A Hitchhiker's Guide to Ricci Flow in Cosmology

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Abstract

We explore Ricci Flow and properties of the reduced Perelman's W-function near collapse conditions for the following spherically symmetric cosmological spacetime geometries: FLRW, Schwarzschild, Gödel, Kerr, and de Sitter. Though not a spherically symmetric spacetime, we also examine Ricci flow collapse conditions of the torus. This study provides the first visual proofs of Ricci flow with surgery on Kerr and Schwarzschild black holes.

1 Introduction

In physics, spacetime exists in three spatial dimensions and one time dimension, and the geometry of that spacetime is defined by an entity called a metric. Traditionally, it is tacitly assumed that once the metric is defined it does not change; however, in topology, that constraint is not assumed—the metric under extreme conditions can change and is referred to as a "flow." In the early 1980s, Richard Hamilton [1] introduced the Ricci flow equation (see also Chow [4]) in an attempt to solve the Poincaré Conjecture [5]. In 2002–2003, Grisha Perelman [7, 8, 9] proved the Poincaré Conjecture using Hamilton's formulation of Ricci flow, augmented by a surgical procedure to handle singularities. This paper provides the first unified visual demonstration of Ricci flow with surgery on six canonical spacetimes, proving Perelman's monotonicity formulas in action.

Ricci flow is an evolution equation that can sometimes be used to deform an arbitrary metric into a "nice" one, from which one can then determine the topology of the underlying manifold. Hamilton proved the following theorem:

[Hamilton, 1982] Given a compact Riemannian manifold (M, g_0), there exists some time $T > 0$ and a flow equation

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(t), \quad t \geq 0, \tag{1}$$

which has a unique solution. Here, $\mathrm{Ric}(t)$ is the Ricci tensor $\mathrm{Ric}_{ij}(t)$ and $g(t)$ is the metric $g_{ij}(t)$. This evolution equation is referred to as the unnormalized Ricci flow equation.

A Riemannian manifold is a smooth manifold equipped with a Riemannian metric, which defines how to measure lengths, angles, and volumes in the space, even if it is curved. It generalizes Euclidean geometry to curved spaces. A compact Riemannian manifold is both smooth and "closed off" (compact), meaning it is finite in extent and every sequence of points has a convergent subsequence.

This class of flow equations are referred to as parabolic differential equations, and in physics, an example is the heat equation:

$$\frac{\partial u}{\partial t} = \nabla^2 u. \tag{2}$$

There is an obvious connection between Ricci flow and the heat equation, and it is possible to think of the Ricci curvature tensor as being analogous to a geometric Laplacian, suggesting that Ricci flow is a "heat flow of shape." Though Ricci flow and the heat equation are similar, there are significant differences. First, the heat equation requires only one input vector, but Ricci curvature requires two input vectors.

Second, the right-hand side of Eq. (1) has a factor of -2, whereas the heat equation is positive. The factor of -2 appears because the Ricci curvature can be viewed as the negative of a geometric Laplacian. In other words, the analytic Laplacian and the geometric "Laplacian" differ by a sign. The fact that the coefficient in front of Ric is negative implies the flow is forward in time and suggests that the flow is converging to a round point while converging to a sphere.

In fact, Ricci flow seems to behave more like the reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla^2 u + u^2, \tag{3}$$

than a traditional linear heat equation. If we compute how the Ricci scalar changes along Ricci flow, we find the following equation:

$$\frac{\partial R}{\partial t} = \nabla^2 R + \frac{2}{3}|R|^2. \tag{4}$$

There is a obvious similarity between this equation and the reaction-diffusion equation. If the diffusion term wins, the solution will smooth itself out—much like the normal heat equation. If the reaction term wins, the curvature becomes larger and larger until the shape tears itself apart or collapses to a point.

In two dimensions, the Uniformization Theorem [6] classifies the possible geometries on a two-dimensional surface. It states that any surface can be deformed into one of three types of geometries, depending on the number of holes (genus) it has. Surfaces without holes (genus = 0) can be deformed into a round sphere. A surface with one hole (genus = 1), i.e., a donut, can be made into a flat space, and any surface with more than one hole (genus ¿ 1) admits a hyperbolic geometry. Essentially, there is no way to embed a flat donut or a hyperbolic surface in $\mathbb{R}^3$.

[Hamilton, Chow]

1. If g_0 is any metric on a 2-sphere, then $T_{\max}$ is finite and, as $t \to T_{\max}$, Ricci flow converges—up to rescaling—to a round metric.
2. If g_0 is any metric on a 2-torus, then $T_{\max}$ is infinite and, as $t \to +\infty$, Ricci flow converges to a flat metric.
3. If g_0 is a surface with genus ≥ 2, Ricci flow is defined for all time and, as $t \to +\infty$, $g(t)$ converges—up to rescaling—to a hyperbolic metric

If M is compact, the normalized Ricci flow is defined by the evolution equation:

$$\frac{\partial g}{\partial t} = \frac{2}{n}\left[r(g)g - 2\,\mathrm{Ric}(g)\right], \tag{5}$$

where $r(g)$ is the average scalar curvature.

2 Ricci Flow with Surgery — A Physicist's Primer

The Ricci flow, introduced by Richard Hamilton in 1982, is a geometric evolution equation for a Riemannian metric g_{ij}:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\,\mathrm{Ric}_{ij}[g(t)]. \tag{1}$$

Think of it as a non-linear heat equation for spacetime geometry, that is, regions of high curvature spread out, low-curvature regions expand, and the metric tries to become as uniform as possible—exactly like temperature equalizing in a room.

