Equivalence of Thermodynamic Potential Equations and Special Relativity: A Gravitational-Thermodynamization Perspective

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Abstract

Thermodynamic potentials provide equivalent descriptions of equilibrium systems through Legendre transforms and associated Maxwell relations. Motivated by recent developments in the "thermodynamization" of gravitational dynamics and by the RVB–residue method viewpoint (attributed to Wen-Xiang Chen), we propose a unified framework comparing two notions of equivalence: (i) the algebraic equivalence classes of thermodynamic potential equations, and (ii) the covariance constraints of special relativity. We formulate a map between potential-based contact geometry identities in thermodynamics and invariant structures in relativity via a variational "residue" extraction on constrained manifolds. Conceptually, we show that the equivalence of different thermodynamic potentials can be viewed as a gauge-like redundancy, while Lorentz covariance in relativity acts as a compatibility condition selecting physically admissible gauges. The framework is illustrated on relativistic ideal fluids and a simple horizon thermodynamics toy model.

Keywords: thermodynamic potentials; Legendre transform; Maxwell relations; special relativity; gravitational thermodynamics; contact geometry; residue methods

1 Introduction

Thermodynamics admits multiple, mutually equivalent representations of the same equilibrium physics (internal energy, enthalpy, Helmholtz free energy, Gibbs free energy), which are interrelated by Legendre transforms [1]. Each choice of thermodynamic potential simply shifts which variables are considered "natural," yet all such potentials contain the same information about the system's state [1]. Special relativity, on the other hand, constrains physical laws to be invariant under Lorentz transformations – i.e. the laws must hold in any inertial frame. Different inertial frames are related by Lorentz transforms and represent the same physics, just in different coordinates. In this sense, one can draw an analogy between (a) the equivalence of thermodynamic potentials (same physics, different thermodynamic representation) and (b) the equivalence of inertial frames in relativity (same physics, different reference frames).

Parallel developments in theoretical physics have suggested that gravity and spacetime dynamics have deep thermodynamic interpretations. In particular, a line of research interprets aspects of gravity as emergent thermodynamics – sometimes called the thermodynamization of gravity – wherein Einstein's field equations or horizon dynamics can be rewritten as thermodynamic relations (e.g. laws of horizon mechanics) [6]. Seminal work by Unruh, Jacobson, Padmanabhan and others showed that one can associate thermodynamic properties with horizons or local Rindler frames [6]. For example, Jacobson (1995) demonstrated that the Einstein equation can be derived from the Clausius relation $\delta Q = T\,\delta S$ applied to local horizon "patches," effectively viewing the Einstein equation as a thermodynamic equation of state [6]. Padmanabhan (2010) further reviewed how gravitational dynamics exhibit thermodynamic behavior, reinforcing the idea that "gravity has thermodynamical aspects" on horizons (e.g. associating entropy and temperature to spacetime horizons) [7].

Finally, we incorporate the conceptual ingredients of the RVB–residue method (ascribed here to Wen-Xiang Chen) as a tool in our analogy. In Chen's approach to black hole thermodynamics, a certain contour integral residue corresponds to a correction term in Hawking temperature calculations [11, 12]. Interpreting this in a broader context, we treat "residues" as a way to extract invariant physical content from a description that might have redundant degrees of freedom. In our setting, these residues will correspond to quantities that remain invariant under changes of thermodynamic potential or changes of reference frame (for example, entropy, or scalar invariants formed from four-vectors like u^μ and $T^{\mu\nu}$).

In this paper, we outline a framework to compare thermodynamic-potential equivalences with relativistic frame equivalences. We show that changing thermodynamic potentials is analogous to switching inertial frames, and we identify the invariant content ("residues") that persists across these changes. This analogy is explored through the language of contact geometry (for thermodynamics) and Lorentz-covariant tensor formulations (for relativity). In summary, the equivalence of different thermodynamic potentials can be viewed as a kind of gauge redundancy in the description of a system, while the requirement of Lorentz covariance in relativity serves as a compatibility condition that restricts the allowed gauges to those that are physically admissible. We illustrate these ideas with examples, including a relativistic ideal fluid and a toy model of horizon thermodynamics.

2 Thermodynamic Potentials as Equivalent Equations

2.1 Legendre Structure

Consider an equilibrium thermodynamic system characterized by an internal energy function $U(S, V, N)$, where S is entropy, V is volume, and N is particle number (for simplicity, a single particle species). The first law of thermodynamics in differential form is:

$$dU = T\,dS - p\,dV + \mu\,dN, \tag{1}$$

where the intensive quantities are defined as usual by partial derivatives of U: the temperature $T = \partial U/\partial S$, pressure $p = -\,\partial U/\partial V$, and chemical potential $\mu = \partial U/\partial N$. From $U(S, V, N)$ one can perform Legendre transformations to define other potentials better suited for different constraints [3, 4]. The four most common thermodynamic potentials are:

- Internal Energy $U(S, V, N)$ (natural variables S, V, N)
- Helmholtz Free Energy $F(T, V, N) = U - TS$ (natural variables T, V, N)
- Enthalpy $H(S, p, N) = U + pV$ (natural variables S, p, N)
- Gibbs Free Energy $G(T, p, N) = U - TS + pV$ (natural variables T, p, N)

Each of these is related by a Legendre transform. For example, $F = U - TS$ trades the entropy variable for temperature, etc. The differentials of these potentials are:

$$dF = -S\,dT - p\,dV + \mu\,dN, \tag{2}$$

$$dH = T\,dS + V\,dp + \mu\,dN, \tag{3}$$

$$dG = -S\,dT + V\,dp + \mu\,dN, \tag{4}$$

each obtained by the corresponding Legendre transform of dU. All of these potentials are mathematically equivalent in the sense that each contains the full information of the equation of state of the system. In fact, since Legendre transforms are invertible mappings, any one potential can be obtained from another, so they "contain the same information" about the thermodynamic system [1]. The choice of which potential to use is merely one of convenience for the experimental or theoretical context [1] (for instance, using F if temperature T is controlled instead of entropy S).

2.2 Maxwell Relations as Integrability Constraints

The family of Maxwell relations emerges from the equality of mixed second derivatives (Clairaut's theorem) applied to state functions that are twice-differentiable. Because the differentials of the potentials (like dU, dF, etc.) are exact differentials, the order of partial differentiation is symmetric. This yields relations such as:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V, \tag{5}$$

and so on for each pair of conjugate variables. These are classic Maxwell relations, and they are essentially compatibility conditions ensuring that a single well-defined thermodynamic potential function exists. In general, "Maxwell's relations are a set of equations derivable from the symmetry of second derivatives and the definitions of the thermodynamic potentials." They codify the fact that, for example, in the energy representation $U(S, V, N)$, the mixed second partials $\partial^2 U/(\partial S\, \partial V)$ must equal $\partial^2 U/(\partial V\, \partial S)$. This symmetry directly leads to constraints like those above. Physically, the Maxwell relations allow one to relate measurable quantities to usually unmeasurable ones (e.g. relating entropy changes to pressure-volume fluctuations), ensuring the consistency of different thermodynamic descriptions.

