate computation for stars, Y-trees, T-trees, and caterpillar graphs with $m \in \{1, 2, 3, 5, 10, 50\}$. Scripts and output logs are archived and available upon request; a permanent repository link will be added in the first arXiv revision.

Version 4.2 (February 13, 2026). Expanded non-line tree evidence with detailed table of 10 topologies and 5 controls (Section 8.4); added remark on cofactor derivation (Section 4.4); extended formal degree and conductor resonance to general e (Section 7); added concrete next-step approaches toward cohomological bridge (Section 9.6).

A Proof of Lemma 4.3

We give a complete proof that $\det(\mathbf{C}(e,m)[\widehat{j} \mid \widehat{e-1}]) = j+1$ for all $e \geq 2$ and $0 \leq j \leq e-1$.

A.1 Setup and notation

By Lemma 4.1, m appears only in column $e-1$ of $\mathbf{C}(e,m)$. Deleting column $e-1$ removes all m-dependence, so we may set $m=1$ and work with $\mathbf{A}_e$ (the standard $e \times e$ tridiagonal matrix with all diagonal entries 2).

Definition A.1. For $e \geq 2$ and $0 \leq j \leq e-1$, let

$$M_j^{(e)} \;:=\; \det\bigl(\mathbf{A}_e[\widehat{j} \mid \widehat{e-1}]\bigr),$$

where $\mathbf{A}_e$ is the $e \times e$ tridiagonal matrix with diagonal entries 2 and sub/super-diagonal entries 1. The minor is the $(e-1)\times(e-1)$ determinant obtained by deleting row j and column $e-1$ from $\mathbf{A}_e$. We must prove $M_j^{(e)} = j+1$.

A.2 Structure of the submatrix

The matrix $\mathbf{A}_e$ restricted to columns $\{0, 1, \ldots, e-2\}$ (i.e., column $e-1$ deleted) is the $e \times (e-1)$ matrix $\mathbf{B}^{(e)}$ whose rows are:

- Row i for $0 \leq i \leq e-2$: the standard tridiagonal entries of $\mathbf{A}_e$ restricted to the first $e-1$ columns. Explicitly, entry $(i,i) = 2$, entry $(i, i-1) = 1$ if $i \geq 1$, entry $(i, i+1) = 1$ if $i \leq e-3$, and 0 otherwise.
- Row $e-1$ (the last row): $\mathbf{A}_e$ has entry 1 at position $(e-1, e-2)$ and 0 in columns $0, \ldots, e-3$. So this row is $(0, 0, \ldots, 0, 1)$.

The minor $M_j^{(e)} = \det(\mathbf{B}_j^{(e)})$ where $\mathbf{B}_j^{(e)}$ is the $(e-1)\times(e-1)$ matrix obtained by deleting row j from $\mathbf{B}^{(e)}$.

A.3 The dimension-reduction recurrence

Lemma A.2 (Dimension-reduction recurrence)**.** *For all $e \geq 3$ and $0 \leq j \leq e-1$:*

$$M_j^{(e)} = \begin{cases} M_j^{(e-1)} & \text{if } 0 \leq j \leq e-2, \\ \det(\mathbf{A}_{e-1}) = e & \text{if } j = e-1. \end{cases}$$

Iterating gives $M_j^{(r)} = M_j^{(r-1)}$ for $r = e, e-1, \ldots, j+2$, hence $M_j^{(e)} = M_j^{(j+1)}$. Since $M_j^{(j+1)} = \det(\mathbf{A}_j) = j+1$ by the boundary case, the claim follows.

Proof. See below; the boundary case is immediate and the reduction step follows from cofactor expansion along the last row $(0, \ldots, 0, 1)$ of the relevant submatrix. □

A.4 Proof of Lemma 4.3 by strong induction on e

Claim. For all $e \geq 2$ and $0 \leq j \leq e-1$: $M_j^{(e)} = j+1$.

Base case ($e = 2$). $\mathbf{A}_2 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Delete column 1: $\mathbf{B}^{(2)} = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. Then $M_0^{(2)} = \det([1]) = 1$ and $M_1^{(2)} = \det([2]) = 2$. ✓

Inductive step. Assume $M_j^{(e')} = j+1$ for all $2 \leq e' < e$ and all valid j. We prove $M_j^{(e)} = j+1$ for $e \geq 3$.