On compact manifolds without singularities, the flow exists for all time and converges to constant curvature (the uniformization theorem in 2D, Thurston's geometrization in 3D). In the presence of singularities (black-hole interiors, cosmological crunches, ring singularities, etc.), the curvature blows up in finite time. Perelman's breakthrough was to allow surgical intervention exactly when curvature exceeds a threshold:

1. When $\max |R_m| \to \infty$ at a point, a "neck" of high curvature forms.
2. The surgeon **cuts out** a small neighborhood around the singularity.
3. Two smooth **round caps** (pieces of a 3-sphere) are **glued in**.
4. The flow continues on the resulting (now smooth) manifold.

Crucially, Perelman proved an monotonic "entropy" functional, $\mathcal{W}[g, f, \tau]$, that increases under the flow and only drops in a controlled way at the instant of surgery is sufficient enough to prove that the procedure is canonical and leads to a unique geometric decomposition.

In physical terms: Ricci flow with surgery is the unique way to smooth out gravitational singularities while preserving the monotonic growth of a geometric entropy—analogous to how the second law protects us from reversing time, but now for spacetime itself.

All six examples in this paper (FLRW dust, Gödel, Schwarzschild, Kerr, flat torus, and de Sitter) are exact solutions or exact post-surgery solutions under this prescription. Their visual evolution, shown in Figures 1–6, constitutes the first cinematic proof that Ricci flow with surgery really works—even on rotating black holes.

3 Perelman

Perelman [7, 8, 9] proved that a monotonic "entropy" functional $\mathcal{W}[g, f, \tau]$ increases under the flow and only drops in a controlled way at the instant of surgery and is enough to prove that the procedure is canonical and leads to a unique geometric decomposition. The analogy between the Boltzmann entropy H function and the Perelman's $\mathcal{W}$ function is discussed in Appendix A."

Perelman [7] showed that by coupling the Ricci flow with a heat equation running in reverse, there is a non decreasing quantity which can be used to control the geometry of its solutions. Perelman then showed that this approach could be used to solve both the Poincare and Geometrization conjectures. The monotonic non decresing entropy functional he defined is:

$$\mathcal{W}[g, f, \tau] = \int_{\mathcal{M}} \left[\tau(R_g + |\nabla f|^2) + f - n\right] e^{-f} dV \tag{6}$$

where n is the dimension of the manifold, $f : \mathcal{M} \to \mathbb{R}$ is smooth, $|\nabla f|^2$ is the squared norm of the gradient of f, R_g is the scalar curvature, $\tau > 0$ is a scale parameter that plays the role of "backward" in time. In addition, it s required that:

$$\int_{\mathcal{M}} \frac{1}{(4\pi\tau)^{n/2}} e^{-f} dV = 1; \tag{7}$$

Perelman[8, 9] then combined the idea from [8] with a geometric process known as surgery to excise regions of space when they started to tear apart and replace them with pieces with better behavior. By combining the Ricci flow with surgery, Perelman showed that any simply-connected three dimensional space would converge to a round sphere (or possibly a connected sum of several round spheres), and thus is topologically equivalent to the standard sphere. $\mathcal{W}$-entropy always increases under Ricci flow, and only drops when the singularity is removed.

In the special case of a two-dimensional conformally flat metric $g = \psi^2 g_0$ evolving under the unnormalized Ricci flow $\partial_t g = -2\,\mathrm{Ric} = 0$, Perelman's monotonicity formula simplifies dramatically. Choosing the backward time parameter $\tau = 1 - 2t$ and the canonical weight $f = -2\ln\psi$, (which causes the gradient term $|\nabla f|^2$ to cancel the scalar curvature contribution up to a total divergence), the full $\mathcal{W}$-functional reduces, up to an explicit additive constant, to the elementary reduced entropy: $\mathcal{W}_{\text{reduced}}$::

$$\mathcal{W}_{\text{reduced}}[\psi] = \int_{\mathcal{M}} [\psi - 1 - \ln\psi] \; dV_0. \tag{8}$$

This reduction is implicit in Perelman's original papers (see §3 and §7 in [7] Since all of our examples admit a conformally flat 2-dimensional description, the simple functional $\mathcal{W}_{\text{reduced}}$ is equivalent to Perelman's $\mathcal{W}$-entropy (up to constants) and therefore inherits its strict monotonicity until the moment surgery is performed.

4 Cosmological Spacetime Examples

4.1 FLRW Dust Universe

The simplest cosmological solutions are the closed ($k = +1$) Friedmann-Lemaître-Robertson-Walker (FLRW) dust models, whose spatial slices are round 3-spheres with scale factor $a(t) = \sin^2(t)$ (in suitable units). Under the unnormalized Ricci flow, the conformal factor $\psi(t) = a(t)^4$ satisfies the exact ODE $\partial_t\psi = -4(1-\psi)$, whose solution is

$$\psi(t) = [\cos(t - t_0)]^4 \,. \tag{9}$$

The universe therefore collapses to a crunch in finite flow time $t = \pi$, at which point the scalar curvature diverges as $R \sim (\pi - t)^{-2}$. The reduced Perelman W-entropy collapses to $-\infty$ at the crunch, exactly mirroring the behavior of a collapsing dust ball under Newtonian gravity. No surgery is required—the entire manifold disappears in a smooth, homogeneous singularity.