In summary, the internal consistency of the thermodynamic potential formulations is guaranteed by these integrability conditions (Maxwell relations). They ensure that no matter which potential or variables we choose, we are describing a single underlying equation of state without contradiction.

2.3 Contact Geometry Viewpoint

Thermodynamics can be given a geometric formulation using contact geometry, which naturally encodes the Legendre transform structure. One considers an odd-dimensional contact manifold $(\mathcal{M}, \eta)$, where η is a one-form called the contact form. For the case of our simple system, we can choose coordinates (U, S, V, N, T, p, μ) on a 7-dimensional space (or reduce to 5-dimensional if N is fixed, etc.), and define a contact form:

$$\eta = dU - T\, dS + p\, dV - \mu\, dN. \tag{6}$$

The physical equilibrium states of the system lie on a Legendre submanifold of this contact manifold, defined by the condition $\eta = 0$ along that submanifold. Intuitively, $\eta = 0$ imposes the first law $dU = T\, dS - p\, dV + \mu\, dN$ and thus selects the physical relationships among the differentials.

A Legendre transform in this picture is realized as a contactomorphism – a smooth map on the contact manifold that preserves the contact form up to scaling. In practical terms, performing a Legendre transform (switching from U to F, for example) corresponds to a change of coordinates that keeps the form of η invariant in structure. In contact-geometric language, different thermodynamic potentials correspond to different choices of Legendre submanifold (different "charts" of the state space), but all such choices are related by contact transformations that preserve the essential thermodynamic content. As Mrugala (1978) and others have shown, "a contact transformation maps one Legendre submanifold of the contact space into another", representing different thermodynamic representations of the same physics [4]. Thus, the Legendre symmetry is a gauge-like freedom in the thermodynamic description: it allows us to switch potentials without changing the physical predictions. Invariants under these transformations (for example, combinations like $dU - T\, dS + p\, dV - \mu\, dN = 0$) represent the true physical relationships, independent of the choice of thermodynamic potential.

3 Special Relativity as an Equivalence Principle

3.1 Lorentz Covariance and Invariants

In special relativity, all physical laws must obey Lorentz covariance – meaning they have the same form in every inertial frame related by a Lorentz transformation. The Minkowski spacetime interval is the prototypical invariant:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{7}$$

which remains the same for all observers. Lorentz transformations preserve this interval. In an arbitrary inertial frame with coordinates $x^\mu = (ct, x, y, z)$ (using metric signature $- + + +$), a Lorentz transformation to a new frame x'^μ is a linear map that leaves s^2 unchanged. Frame equivalence thus means if one observer measures a time separation Δt and spatial separation Δx^i between two events, and another observer in relative motion measures $\Delta t'$ and $\Delta x'^i$, the combination $-c^2(\Delta t')^2 + |\Delta \mathbf{x}'|^2$ will equal the original s^2. In other words, the physics is encoded in Lorentz-invariant quantities.

For a concrete example, consider a relativistic fluid. The energy–momentum tensor of a perfect fluid (zero viscosity and heat conduction) is given by:

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu}, \tag{8}$$

where ε is the energy density in the rest frame of the fluid, p is the pressure, u^μ is the fluid four-velocity (with $u^\mu u_\mu = -1$), and $g^{\mu\nu}$ is the Minkowski metric tensor. This form of $T^{\mu\nu}$ is the same in any frame – it is a tensor equation that encapsulates energy density and pressure in a covariant way. In the fluid's rest frame, $T^{\mu\nu}$ becomes $\mathrm{diag}(\varepsilon, p, p, p)$ (for $\mu, \nu = 0, 1, 2, 3$), and the given formula ensures that this diagonal form holds in that frame while guaranteeing the correct Lorentz-transformed components in any other frame. Alongside $T^{\mu\nu}$, one can define a particle number current four-vector $N^\mu = n\, u^\mu$, where n is the number density in the fluid rest frame. These objects obey conservation laws $\partial_\mu T^{\mu\nu} = 0$ (energy-momentum conservation) and $\partial_\mu N^\mu = 0$ (particle number conservation) in the absence of external sources.

A crucial aspect of special relativity is identifying scalar invariants. For example, the rest mass m of a particle can be obtained from its four-momentum P^μ via $P_\mu P^\mu = -\, m^2 c^2$, which is Lorentz-invariant (all observers compute the same value for this combination). Similarly, in the fluid context, quantities like ε, p, and n are scalar fields (the numerical values differ by frame, but when properly related via equations of state they represent invariants in the rest frame). The equations of physics are often most elegantly expressed in a manifestly covariant (tensorial) form, which makes the invariances under frame changes transparent.

3.2 Thermodynamic Variables as Covariant Fields

Given the fluid example, we can introduce covariant thermal fields. Define the inverse temperature four-vector β^μ and a dimensionless chemical potential ratio α as:

$$\beta^\mu = \frac{u^\mu}{T}, \quad \alpha = \frac{\mu}{T}, \tag{9}$$

where T is the local temperature and μ the local chemical potential as measured in the fluid rest frame. The vector β^μ can be thought of as a four-velocity weighted by temperature (sometimes called the thermal Killing vector field in equilibrium configurations). It points in the direction of the fluid's four-velocity and has magnitude $1/T$. In global thermal equilibrium (with no acceleration or rotation of the fluid), β^μ will be a Killing vector field, and α will be constant throughout the system. These quantities enter covariant formulations of thermodynamics and statistical mechanics. For instance, the grand-canonical equilibrium distribution in relativity can be written using β^μ and α as $\exp[-\beta_\mu P^\mu + \alpha N]$ for a system with four-momentum P^μ and particle number N. Here $\beta_\mu P^\mu = (1/T)u_\mu P^\mu$ is essentially $E/T - \mathbf{u}\cdot\mathbf{P}/T$ which in the rest frame reduces to E/T. The introduction of β^μ ensures this expression is Lorentz-invariant; all observers will agree on the scalar value of $\beta_\mu P^\mu$ if the system is in equilibrium.

In summary, specifying thermodynamic state in a covariant way involves fields like $T(x)$, $u^\mu(x)$, $\mu(x)$ or equivalently $\beta^\mu(x)$, $\alpha(x)$. The conditions of equilibrium can be written as requirements on these fields (e.g. β_μ being a Killing field means $\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu = 0$ and so on, which encodes constant T and no flow of heat). The advantage is that these formulations make the Lorentz-frame equivalence explicit: changing frame corresponds to a Lorentz transform of β^μ and other vectors, but all physically meaningful statements are those that can be made with scalars like $\beta_\mu T^{\mu\nu}u_\nu$ or $\nabla_\mu S^\mu$, etc., which are invariant in value.

3.3 No Rest Frame for Light

In special relativity, photons do not admit a rest frame. Any attempt to perform a Lorentz transformation to a frame comoving with light leads to

$$\omega' \to 0, \qquad E' = \hbar\omega' \to 0, \tag{10}$$

where ω is the photon frequency. Thus, "light at rest" should be understood as a *limiting configuration* characterized by vanishing frequency, vanishing energy flow, and absence of causal exchange.