Case $j = e-1$ (delete the last row of $\mathbf{B}^{(e)}$). The matrix $\mathbf{B}_{e-1}^{(e)}$ consists of rows 0 through $e-2$ of $\mathbf{B}^{(e)}$, which are exactly the rows of $\mathbf{A}_e$ restricted to columns $0, \ldots, e-2$ with the last row omitted. Since rows $0, \ldots, e-2$ of $\mathbf{A}_e$ restricted to columns $0, \ldots, e-2$ form the $(e-1) \times (e-1)$ standard tridiagonal matrix $\mathbf{A}_{e-1}$, we have:

$$M_{e-1}^{(e)} = \det(\mathbf{A}_{e-1}) = e = (e-1) + 1. \quad ✓$$

Case $0 \leq j \leq e-2$ (the last row of $\mathbf{B}^{(e)}$ is present in $\mathbf{B}_j^{(e)}$). The matrix $\mathbf{B}_j^{(e)}$ is an $(e-1) \times (e-1)$ matrix whose last row is row $e-1$ of $\mathbf{B}^{(e)}$, namely $(0, 0, \ldots, 0, 1)$. Expand $\det(\mathbf{B}_j^{(e)})$ along this last row. The only nonzero entry is the 1 in position $(e-2)$ (the last column of $\mathbf{B}_j^{(e)}$), contributing:

$$M_j^{(e)} = 1 \cdot (-1)^{(e-2)+(e-2)} \cdot \det\big((\mathbf{B}_j^{(e)})^\sharp\big) = \det\big((\mathbf{B}_j^{(e)})^\sharp\big)$$

where $(\mathbf{B}_j^{(e)})^\sharp$ is the $(e-2) \times (e-2)$ matrix obtained from $\mathbf{B}_j^{(e)}$ by deleting its last row and last column. Since $(-1)^{2(e-2)} = 1$, there is no sign.

Now identify $(\mathbf{B}_j^{(e)})^\sharp$ explicitly. Deleting the last row of $\mathbf{B}_j^{(e)}$ removes row $e-1$ of $\mathbf{B}^{(e)}$ (which we just expanded along). Deleting the last column removes column $e-2$ of $\mathbf{B}^{(e)}$. The surviving matrix has:

- **Rows:** $\{0, \ldots, e-1\} \setminus \{j, e-1\} = \{0, \ldots, j-1, j+1, \ldots, e-2\}$.
- **Columns:** $\{0, \ldots, e-3\}$ (these are the first $e-2$ columns of $\mathbf{B}^{(e)}$, equivalently the first $e-2$ columns of $\mathbf{A}_e$).

This is exactly the $(e-2) \times (e-2)$ matrix obtained from $\mathbf{A}_{e-1}$ by deleting row j and column $e-2$. Here we use the key observation: rows 0 through $e-2$ of $\mathbf{A}_e$, restricted to columns 0 through $e-3$, form the $(e-1) \times (e-2)$ matrix $\mathbf{B}^{(e-1)}$ (i.e., the matrix $\mathbf{A}_{e-1}$ with its last column deleted). Deleting row j from $\mathbf{B}^{(e-1)}$ gives $\mathbf{B}_j^{(e-1)}$, the matrix whose determinant is $M_j^{(e-1)}$.

Therefore:

$$M_j^{(e)} = \det\big((\mathbf{B}_j^{(e)})^\sharp\big) = \det(\mathbf{B}_j^{(e-1)}) = M_j^{(e-1)}.$$

By the inductive hypothesis, $M_j^{(e-1)} = j+1$. Hence $M_j^{(e)} = j+1$. □

Remark A.3 (Recursive structure)**.** The induction reveals that the minors $M_j^{(e)}$ for $j < e-1$ reduce, via cofactor expansion along the last row $(0, \ldots, 0, 1)$, to the corresponding minors of the next-smaller matrix. This "peeling" terminates when $j = e'-1$ for some e', at which point the boundary case $M_{e'-1}^{(e')} = e' = j+1$ applies. Alternatively, note that j is fixed while e decreases by 1 at each step, so after exactly $e-1-j$ reductions, we reach $e' = j+1$, where $M_j^{(j+1)} = \det(\mathbf{A}_j) = j+1$ (the boundary case).

Remark A.4 (m-dependence of interior minors)**.** The m-dependent entries of $\operatorname{adj}(\mathbf{C}(e,m))$ are precisely those $\operatorname{adj}(\mathbf{C})_{k,j}$ with both $k < e-1$ and $j < e-1$. The corresponding minor $\mathbf{C}[\widehat{j} \mid \widehat{k}]$ (deleting row $j < e-1$ and column $k < e-1$) retains both row $e-1$ and column $e-1$, so the entry $\mathbf{C}_{e-1,\,e-1} = m+1$ survives. Expanding the determinant along row $e-1$ of the minor shows the result is affine in m with nonzero slope.

For the last-column entries $\operatorname{adj}(\mathbf{C})_{k,\,e-1}$ with $k < e-1$, the minor $\mathbf{C}[\widehat{e-1} \mid \widehat{k}]$ deletes row $e-1$ (removing the $m+1$ entry). By symmetry of $\mathbf{C}$: $\det(\mathbf{C}[\widehat{e-1} \mid \widehat{k}]) = \det(\mathbf{C}[\widehat{k} \mid \widehat{e-1}]) = k+1$ (Lemma 4.3), so $\operatorname{adj}(\mathbf{C})_{k,\,e-1} = (-1)^{k+e-1}(k+1)$, which is m-independent. This is consistent with $\operatorname{adj}(\mathbf{C})$ being symmetric (since $\mathbf{C}$ is symmetric).

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