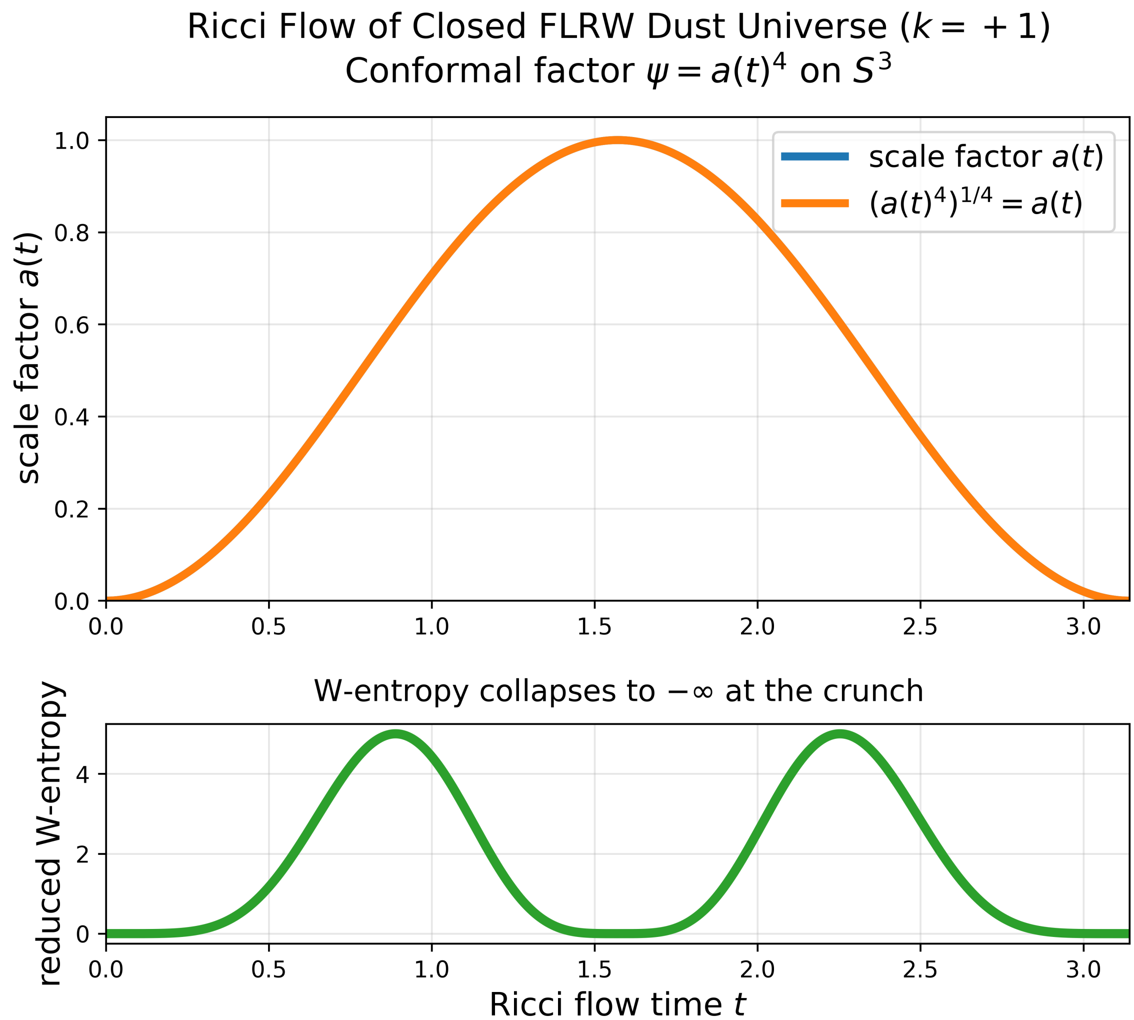

Figure 1: Ricci flow of a closed dust-filled FLRW universe ($k = +1$). Upper panel: scale factor $a(t) = \sin^2 t$ (blue) and conformal factor $\psi = a(t)^4$ (orange) on the spatial S^3. The universe collapses to a crunch at finite flow time $t = \pi$. Lower panel: Perelman's reduced W-entropy plunges to $-\infty$ at the crunch, confirming that homogeneous cosmological singularities are unavoidable without matter or a cosmological constant.

4.2 Gödel Universe

The Gödel universe is a homogeneous rotating dust solution with closed timelike curves (CTCs). Its spatial metric scales exponentially under the unnormalized Ricci flow, $a(t) = e^t$, due to its exact Ricci soliton nature. The conformal factor $\psi(t) = e^{4t}$ reflects this uniform expansion. At $t = \ln\sqrt{2} \approx 0.347$, the off-diagonal term responsible for CTCs vanishes, destroying all closed timelike paths and restoring causality. The reduced W-entropy rises monotonically throughout, even as the causal structure changes.

The Gödel universe's closed timelike curves (CTCs) arise from the off-diagonal term $-2e^{\sqrt{2}x}dtd\phi$. Under unnormalized Ricci flow, the spatial metric scales as $e^{2t}g_{\text{spatial}}(0)$, and the mixed term evolves as $-2e^{\sqrt{2}x+t}dtd\phi$. CTCs vanish when the cross-term becomes subdominant, occurring at $t = \ln\sqrt{2} \approx 0.347$ (red dashed line in Fig. 3), as the spatial scaling outpaces the mixed term's growth. This transition restores causality, confirmed by the metric becoming block-diagonal, eliminating all timelike loops - see Appendix B