Temperature is not defined by motion itself, but by the availability of exchangeable microscopic energy. In statistical mechanics, a nonzero temperature implies a characteristic energy scale

$$\langle E\rangle \sim k_B T. \tag{11}$$

If a radiation field carries no exchangeable energy, no spectral structure, and no entropy flux, its effective temperature must vanish:

$$\langle E\rangle = 0 \quad \Rightarrow \quad T = 0. \tag{12}$$

The occupation number of photons in thermal equilibrium is given by

$$n(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1}. \tag{13}$$

If one takes the limit $\omega \to 0$ while keeping $T > 0$, then

$$n(\omega) \to \infty, \tag{14}$$

leading to an infrared divergence and infinite energy density. The only way to maintain a finite, well-defined thermodynamic state in the zero-frequency limit is

$$\omega \to 0 \quad \text{and} \quad T \to 0. \tag{15}$$

Hence, zero-frequency radiation necessarily corresponds to zero temperature.

Consider thermal radiation with temperature T measured by an observer. Under a Lorentz boost with velocity v along the propagation direction, the frequency transforms as

$$\omega' = \gamma(1 - v)\,\omega. \tag{16}$$

As $v \to c$, one finds $\omega' \to 0$. Since the temperature of radiation scales linearly with frequency under Lorentz transformations,

$$T' \propto \omega', \tag{17}$$

it follows that

$$T' \to 0. \tag{18}$$

According to the third law of thermodynamics, when the number of accessible microstates ceases to change, the entropy variation vanishes and the temperature approaches zero. In the zero-frequency limit of radiation:

- no causal energy exchange occurs,
- no spectral degrees of freedom remain,
- entropy flow is frozen.

Therefore, the effective thermodynamic description is

$$T = 0, \qquad \mathrm{d}S = 0. \tag{19}$$

The statement that "light at rest has zero temperature" should be understood as a thermodynamic limit rather than a kinematical one. When radiation loses its spectral content, energy flow, and entropy exchange, its only consistent thermodynamic assignment is absolute zero temperature. This interpretation is fully compatible with statistical mechanics, special relativity, and the third law of thermodynamics.

4 One-to-One Correspondence: Thermodynamic Potentials vs. Special Relativity

We now build a detailed analogy between the thermodynamic potential framework and the special relativistic framework. The correspondence operates at two levels: (1) the relation between different representations (potentials ↔ frames), and (2) the associated invariant content (quantities independent of representation choice).

4.1 Dictionary of State Spaces

Thermodynamics: The equilibrium state space of a simple thermodynamic system can be thought of as an abstract manifold parameterized by extensive variables (and their conjugates). For example, one can use (S, V, N) or alternatively (T, V, N), etc., depending on the chosen potential. There is a contact form η (as given above) and equilibrium states lie on a Legendre submanifold (satisfying $\eta = 0$). The state postulate ensures that giving two independent variables (in single-component systems) fixes the third via an equation of state.

Special Relativity: The "state space" for kinematics is Minkowski spacetime plus the velocity space. An event is given by coordinates x^μ, and a timelike four-velocity u^μ characterizes an observer or a comoving fluid element at that event. The natural invariant structure is the Minkowski metric $g_{\mu\nu}$ which yields invariant intervals $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The space of physically allowed four-velocities is constrained by $u^\mu u_\mu = -1$. In a dynamical context, one may extend this to include the space of stress–energy tensors or other fields, but the key is that any "state" description must be in terms of Lorentz tensors to be meaningful for all observers.

In short, thermodynamic state space (with contact structure) plays the role analogous to spacetime with Minkowski structure. In thermodynamics the analogous invariant "metric" structure is encoded in the fundamental differential form η, which remains invariant under Legendre changes (up to sign or conformal factors), just as ds^2 remains invariant under coordinate Lorentz transformations.

4.2 Potential Choice $\leftrightarrow$ Frame Choice

A central part of our analogy is that choosing a different thermodynamic potential is structurally similar to choosing a different inertial frame in relativity. Both are changes of representation that leave the underlying physics unchanged.

Thermodynamics: Changing the thermodynamic potential (say from $U(S,V,N)$ to $F(T,V,N)$) is achieved by a Legendre transform. This is essentially a change of variables (from S to T in this case) on the equilibrium manifold. The new potential has a different set of "natural" independent variables, but describes the same physical state of the system in a different way. For example, (U,S,V,N) and (F,T,V,N) contain equivalent information [1]; one can go back and forth between them as needed. This freedom reflects the fact that one can choose different thermodynamic ensembles or energy representations, but predictions (like values of measurable quantities) do not depend on which potential was used in intermediate calculations.

Special Relativity: Changing the inertial frame (from coordinates x^μ to x'^μ moving at some relative velocity) is a Lorentz transformation. This changes the coordinate description of physical events and the components of vectors/tensors, but it's just a different point of view of the same reality. For example, an electric field and magnetic field measured in one frame may transform into a combination of fields in another frame, but the underlying electromagnetic field (the Faraday tensor) is the same entity. Observers in frames A and B will agree on invariant facts (like $E^2 - B^2$ for electromagnetic fields, or the lifetime of a particle, etc.), even though their coordinate descriptions differ.

To put it succinctly:

Mapping A (representation redundancy):

- Thermodynamics: (U,S,V,N) vs. (F,T,V,N) vs. (H,S,p,N) vs. (G,T,p,N) are all different representations of the same thermodynamic state, related by Legendre transforms. This is a redundancy in description – one can convert between these representations freely with no loss of information [1].
- Relativity: $(t,\mathbf{x})$ vs. $(t',\mathbf{x}')$ are different inertial frame coordinates for the same physical scenario, related by Lorentz transforms. This is again a redundancy – physics doesn't depend on the absolute choice of inertial frame (this is the principle of relativity).

Each Legendre transform in thermodynamics plays a role analogous to a Lorentz boost or rotation in spacetime. Indeed, just as one can derive one representation's formulae from another via a well-defined transformation, one can obtain one frame's quantities by Lorentz-transforming those of another frame.

We can also see a parallel in how these transformations "shuffle" components:

Mapping B (generating functions and component mixing):

- In thermodynamics, performing a Legendre transform swaps an independent variable with its conjugate. For example, the transformation to Helmholtz free energy swaps S (independent in U) with T (its conjugate) as the new independent variable. In the differential $dU = T\,dS - p\,dV + \mu\,dN$, one can see T paired with S. Switching to dF, one gets $-S\,dT$ instead: the roles of S and T have exchanged (one is now coefficient, the other differential). Similarly, swapping V with $-p$ leads to enthalpy H, etc. These changes are like using different generating functions for the contact structure, but they preserve the overall content (the "contactomorphism" property).
- In special relativity, a Lorentz transformation mixes time and space components. For example, a boost in the x-direction of velocity v will transform the time coordinate and the x-coordinate as $t' = \gamma(t - vx/c^2)$, $x' = \gamma(x - vt)$, while y and z remain the same. This is a linear recombination of components. The effect is analogous to how a Legendre transform mixes variables: some quantities that were "time-like" might become "space-like" components in a new basis (e.g.,

energy and momentum mix under boosts, just as entropy and temperature mix under a Legendre transform).

In both cases, the transformation has an associated generator (in thermodynamics, the generating function for the Legendre transform; in relativity, the generator of the Lorentz group like a boost or rotation operator). But importantly, these transformations do not change the underlying invariants (contact structure η for thermodynamics, metric $g_{\mu\nu}$ for relativity).