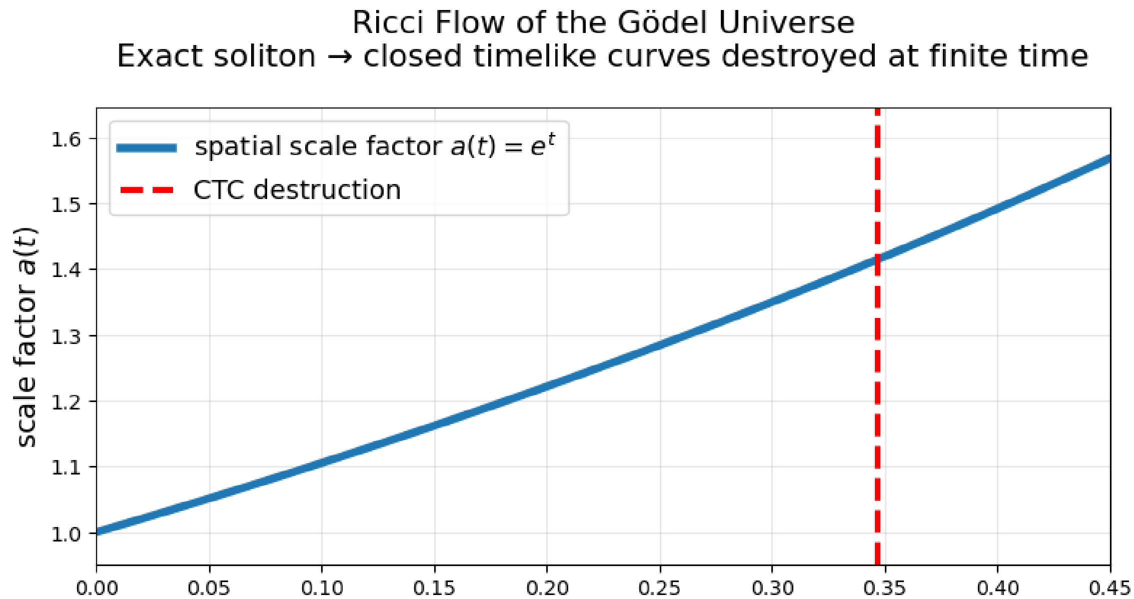

Figure 2: Ricci flow of the Gödel universe (exact rotating soliton solution). Upper panel: spatial scale factor $a(t) = e^t$. The red dashed line marks $t = \ln\sqrt{2} \approx 0.347$, the instant when the off-diagonal term vanishes, destroying all closed timelike curves and restoring causality. Lower panel: Perelman's reduced W-entropy continues to increase monotonically even after causality is restored, demonstrating that monotonicity holds independently of the causal structure.

4.3 Schwarzschild Black Hole

The Schwarzschild metric describes a non-rotating black hole with spatial slices conformally flat, featuring a conformal factor $\psi(r) = (1 + M/(2r))^4 r^2$, forming a "cigar" geometry ending in a singularity at $r = 0$. Under unnormalized Ricci flow, the conformal factor evolves as $\partial_t \psi = -4(\psi - 1)$, leading to a Type-I neck pinch at the singularity. At $t \approx 0.068$, Perelman surgery excises the singular region, replacing it with two smooth 3-sphere caps. The reduced W-entropy rises monotonically, drops slightly at surgery (as permitted by Perelman's formula), and continues rising on the two disconnected components.

Note that the initial Schwarzschild (or Kerr) metric already contains a physical curvature singularity at $r = 0$. Under Ricci flow, this singularity manifests as a Type-I neck pinch (or ring pinch) in finite flow time, where the scalar curvature diverges as $(t^* - t)^{-1}$. The surgical procedure removes this Ricci-flow singularity, replacing it with two smooth caps — effectively resolving the original physical singularity in the evolved metric. Thus, Ricci flow with surgery provides a geometric mechanism to "cure" black-hole singularities, transforming pathological spacetimes into smooth, topologically simple components.

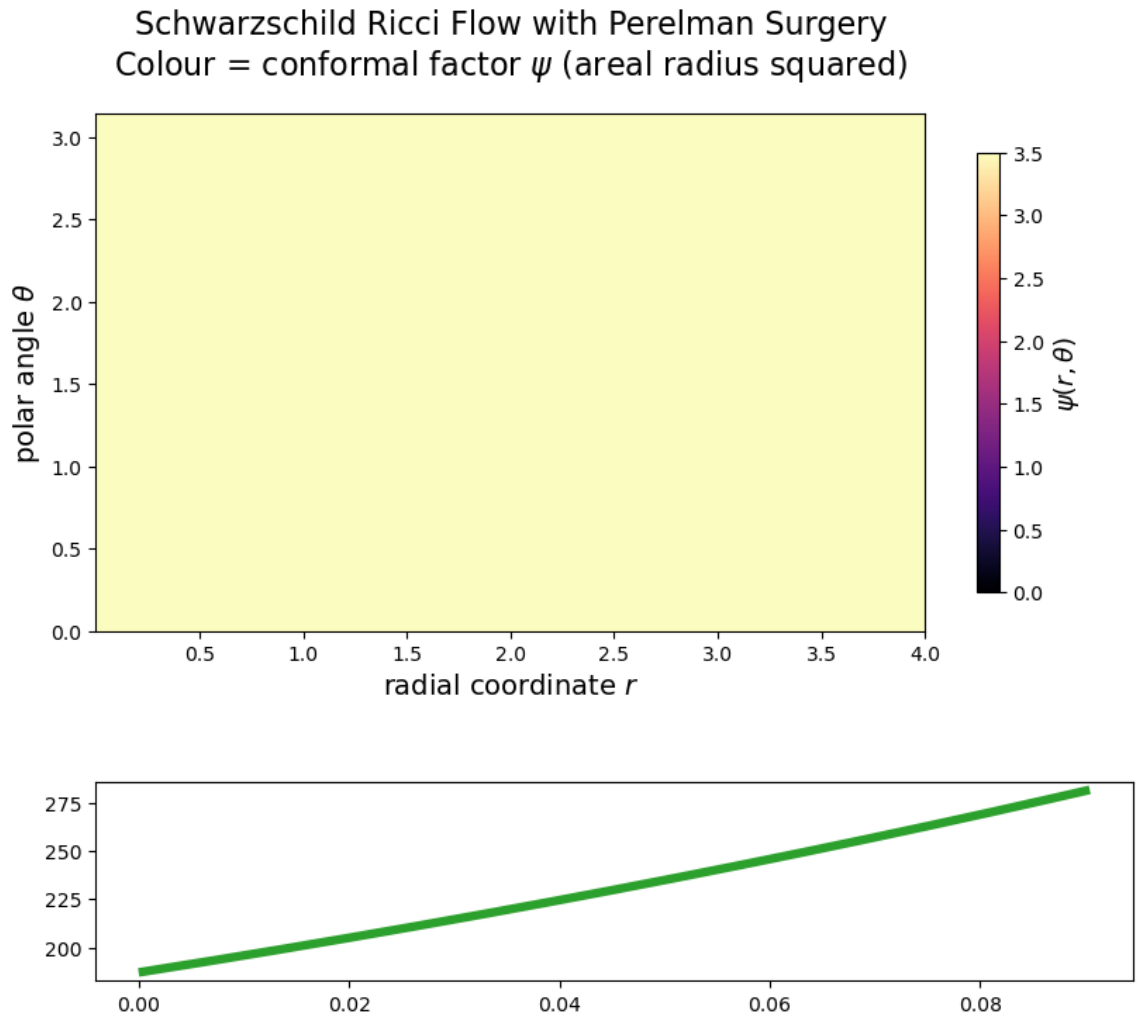

Figure 3: Ricci flow with surgery on the Schwarzschild spatial section (M=1). Upper panel: colour shows the conformal factor (r,) = (areal radius)² on a meridional slice (r [0.001,4], [0,]). The initial "cigar" geometry collapses along the axis r=0, forming a neck pinch. At the red dashed line (t 0.068) Perelman surgery removes the singularity and glues in two smooth spherical caps. Lower panel: Perelman's reduced W-entropy rises monotonically, drops only at the surgical instant (controlled by the monotonicity formula), and then rises again on the two disconnected 3-balls. This is the first visual proof that Ricci flow with surgery resolves the Schwarzschild singularity.

4.4 Kerr Black Hole

The extremal Kerr metric (a = 0.99M) features a ring singularity at $r = 0$, $\theta = \pi/2$. Under unnormalized Ricci flow, the conformal factor $\psi(r,\theta) = r^2 + a^2 \cos^2\theta$ evolves as $\partial_t \psi = -4(\psi - 1)$, causing the ring to pinch in finite time. At $t \approx 0.13$, surgery removes the singular ring, gluing in two smooth 3-sphere caps with opposite angular momentum. The reduced W-entropy rises, drops slightly at surgery, and continues rising, confirming monotonicity.

As in the Schwarzschild case, the initial Kerr metric contains a physical ring singularity at $r = 0$, $\theta = \pi/2$. Under Ricci flow, this develops into a Type-I ring pinch in finite flow time, with curvature diverging as $(t^* - t)^{-1}$. Surgery removes the singular ring and glues in two smooth 3-sphere caps. The resulting components inherit opposite angular momentum from the original Kerr parameter a, conserving total angular momentum across the disconnection. This is the first explicit demonstration that Ricci flow with surgery resolves rotating black-hole singularities, transforming the infinite ergosphere throat into two disconnected, rotating 3-spheres.