4.3 Conjugate Pairs ↔ Four-Vector Components

Thermodynamics is built on pairs of *conjugate variables*: for each extensive quantity there exists an intensive conjugate. Typical examples include (S, T), $(V, -p)$, and (N, μ). The product of each conjugate pair carries the dimension of energy: $T\,\mathrm{d}S$, $p\,\mathrm{d}V$, and $\mu\,\mathrm{d}N$ all have units of energy. Accordingly, in the fundamental thermodynamic relation

$$\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V + \mu\,\mathrm{d}N, \tag{20}$$

each term appears explicitly as a product of a conjugate pair.

In special relativity (and in field theories more generally), natural pairings arise through Lorentz-covariant contractions. A four-vector A^μ and a covector B_μ produce a Lorentz scalar upon contraction,

$$A_\mu B^\mu. \tag{21}$$

For example, the contraction $u_\mu T^{\mu\nu}$ provides a covariant description of the energy–momentum density measured in a frame moving with four-velocity u^μ. Similarly, the scalar $p_\mu u^\mu$ represents the relativistic generalization of "energy = momentum dotted with velocity."

The analogy with thermodynamics can be pushed further. Consider the Gibbs free energy,

$$G = U - TS + pV, \tag{22}$$

which enters the grand-canonical ensemble through the Boltzmann factor

$$\exp\left[-\frac{U - TS + pV}{T}\right] = \exp\left(-\frac{U}{T} + S - \frac{pV}{T}\right). \tag{23}$$

On the other hand, a covariant equilibrium distribution in relativistic statistical mechanics can be written as

$$\exp(-\beta_\mu P^\mu + \alpha N) = \exp\left(-\frac{u_\mu P^\mu}{T} + \frac{\mu N}{T}\right), \tag{24}$$

where $\beta_\mu \equiv u_\mu/T$ and $\alpha \equiv \mu/T$.

In the rest frame of u^μ (with $u^\mu = (1, 0, 0, 0)$ in units $c = 1$), the contraction reduces to

$$u_\mu P^\mu = E, \tag{25}$$

so that $-u_\mu P^\mu/T = -E/T$, directly paralleling the thermodynamic factor $-U/T$. The term $\alpha N = \mu N/T$ plays the role of the chemical contribution. The entropy contribution arises from the multiplicity of microstates,

$$S = \ln \Omega, \tag{26}$$

which explains the appearance of $\exp(S)$ in the statistical weight.

The central observation is therefore the following: *thermodynamic conjugate products* (intensive × extensive) correspond structurally to *Lorentz-invariant scalar contractions* of vectors. In this sense, the Gibbs entropy and free energies can be viewed as covariant pairings between β_μ and the total four-momentum P^μ, together with the scalar pairing between α and the particle number N.

In more intuitive terms, the structure

$$\text{intensive} \times \text{extensive}$$

in thermodynamics plays a role analogous to the

$$\text{time component} \times \text{time basis} \ + \ \text{space component} \times \text{space basis}$$

structure underlying relativistic invariants. Changing thermodynamic potentials reshuffles intensive–extensive pairings, just as changing reference frames reshuffles time and space components. In both cases, physical observables emerge only through invariant combinations that treat each conjugate pair on an equal footing.

4.4 Maxwell Relations ↔ Integrability/Compatibility in SR

Earlier we identified Maxwell relations as arising from the equality of mixed partial derivatives, ensuring integrability of the thermodynamic potentials. We can draw a loose analogy to compatibility conditions in relativistic field theory. For example, consider electromagnetic fields: the existence of a four-potential A_μ such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ requires that the field tensor $F_{\mu\nu}$ satisfy the identity $\partial_{[\lambda} F_{\mu\nu]} = 0$ (the square brackets denote antisymmetrization). This is a type of integrability condition (specifically, it's the Bianchi identity for the electromagnetic field, ensuring that F comes from a potential A). In general relativity, the Bianchi identities $\nabla_{[\alpha} R_{\mu\nu]\rho\sigma} = 0$ ensure the consistency of Einstein's field equations with energy-momentum conservation. In fluid dynamics, the condition of irrotational flow can be seen as an integrability condition on the velocity potential, etc.

The analogy can be framed as:

- Thermodynamics: Mixed second derivatives commute $\Longrightarrow$ state functions exist (integrability) $\Longrightarrow$ Maxwell relations. If a would-be "thermodynamic" relation violated a Maxwell relation, it would indicate no true thermodynamic potential underlies those variables (i.e. an inconsistency).
- Relativity: Demanding a global tensor potential (or certain symmetry) $\Longrightarrow$ integrability conditions. For instance, demanding the existence of a scalar potential for a conservative force field means the curl must vanish; demanding the existence of a coordinate system covering a region smoothly implies certain curvature integrability conditions, etc. In special relativity specifically, covariance imposes constraints: e.g., the structure of Maxwell's equations with magnetic monopoles absent gives $\nabla \cdot \mathbf{B} = 0$ and $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$, which are analogous to integrability (no magnetic charge, Faraday's law) – these are essentially the Bianchi identity for the electromagnetic field.

While these two domains are quite different in content, the philosophical similarity is that both Maxwell relations and relativistic integrability conditions act as consistency checks. They ensure that a given "change of representation" (be it potential or frame) does not lead to contradictions. In relativity, if one tried to impose equations that were not Lorentz-covariant, different frames would see different physics – an unacceptable situation. In thermodynamics, if one wrote down an equation of state that violated Maxwell relations, it would mean one's partial derivatives are inconsistent – an unphysical situation. Both sets of conditions enforce that different perspectives glue together into one coherent description.

4.5 Invariants: Legendre Invariants ↔ Lorentz Scalars

Ultimately, the true physical content in each theory lies in quantities that are invariant under the respective transformations:

- Thermodynamics: Invariants under Legendre transforms are those combinations that do not depend on which potential is used. A simple example is the equation of state itself (e.g. a relation $f(p, V, T) = 0$). Whether one works in $U(S, V)$ or $F(T, V)$, the equation of state $p(V, T)$ is the same physical constraint. Another example: the Maxwell relations themselves are invariant statements – they hold regardless of representation. Geometrically, one might say that a contact invariant is an object or value that remains the same for all contactomorphisms (Legendre transforms). These could include, for instance, the values of measurable quantities like compressibility, heat capacity (which are derivatives that can be expressed in any representation if done carefully), or more abstractly the Pfaffian form class of η.

- Special Relativity: Invariants under Lorentz transformations are Lorentz scalars. Classic examples are the rest mass m of a particle, the spacetime interval s^2 between events, the magnitude of the electric charge, the norms and scalar products of four-vectors like $u_\mu u^\mu = -1$, or $E^2 - p^2c^2$ for a particle (which equals m^2c^4). In the fluid context, ε, p, and n are scalar fields (though their measured values change with frame, one can think of them as evaluated in the local rest frame of the fluid). Another important invariant is entropy: all observers compute the same total entropy for an isolated system. The second law of thermodynamics (entropy non-decrease) is a Lorentz-invariant statement in the sense that if $dS/d\tau \geq 0$ in one frame, it holds in all frames (entropy is a scalar – you can't have one observer measure a different total entropy change than another; they may disagree on rates per time, but not on the total change between two equilibrium states).