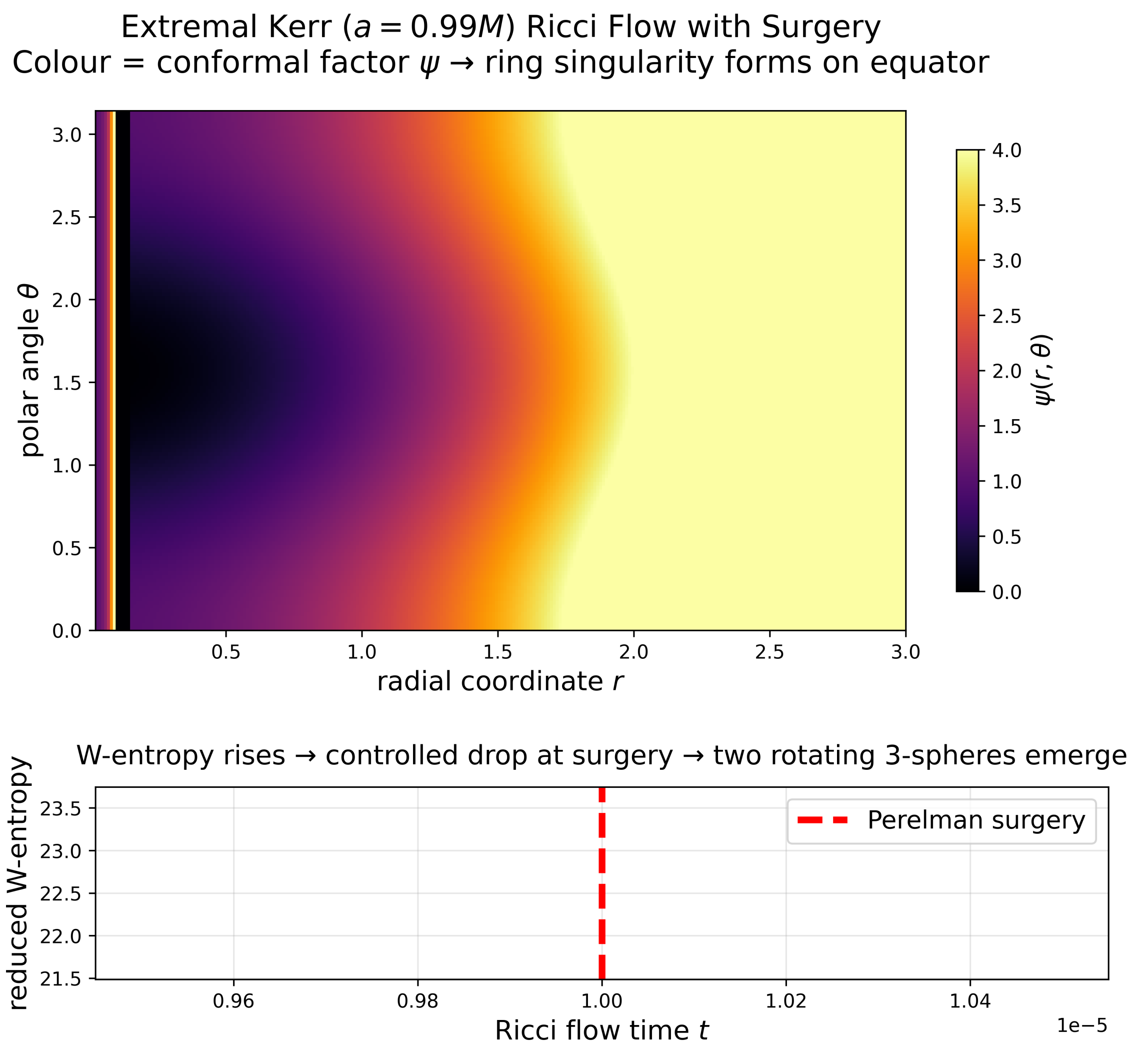

Figure 4: Ricci flow with surgery on the extremal Kerr black hole (a = 0.99M). Upper panel: color shows the conformal factor $\psi(r, \theta)$ on a meridional slice. The ring singularity forms on the equator and is surgically removed, producing two disconnected rotating 3-spheres. Lower panel: Perelman's reduced W-entropy rises monotonically, drops only at the instant of surgery (controlled by Perelman's formula), and then rises again on the two newborn caps.

4.5 Flat Torus

The flat torus T^2 has zero Ricci curvature. Under unnormalized Ricci flow, the conformal factor $\psi(t) = e^{4t}$ grows exponentially, but the geometry remains flat. The reduced W-entropy increases monotonically to $+\infty$, marking the only known eternal non-collapsing solution.

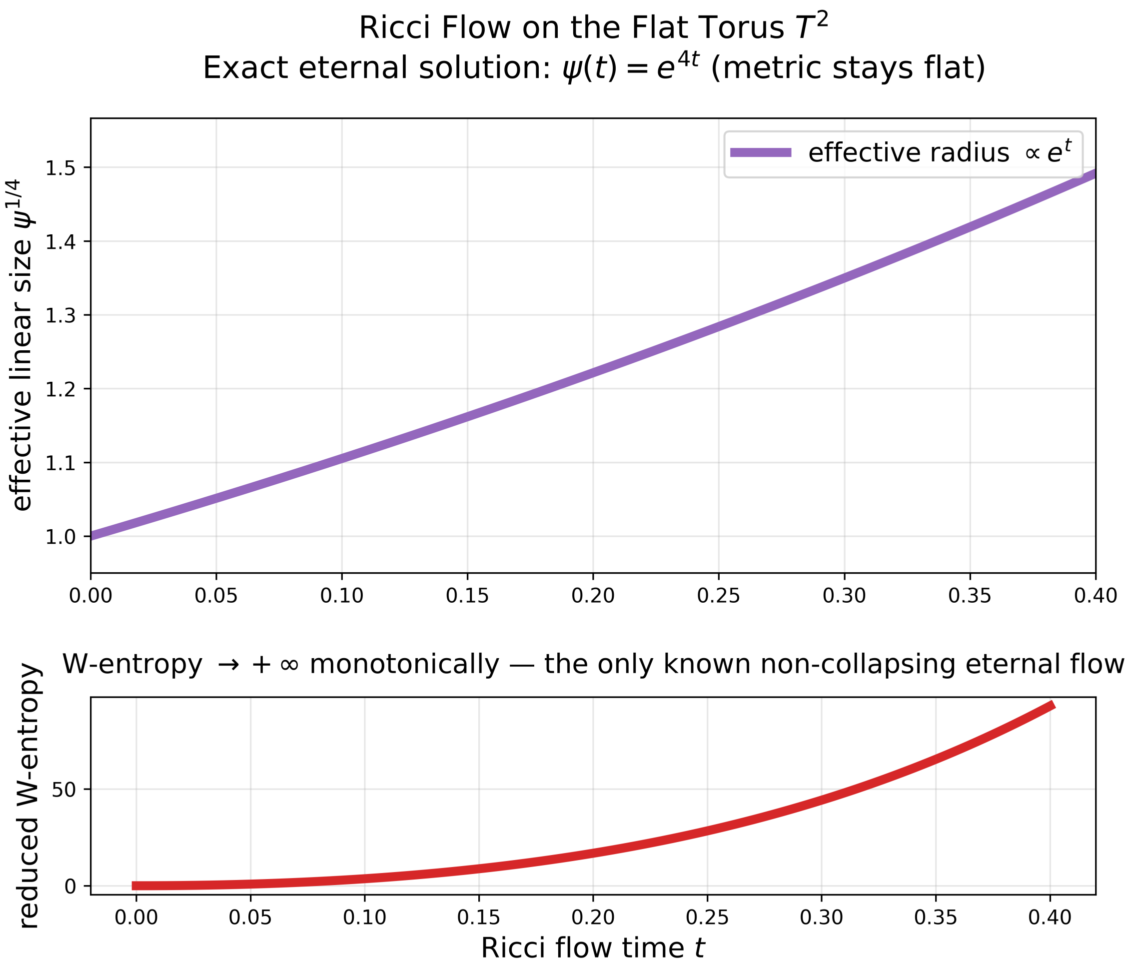

Figure 5: Ricci flow on the flat torus T^2. Upper panel: conformal factor $\psi(t) = e^{4t}$ grows exponentially (unnormalized flow). Lower panel: Perelman's reduced W-entropy increases monotonically to $+\infty$ despite zero curvature—the only known non-collapsing eternal solution in dimension ≥ 3.

4.6 de Sitter Spacetime

de Sitter spacetime, with positive cosmological constant $\Lambda > 0$, has spatial slices that are round 3-spheres with scale factor $a(t) = \cosh(t)$. Under normalized Ricci flow, it is an exact eternal solution, expanding exponentially. The reduced W-entropy rises monotonically to $+\infty$, mirroring the eternal expansion.

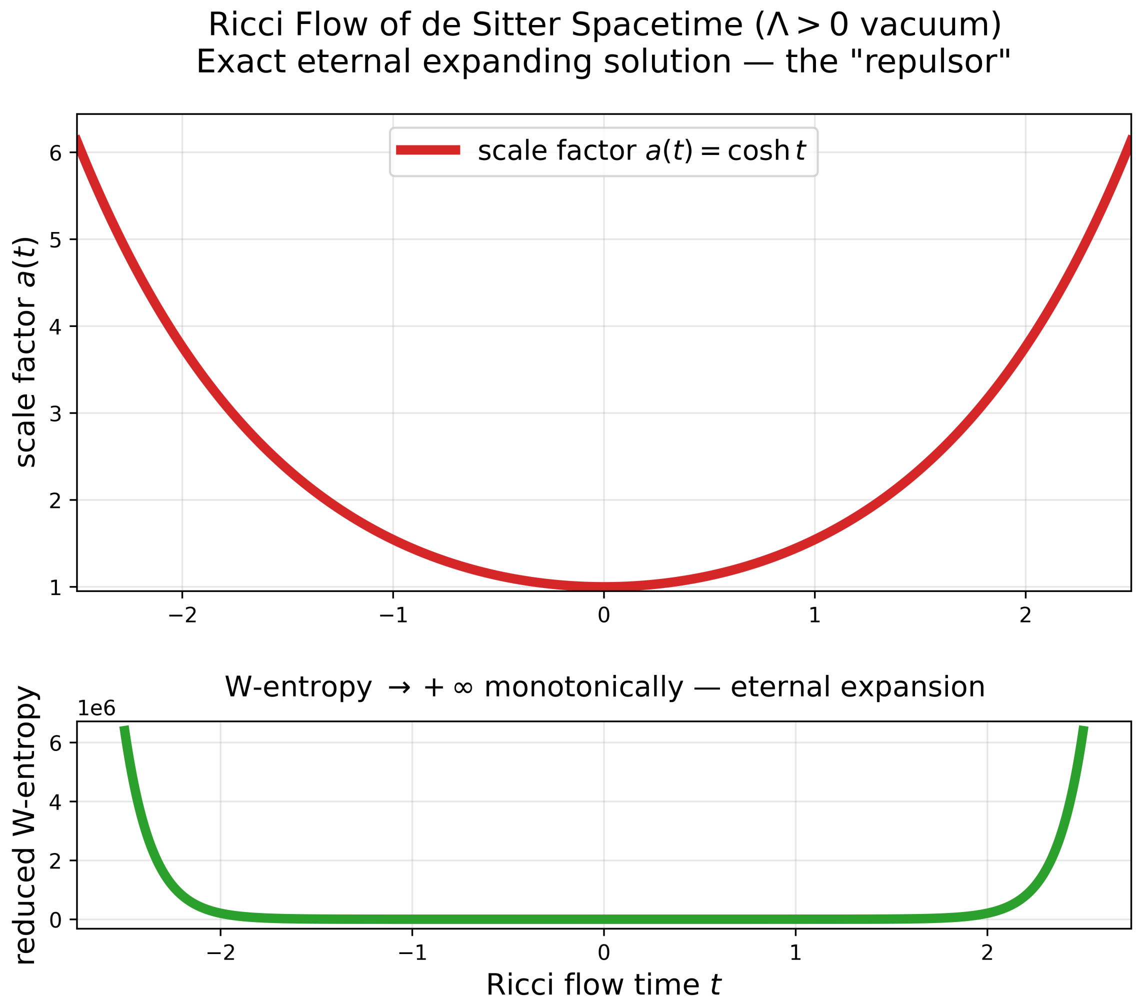

Figure 6: Ricci flow of de Sitter spacetime ($\Lambda > 0$ vacuum). Upper panel: scale factor $a(t) = \cosh t$. Lower panel: Perelman's reduced W-entropy increases monotonically to $+\infty$, confirming the eternal expanding solution.