In our analogy, Legendre invariants correspond to Lorentz scalars. For example, the condition for thermodynamic stability can be phrased in terms of Hessians (second derivatives of energy) being positive/negative definite – this is invariant under Legendre transforms (it basically moves to a corresponding condition in the new variables). Likewise, causality or energy conditions in relativity (e.g., energy density positive, u^μ being timelike) are conditions on Lorentz scalars or invariants (e.g., T^{00} in any rest frame > 0 etc.).

To give a concrete pairing: the statement $u_\mu u^\mu = -1$ is Lorentz-invariant (it's the normalization of the four-velocity), which corresponds to the statement that thermodynamic differentials define a surface (Legendre submanifold) satisfying $\eta = 0$ – an invariant condition describing the equilibrium state. Both are single constraints on the system that do not depend on representation/frame.

4.6 Mass–Energy Equivalence $E = mc^2 \leftrightarrow$ Thermodynamic Potential Equivalences

The famous mass–energy equivalence from relativity states that an object's rest energy E_0 is mc^2. This is often written as $E = mc^2$ when referring to the rest frame energy. How can we draw an analogy of this with thermodynamic potentials? We propose the following interpretative analogy:

- In relativity, $E = mc^2$ highlights that mass can be viewed as "energy content" and that energy measured in one frame can partly be kinetic when viewed from another frame. The total energy in a moving frame is $E = \gamma mc^2$, which can be split into rest energy mc^2 and kinetic energy K. All observers agree on the value of m (an invariant), but not on the split between rest and kinetic energy – that split is frame-dependent.

- In thermodynamics, switching from one potential to another changes how the "energy" is split among terms. For instance, $U = F + TS$ splits internal energy into "free energy" F and the TS term which one might call "bound energy" (energy stored as heat at temperature T that is not available to do work). This decomposition is analogous to splitting E into rest energy and kinetic or other forms; it depends on the representation (the ensemble) rather than an invariant difference.

We can line up specific formulas:

1. Existence of a preferred invariant: In relativity, the invariant mass m is encoded via $P_\mu P^\mu = -m^2c^2$. In thermodynamics, there isn't a single scalar number that's invariant under Legendre transform (since energy itself changes when going to F, etc.), but one can say the "equation of state" or the fundamental relation is invariant. If one were to write something like $dU - TdS + pdV - \mu dN = 0$, that is an invariant one-form equation characterizing the system. We might liken mc^2 (rest energy) to the internal energy in one special representation (perhaps U itself in the energy representation).

2. Convertibility and bookkeeping: $E = E_0 + K$ (total energy = rest energy + kinetic energy) is analogous to how when we change from U to F, we subtract off TS. In one representation, part of the energy is explicit; in another, it's accounted for differently. The term TS is somewhat like "internal heat content" which is not free to do work, whereas F is the part available for work at constant T (hence "free energy"). Similarly, mc^2 is an internal energy of matter that isn't apparent as dynamical energy unless converted, whereas kinetic energy is the additional "free" energy due to motion. In this sense, TS in thermodynamics plays the role of "locked-up energy" (needed to maintain temperature T), just as mc^2 is locked-up energy of mass. Under change of representation/frame, these terms move from one side to another. For instance:

3. Relativity: $E = mc^2 + K$ (with $K = \frac{1}{2}mv^2 + \cdots$ for low v). If you go to the rest frame, $K = 0$ and $E = E_0$. In a moving frame, some of the energy is kinetic.

4. Thermodynamics: $U = F + TS$. In the entropy representation, U is all energy. In the Helmholtz representation, F is the part that can be used to do work, while TS is energy in the form of heat. If you go to a different ensemble (representation), the value of the potential changes by subtracting or adding terms like TS or pV.

5. Invariants and fundamental equations: The energy–momentum relation $E^2 - (pc)^2 = (mc^2)^2$ is an invariant equation relating energy, momentum, and mass. It can be rewritten as $P_\mu P^\mu + m^2c^2 = 0$. In thermodynamics, an analogous "invariant" equation is encapsulated by the contact form: $dU - TdS + pdV - \mu dN = 0$. This is independent of representation; if we Legendre-transform to dF, the form becomes $dF + SdT + pdV - \mu dN = 0$, which is essentially the same condition expressed in new variables. So one could write a symbolic invariant as $\eta \equiv dU - TdS + pdV - \mu dN \approx 0$. The "$\approx 0$" indicates it vanishes on the equilibrium (Legendre sub)manifold. This is akin to saying $P_\mu P^\mu + m^2c^2 = 0$ (which vanishes for the on-shell physical momentum).

To make the correspondence more explicit, we can tabulate some matching equations:

Relativity (Lorentz invariants)	Thermodynamics (Legendre invariants)
Rest energy: $E_0 = mc^2$	Internal energy & free energy: $U = F + TS$
Kinetic energy addition: $E = mc^2 + K$	Potential transform: $U = G + TS - pV$, etc.
Energy–momentum invariant: $E^2 - p^2c^2 = m^2c^4$	Contact invariant: $dU - TdS + pdV - \mu dN = 0$ (equilibrium condition)

Table 1: Correspondence between relativity and thermodynamics invariants.

The first line pairs mass-energy equivalence with the expression of one potential in terms of another plus conjugate terms. The second line notes how adding kinetic energy is like adding or removing conjugate terms in potentials. The third line pairs fundamental invariant equations (conservation or defining relations).

In words: Mass–energy equivalence ensured that what was previously thought to be separate (mass vs energy) are in fact convertible bookkeeping of one underlying quantity. Likewise, the equivalence of potentials says what might seem like different energies (enthalpy H, free energy F, etc.) are largely bookkeeping variations – they are inter-convertible and not fundamentally different in content.

4.7 A Compact Correspondence Table

To summarize the analogies, we provide a correspondence between key concepts in the thermodynamic potential framework and in special relativity:

This table highlights the conceptual equivalence: a change of thermodynamic potential is like a change of reference frame, and the underlying physics must be described in ways that do not depend on either choice (hence the importance of invariants in both cases).

5 Gravitational Thermodynamization Bridge

With the analogies in place, we now discuss how gravity – specifically general relativity – can serve as a bridge between thermodynamic and relativistic concepts. The idea is that in certain regimes, Einstein's equations or other gravitational laws mimic thermodynamic relations, allowing us to apply the above correspondence in a new context.

5.1 Local Horizon Thermodynamics (Toy Statement)

A key insight from Jacobson's 1995 work [6] was that Einstein's field equation $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$ can be derived by demanding that the Clausius relation $\delta Q = T\delta S$ holds for all local Rindler horizon patches, with δQ identified as the energy flux through the horizon, T as the Unruh temperature seen by an accelerated observer, and S proportional to horizon area (with k_B , $\hbar$, G factors etc. set to 1 in suitable units) [6]. In simpler terms, Einstein's equation emerges as an equation of state when one assigns thermodynamic properties (temperature, entropy) to spacetime horizons. This is often encapsulated by the slogan "thermodynamics of spacetime" [6].

Many works have since elaborated on this idea. For instance, Padmanabhan (2010) emphasizes that "one can associate thermodynamic properties with horizons", and that gravitational dynamics can be interpreted in terms of thermodynamic potentials and relations [7]. The first law of black hole mechanics (which in physical units reads $dM = \frac{\kappa}{8\pi G}dA + \Omega\, dJ + \Phi\, dQ$ for a stationary black hole, where κ is surface gravity, A horizon area, Ω angular velocity, J angular momentum, Φ horizon electrostatic potential, Q charge) is a direct analog of the first law of thermodynamics $dU = TdS +$ work terms. In fact, with $T_{\mathrm{H}} = \kappa/2\pi$ and $S = A/4G$, the black hole first law becomes $d(Mc^2) = T_{\mathrm{H}}\, dS + \Omega\, dJ + \Phi\, dQ$, which is formally a thermodynamic identity.

For our purposes, we take as a toy model the idea that gravitation can be "thermodynamized", meaning there exist representations of gravitational field equations that look like ordinary thermodynamic equations. For example, one might write something akin to $\delta Q = T\delta S +$ (work terms) for a patch of horizon, or rewrite the Friedmann equations for an expanding universe in the form of a first law of thermodynamics for the Hubble horizon, etc. These are not fundamental laws but re-interpretations, yet they suggest a deep analogy between the invariances in general relativity and the invariances in thermodynamics.

5.2 Potential Equivalence as Redundancy in Gravitational Thermodynamics

If gravity has a thermodynamic interpretation, one can ask: does changing the form of the gravitational "thermodynamic potential" correspond to anything meaningful? In other words, are there different but equivalent ways to express the gravitational field equations, just like U, F, H, G are different but equivalent potentials?

Indeed, in the gravity context, one can sometimes reformulate the field equations in alternate ways that resemble the first law with different identifications. For example, one can define an "enthalpy" related to black holes (some authors refer to the mass of AdS black holes as enthalpy, since $dM = TdS + VdP$ when considering a cosmological constant term) – this is part of extended black hole thermodynamics where Λ is treated as pressure and V as volume of the black hole. Similarly, one could choose to work with an energy defined at infinity (ADM mass) versus a quasilocal energy at the horizon (Komar energy), etc. These are analogous to choosing different zero-point energies or different energy representations.

If these various representations are truly equivalent (connected by Legendre-like transforms or other mathematical mappings), then we have a redundancy akin to thermodynamic potentials. The covariance (diffeomorphism invariance) of general relativity plays a role analogous to Lorentz invariance in special relativity, ensuring that any such reformulation must respect the underlying geometric invariants (like horizon area, surface gravity, etc. which are invariantly defined). Only certain transformations of the gravitational "first law" will be allowed – those that do not violate the general covariance or physical invariants.

For example, one representation might treat the area A as a fundamental variable (as in Bekenstein–Hawking entropy S), while another might treat the energy as fundamental (like the Misner–Sharp energy in cosmology). If they are Legendre pairs in a gravitational thermodynamic sense, then moving from one to the other is like shifting frames of description. The question becomes: what are the invariant quantities under these shifts? The expectation is that things like the horizon area (or entropy), surface gravity (or temperature), and conserved charges (mass, angular momentum, charge) are invariant content – just expressed differently in each "potential." Meanwhile, intermediate constructs (like specific heat at constant pressure vs constant volume, in analogy, or different definitions of energy) might change.

In our analogy, we consider that thermodynamic potential choice (in gravity) $\leftrightarrow$ gauge choice or coordinate choice (in spacetime). The physical laws must ultimately not depend on these choices, which leads us to propose a unifying viewpoint using the residue method.

6 RVB–Residue Method Viewpoint and a Unifying Formalism

6.1 Residue Extraction of Invariant Content

The RVB method introduced by Wen-Xiang Chen and collaborators in the context of black hole temperature calculations provides a technique to isolate certain contributions via contour integrals [12]. Specifically, Chen's work showed that an extra term in the Hawking temperature for some modified gravity black holes can be attributed to a residue of a pole in the complex plane associated with the metric function [12]. This residue was invariant in the sense that it did not depend on the arbitrary choice of coordinate parametrization of the solution (it was a physical effect).

We adopt a generalized notion of "residue" as an operation $\mathcal{R}$ that picks out the invariant core of a formulation. Imagine we have a family of descriptions of a system, denoted generally by Φ, which could be a thermodynamic potential or some gravitational functional (like an action or energy expression). These descriptions might differ by terms that are analogs of gauge terms or Legendre transform terms. We define $\mathcal{R}[\Phi]$ to be an operation that filters out the representation-dependent parts and retains only the common invariant part.

In complex analysis, the residue at a pole is a contour integral that yields a unique coefficient independent of path deformation – an invariant of sorts. By analogy, $\mathcal{R}[\Phi]$ should be constructed such that if Φ_1 and Φ_2 are two representations related by a legitimate transformation (contactomorphism or Lorentz transform or diffeomorphism), then $\mathcal{R}[\Phi_1] = \mathcal{R}[\Phi_2]$. If, however, one representation included a physically distinct term (not related by the symmetry transformation), the residue extraction would reveal a difference.

One way to formalize this is using differential forms and integrals: consider an extended phase space (for thermodynamics) or spacetime-manifold bundle (for relativity) where a closed form ω can be defined such that integrating ω over a certain cycle Γ yields an invariant. For example, ω might be related to η in contact space, and Γ could be a loop that enforces a constraint (like returning to the same thermodynamic state). Then $\oint_\Gamma \omega$ could be zero for all equivalent potentials (meaning no net change in a state function around a cycle), but if a formulation is inconsistent, this integral might not vanish.

For a more concrete example: in thermodynamics, one could take $\omega = \nabla T \wedge \nabla V \wedge \nabla p$ in an extended space and integrate over a suitable surface to test Maxwell relations. If Maxwell relations hold (representation is consistent), the integral is zero; if not, a non-zero "residue" shows an inconsistency.

In the gravitational context, one might integrate the difference between two sides of an assumed first-law identity over a closed surface. For instance, if one writes $\delta Q - T\delta S - \delta W = 0$ for work term δW, integrating this around a cycle might expose if an extra term (residue) appears.

While the details are beyond our scope here, the idea is that residue = invariant content and the RVB method's success in black hole thermodynamics suggests using it as an analogy for identifying invariant quantities in our thermodynamics–relativity correspondence. Chen's residue effectively extracted a part of Hawking temperature that was overlooked by naive calculation but was essential for consistency [12]. In our analogy, a "residue" could be something like entropy production or a particular invariant scalar that all frames/potentials agree on.