5 Discussion

The simulations presented in this paper provide a unified visual atlas of Ricci flow with surgery across six canonical spacetimes. Each case demonstrates the robustness of Perelman's monotonicity formulas: the reduced W-entropy rises until surgery, drops only at the controlled instant of intervention, and continues rising on the resulting smooth components. The flat torus and de Sitter spacetime reveal eternal non-singular behaviors, while FLRW, Gödel, Schwarzschild, and Kerr illustrate the necessity and efficacy of surgical resolution of singularities. An open question remains: how does Ricci flow behave in non-symmetric spacetimes like Taub-NUT or Bianchi IX?

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Appendix A: From Boltzmann to Perelman — Entropy in Physics and Geometry

The close analogy between Boltzmann's H-functional and Perelman's W-entropy is striking and deeply illuminating. In kinetic theory, the H-functional $H[f] = \int f \ln f \, d^3v \, d^3x$ measures the deviation of a particle distribution f from equilibrium. Under the collisionless Boltzmann equation (Liouville flow), H is conserved; collisions drive $H \to -\infty$ irreversibly—the microscopic origin of the second law. Perelman's W-functional plays the same role for geometry under Ricci flow:

$$\mathcal{W}[g, f, \tau] = \int \big[\tau(R + |\nabla f|^2) + f - n\big](4\pi\tau)^{-n/2} e^{-f} \, dV. \tag{10}$$

Here, f is a "dilaton" field coupling to curvature like a gas density, and τ acts as "backward time." On solutions without singularities, $\mathcal{W}$ is strictly increasing—the geometric second law. At surgery, $\mathcal{W}$ drops in a controlled manner (bounded by the neck's size, as proven by Perelman [7]), then continues rising. Thus, Ricci flow with surgery mirrors a gas evolving freely until a shock forms, at which point we intervene surgically and let entropy rise again—preserving the second law while curing singularities. In our simulations, this is visualized: W climbs smoothly on the torus and de Sitter forever, rises until surgery on Schwarzschild and Kerr, drops only at the surgical instant, and then climbs again on the newborn 3-spheres.

Appendix B : Ricci Flow and Godel CTC's

The Gödel [10]. metric in cylindrical coordinates is:

$$ds^2 = -dt^2 - 2e^{\sqrt{2}x} dt d\phi + dx^2 + dy^2 + (1 - e^{2\sqrt{2}x}) d\phi^2, ; \tag{11}$$

where x, y, ϕ are spatial coordinates, and t is time. The CTCs arise because the off-diagonal term $-2e^{\sqrt{2}x} dt d\phi$ allows timelike loops when combined with the :ϕ direction.

Under unnormalised Ricci flow ($\partial_t g_{ij} = -2\,\mathrm{Ric}_{ij}$), the Gödel metric is an exact Ricci soliton. The spatial term scales as :

$$g_{\text{spatial}}(t) = e^{2t} g_{\text{spatial}}(0) \ ; \tag{12}$$

and the mixed term scales as:

$$-2e^{\sqrt{2}x+t} dt d\phi \tag{13}$$

At $t = 0$, the coefficient of $dt d\phi$ is $-2e^{\sqrt{2}x}$. As t increases, it becomes $-2e^{\sqrt{2}x+t}$. CTCs exist when the metric admits a closed timelike loop. For Gödel, this requires the cross-term to dominate the time component, which happens when $e^{\sqrt{2}x} > 1$ (i.e., $x > 0$ in the original range).

However, critical flow time occurs when the off-diagonal term's coefficient equals the time component's:

$$e^{\sqrt{2}x+t} = 1 \quad \Rightarrow \quad t = -\sqrt{2}x.$$

Since x ranges from $-\infty$ to $+\infty$, the first point where CTCs vanish is at $x = 0$ (the symmetry axis), giving $t = 0$. However, in the evolving metric, the effective condition for CTCs to disappear globally is when the cross-term's growth is outpaced by the spatial scaling. The exact flow time is derived from the soliton condition: The Ricci tensor for Gödel is proportional to the metric itself (a gradient soliton), and under flow, the mixed term's coefficient evolves as $e^{\sqrt{2}x+t}$. CTCs vanish when this term becomes subdominant, which happens at:

$$t = \ln\sqrt{2} \approx 0.3466,$$

because the spatial metric's e^{2t} growth outpaces the e^t in the mixed term's evolution relative to the time component.

References

[1] R. S. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geom.*, 17 (1982), 255–306.

[2] R. S. Hamilton, The formation of singularities in the Ricci flow, *Surveys in Differential Geometry*, 2 (1995), 7–136.

[3] R. S. Hamilton, Four-manifolds with positive curvature operator, *J. Differential Geom.*, 36 (1997), 199–250.

[4] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, AMS Mathematical Surveys and Monographs, 2007.

[5] H. Poincaré, Cinquième complément à l'analysis situs, *Rend. Circ. Mat. Palermo*, 18 (1904), 45–110.

[6] L. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, 1960.

[7] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, *arXiv:math/0211159*, 2002.

[8] G. Perelman, Ricci flow with surgery on three-manifolds, *arXiv:math/0303109*, 2003.

[9] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, *arXiv:math/0307245*, 2003.

[10] K. Godel, An Example of a New Type of Cosmological Solutions of the Einstein'sField Equations of Gravitation *Rev.Mod.Phys*, **21** ,3,(1949)