6.2 Equivalence Theorem (Schematic)

Using the above idea, we can state a schematic theorem: Theorem (Schematic Equivalence): Suppose we have: (i) a contact manifold structure encoding a set of thermodynamic identities, and (ii) a Lorentz-covariant formulation of the corresponding dynamical system (or emergent gravity system). Let Φ_λ be a family of representations of the system's equations, where each Φ_λ could be a thermodynamic potential (for different choices of thermodynamic representation) or a gravitational identity derived under different gauge or horizon setups. If all Φ_λ in the family are related by transformations that preserve the fundamental invariant structures (contactomorphisms in the thermodynamic picture and Lorentz/diffeomorphism transformations in the relativistic picture), then for a suitably defined residue operation $\mathcal{R}$, we have:

$$\mathcal{R}[\Phi_\lambda] = \mathcal{R}[\Phi_0], \tag{27}$$

for any reference Φ_0 and all λ. Conversely, if for some $\Phi_{\lambda'}$ we find $\mathcal{R}[\Phi_{\lambda'}] \neq \mathcal{R}[\Phi_0]$, then the representation $\Phi_{\lambda'}$ is not physically equivalent to Φ_0 – it has introduced a genuine physical difference rather than a mere gauge/Legendre transform difference.

Proof Sketch: The core idea is that the residue $\mathcal{R}$ is chosen to capture exactly the combination of terms that remain invariant under the allowed transformations. For instance, $\mathcal{R}$ could be an integral over a closed path in the space of states that yields zero if η is exact (which it is for all true Legendre transforms). In relativity, $\mathcal{R}$ might involve integrating a quantity like $(\delta Q - T\delta S)$ over a closed null surface, which should yield zero if Einstein's equation holds as an equation of state. Provided the transformations between Φ_λ are symmetry operations (i.e., they do not change the physical content), the value of the integral (or other residue measure) does not change. If a transformation is not allowed (not a true symmetry of the system's physical content), it will alter the result of $\mathcal{R}$, signaling inequivalence.

This theorem is schematic because one needs to define precisely what $\mathcal{R}$, ω, Γ, etc., are in each concrete situation. In Chen's case [12], $\mathcal{R}$ was literally the evaluation of $2\pi i$ times a contour integral encircling a pole in the complex plane of the radial coordinate (related to the black hole horizon). In a general thermodynamic setting, one might define $\mathcal{R}$ by taking the difference between two potentials evaluated along a reversible cycle. The main point is, if something is a true redundancy (gauge/representation choice), it will have no observable residue.

This approach gives a method to test if a proposed "thermodynamization" of a physical law is legitimate: if two forms of the law yield different $\mathcal{R}$, they are not just different representations – they predict different physical outcomes, hence one or both could be invalid or they describe different regimes.

7 Worked Example: Relativistic Ideal Fluid

To illustrate the ideas in a tangible scenario, consider a simple relativistic ideal fluid. Its local thermodynamic state is characterized by energy density ε, pressure p, entropy density s, particle number density n, and temperature T, chemical potential μ (all defined in the fluid's local rest frame). The Gibbs relation (first law for a local fluid element) is:

$$d\varepsilon = Tds - pd(1/n) + \mu dn, \tag{28}$$

for a fluid element (here $V = 1$ for unit volume). This can be combined with the Euler relation for extensive quantities (assuming ε, s, n are densities of extensive quantities) to also give $\varepsilon + p = Ts + \mu n$ (this is the thermodynamic identity for homogeneous functions of first order). Equivalently, $p = -\varepsilon + Ts + \mu n$.

Now, the fluid's dynamics are given by local conservation laws: $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu N^\mu = 0$, where (as before) $T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}$ and $N^\mu = nu^\mu$. In addition, if the fluid is non-dissipative and in equilibrium, one has an entropy current $S^\mu = su^\mu$ which is also conserved: $\nabla_\mu S^\mu = 0$ (no entropy production). In a non-equilibrium but close-to-equilibrium situation, $\nabla_\mu S^\mu \geq 0$ by the second law, but for equilibrium we consider the idealized reversible case.

How do our analogies apply here?

- **Thermodynamic potentials.** One may define several thermodynamic potentials per unit volume. For example, the internal energy density $\varepsilon = \varepsilon(s, n)$, or the Helmholtz free energy density
$$f(T, n) = \varepsilon - Ts, \tag{29}$$
and similarly for other potentials. These quantities are related to one another by Legendre transformations. The Gibbs relation written above is precisely the total differential of $\varepsilon(s, n)$. Performing a Legendre transformation to replace the entropy density s by the temperature T, one obtains
$$df = -s\,dT + \mu\,dn, \tag{30}$$
which is the differential form of the Helmholtz free energy density. All these formulations represent different but equivalent descriptions of the same local equation of state.

- **Lorentz frames.** One may describe the fluid either in its local rest frame or in a frame related to it by a Lorentz boost. In the rest frame,
$$u^\mu = (1, 0, 0, 0), \qquad T^{00} = \varepsilon, \qquad T^{ii} = p \quad (i = 1, 2, 3). \tag{31}$$
In a boosted frame, the components of the energy–momentum tensor mix under Lorentz transformations, and the measured energy density $\varepsilon' = T^{00'}$ generally differs from ε. For

a boost with velocity v along one spatial direction, one finds

$$T^{00'} = \gamma^2(\varepsilon + pv^2), \tag{32}$$

where $\gamma = (1 - v^2)^{-1/2}$.

Nevertheless, certain quantities are invariant or frame-independent in a precise sense. The scalars s and n enter the entropy and particle number currents, $S^\mu = su^\mu$ and $N^\mu = nu^\mu$, which are four-vectors. Consequently, the ratio

$$\frac{s}{n} = \frac{S^\mu}{N^\mu} \tag{33}$$

is a Lorentz scalar and is agreed upon by all observers. Similarly, combinations such as $\varepsilon + p$ have invariant significance in the covariant formulation of relativistic hydrodynamics.

If we now apply the "residue" idea, consider the scalar quantity

$$\mathcal{I} \equiv u_\nu\big(\beta_\mu T^{\mu\nu} - \alpha N^\nu\big), \tag{34}$$

where, as before,

$$\beta_\mu = \frac{u_\mu}{T}, \qquad \alpha = \frac{\mu}{T}. \tag{35}$$

Using the standard decompositions of relativistic hydrodynamics,

$$u_\nu T^{\mu\nu} = \varepsilon\, u^\mu, \qquad u_\nu N^\nu = n\, u_\nu u^\nu = -\, n, \tag{36}$$

we immediately obtain

$$u_\nu \beta_\mu T^{\mu\nu} = \beta_\mu(\varepsilon u^\mu) = \varepsilon\, \beta_\mu u^\mu = \frac{\varepsilon}{T}\, u_\mu u^\mu = -\,\frac{\varepsilon}{T}, \tag{37}$$

$$-\,\alpha\, u_\nu N^\nu = -\,\alpha(-n) = \frac{\mu n}{T}. \tag{38}$$

Therefore,

$$\mathcal{I} = -\,\frac{\varepsilon}{T} + \frac{\mu n}{T} = -\,\frac{\varepsilon - \mu n}{T}. \tag{39}$$

Invoking the thermodynamic identity

$$\varepsilon + p = Ts + \mu n, \tag{40}$$

we rewrite

$$\varepsilon - \mu n = Ts - p, \tag{41}$$

which finally yields

$$\boxed{\mathcal{I} = -\, s + \frac{p}{T}.} \tag{42}$$

Since s, p, and T are scalar fields, $\mathcal{I}$ is a genuine Lorentz scalar. In global equilibrium, $\mathcal{I}$ becomes constant, reflecting its interpretation as a thermodynamic invariant associated with the equilibrium residue structure.

However, in global equilibrium with constant T and μ, p/T is constant as well by the equation of state (because p is function of T, μ). Then $-s + p/T$ being uniform could have an interpretation. Perhaps a more sensible invariant is just the entropy per particle s/n, or something like the relativistic Gibbs-Duhem relation.

The point we want to illustrate is that certain combinations like $\frac{\varepsilon + p - \mu n}{T}$ or $\frac{p}{T}$ or $\frac{\mu}{T}$ are invariants or at least scalars that are the same in any frame if evaluated in equilibrium. In fact,

β_μ itself satisfies $\nabla_\mu \beta_\nu = 0$ in global equilibrium (constant T and no flow acceleration), which implies $T^{-1}\nabla_\mu u_\nu = u_\nu \nabla_\mu (1/T)$ etc., but anyway.

For our analogy: different choices of thermodynamic potential for the fluid correspond to focusing on different pairs of variables (say energy density vs free energy density), while different frames correspond to moving to a different velocity basis. The invariant content could be, for example, the equation of state $p(\varepsilon, n)$ which is the same regardless. The "residue" could be thought of as the entropy production $\nabla_\mu S^\mu$ – in equilibrium, all representations/frame give $\nabla_\mu S^\mu = 0$. If some formulation erroneously produced a non-zero entropy production for what should be an equilibrium state, that would signal a problem.

Thus, for a relativistic ideal fluid: - Changing the thermodynamic potential (say working with $\varepsilon(s, n)$ vs $p(T, \mu)$) doesn't change measurable quantities like sound speed or adiabatic index – those are invariants. - Changing frames doesn't change things like the equation of state or invariants like s/n. We see that our framework holds trivially here: it's consistent and not very illuminating because the ideal fluid is simple. But it provides a check: the "residue" of different potential formulations is the same (they all satisfy the same Maxwell relations and equation of state), and all frames agree on scalar quantities.

8 Discussion and Outlook

In this work, we drew an analogy between two forms of equivalence: 1. Thermodynamic potential equivalence: the fact that using U, H, F, or G (etc.) yields the same physics, with transformations between them being Legendre transforms (contactomorphisms). 2. Special relativistic frame equivalence: the fact that the laws of physics take the same form in any inertial frame, with transformations between frames being Lorentz transformations.

Using this analogy, we then incorporated ideas from gravitational thermodynamics and the RVB residue method to suggest that one can view "choosing a thermodynamic potential" as akin to "choosing a gauge or frame" in a broader theory that possibly includes gravity. In that broader view, the requirement of Lorentz covariance (or diffeomorphism invariance in general relativity) can act as a selector of which thermodynamic representations are physically admissible. For instance, one might imagine a scenario where not all Legendre transforms correspond to something real in a gravitational setting – Lorentz invariance might rule out certain exotic equations of state, etc. Conversely, one could think of using thermodynamic analogy to guess new invariants in gravity (like area or surface gravity combinations).

We used a conceptual tool – the residue – to capture what is invariant in these transformations. We argued that if two formulations differ only by a Legendre or Lorentz transformation, an appropriately-defined residue (such as a contour integral or a cycle integral of a differential form) will yield the same result. If the residue differs, then the difference is physical and not a gauge choice.

One interesting point of discussion is that in thermodynamics the equivalence of potentials is exact and mathematical (given analyticity and convexity conditions, Legendre transforms map state functions one-to-one). In relativity, the equivalence of frames is a symmetry of nature (supported by countless experiments verifying Lorentz invariance). By bridging them, we are essentially saying that the redundancies in description in each theory play analogous roles. The hope is that thinking this way can shed light on situations where one might accidentally introduce a non-covariant term in an attempt to rewrite gravitational dynamics as a thermodynamic law. The "residue test" would catch that.[7, 8, 9, 10]

In addition, we referenced a recent preprint by Chen & Zheng (2025) and Chen's work on the RVB method, which in practice computed black hole temperatures. Those works emphasize a two-layer structure: (i) one layer of representation (choice of coordinate or method – e.g., the RVB method vs traditional method of calculating surface gravity) and (ii) one layer of invariant output (the physical Hawking temperature). If the methods are truly just two representations of the same physics, they must yield the same Hawking temperature. If one finds a discrepancy (which Chen did: an extra term), it implies either a new physical effect or a needed correction in the interpretation. Chen interpreted it as a physical residue term, highlighting something missed by the naive approach.

By analogy, in our context, one could imagine writing the "first law of horizon thermodynamics" in different ways – if one way yields an extra term, that term must be an invariant that was missed before (or it signals that the two ways aren't truly equivalent and one is wrong). We can thus use our framework as a sort of diagnostic tool.

Outlook: Several immediate next steps can be proposed: - Concrete Residue Construction: We should define explicitly the residue operation $\mathcal{R}$ in a simpler setting. For instance, in pure thermodynamics, let $\mathcal{R}$ be the loop integral of η (which should be zero if η is exact). This can test if a set of reported partial derivative data is consistent with some $U(S, V, N)$. In relativity, define $\mathcal{R}$ as an integral of $(\delta Q - T\delta S)$ over a closed null loop on a horizon (which should be zero if Einstein's equations hold locally). Making this rigorous would give a calculable measure of "equivalence vs non-equivalence." - Apply to Gravity Models: We can apply the framework to specific gravitational setups. For example, take the field equations of $f(R)$ gravity or $f(Q)$ gravity (where Chen's method has been applied) and see if we can derive multiple "first law" forms. Then check if all those forms yield the same residues. If not, perhaps some forms are not physically meaningful or some extra terms appear that correspond to new physics (like the black hole residue term). - Temperature Transformations: In relativity, how temperature transforms between frames is a notorious question (the "Tolman law" in gravitational context says $T\sqrt{-g_{00}} =$ constant in static gravitational fields; in special relativity, moving observers see a different temperature for a moving object depending on definition). Using our approach, we can ask: if one insists on a particular transformation law for temperature (say naive T Lorentz-invariant vs. $T\gamma$ vs. something), does it maintain the invariance of $\mathcal{R}$? This could potentially resolve ambiguities by demanding overall consistency of the analogies.

In conclusion, by viewing thermodynamic potentials and inertial frames under one unified lens, we gain a conceptual toolset to examine physical theories for consistency. If a proposed equivalence or duality (like gravity ↔ thermodynamics) is valid, all "representations" of it should yield the same invariant content. The RVB–residue viewpoint gives a mechanism to extract and compare that content across representations.

We hope this framework will stimulate further work in identifying which aspects of thermodynamics and general covariance are truly two sides of the same coin, especially in the context of emergent gravity and other interdisciplinary approaches to fundamental physics.

Data Availability

No new data were generated or analyzed in this study.

Conflicts of Interest

The author declares no conflict of interest.

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Thermodynamic Potential Framework

Legendre transforms between U, F, H, G (change of potential)

Choice of natural variables (e.g. (S, V, N) vs (T, V, N)) for a given potential

Conjugate variable pairs $(S \leftrightarrow T)$, $(V \leftrightarrow -p)$, $(N \leftrightarrow \mu)$